Mirror Symmetry for Gromov-Witten Invariants of a Quintic Threefold

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From String Theory to Gromov-Witten Theory

Mirror Symmetry Principle of String Theory produces predictions for GW-Invariants

- especially for Calabi-Yau 3-fold
- especially for quintic 3-fold $X_5 \subset \mathbb{P}^4$
  $X_5 =$ degree 5 hypersurface in $\mathbb{P}^4$

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Some Predictions of String Theory

- Candelas-de la Ossa-Green-Parkes'91: $g=0$ for X_5
- Bershadsky-Cecotti-Ooguri-Vafa'93 (BCOV): $g=1$ for X_5
- Huang-Klemm-Quackenbush'06: $g\le 52$ for X_5
- Klemm-Pandharipande'07: $g=1$ for X_6
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Theorem (Givental'96, Lian-Liu-Yau'97,........∼'00)

$g=0$ predict. holds for X_5; generalizes to other hypersurfaces

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A Curious Identity for $n = 3$

- X_3 = cubic in $\mathbb{P}^2$, smooth curve of genus 1
- genus 1 GWs $\longleftrightarrow$ counts of unbranched covers
- comparison with $n = 3$ case of $g = 1$ thm gives identity for

$$\mathbb{I}_0(q) \equiv 1 + \sum_{d=1}^{\infty} q^d \frac{(3d)!}{(d!)^3}, \quad \mathbb{I}_1(q) \equiv \sum_{d=1}^{\infty} q^d \left(\frac{(3d)!}{(d!)^3} \sum_{r=d+1}^{3d} \frac{3}{r} \right)$$

- With $Q \equiv q \cdot e^{\mathbb{I}_1(q)/\mathbb{I}_0(q)}$,

$$q^3 (1 - 27q) \mathbb{I}_0(q)^{12} = Q^3 \prod_{d=1}^{\infty} \left(1 - Q^{3d}\right)^{24}$$

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Approach to GWs of X_n

Step 1: relate GWs of $X_n \subset \mathbb{P}^{n-1}$ to GWs of $\mathbb{P}^{n-1}$

Step 2: use $(\mathbb{C}^*)^n$-action on $\mathbb{P}^{n-1}$ to compute each GW by localization

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Base Spaces

- $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^{n-1},d) = \{\text{deg. } d \text{ genus-}g\ k\text{-pointed maps to } \mathbb{P}^{n-1}\}$
- $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^{n-1},d) \subset \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^{n-1},d)$ main irred. component
  closure of $\{[u\colon \Sigma \longrightarrow \mathbb{P}^{n-1}] : \Sigma \text{ is smooth}\}$
- $\widetilde{\mathfrak{M}}^0_{g,k}(\mathbb{P}^{n-1},d) \longrightarrow \overline{\mathfrak{M}}^0_{g,k}(\mathbb{P}^{n-1},d)$ natural desingularization
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From $X_n \subset \mathbb{P}^{n-1}$ to $\mathbb{P}^{n-1}$

$$\begin{array}{ccc} \mathcal{L} \equiv \mathcal{O}(n) & & \widetilde{\mathfrak{M}}^0_{g,k}(\mathcal{L}, d) \\ \downarrow \pi & & \\ \mathbb{P}^{n-1} & & \widetilde{\mathfrak{M}}^0_{g,k}(\mathbb{P}^{n-1}, d) \end{array}$$

$g = 1$ Hyperplane Property: sufficient to compute

$$F(Q) \equiv \sum_{d=1}^{\infty} Q^d \int_{\widetilde{\mathfrak{M}}^0_{1,1}(\mathbb{P}^{n-1},d)} e(\widetilde{\mathcal{V}}_{1,d})\,\mathrm{ev}_1^* x$$

$x \in H^2(\mathbb{P}^{n-1})$ hyperplane class

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Torus Actions

- $\mathbb{T} \equiv (\mathbb{C}^*)^n$ acts on $\mathbb{P}^{n-1}$ (with n fixed pts)
- $\Longrightarrow$ on $\widetilde{\mathcal{V}}_{g,d} \longrightarrow \widetilde{\mathfrak{M}}^0_{g,k}(\mathbb{P}^{n-1}, d)$ by composition with simple fixed loci
- $\Longrightarrow$ Atiyah-Bott Localization Thm reduces

$$\int_{\widetilde{\mathfrak{M}}^0_{g,k}(\mathbb{P}^{n-1},d)} e(\mathcal{V}_{g,d})\eta$$

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- split genus 1 graphs into many genus 0 graphs at special vertex
- make use of good properties of genus 0 numbers to eliminate infinite sums
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Genus 0 Data

What we know

- $H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) = \mathbb{Q}[x,\alpha_1,\ldots,\alpha_n]/\prod_k(x-\alpha_k)$
- With $\mathrm{ev}_1,\mathrm{ev}_2 : \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d) \longrightarrow \mathbb{P}^{n-1}$,
  - Givental'96:

$$Z^*(\hbar,x,Q) \equiv \sum_{d=1}^{\infty} Q^d \mathrm{ev}_{1*}\left(\frac{e(\mathcal{V}_{0,d})}{\hbar-\psi_1}\right) \in \mathbb{Q}(x,\alpha)[[\hbar^{-1},Q]]$$

  - Z'07:

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What we know

- $H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) = \mathbb{Q}[x, \alpha_1, \ldots, \alpha_n] / \prod_k (x - \alpha_k)$
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$\mathcal{Z}_i^* \equiv \mathcal{Z}(x=\alpha_i)$ satisfies: for all $a \geq 0$

$$\sum_{m=2}^{\infty} \frac{1}{m(m-1)} \sum_{\substack{\sum a_l = m-2-a \\ a_l \geq 0}} \frac{(-1)^{a_l}}{a_l!} \mathfrak{R}_{\hbar=0}\{\hbar^{-a_l} \mathcal{Z}_i^*(\hbar)\} = a!\, \mathfrak{R}_{\hbar=0}\{\hbar^{a+1} \mathcal{Z}_i^*(\hbar)\} \quad \boxed{a}$$

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Each genus 1 graphs breaks at special node into genus 0 strands:

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