Enumerative Geometry: from Classical to Modern

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Summary

- Classical enumerative geometry: examples
- Modern tools: Gromov-Witten invariants
  counts of holomorphic maps
- Insights from string theory:
  - quantum cohomology: refinement of usual cohomology
  - mirror symmetry formulas
    duality between symplectic/holomorphic structures
  - integrality predictions for GW-invariants
    geometric explanation yet to be discovered

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What is Classical EG about?

How many geometric objects satisfy given geometric conditions?

objects = curves, surfaces, ...

conditions = passing through given points, curves,...
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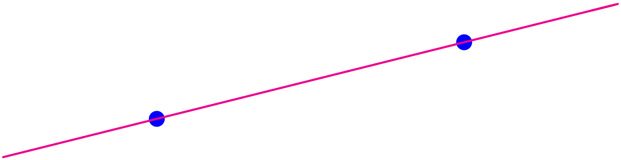

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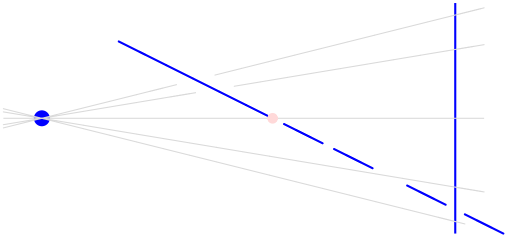

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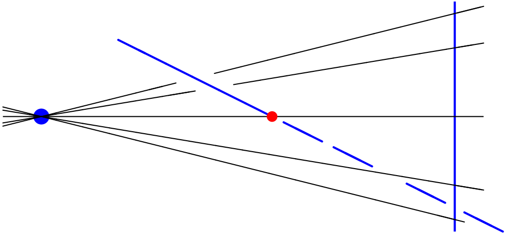

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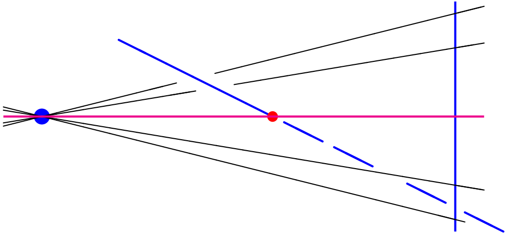

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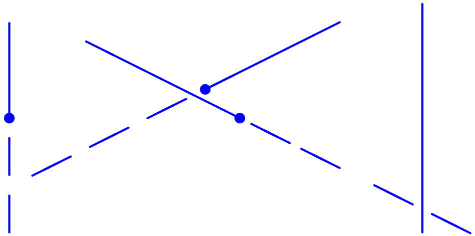

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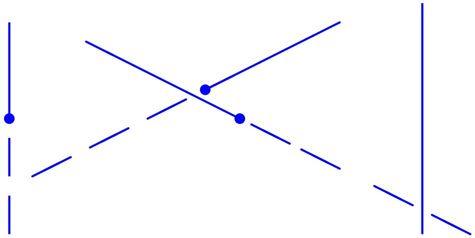

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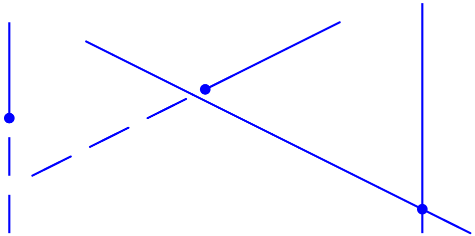

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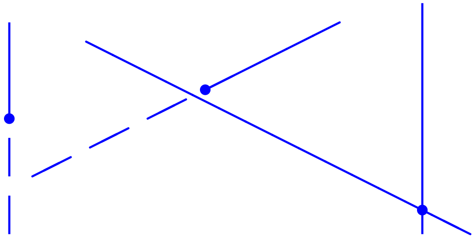

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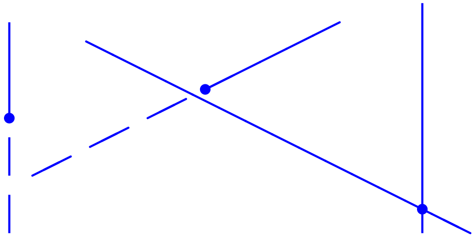

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General Line Counting Problems

All/most line counting problems in vector space V reduce to computing intersections of cycles on

$$G(2, V) \equiv \{2 \text{ dim linear subspaces of } V\} \cong \{(\text{affine}) \text{ lines in } V\}$$

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Q: How many lines pass through 2 distinct points?

A line in the plane is described by $(A,B,C)\neq 0$:

$$Ax+By+C=0.$$

(A,B,C) and (A',B',C') describe the same line iff

$$(A',B',C')=\lambda(A,B,C)$$

$$\therefore\quad \{\text{lines in }(x,y)\text{-plane}\}=\{\text{1dim lin subs of }(A,B,C)\text{-space}\}$$
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degree d curve in (x,y)-plane

$\equiv$ 0-set of nonzero degree d polynomial in (x,y)

polynomials Q and Q' determine same curve iff $Q' = \lambda Q$

coefficients of Q is $\binom{d+2}{2}$ $\quad\Longrightarrow$

$$\{\text{deg d curves in } (x,y)\text{-plane}\} = \left\{\text{1dim lin subs of } \binom{d+2}{2}\text{-dim v.s.}\right\}$$

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"Passing thr a point" = 1 linear eqn on coefficients of Q

$\Longrightarrow$ get hyperplane in $\binom{d+2}{2}$-dim v.s. of coefficients

$$\{\text{deg d curves in } (x,y)\text{-plane thr. } (x_i,y_i)\} \approx \mathbb{P}^{N(d)-1} \subset \mathbb{P}^{N(d)}$$

intersection of $\binom{d+2}{2}-1$ HPs in $\binom{d+2}{2}$-dim v.s. is 1 1dim lin subs

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∃! degree d plane curve thr $N(d) \equiv \binom{d+2}{2} - 1$ general pts

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Typical Enumerative Problems

Count complex curves = (singular) Riemann surfaces Σ

of fixed genus g, fixed degree d

in $\mathbb{C}^n$, $\mathbb{C}P^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \ldots \sqcup \mathbb{C}^0$

in a hypersurface $Y \subset \mathbb{C}^n, \mathbb{C}P^n$ (0-set of a polynomial)

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Genus 0 Plane Quartics thr 11 pts

$n_4 = \#$ plane quartics thr 11 pts with 3 non-separating nodes

Zeuthen'1870s: $n_4 = 620 = 675 - 55$

$3! \cdot 675$ = euler class of rank 9 vector bundle over $\mathbb{C}P^3 \times (\mathbb{C}P^2)^3$
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$\mathbb{C}P^3$ = quartics thr 11 pts; $\mathbb{C}P^2$ = possibilities for i-th node

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Kontsevich's Formula (Ruan-Tian'1993)

$n_d \equiv$ # genus 0 degree d plane curves thr. (3d-1) general pts
$n_1 = 1$

$$n_d = \frac{1}{6(d-1)} \sum_{d_1+d_2=d} \left(d_1 d_2 - 2\frac{(d_1-d_2)^2}{3d-2} \right) \binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}$$

$n_2 = 1,\ n_3 = 12,\ n_4 = 620,\ n_5 = 87,304,\ n_6 = 26,312,976,\ \ldots$

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$$n_d = \frac{1}{6(d-1)} \sum_{d_1+d_2=d} \left(d_1 d_2 - 2\frac{(d_1-d_2)^2}{3d-2} \right) \binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}$$

$n_2 = 1$, $n_3 = 12$, $n_4 = 620$, $n_5 = 87,304$, $n_6 = 26,312,976$, ...

Gromov's 1985 paper

Consider equivalence classes of maps $f\colon (\Sigma, j) \longrightarrow \mathbb{C}P^n$
(Σ, j)=connected Riemann surface, possibly with nodes

$f : (\Sigma, j) \longrightarrow \mathbb{C}P^n$ and $f' : (\Sigma', j') \longrightarrow \mathbb{C}P^n$ are equivalent if
$f = f' \circ \tau$ for some $\tau\colon (\Sigma, j) \longrightarrow (\Sigma', j')$

$f\colon (\Sigma, j) \longrightarrow \mathbb{C}P^n$ is stable if

$$\mathrm{Aut}(f) \equiv \{\tau\colon (\Sigma, j) \longrightarrow (\Sigma, j) | f \circ \tau = f\} \qquad \text{is finite}$$

non-constant holomorphic $f\colon (\Sigma, j) \longrightarrow \mathbb{C}P^n$ is stable iff
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Gromov's Compactness Theorem

genus of $f : (\Sigma, j) \longrightarrow \mathbb{C}P^n$ is # of holes in Σ ($\geq g(\Sigma)$)

degree d of $f \equiv |f^{-1}(H)|$ for a generic hyperplane:

$$f_*[\Sigma] = d[\mathbb{C}P^1] \in H_2(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[\mathbb{C}P^1]$$

Theorem: With respect to a natural topology,

$$\overline{\mathfrak{M}}_g(\mathbb{C}P^n, d) \equiv \{[f : (\Sigma, j) \longrightarrow \mathbb{C}P^n] : g(f) = g,\ d(f) = d,\ f \text{ holomor}\}$$

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Maps vs. Curves

Image of holomorphic $f\colon (\Sigma, j) \longrightarrow \mathbb{C}P^n$ is a curve

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$$= \#\text{ degree d } f\colon (S^2, j) \longrightarrow \mathbb{C}P^2 \text{ s.t. } p_i \in f(\mathbb{C}P^1)$$
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Physics Insight I: Quantum (Co)homology

Use counts of genus 0 maps to $\mathbb{C}P^n$ to deform $\cup$-product on H^*,

$$H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/x^{n+1}, \qquad x^a \cup x^b = x^{a+b},$$

to $*$-product on $H^*(\mathbb{C}P^n)[q_0, \ldots, q_n]$

$x^a * x^b = x^{a+b} +$ q-corrections counting genus 0 maps thr. $\mathbb{C}P^{n-a}$, $\mathbb{C}P^{n-b}$

Theorem (McDuff-Salamon'93, Ruan-Tian'93, ...)

The product $*$ is associative

$*$ generalizes to all cmpt algebraic/symplectic manifolds

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Associativity of quantum multiplication is equivalent to

- Kontsevich's formula for $\mathbb{C}P^2$, extension to $\mathbb{C}P^n$
- gluing formula for counts of genus 0 maps

Remark: Classical proof of Kontsevich's formula for $\mathbb{C}P^2$ only: Z. Ran'95, elaborating on '89

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Other Enumerative Applications of Stable Maps

- Genus 0 with singularities: Pandharipande, Vakil, Z.-
- Genus 1: R. Pandharipande, Ionel, Z.-
- Genus 2,3: Z.-

Gromov-Witten Invariants

$Y = \mathbb{C}P^n$, = hypersurface in $\mathbb{C}P^n$ (0-set of a polynomial),...
$\mu_1, \ldots, \mu_k \subset Y$ cycles

$$\mathrm{GW}^Y_{g,d}(\mu) \equiv \text{"\#"}\{[f\colon (\Sigma, j) \longrightarrow Y] \in \overline{\mathfrak{M}}_g(Y,d) \colon f(\Sigma) \cap \mu_i \neq \emptyset\}$$

$g=0$, $Y=\mathbb{C}P^n$: $\overline{\mathfrak{M}}_g(Y,d)$ is smooth, of expected dim, "#"=#

Typically, $\overline{\mathfrak{M}}_g(Y,d)$ is highly singular, of wrong dim

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Example: Quintic Threefold

$Y_5 \subset \mathbb{C}P^4$ 0-set of a degree 5 polynomial Q

Schubert Calculus: Y_5 contains 2,875 (isolated) lines

S. Katz'86 (via Schubert): Y_5 contains 609,250 conics

For each line $L \subset Y_5$ and conic $C \subset Y_5$,

$$\{[f:(\Sigma,j)\longrightarrow Y_5]\in\overline{\mathfrak{M}}_0(Y_5,2): f(\Sigma)\subset L\}\approx\overline{\mathfrak{M}}_0(\mathbb{C}P^1,2)$$
$$\{[f:(\Sigma,j)\longrightarrow Y_5]\in\overline{\mathfrak{M}}_0(Y_5,2): f(\Sigma)\subset C\}\approx\overline{\mathfrak{M}}_0(\mathbb{C}P^1,1)$$

are connected components of $\overline{\mathfrak{M}}_0(Y_5,2)$ of dimensions 2 and 0

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Expected dimension of $\overline{\mathfrak{M}}_0(Y_5, d)$

$Y_5 = Q^{-1}(0) \subset \mathbb{C}P^4$ for a degree 5 polynomial Q

$$\Longrightarrow \quad \overline{\mathfrak{M}}_0(Y_5, d) = \big\{[f\colon (\Sigma, j) \longrightarrow \mathbb{C}P^4] \in \overline{\mathfrak{M}}_0(\mathbb{C}P^4, d) : Q \circ f = 0\big\}$$

holomorphic degree d $f\colon \mathbb{C}P^1 \longrightarrow \mathbb{C}P^4$ has the form

$$f([u, v]) = \big[R_1(u, v), \ldots, R_5(u, v)\big]$$

$R_1, \ldots, R_5 =$ homogeneous polynomials of degree d

$$\Longrightarrow \qquad \dim \overline{\mathfrak{M}}_0(\mathbb{C}P^4, d) = 5 \cdot (d+1) - 1 - 3$$

$Q \circ f$ is homogen of degree $5d$

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Expected dimension of $\overline{\mathfrak{M}}_0(Y_5,d)$

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A more elaborate computation gives

$$\dim^{vir}\overline{\mathfrak{M}}_g(Y_5,d) = 0 \qquad \forall\, g$$

$\Longrightarrow$ want to define

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GW-Invariants of $Y_5 \subset \mathbb{C}P^4$

$$\overline{\mathfrak{M}}_g(Y_5,d) = \big\{[f\colon (\Sigma,j) \longrightarrow Y_5] \big|\, g(f)=g, d(f)=d,\ \bar\partial_j f = 0\big\}$$

$$\bar\partial_j f \equiv df + J_{Y_5} \circ df \circ j$$

$$N_{g,d} \equiv \big|\overline{\mathfrak{M}}_g(Y_5,d)\big|^{vir}$$
$$\equiv \#\big\{[f\colon (\Sigma,j) \longrightarrow Y_5] \big|\, g(f)=g, d(f)=d,\ \bar\partial_j f = \nu(f)\big\}$$

ν = small generic deformation of $\bar\partial$-equation

ν multi-valued $\Longrightarrow N_{g,d} \in \mathbb{Q}$

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$$N_{g,d} \equiv \big|\overline{\mathfrak{M}}_g(Y_5,d)\big|^{vir}$$

$$\equiv \#\big\{[f\colon(\Sigma,j)\longrightarrow Y_5] \,\big|\, g(f)=g,\, d(f)=d,\, \bar\partial_j f = \nu(f)\big\}$$

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GW-Invariants of $Y_5 \subset \mathbb{C}P^4$

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What is special about Y_5?

Y_5 is Calabi-Yau 3-fold:

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A-Model partition function for Calabi-Yau 3-fold Y —MIRROR principle— B-Model partition function for mirror (family) of Y

generating function for GWs of Y: $F_g^A(q) = \sum_{d=1}^{\infty} N_{g,d} q^d$

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B-Side Computations for $Y = Y_5$

- Candelas-de la Ossa-Green-Parkes'91
  construct mirror family, compute F_0^B
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Mirror Symmetry Predictions and Verifications

Predictions

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Theorem (Givental'96, Lian-Liu-Yau'97,..........∼'00)

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General Approach to Verifying $F_g^A = F_g^B$ (works for $g = 0, 1$)

Need to compute each $N_{g,d}$ and all of them (for fixed g):

Step 1: relate $N_{g,d}$ to GWs of $\mathbb{C}P^4 \supset Y_5$

Step 2: use $(\mathbb{C}^*)^5$-action on $\mathbb{C}P^4$ to compute each $N_{g,d}$ by localization

Step 3: find some recursive feature(s) to compute $N_{g,d}$ $\forall d$ $\Longleftrightarrow F_g^A$

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From $Y_5 \subset \mathbb{C}P^4$ to $\mathbb{C}P^4$

$\mathcal{L}$ $\qquad$ $\overline{\mathfrak{M}}_g(\mathcal{L}, d)$

$Y_5 \equiv Q^{-1}(0) \subset \mathbb{C}P^4$ $\qquad$ $\overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)$

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$$\tilde{\pi}\big([\xi : \Sigma \longrightarrow \mathcal{L}]\big) = [\pi \circ \xi : \Sigma \longrightarrow \mathbb{C}P^4]$$

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This suggests: *Hyperplane Property*

$$N_{g,d} \equiv \left|\overline{\mathfrak{M}}_g(Y_5, d)\right|^{vir} \equiv \left|\tilde{Q}^{-1}(0)\right|^{vir}$$

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$$\stackrel{?}{=} \left\langle e(\mathcal{V}_{g,d}), [\overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)]^{vir} \right\rangle$$

Genus 0 vs. Positive Genus

$g = 0$ everything is as expected:

- $\overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)$ is smooth
- $[\overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)]^{vir} = [\overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)]$
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Genus 1 Analogue

Thm. A (J. Li–Z.'04): HP holds for reduced genus 1 GWs

$$\left|\overline{\mathfrak{M}}_1^0(Y_5,d)\right|^{vir} = e(\mathcal{V}_{1,d}) \cap \overline{\mathfrak{M}}_1^0(\mathbb{C}P^4,d).$$

This generalizes to complete intersections $Y \subset \mathbb{C}P^n$.

- $\overline{\mathfrak{M}}_1^0(\mathbb{C}P^4,d) \subset \overline{\mathfrak{M}}_1(\mathbb{C}P^4,d)$ main irred. component
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$$\text{Thm. A} \implies N^0_{1,d} \equiv \left|\overline{\mathfrak{M}}^0_1(Y_5,d)\right|^{vir} = \int_{\overline{\mathfrak{M}}^0_1(\mathbb{C}P^4,d)} e(\mathcal{V}_{1,d})$$

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Thm. B (Z.'04,'07): $N_{1,d} - N^0_{1,d} = \frac{1}{12} N_{0,d}$

This generalizes to all symplectic manifolds:

$$[\text{standard}] - [\text{reduced genus 1 GW}] = F(\text{genus 0 GW})$$

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Torus Actions

- $(\mathbb{C}^*)^5$ acts on $\mathbb{C}P^4$ (with 5 fixed pts)
- $\Longrightarrow$ on $\overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)$ (with simple fixed loci)
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$g=0$: Atiyah-Bott Localization Thm reduces $\int$ to $\sum_{graphs}$

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Genus 1 Bypass

Thm. C (Vakil–Z.'05): $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{C}P^4, d)$ admit natural desingularizations:

$$\begin{array}{ccc} \widetilde{\mathcal{V}}_{1,d} & \longrightarrow & \mathcal{V}_{1,d} \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{M}}_1^0(\mathbb{C}P^4, d) & \longrightarrow & \overline{\mathfrak{M}}_1^0(\mathbb{C}P^4, d) \end{array}$$

$$\Longrightarrow \qquad \int_{\overline{\mathfrak{M}}_1^0(\mathbb{C}P^4,d)} e(\mathcal{V}_{1,d}) = \int_{\widetilde{\mathfrak{M}}_1^0(\mathbb{C}P^4,d)} e(\widetilde{\mathcal{V}}_{1,d})$$

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Computation of Genus 1 GWs of CIs

Thm. C generalizes to all $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{C}P^n, d)$:

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- split genus 1 graphs into many genus 0 graphs at special vertex
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