Degenerate Contributions, Enumerative Geometry, and Gromov-Witten Invariants

slides to appear at

`http://math.sunysb.edu/~azinger`

more details in expository notes

- General Topological Background
- Degenerate Loci of Vector Bundle Sections
- Examples from Enumerative Geometry and GW-Invariants
- Computation of Degenerate Contributions

Background Topology

X top. space $\longrightarrow H_k(X;\mathbb{Z})$, $\quad \mathbb{Z}$-modules

(1) $H_0(X;\mathbb{Z}) \xrightarrow{\pm|\cdot|} \mathbb{Z}$, $\quad$ pt count
(2) M cmpt mfld: $H_k(M;\mathbb{Z}) = \{k\text{−pseudocycles}\}/\sim$

- $A \subset M$: $\dim A \le l$ if
  $\exists h\colon Z^l_{\text{cmpt}} \xrightarrow{\text{smooth}} M$ s.t. $A \subset \operatorname{Im} h$
- $f\colon Y \xrightarrow{\text{cont}} M$: $\operatorname{Bd} f = \bigcap_{K_{\text{comp}} \subset Y} \overline{f(Y - K)} \subset M$
  $Y_{\text{comp}} \Longrightarrow \operatorname{Bd} f = \emptyset$
- k-pseudocycle: $f\colon Y^k \xrightarrow{\text{smooth}} M$ s.t.
  $\dim \operatorname{Bd} f \le k-2$

Examples

(1) $Y^k_{\mathrm{cmpt}} \subset M$: $\quad \iota_Y : Y \hookrightarrow M$ k-pseudocycle

$\longrightarrow [Y] = [k-\text{pseudocycle } \iota_Y] \in H_k(M;\mathbb{Z})$

(2) $Y_{\mathrm{cmpt}} \subset M$ k-"variety": $Y = Y^k_{\mathrm{main}} \sqcup \bigsqcup_{\alpha \in \mathcal{A}} Y_\alpha$

$\mathcal{A}$ finite, $Y^k_{\mathrm{main}}, Y_\alpha$ smooth, $\dim Y_\alpha \leq k-2$

$\longrightarrow [Y] = [\iota_Y] \in H_k(M;\mathbb{Z})$

Vector Bundles

$$\begin{tikzcd} \mathbb{C}^n \arrow[r] & V \arrow[d, "\pi"'] \\ & M^m \arrow[u, bend right, "s"'] \end{tikzcd}$$

v.b. $\longleftrightarrow \mathcal{U}_{\text{small}} \subset M \implies \pi^{-1}(\mathcal{U}) \approx \mathcal{U} \times \mathbb{C}^n$

$\pi \circ s = \mathrm{id}_M$ $\qquad \supset \underbrace{\mathcal{U} \times 0}_{0-\text{set}}$

s generic (transverse to 0-set in V)

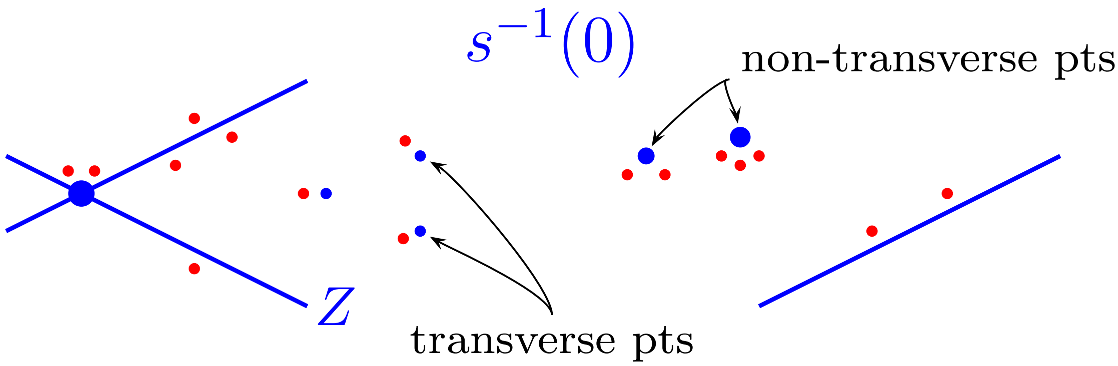
$\implies \big[\underbrace{s^{-1}(0)}_{\text{sub-mfld}}\big] = \mathrm{PD}_M e(V) \in H_{m-2n}(M;\mathbb{Z})$

$\implies [s^{-1}(0)]$ indep. of s generic; top. inv. of V
often computable

$m = 2n$: s transv. $\implies s^{-1}(0)$ fnt set of pts w. $+/-$
$\longrightarrow \pm|s^{-1}(0)| \in \mathbb{Z}$ indep. of s
signed card. of $s^{-1}(0)$=pt count for $[s^{-1}(0)]$

each pt of $\mathbf{s^{-1}(0)}$ contributes ± 1

What if s not transverse?



Perturb s to transverse $s+\nu$

$\{s+\nu\}^{-1}(0) \subset M$ lies near $s^{-1}(0)$ if ν small

$\mathcal{C}_Z(s) \equiv {}^{\pm}\big|\{s+\nu\}^{-1}(0) \cap \mathcal{U}_Z\big|$ if

$\mathcal{U}_Z$=small neighbd of Z; $\mathcal{C}_Z(s)$ indep. of ν gen

$\mathcal{C}_Z(s) = s$-contribution of Z to $\mathrm{PD}_M e(V)$

$$\sum_{Z \subset s^{-1}(0)} \mathcal{C}_Z(s) = \mathrm{PD}_M e(V)$$

How $\mathcal{C}_Z(s)$ arise

1. Find $n_d = \#\{$deg.-d genus-0 curves thr. $3d-1$ pts in $\mathbb{P}^2 \equiv \mathbb{C}P^2$ (or $\mathbb{C}^2$)$\}$

2. Find $n_d^c = \#\{$deg.-d genus-0 curves with a cusp thr. $3d-2$ pts in $\mathbb{P}^2\}$

   more generally: other singularities, in $\mathbb{P}^n$

3. Compute pos.-genus enum. inv. (curve counts) from GW-inv. $\longleftarrow$ "PD$e(V)$"

In all cases:

$$\boxed{\# = \text{"PD}e(V)\text{" - } \mathcal{C}_Z(s)}$$

fixed V, s; $\quad Z=$ "boundary loci"

$n_1 = \#\{\text{lines thr. 2 pts in } \mathbb{C}^2\} = 1$

deg.-d curve in $\mathbb{C}^2$=zero-set of deg.-d polyn. in x,y
deg.-d curve in $\mathbb{P}^2$=zero-set of deg.-d homogen. polyn. in X,Y,Z

$$\{\text{deg.-}d \text{ curves in } \mathbb{P}^2\}=\{\ldots\}/\sim \qquad P\sim cP, c\in\mathbb{C}^*$$
$$=(\mathbb{C}^N-0)/\mathbb{C}^* = \mathbb{P}^{N-1} \qquad N=\binom{d+2}{2}$$

polyn. $P \longleftrightarrow$ curve $\mathcal{C}_P$
$P(X_0,Y_0,Z_0)=0 \iff [X_0,Y_0,Z_0]\in\mathcal{C}_P$
lin. cond. on $P \Longrightarrow$ sol. space=hyperpl. in $\mathbb{C}^N$, $\mathbb{P}^{N-1}$

$$\Longrightarrow \mathfrak{X}_d \equiv \{\text{deg.-}d \text{ curves thr. } 3d-1 \text{ gen. pts}\}$$
$$\approx \mathbb{P}^{(N-1)-(3d-1)} = \mathbb{P}^{g_d} \qquad g_d=\binom{d-1}{2}$$

for generic $\mathcal{C}\in\mathfrak{X}_d$, $g(\mathcal{C})=g_d$

$d=2 \quad g=0 \qquad \mathfrak{X}_2 = pt \Longrightarrow n_2=1$

$d=3 \quad g=1 \qquad \mathfrak{X}_3 \approx \mathbb{P}^1$

need 1 node to make genus-0 $\Longrightarrow$ 1 cond.

$d=4 \quad g=3 \qquad \mathfrak{X}_4 \approx \mathbb{P}^3$

need 3 nodes to make genus-0 $\Longrightarrow$ 3 cond.

$$\begin{aligned} n'_d &= \big|\{\mathcal{C}\in\mathfrak{X}_d : \mathcal{C} \text{ has } g_d \text{ nodes}\}\big| \\ &= \Big|\Big\{(\mathcal{C},x_1,\ldots,x_{g_d})\in\mathfrak{X}_d\times(\mathbb{P}^2)^{g_d} : \underbrace{x_i \text{ node of } \mathcal{C}}_{\substack{(\mathcal{C},x_i)\in s_i^{-1}(0) \\ s_i \text{ section of rk-3 v.b.} \\ \longrightarrow \mathfrak{X}_d\times\mathbb{P}^2_i}}, x_i\neq x_j\Big\}\Big| \\ &= \Big|\underset{\substack{\text{section of rank } 3g_d \text{ v.b.} \\ V_d\longrightarrow\mathfrak{X}_d\times(\mathbb{P}^2)^{g_d}}}{s}{}^{-1}(0)\cap\mathfrak{X}_d\times\big((\mathbb{P}^2)^{g_d}-\underset{x_i=x_j \text{ for some } i\neq j}{\Delta_{\text{big}}}\big)\Big| \end{aligned}$$

$$\boxed{n'_d = \mathrm{PD}_{\mathfrak{X}_d \times (\mathbb{P}^2)^{g_d}} e(V_d) - \mathcal{C}_{\mathfrak{X}_d \times \Delta_{\mathrm{big}}}(s)}$$

$d = 3 \quad g_d = 1 \quad \Longrightarrow \Delta_{\mathrm{big}} = \emptyset \longrightarrow n_3 = n'_3 = 12$

$d = 4 \quad g_d = 3 \quad \Longrightarrow \dim_{\mathbb{C}} \Delta_{\mathrm{big}} = 2$

can compute $\mathcal{C}_{\mathfrak{X}_d \times \Delta_{\mathrm{big}}}(s) \longrightarrow n'_4 = 3! \cdot 675$

includes unions of a line with a genus-1 cubics

$\binom{11}{2} = 55$ of these $\longrightarrow n_4 = 620$ ($\sim$1870s)

Special Case of

$$\boxed{\# = \text{“PD}e(V)\text{” - } \mathcal{C}_Z(s)}$$

fixed V, s; $\quad Z=$ “boundary loci”

n_d: classical approach gets hard fast with d

n_5 unknown until 1990s

Better: n_d=genus-0 deg.-d GW-inv. for $\mathbb{P}^2$

$\longrightarrow$ Recursion for n_d (and genus-0 counts for $\mathbb{P}^n$)

Kontsevich-Manin, Ruan-Tian

positive-genus invariants:

curve counts $\neq$ GWs $=$ "PD$e(V)$"

(motivates Question 3 above)

Construction of GW_g for $\mathbb{P}^n$

$$\Gamma_J^{0,1}(\Sigma,u)=\{\eta\colon T\Sigma\longrightarrow u^*T\mathbb{P}^n \mid J\eta=-\eta j\}$$

cmplx str. on $\mathbb{P}^n$

$$\begin{tikzcd} \Gamma_J^{0,1} \arrow[d] \\ \mathfrak{X}_g(\mathbb{P}^n,d) \arrow[u, bend right, "\bar\partial_J"'] \end{tikzcd}$$

$$\bar\partial_J(\Sigma,u) = \tfrac{1}{2}\big(du+J\circ du\circ j\big)$$

$$\mathfrak{X}_g(\mathbb{P}^n,d) = \{\text{smooth maps } u\colon \Sigma\longrightarrow\mathbb{P}^n\}$$

$\Sigma=(\Sigma,j)$ Riemann surface
may have simple nodes

GW_g = (PD of) "Euler class" of $\Gamma_J^{0,1}$, w.r.t. $\bar\partial_J$

$$\boxed{\mathrm{GW}_g\equiv\big[\{\bar\partial_J+\nu\}^{-1}(0)\big]\in H_{\mathrm{ind}\,\bar\partial_J}\big(\underbrace{\mathfrak{X}_g(\mathbb{P}^n,d)}; \mathbb{Q}\big)}$$

small generic
perturbation

can replace by
small neighb. of $\bar\partial_J^{-1}(0)$

$$\begin{aligned} \nu \text{ small} \quad &\Longrightarrow\quad \{\bar\partial_J+\nu\}^{-1}(0) \text{ cmpt}\\ \nu \text{ generic} \quad &\Longrightarrow\quad \{\bar\partial_J+\nu\}^{-1}(0) \text{ "variety"} \end{aligned}$$

Curve Counts vs. GWs

$$\mathfrak{X}_g^0(\mathbb{P}^n,d) \equiv \big\{(\Sigma,u)\in\mathfrak{X}_g(\mathbb{P}^n,d)\colon \Sigma \text{ smooth}\big\}$$

- $(\Sigma,u)\in\bar{\partial}_J^{-1}(0)\cap\mathfrak{X}_g^0(\mathbb{P}^n,d)$
  $\Longrightarrow\ u(\Sigma)$ genus-g curve in $\mathbb{P}^n$
- $\bar{\partial}_J$ transverse to 0-set along $\mathfrak{X}_g^0(\mathbb{P}^n,d)$
  if $g=1,2$, $d\geq 1$; in other cases

$$\begin{aligned}\text{curve counts} &\longleftrightarrow \bar{\partial}_J^{-1}(0)\cap\mathfrak{X}_g^0(\mathbb{P}^n,d)\\ &\longleftrightarrow \mathrm{GW}_g - \mathcal{C}_{\mathfrak{X}_g(\mathbb{P}^n,d)-\mathfrak{X}_g^0(\mathbb{P}^n,d)}(\bar{\partial}_J)\end{aligned}$$

$\longrightarrow$ compute $\mathcal{C}_{\mathfrak{X}_g-\mathfrak{X}_g^0}(\bar{\partial}_J)$ $\longrightarrow$ get enum. inv.
works for $g=1$; for $g\geq 2$ in some cases
Ionel, Z.; AG: Pandharipande, Katz-Qin-Ruan/Z.

Special Case of

$$\boxed{\# = \text{“PD}e(V)\text{” - } \mathcal{C}_Z(s)}$$

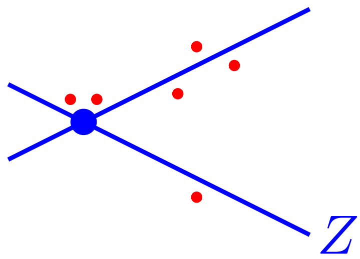
fixed V, s; $\quad Z=$ “boundary loci”

How to Compute Degenerate Contributions

$=$ Contributions from "Bad" Loci

$$\begin{tikzcd} \mathbb{C}^n \arrow[r] & V \arrow[d] \\ & M^{2n} \arrow[u, bend right, "s"'] \end{tikzcd}$$

$Z \subset s^{-1}(0)$
comp. of $s^{-1}(0)$



$$\mathcal{C}_Z(s) \equiv {}^{\pm}\big|\{s+\nu\}^{-1}(0) \cap \mathcal{U}_Z\big|$$

if $\mathcal{U}_Z$=small neighbd of Z

to compute $\mathcal{C}_Z(s)$, need to know:
derivatives of s in directions normal to Z

Easy Cases

1. $p \in s^{-1}(0)$ transverse zero $\Longrightarrow \mathcal{C}_p(s) = \pm 1$
2. $Z \subset s^{-1}(0)$ smooth, $\nabla s\colon \mathcal{N}Z \longrightarrow V$ injective
   $\Longrightarrow \; \mathcal{C}_Z(s) = \mathrm{PD}_Z e(V/\mathrm{Im}\,\nabla s) \equiv \langle c(V)/c(\mathcal{N}Z), Z\rangle$

General Case

Approach 1 (Fulton-MacPhearson)

(Global) Excess Intersection Theory

- work with entire component Z (cmpt)
- blow up to get to Easy Case 2
- AG $\Longrightarrow$ rigid

Approach 2

Local Excess Intersection "Theory"

- work with smooth strata Z_i of Z (non-cmpt)
- $\mathcal{C}_{Z_i}(s)$ often well-defined; $\mathcal{C}_Z(s) = \sum_i \mathcal{C}_{Z_i}(s)$
- topology + analysis $\Longrightarrow$ not rigid
  applicable outside of AG, in ∞-dim settings

Approach 2

Stratify $Z = \bigsqcup_i Z_i$ s.t.

- Z_i is smooth and of even codim
- near Z_i, $s \approx \alpha_i$ where
  - $\alpha_i : \mathcal{N}Z_i \longrightarrow V$ nondegen. polyn.
  - for some $\mathcal{N}Z_i^+ \subset \mathcal{N}Z_i$ and $V_i^+ \subset V$ sub-v.b. $\alpha_i : \mathcal{N}Z_i^+ \xrightarrow{\approx} V_i^+$
  - $\alpha_i : \mathcal{N}Z_i/\mathcal{N}Z_i^+ \longrightarrow V/V_i^+$ extends to some $\alpha_i^- : \mathcal{N}Z_i^- \longrightarrow V_i^-$ over $\bar{Z}_i$

such $Z = \bigsqcup_i Z_i$ often exists; always in AG

Zeros of Polynomial Maps

$$\begin{tikzcd}
E^k \arrow[rr, "\alpha"] \arrow[dr] & & \mathcal{O}^{k+m} \arrow[dl] \\
& \bar{Z}^{2m}_{\text{cmpt}} \arrow[ur, bend right, "\bar{\nu}"'] &
\end{tikzcd}$$

$\alpha + \bar{\nu} : E \longrightarrow V$ "const term"

$\bar{\nu}$ generic $\Longrightarrow \{\alpha+\bar{\nu}\}^{-1}(0)$ 0-dim mfld

$N(\alpha) \equiv^{\pm} \left|\{\alpha+\bar{\nu}\}^{-1}(0)\right|$ indep. of $\bar{\nu}$

The Computation Scheme

Proposition 1: $\mathcal{C}_{Z_i}(s) = N(\alpha_i^-)$

Proposition 2: If $\alpha|_Z$ is nondegenerate,

$$N(\alpha) = f(\text{deg. str. of } \alpha, \mathcal{O}, E) - \mathcal{C}_{\bar{Z}-Z}(\alpha^\perp)$$

section induced by α

Example: α linear $\Longrightarrow f = \langle c(\mathcal{O})/c(E), \bar{Z}\rangle$

Props 1+2 generalize Easy Case 2 above ($Z = Z_i$)

Props 1+2 reduce rank of V by 1 or more
$\Longrightarrow$ compute $\mathcal{C}_Z(s)$ in finitely many steps

Concluding Remarks

regarding *Local Excess Intersection "Theory"*

General Questions

- elaborate formulation?
- more topological reinterpretation?
- equivariant reinterpretation?
- algebraic geometry reinterpretation?

Potential General Applications

- *Localization Theorem* for singular setting?
- *Localization Theorem* in virtual setting?
  in AG situation: Graber-Pandharipande'95
  no symplectic-topology analogue still