On the Geometry of Genus 1 Gromov-Witten Invariants

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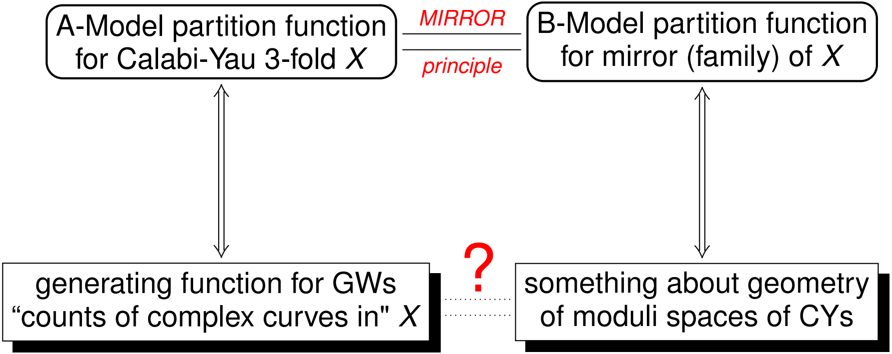
From String Theory to Enumerative Geometry



“Simplest" Calabi-Yau 3-fold

- quintic 3-fold $X_5 =$ degree 5 hypersurface in $\mathbb{P}^4$
- expected # of genus g degree d curves is finite: $n_{g,d}$
- genus g degree d GW-invariant $N_{g,d}$ is made up of $n_{h,d}$
- A-model partition function:

$$F_g^A(q) = \sum_{d=1}^{\infty} N_{g,d} q^d .$$

- B-model partition function F_g^B “measures" geometry of moduli spaces of CYs

B-Side Computations

- Candelas-de la Ossa-Green-Parkes'91
  construct mirror family, compute F_0^B
- Bershadsky-Cecotti-Ooguri-Vafa'93 (BCOV)
  compute F_1^B using physics arguments
- Fang-Z. Lu-Yoshikawa'03 compute F_1^B mathematically
- Huang-Klemm-Quackenbush'06
  compute F_g^B, $g \leq 52$ using physics

Mirror Symmetry Predictions and Verifications

Predictions

$$F_g^A(q) \equiv \sum_{d=1}^{\infty} N_{g,d} q^d \stackrel{?}{=} F_g^B(q).$$

Theorem (Givental'96, Lian-Liu-Yau'97,.........∼'00)

$g = 0$ predict. of Candelas-de la Ossa-Green-Parkes'91 holds

Theorem (Z.'07)

$g = 1$ predict. of Bershadsky-Cecotti-Ooguri-Vafa'93 holds

General Approach to Verifying $F_g^A = F_g^B$ (works for $g = 0, 1$)

Need to compute each $N_{g,d}$ and all of them (for fixed g):

Step 1: relate $N_{g,d}$ to GWs of $\mathbb{P}^4 \supset X_5$

Step 2: use $(\mathbb{C}^*)^5$-action on $\mathbb{P}^4$ to compute each $N_{g,d}$ by localization

Step 3: find some recursive feature(s) to compute $N_{g,d}\ \forall d \iff F_g^A$

GW-Invariants of $X_5 \subset \mathbb{P}^4$

$$\overline{\mathfrak{M}}_g(X_5,d) = \big\{[u\colon \Sigma \longrightarrow X_5] \big|\ g(\Sigma)=g, \deg u=d,\ \bar\partial u = 0\big\}$$

$$\begin{aligned} N_{g,d} &\equiv \deg\big[\overline{\mathfrak{M}}_g(X_5,d)\big]^{vir} \\ &\equiv \#\big\{[u\colon \Sigma \longrightarrow X_5] \big|\ g(\Sigma)=g, \deg u=d,\ \bar\partial u = \nu(u)\big\} \end{aligned}$$

ν = small generic deformation of $\bar\partial$-equation

From $X_5 \subset \mathbb{P}^4$ to $\mathbb{P}^4$

$$\begin{array}{ccc} \mathcal{L} \equiv \mathcal{O}(5) & & \mathcal{V}_{g,d} \equiv \overline{\mathfrak{M}}_g(\mathcal{L}, d) \\ s \uparrow\downarrow \pi & & \tilde{s} \uparrow\downarrow \tilde{\pi} \\ X_5 \equiv s^{-1}(0) \subset \mathbb{P}^4 & & \overline{\mathfrak{M}}_g(X_5, d) \equiv \tilde{s}^{-1}(0) \subset \overline{\mathfrak{M}}_g(\mathbb{P}^4, d) \end{array}$$

$$\tilde{\pi}\big([\xi\colon \Sigma \longrightarrow \mathcal{L}]\big) = [\pi\circ\xi\colon \Sigma \longrightarrow \mathbb{P}^4]$$
$$\tilde{s}\big([u\colon \Sigma \longrightarrow \mathbb{P}^4]\big) = [s\circ u\colon \Sigma \longrightarrow \mathcal{L}]$$

From $X_5 \subset \mathbb{P}^4$ to $\mathbb{P}^4$

$$\begin{array}{ccc} \mathcal{L} \equiv \mathcal{O}(5) & & \mathcal{V}_{g,d} \equiv \overline{\mathfrak{M}}_g(\mathcal{L}, d) \\ s \uparrow \downarrow \pi & & \tilde{s} \uparrow \downarrow \tilde{\pi} \\ X_5 \equiv s^{-1}(0) \subset \mathbb{P}^4 & & \overline{\mathfrak{M}}_g(X_5, d) \equiv \tilde{s}^{-1}(0) \subset \overline{\mathfrak{M}}_g(\mathbb{P}^4, d) \end{array}$$

This suggests: *Hyperplane Property*

$$N_{g,d} \equiv \deg \left[\overline{\mathfrak{M}}_g(X_5, d)\right]^{vir} \equiv \, ^{\pm}\left|\tilde{s}^{-1}(0)\right|$$
$$\stackrel{?}{=} \left\langle e(\mathcal{V}_{g,d}), \overline{\mathfrak{M}}_g(\mathbb{P}^4, d)\right\rangle$$

Genus 0 vs. Positive Genus

$g = 0$ everything is as expected:

- $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ is smooth
- $[\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)]^{vir} = [\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)]$
- $\mathcal{V}_{0,d} \longrightarrow \overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ is vector bundle
- hyperplane prop. makes sense and holds

$g \geq 1$ none of these holds

Genus 1 Analogue

Thm. A (J. Li–Z.'04): HP holds for reduced genus 1 GWs

$$\big[\overline{\mathfrak{M}}_1(X_5,d)\big]^{vir} = e(\mathcal{V}_{1,d}) \cap \overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d).$$

This generalizes to complete intersections $X \subset \mathbb{P}^n$.

- $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d) \subset \overline{\mathfrak{M}}_1(\mathbb{P}^4,d)$ main irred. component closure of $\big\{[u\colon \Sigma \longrightarrow \mathbb{P}^4] \in \overline{\mathfrak{M}}_1(\mathbb{P}^4,d) : \Sigma \text{ is smooth}\big\}$
- $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d)$ not vector bundle, but $e(\mathcal{V}_{1,d})$ well-defined (0-set of generic section)

Standard vs. Reduced GWs

$$\text{Thm. A} \implies N^0_{1,d} \equiv \deg\,[\overline{\mathfrak{M}}^0_1(X,d)]^{vir} = \int_{\overline{\mathfrak{M}}^0_1(\mathbb{P}^4,d)} e(\mathcal{V}_{1,d})$$

$$\overline{\mathfrak{M}}^0_1(X,d) \equiv \overline{\mathfrak{M}}^0_1(\mathbb{P}^4,d) \cap \overline{\mathfrak{M}}_1(X,d)$$

Thm. B (Z.'04,'07): $N_{1,d} - N^0_{1,d} = \frac{1}{12} N_{0,d}$

This generalizes to all symplectic manifolds:

$$[\text{standard}] - [\text{reduced genus 1 GW}] = f(\text{genus 0 GW})$$

$\therefore$ to check BCOV, enough to compute $\int_{\overline{\mathfrak{M}}^0_1(\mathbb{P}^4,d)} e(\mathcal{V}_{1,d})$

Torus Actions

- $(\mathbb{C}^*)^5$ acts on $\mathbb{P}^4$ (with 5 fixed pts)
- $\Longrightarrow$ on $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ (with simple fixed loci)
  and on $\mathcal{V}_{g,d} \longrightarrow \overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$
- $\int_{\overline{\mathfrak{M}}_g^0(\mathbb{P}^4,d)} e(\mathcal{V}_{g,d})$ localizes to fixed loci

  $g = 0$: Atiyah-Bott Localization Thm reduces $\int$ to $\sum_{graphs}$

  $g = 1$: $\overline{\mathfrak{M}}_g^0(\mathbb{P}^4, d), \mathcal{V}_{g,d}$ singular $\Longrightarrow$ AB does not apply

Genus 1 Bypass

Thm. C (Vakil–Z.'05): $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)$ admit natural desingularizations:

$$\begin{array}{ccc} \widetilde{\mathcal{V}}_{1,d} & \longrightarrow & \mathcal{V}_{1,d} \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{M}}_1^0(\mathbb{P}^4, d) & \longrightarrow & \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d) \end{array}$$

$$\Longrightarrow \qquad \int_{\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d)} e(\mathcal{V}_{1,d}) = \int_{\widetilde{\mathfrak{M}}_1^0(\mathbb{P}^4,d)} e(\widetilde{\mathcal{V}}_{1,d})$$

Computation of Genus 1 GWs of CIs

Thm. C generalizes to all $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d)$:

$$\begin{array}{ccc} \mathcal{L} \equiv \mathcal{O}(a) & & \mathcal{V}_{1,d} \equiv \overline{\mathfrak{M}}_{1,k}(\mathcal{L}, d) \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ \mathbb{P}^n & & \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n, d) \end{array}$$

∴ Thms A,B,C provide an algorithm for computing genus 1 GWs of complete intersections $X \subset \mathbb{P}^n$

Computation of $N_{1,d}$ for all d

- split genus 1 graphs into many genus 0 graphs at special vertex
- make use of good properties of genus 0 numbers to eliminate infinite sums
- extract non-equivariant part of elements in $H^*_{\mathbb{T}}(\mathbb{P}^4)$

Key Geometric Foundation

A Sharp Gromov's Compactness Thm in Genus 1 (Z.'04)

- describes limits of sequences of pseudo-holomorphic maps
- describes limiting behavior for sequences of solutions of a $\bar{\partial}$-equation with limited perturbation
- allows use of topological techniques to study genus 1 GWs

Main Tool

Analysis of Local Obstructions

- study obstructions to smoothing pseudo-holomorphic maps from smooth domains
- not just potential existence of obstructions