Reduced Genus-One
Gromov-Witten Invariants
and
Applications

- **Description** and **Dfn** of GW-Invariants
- **“Defects”** of Pos.-Genus Invariants
- **“Fix”** for Genus-One Invariants

`http://math.sunysb.edu/∼azinger`

GW-Invariants

(X,ω,J) (almost) Kahler cmpt mnfd

$$\rightsquigarrow \quad \boxed{\mathrm{GW}^{A}_{g,k}(\mu)\in\mathbb{Q}}$$

$$g,k\geq 0,\quad A\in H_2(X),\quad \mu=(\mu_1,\ldots,\mu_k)\in H^*(X)^{\oplus k}$$

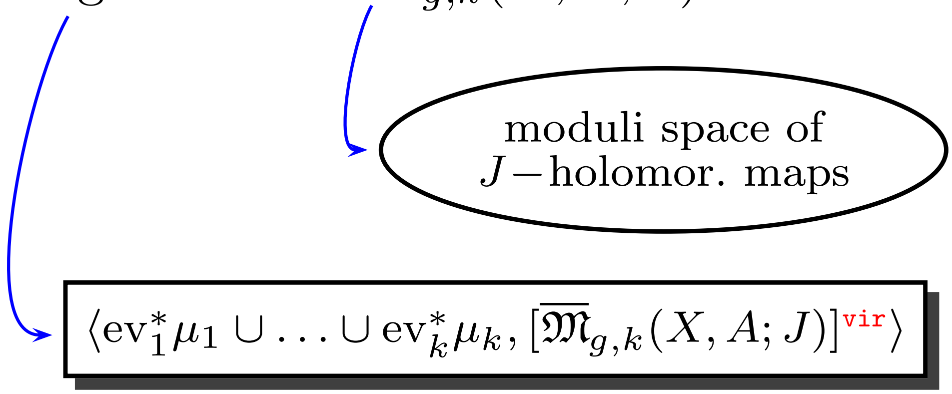
$\underset{\text{roughly}}{\overset{\text{very}}{\sim}}$ counts of J-holomor. `curves in` X

$$C\subset X,\quad \dim_{\mathbb{R}}C=2,\quad J\,TC=TC\subset TX$$

$\underset{\text{roughly}}{\sim}$ counts of J-holomor. `maps into` X

$$u\colon \underbrace{(\Sigma, j)}_{\substack{\text{Riemann surf.}\\ \text{of genus } g}} \longrightarrow (X, J), \qquad \underbrace{\bar{\partial}_J u \equiv du + J\circ du\circ j \;=\; 0}_{\text{Cauchy-Riemann eqn.}}$$

$\underset{\text{really}}{\sim}$ integrals over $\overline{\mathfrak{M}}_{g,k}(X, A; J)$



moduli space of
$J-$holomor. maps

$$\boxed{\langle \mathrm{ev}_1^*\mu_1 \cup \ldots \cup \mathrm{ev}_k^*\mu_k, [\overline{\mathfrak{M}}_{g,k}(X, A; J)]^{\texttt{vir}}\rangle}$$

$$\mathrm{ev}_i : \overline{\mathfrak{M}}_{g,k}(X, A; J) \longrightarrow X, \quad (\Sigma, x_1, \ldots, x_k; u) \longrightarrow u(x_i)$$

Standard Fact from Topology

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r] & V \arrow[d] \\
& M^m \arrow[u, bend right, "\varphi"']
\end{tikzcd}$$

If φ is generic,

$$\boxed{\left[\varphi^{-1}(0)\right] = \mathrm{PD}_X e(V) \in H_{m-2n}(M)}$$

Dfn of $\mathrm{GW}^A_{g,k}$ **or** $\big[\overline{\mathfrak{M}}_{g,k}(X,A;J)\big]^{\mathbf{vir}}$

$$\begin{tikzcd} \Gamma_J^{0,1} \arrow[d] \\ \mathfrak{X}_{g,k}(X,A) \arrow[u, bend right, "\bar\partial_J"'] \end{tikzcd}$$

$$\Gamma_J^{0,1}(\Sigma,u)=\{\eta\colon T\Sigma\longrightarrow u^*TX\mid J\eta=-\eta j\}$$

$$\mathfrak{X}_{g,k}(X,A)=\{\text{smooth maps } u\colon \Sigma\longrightarrow X\}$$

may have simple nodes

$$\mathrm{GW}^A_{g,k}=\text{(PD of) “Euler class” of } \Gamma_J^{0,1},\ \text{w.r.t. } \bar\partial_J$$

$$\boxed{\mathrm{GW}^A_{g,k}\equiv\big[\{\bar\partial_J+\nu\}^{-1}(0)\big]\in H_{\mathrm{ind}\,\bar\partial_J}\big(\underbrace{\mathfrak{X}_{g,k}(X,A)};\mathbb{Q}\big)}$$

small generic perturbation

can replace by small neighb. of $\bar\partial_J^{-1}(0)=\overline{\mathfrak{M}}_{g,k}(X,A;J)$

Good Things about Genus-ZERO GWs

- **Hyperplane** Property

- **Enumerative** Property:

(X, J) regular (e.g. $\mathbb{P}^n$, Fano) $\Longrightarrow$

$$\boxed{\text{genus-0 GWs} = \text{genus-0 enumerative invariants}}$$

$\Longrightarrow$ can count rational curves:

Kontsevich-Manin, Ruan-Tian, ...

Hyperplane Property

$X=\mathbb{P}^n$, $\quad Y^{n-1}$ = hypersurface of degree $a\in\mathbb{Z}^+$

$$\begin{tikzcd} \mathcal{L}=\gamma^{*\otimes a} \arrow[d] \\ \mathbb{P}^n \arrow[u, bend right, "s\ \substack{\text{holom.}\\ \text{section}}"'] \end{tikzcd}$$

$\gamma\longrightarrow\mathbb{P}^n$ taut. line bundle

$$\boxed{Y=s^{-1}(0)}$$

$$\overline{\mathfrak{M}}_{g,k}(Y,d)=\left\{(\Sigma,u)\in\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d):\ \underbrace{u(\Sigma)\subset Y}_{s\circ u=0\in H^0(\Sigma;u^*\mathcal{L})}\right\}$$

$$\begin{tikzcd} \mathcal{V}_{g,k} \arrow[d] \\ \overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d) \arrow[u, bend right, "\tilde{s}"'] \end{tikzcd}$$

$$\mathcal{V}_{g,k}(\Sigma,u)=H^0(\Sigma;u^*\mathcal{L})$$

$$\tilde{s}(\Sigma,u)=s\circ u$$

$$\boxed{\overline{\mathfrak{M}}_{g,k}(Y,d)=\tilde{s}^{-1}(0)}$$

$$\boxed{\left[\overline{\mathfrak{M}}_{\mathbf{0},k}(Y,d)\right]^{\mathrm{vir}}=\mathrm{PD}e(\mathcal{V}_{\mathbf{0},k})\in H_*\big(\overline{\mathfrak{M}}_{\mathbf{0},k}(\mathbb{P}^n,d)\big)}$$

$\Longrightarrow$ compute $\mathrm{GW}^{Y,A}_{\mathbf{0},k}$ by Atiyah-Bott

verify genus-**0** mirror symmetry prediction

“Defects” of $\mathbf{g}\geq 1$ Invariants

- **No** Enumerative Property:

  $\boxed{\text{genus-}g\text{ GWs}\neq\text{genus-}g\text{ enum. invariants}}$

  even for regular (X,J)

- **No** Hyperplane Property:

  $$\boxed{\big[\overline{\mathfrak{M}}_{g,k}(Y,d)\big]^{\mathrm{vir}}\neq\mathrm{PD}\underbrace{e(\mathcal{V}_{g,k})}\in H_*\big(\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)\big)}$$

  also for $e(H^0-H^1)\equiv\mathrm{ind}\,\bar{\partial}_{\mathcal{L}}$

Rough Reason

- $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ **too big**

Fix (g = 1)

- **cut** $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ down
- define **new** invariants
- **relate** them to standard GWs

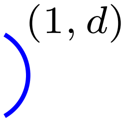
Structure of $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$

$=$"expected" structure of $\overline{\mathfrak{M}}_{1,k}(X,A;J)$

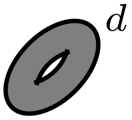
Main Strata of $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$

$$\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d)=\{u:\text{smooth }\Sigma\longrightarrow\mathbb{P}^n\}$$

$=\Big\{$

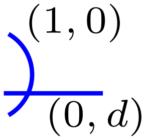


$\Big\}=\Big\{$

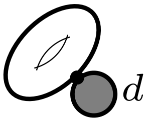


$\Big\}$

$\mathfrak{M}^1_{1,k}(\mathbb{P}^n,d)=\Big\{$

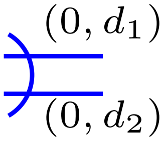


$\Big\}=\Big\{$

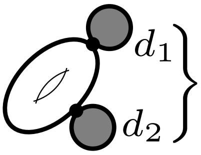


$\Big\}$

$\mathfrak{M}^2_{1,k}(\mathbb{P}^n,d)=\Big\{$



$\Big\}=\Big\{$



$\Big\}$

$d_1+d_2=d,\ \ d_1,d_2>0$

$$\mathfrak{M}^3_{1,k}(\mathbb{P}^n,d),\ \mathfrak{M}^4_{1,k}(\mathbb{P}^n,d),\ \ldots,\ \mathfrak{M}^n_{1,k}(\mathbb{P}^n,d)$$

- $\mathfrak{M}^0_{1,k}(\mathbb{P}^n, d) \sim$ genus-1 curves in $\mathbb{P}^n$

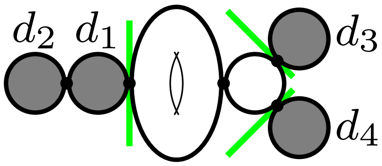
- $l > 0$, $\mathfrak{M}^l_{1,k}(\mathbb{P}^n, d) \sim$ genus-0 curves in $\mathbb{P}^n$

$$\begin{aligned}\dim \mathfrak{M}^l_{1,k}(\mathbb{P}^n, d) &= \dim \mathfrak{M}^0_{1,k}(\mathbb{P}^n, d) + (n-l) \\ &\textcircled{\geq} \dim \mathfrak{M}^0_{1,k}(\mathbb{P}^n, d)\end{aligned}$$

$$\Longrightarrow \quad \boxed{\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n, d) \supsetneq \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d)}$$

What is $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \subset \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$?

Example:



$\in \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$

$$d_1+d_2+d_3+d_4=d$$
$$d_1,d_2,d_3,d_4>0$$

if and only if

Span of green tangent lines in $\mathbb{P}^n < 3$

# of first-level effective bubbles

$(\mathbf{X}, \omega, \mathbf{J})$ arbitrary:

$$\overline{\mathfrak{M}}^0_{1,k}(X,A;J) \equiv \Big\{ [\Sigma,u] \in \overline{\mathfrak{M}}_{1,k}(X,A;J) \colon \ \deg u|_{\Sigma_{\mathrm{Prin}}} \neq 0, \\ \text{OR } u \text{ satisfies } \boxed{\text{analogue}} \Big\}$$

algebraic-genus compactification of $\mathfrak{M}^0_{1,k}(X,A;J)$

Theorems

(1) $\overline{\mathfrak{M}}^0_{1,k}(X,A;J)$ is **compact**

(2) $[\overline{\mathfrak{M}}^0_{1,k}(X,A;J)]$ carries **virtual fund. class**

part of zero set of a generic *non*-generic section

(obstruction theory not perfect)

$\Longrightarrow$ get new invariants $\mathrm{GW}^{0;A}_{1,k}(\mu) \in \mathbb{Q}$

(3) if (X,J) regular, then

$\overline{\mathfrak{M}}^0_{1,k}(X,A;J) =$ **closure** of $\mathfrak{M}^0_{1,k}(X,A;J)$

$\Longrightarrow$ **Enumerative Property** holds

$$\boxed{\mathrm{GW}^{\mathbf{0}}_{1,k}\text{'s} = \text{genus-1 enumerative invariants}}$$

$\Longrightarrow$ can count genus-1 curves

(4) $\mathrm{GW}_{1,k}^{A} - \mathrm{GW}_{1,k}^{0;A}$ **determined** by genus-0 GWs

Example: Y 3-fold, $\mu =$ classes on Y

$$\boxed{\mathrm{GW}_{1,k}^{A}(\mu) - \mathrm{GW}_{1,k}^{0;A}(\mu) = \tfrac{2-\langle c_1(TY),A\rangle}{24}\mathrm{GW}_{0,k}^{A}(\mu)}$$

(5) **PD of euler class of** $\mathcal{V}_{1,k} \longrightarrow \overline{\mathfrak{M}}_{1,k}^{0}(\mathbb{P}^n, d; J)$

is well-defined: zero set of a generic section

Theorem A (J. Li, Z.-):
Hyperplane Property Holds

$$\boxed{\left[\overline{\mathfrak{M}}^{\mathbf{0}}_{1,k}(Y,d)\right]^{\mathrm{vir}} = \mathrm{PD}e(\mathcal{V}_{1,k}) \in H_*\big(\overline{\mathfrak{M}}^{\mathbf{0}}_{1,k}(\mathbb{P}^n,d)\big)}$$

$Y \subset \mathbb{P}^n$ smooth hypersurface/complete intersection

Theorem B (R. Vakil, Z.-)

(1) $\exists$ desing. $\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$

(2) $\exists$ desing. $\tilde{\mathcal{V}}_{1,k} \longrightarrow \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ of

$$\mathcal{V}_{1,k} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \rightsquigarrow \boxed{e(\tilde{\mathcal{V}}_{1,k}) \sim e(\mathcal{V}_{1,k})}$$

(3) localiz. data similar to genus-0 case

Application

compute genus-1 GWs for complete intersections
from Thms A,B,(4) + Atiyah-Bott

Results for Quintic 3-Fold

d	1	2	3	4
$\langle \ldots \rangle$	0	$\frac{2{,}875}{32}$	$\frac{49{,}355{,}000}{81}$	$\frac{952{,}691{,}384{,}375}{256}$
$N_1(d)$	$\frac{2{,}875}{12}$	$\frac{407{,}125}{8}$	$\frac{243{,}388{,}750}{9}$	$\frac{366{,}163{,}353{,}125}{16}$
$n_1(d)$	0	0	609,250	$3,721,431,625$

$N_1(d) = \mathrm{GW}^{Y,d}_{1,0}(1)$ $\quad n_1(d) =$ "# of genus-1 deg.-d `curves`"

table agrees with predictions of Bershadsky, etc.'93

Hope

verify genus-1 mirror symmetry prediction
for curves in Calabi-Yau 3-folds

Other Approaches

Gathmann, Maulik-Pandharipande

Conjectural Extensions

(1) can define $\overline{\mathfrak{M}}^{\mathbf{0}}_{g,k}(X,A;J)$, $[\overline{\mathfrak{M}}^{\mathbf{0}}_{g,k}(X,A;J)]^{\mathrm{vir}}$, $\mathrm{GW}^{\mathbf{0}}_{g,k}$ for all g

(2) $\mathrm{GW}^{A}_{g,k} - \mathrm{GW}^{0;A}_{g,k}$ **determined** by genus-g' GWs, with $g' < g$

(3) **PD of euler class** of $\mathcal{V}_{g,k} \longrightarrow \overline{\mathfrak{M}}^{0}_{g,k}(X,A;J)$ is well defined

(4) **Enumerative and Hyperplane** Properties hold for $\mathrm{GW}^{\mathbf{0}}_{g,k}$

Outline of Pf of Thm A

Assume: $J =$ almost complex str. on $\mathbb{P}^n$ s.t.

$$\overline{\mathfrak{M}}^0_{1,k}(Y,\mu) \equiv \big\{u \in \overline{\mathfrak{M}}^0_{1,k}(Y,d;J) : \mathrm{ev}_i(u) \in \mu_i\big\} \subset \mathfrak{M}^0_{1,k}(Y,d;J)$$

and is **finite** (to 1st order)

Then: $\boxed{\mathrm{GW}^{0,d}_{1,k}(Y;\mu) \equiv \ {}^{\pm}\big|\overline{\mathfrak{M}}^0_{1,k}(Y,\mu)\big|}$

Furthermore:

$\tilde{s}$ transverse to 0-set in $\mathcal{V}_{1,k} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,\mu) \quad \Longrightarrow$

$$\boxed{\langle e(\mathcal{V}_{1,k}), \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,\mu)\rangle = \ {}^{\pm}\big|\tilde{s}^{-1}(0)\big| = \big|\overline{\mathfrak{M}}^0_{1,k}(Y,\mu)\big|}$$

Thus:

$$\begin{aligned}\langle \mu, [\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)]^{\mathrm{vir}}\rangle &\equiv \mathrm{GW}^{0,d}_{1,k}(Y;\mu)\\ &= \langle e(\mathcal{V}_{1,k}), \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,\mu)\rangle\\ &\equiv \langle \mu, \mathrm{PD}_{\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)} e(\mathcal{V}_{1,k})\rangle\end{aligned}$$

Admissibility of Assumption:

Y Fano: OK by structure of $\overline{\mathfrak{M}}^0_{1,k}(Y,d;J)$
Y general: OK conjecturally

Solution

use perturbed (in a restricted way) J-holomor. maps u:

$$\bar{\partial}_J u + \nu(u) = 0$$

$\rightsquigarrow$ makes proof the same in all cases

Theorem

euler class of $\mathcal{V}_{1,k} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d)$ is well-defined

Reminder

$$\begin{tikzcd}
\mathcal{L}=\gamma^{*\otimes a} \arrow[d] \\
\mathbb{P}^n
\end{tikzcd}$$

$a \in \mathbb{Z}^+$

$$\mathcal{V}_{1,k}|_{(\Sigma,u)} = H^0(\Sigma; u^*\mathcal{L})$$

Outline of Proof

$$m = \dim_{\mathbb{C}} \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d) \equiv \dim_{\mathbb{C}} \mathfrak{M}^0_{1,k}(\mathbb{P}^n, d)$$

$$r = \dim_{\mathbb{C}} H^0(\Sigma; u^*\mathcal{L}) - \underbrace{\dim_{\mathbb{C}} H^1(\Sigma; u^*\mathcal{L})}_{0 \;\text{ or }\; 1} = \mathrm{ind}_{\mathbb{C}} \bar{\partial}_{\mathcal{L},u}$$

if $(\Sigma, u) \in \mathfrak{M}^0_{1,k}(\mathbb{P}^n, d) \equiv \{(\Sigma, u) \in \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n, d) \colon \Sigma \text{ smooth}\}$

(1) find section $\varphi \colon \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d) \longrightarrow \mathcal{V}_{1,k}$ s.t.

$$\dim_{\mathbb{R}} \varphi^{-1}(0) \cap \underbrace{\partial \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d)}_{\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d) - \mathfrak{M}^0_{1,k}(\mathbb{P}^n, d)} \le 2(m-r)-2$$

(2) show space of such sections is path-connected

φ generic $\Longrightarrow$ $\dim_{\mathbb{R}} \varphi^{-1}(0) \cap \mathfrak{M}^0_{1,k}(\mathbb{P}^n, d) = 2(m-r)$

$+$ bound. cond. $\Longrightarrow$ $\big[\varphi^{-1}(0)\big] \in H_{2(m-r)}\big(\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d)\big)$

Key Proposition

$\exists$ stratification $\partial\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)=\bigsqcup_{\alpha\in A}\mathcal{S}_\alpha$ and for $\alpha\in A$ neighb. $\mathcal{U}_\alpha$ of $\mathcal{S}_\alpha$ in $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ and v.b. $V_\alpha\longrightarrow\mathcal{U}_\alpha$ s.t.

(a) $V_\alpha\subset\mathcal{V}_{1,k}$

(b) $V_\beta|_{\mathcal{S}_\beta}\subset V_\alpha|_{\mathcal{S}_\beta}$ if $\overline{\mathcal{S}}_\alpha\cap\mathcal{S}_\beta\neq\emptyset$

(c) $\mathrm{rk}_{\mathbb{C}}\mathcal{V}_\alpha>\dim_{\mathbb{C}}S_\alpha-(m-r)$

$\Longrightarrow$ if φ is a generic section of $\mathcal{V}_{1,k}\longrightarrow\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$, then $\dim_{\mathbb{R}}\varphi^{-1}(0)\cap\mathcal{S}_\alpha\leq 2(m-r)-2$

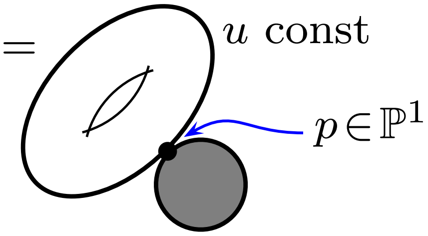
Meaning of Proposition

$\mathrm{rk}_{\mathbb{C}}\mathcal{V}_{1,k}|_{\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d)}=r$

if $\dim_{\mathbb{C}}H^0(\Sigma;u^*\mathcal{L})=r+1$, can find large-dim subspace of sections that extend over a neighb. in $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$

Example 1

$\Sigma =$



$$(\Sigma,u)\in\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \Longrightarrow du|_p=0$$

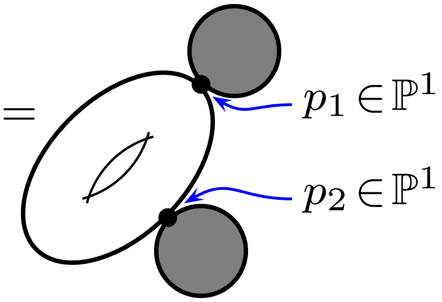
$$\Longrightarrow \dim_{\mathbb{C}}\mathcal{S}_\alpha = m-1$$

$$V_\alpha|_{(\Sigma,u)} = \big\{\xi\in H^0(\Sigma;u^*\mathcal{L}) :\, \underbrace{\nabla^{\mathcal{L}}\xi|_p}_{\text{indep. of } \nabla^{\mathcal{L}} \text{ in } \mathcal{L}\longrightarrow\mathbb{P}^n \text{ b/c } du|_p=0} = 0\big\}$$

$$\mathrm{rk}_{\mathbb{C}} V_\alpha|_{(\Sigma,u)} = (r+1)-1 = r \quad \surd$$

Example 2

$\Sigma =$



$$\mathcal{S}_\alpha \equiv \{du|_{p_1}=0,\ du|_{p_2}=0\}$$

$$\Longrightarrow \dim_{\mathbb{C}}\mathcal{S}_\alpha = m-2-n$$

$$V_\alpha|_{(\Sigma,u)} = \big\{(\xi_1,\xi_2)\in H^0(\Sigma;u^*\mathcal{L}) :\, \nabla^{\mathcal{L}}\xi_i|_{p_i}=0\big\}$$

$$\mathrm{rk}_{\mathbb{C}} V_\alpha|_{(\Sigma,u)} = (r+1)-2 = r-1 \quad \surd$$