Mirror Symmetry for Gromov-Witten Invariants of a Quintic Threefold

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From String Theory to Gromov-Witten Theory

Mirror Symmetry Principle of String Theory produces predictions for GW-Invariants

- especially for Calabi-Yau 3-fold
- especially for quintic 3-fold $X_5 \subset \mathbb{P}^4$
  $X_5 =$ degree 5 hypersurface in $\mathbb{P}^4$

Some Predictions of String Theory

- Candelas-de la Ossa-Green-Parkes'91: $g=0$ for X_5
- Bershadsky-Cecotti-Ooguri-Vafa'93 (BCOV): $g=1$ for X_5
- Huang-Klemm-Quackenbush'06: $g \leq 52$ for X_5
- Klemm-Pandharipande'07: $g=1$ for X_6
  $X_6 =$ degree 6 hypersurface in $\mathbb{P}^5$

Mirror Symmetry Verifications

Theorem (Givental'96, Lian-Liu-Yau'97,........∼'00)

$g=0$ predict. holds for X_5; generalizes to other hypersurfaces

Theorem (Z.'07)

$g=1$ predictions hold for X_5, X_6; generalize to X_n

$X_n =$ degree n hypersurface in $\mathbb{P}^{n-1}$: $c_1(X)=0$

A Curious Identity for $n = 3$

- $X_3 =$ cubic in $\mathbb{P}^2$, smooth curve of genus 1
- genus 1 GWs $\longleftrightarrow$ counts of unbranched covers
- comparison with $n = 3$ case of $g = 1$ thm gives identity for

$$\mathbb{I}_0(q) \equiv 1 + \sum_{d=1}^{\infty} q^d \frac{(3d)!}{(d!)^3}, \quad \mathbb{I}_1(q) \equiv \sum_{d=1}^{\infty} q^d \left(\frac{(3d)!}{(d!)^3} \sum_{r=d+1}^{3d} \frac{3}{r} \right)$$

- With $Q \equiv q \cdot e^{\mathbb{I}_1(q)/\mathbb{I}_0(q)}$,

$$q^3(1 - 27q)\mathbb{I}_0(q)^{12} = Q^3 \prod_{d=1}^{\infty} \left(1 - Q^{3d}\right)^{24}$$

Approach to GWs of X_n

Step 1: relate GWs of $X_n \subset \mathbb{P}^{n-1}$ to GWs of $\mathbb{P}^{n-1}$

Step 2: use $(\mathbb{C}^*)^n$-action on $\mathbb{P}^{n-1}$ to compute each GW by localization

Step 3: find some degree-recursive feature(s) to compute all GWs for fixed genus

Base Spaces

- $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^{n-1},d) = \{\text{deg. } d \text{ genus-}g\ k\text{-pointed maps to } \mathbb{P}^{n-1}\}$
- $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^{n-1},d) \subset \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^{n-1},d)$ main irred. component
  closure of $\big\{[u\colon \Sigma \longrightarrow \mathbb{P}^{n-1}]\colon \Sigma \text{ is smooth}\big\}$
- $\widetilde{\mathfrak{M}}^0_{g,k}(\mathbb{P}^{n-1},d) \longrightarrow \overline{\mathfrak{M}}^0_{g,k}(\mathbb{P}^{n-1},d)$ natural desingularization
  $\widetilde{\mathfrak{M}}^0_{0,k}(\mathbb{P}^{n-1},d) = \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$
- $\mathrm{ev}_i\colon \widetilde{\mathfrak{M}}^0_{g,k}(\mathbb{P}^{n-1},d) \longrightarrow \mathbb{P}^{n-1}$ evaluation at ith marked pt
  $[u\colon \Sigma \longrightarrow \mathbb{P}^{n-1}; x_1,\ldots,x_k] \longrightarrow u(x_i)$

From $X_n \subset \mathbb{P}^{n-1}$ to $\mathbb{P}^{n-1}$

$$\begin{array}{c} \mathcal{L} \equiv \mathcal{O}(n) \\ \big\downarrow \pi \\ \mathbb{P}^{n-1} \end{array} \qquad\qquad \begin{array}{c} \widetilde{\mathcal{V}}_{g,d} \equiv \widetilde{\mathfrak{M}}^0_{g,k}(\mathcal{L}, d) \\ \big\downarrow \tilde{\pi} \\ \widetilde{\mathfrak{M}}^0_{g,k}(\mathbb{P}^{n-1}, d) \end{array}$$

$g = 1$ Hyperplane Property: sufficient to compute

$$F(Q) \equiv \sum_{d=1}^{\infty} Q^d \int_{\widetilde{\mathfrak{M}}^0_{1,1}(\mathbb{P}^{n-1},d)} e(\widetilde{\mathcal{V}}_{1,d}) \mathrm{ev}_1^* x$$

$x \in H^2(\mathbb{P}^{n-1})$ hyperplane class

Torus Actions

- $\mathbb{T} \equiv (\mathbb{C}^*)^n$ acts on $\mathbb{P}^{n-1}$ (with n fixed pts)
- $\Longrightarrow$ on $\widetilde{\mathcal{V}}_{g,d} \longrightarrow \widetilde{\mathfrak{M}}^0_{g,k}(\mathbb{P}^{n-1}, d)$ by composition with simple fixed loci
- $\Longrightarrow$ Atiyah-Bott Localization Thm reduces

$$\int_{\widetilde{\mathfrak{M}}^0_{g,k}(\mathbb{P}^{n-1},d)} e(\mathcal{V}_{g,d})\eta$$

to integrals over fixed loci $\rightsquigarrow \sum_{graphs}$

Summing over all Genus 1 Graphs

- split genus 1 graphs into many genus 0 graphs at special vertex
- make use of good properties of genus 0 numbers to eliminate infinite sums
- extract non-equivariant part of elements in $H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$

Genus 0 Data

What we know

- $H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) = \mathbb{Q}[x, \alpha_1, \ldots, \alpha_n] / \prod_k (x - \alpha_k)$
- With $\mathrm{ev}_1, \mathrm{ev}_2 : \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, d) \longrightarrow \mathbb{P}^{n-1}$,
  - Givental'96:

$$\mathcal{Z}^*(\hbar, x, Q) \equiv \sum_{d=1}^{\infty} Q^d \mathrm{ev}_{1*}\left(\frac{e(\mathcal{V}_{0,d})}{\hbar - \psi_1}\right) \in \mathbb{Q}(x, \alpha)\big[\big[\hbar^{-1}, Q\big]\big]$$

  - Z'07:

$$\widetilde{\mathcal{Z}}^* \equiv \frac{1}{2\hbar_1 \hbar_2} \sum_{d=1}^{\infty} Q^d \big\{\mathrm{ev}_1 \times \mathrm{ev}_2\big\}_* \left(\frac{e(\mathcal{V}_{0,d})}{(\hbar_1 - \psi_1)(\hbar_2 - \psi_2)}\right)$$

Good Properties of $\mathcal{Z}^*$

$\mathcal{Z}_i^* \equiv \mathcal{Z}(x=\alpha_i)$ satisfies: for all $a \geq 0$

$$\sum_{m=2}^{\infty} \frac{1}{m(m-1)} \sum_{\substack{\sum a_l = m-2-a \\ a_l \geq 0}} \frac{(-1)^{a_l}}{a_l!} \mathfrak{R}_{\hbar=0}\left\{\hbar^{-a_l} \mathcal{Z}_i^*(\hbar)\right\}$$
$$= a!\, \mathfrak{R}_{\hbar=0}\left\{\hbar^{a+1} \mathcal{Z}_i^*(\hbar)\right\} \qquad \boxed{a}$$

$\mathfrak{R}_{\hbar=0} \equiv$ residue at $\hbar=0$

Good Properties of $\mathcal{Z}^*$

Lemma 1: $\mathcal{Z} \in Q \cdot \mathbb{Q}(\hbar)[[Q]]$ satisfies $\boxed{a}$ $\forall\, a \geq 0$ iff

$\exists\, \eta \in Q \cdot \mathbb{Q}[[Q]]$ and $\bar{\mathcal{Z}} \in Q \cdot \mathbb{Q}(\hbar)[[Q]]$ regular at $\hbar = 0$ s.t.

$$1 + \mathcal{Z} = e^{\eta/\hbar}\big(1 + \bar{\mathcal{Z}}(\hbar)\big)$$

such $(\eta, \bar{\mathcal{Z}})$ must be unique

Lemma 2: If $\mathcal{Z} \in Q \cdot \mathbb{Q}(\hbar)[[Q]]$ satisfies above, then $\forall\, a \geq 0$

$$\sum_{m=0}^{\infty} \sum_{\substack{\sum a_l = m-a \\ a_l \geq 0}} \frac{(-1)^{a_l}}{a_l!} \mathfrak{R}_{\hbar=0}\left\{\hbar^{-a_l} \mathcal{Z}_i^*(\hbar)\right\} = \frac{\eta^a}{1 + \bar{\mathcal{Z}}(\hbar = 0)}$$

What We Know

If $\mathrm{ev}_1, \mathrm{ev}_2 \colon \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, d) \longrightarrow \mathbb{P}^{n-1}$:

$$\mathcal{Z}^*(\alpha; \hbar, x, Q) \equiv \sum_{d=1}^{\infty} Q^d \mathrm{ev}_{1*}\left(\frac{e(\mathcal{V}_{0,d})}{\hbar - \psi_1}\right)$$

$$\mathcal{A}_i^{(a)} \equiv \sum_{m=0}^{\infty} \sum_{\substack{\sum a_l = m-a \\ a_l \geq 0}} \frac{(-1)^{a_l}}{a_l!} \mathfrak{R}_{\hbar=0}\left\{\hbar^{-a_l} \mathcal{Z}^*(x=\alpha_i)\right\}$$

$$\widetilde{\mathcal{Z}}^* \equiv \frac{1}{2\hbar_1\hbar_2} \sum_{d=1}^{\infty} Q^d \left\{\mathrm{ev}_1 \times \mathrm{ev}_2\right\}_* \left(\frac{e(\mathcal{V}_{0,d})}{(\hbar_1 - \psi_1)(\hbar_2 - \psi_2)}\right)$$

Genus 1 Setup

- What we want to know: if $\mathrm{ev}_1 : \widetilde{\mathfrak{M}}^0_{1,1}(\mathbb{P}^{n-1}, d) \longrightarrow \mathbb{P}^{n-1}$

$$F(Q) \equiv \sum_{d=1}^{\infty} Q^d \, ev_{1*}\big(e(\widetilde{\mathcal{V}}_{1,d})\big)$$

- Atiyah-Boot reduces F to $\sum$ over genus 1 graphs w. special node

From Genus 1 to 0

Each genus 1 graphs breaks at special node into genus 0 strands:

- each genus 0 strand contributes to $\mathcal{Z}^*$, $\widetilde{\mathcal{Z}}^*$, or $\hbar_2^{-2}$-coefficient of $\widetilde{\mathcal{Z}}^*$
- at most one strand contributes to $\widetilde{\mathcal{Z}}^*$, $\mathrm{Coeff}_{\hbar_2^{-2}}(\widetilde{\mathcal{Z}}^*)$ each
- remaining stands make up either Log of something simple or $\mathcal{A}_i^{(a)}$!