

On Gromov-Witten Invariants of a Quintic Threefold and a Rigidity Conjecture

Jun Li* and Aleksey Zinger†

January 1, 2008

Abstract

We show that a widely believed conjecture concerning rigidity of genus-zero and -one holomorphic curves in Calabi-Yau threefolds implies a relation between the genus-one GW-invariants of a quintic threefold in \mathbb{P}^4 and the genus-zero and genus-one GW-invariants of \mathbb{P}^4 . This relation is a special case of a general formula for the genus-one GW-invariants of complete intersections obtained in a previous paper. In contrast to the general case, this paper's derivation is more geometric and makes direct use of the rigidity property. Thus, it provides further evidence for the rigidity conjecture in low genera. On the other hand, this paper also suggests a potential way of disproving the less commonly believed generalization of the rigidity conjecture to arbitrary genus.

Contents

1	Introduction	2
1.1	Summary	2
1.2	Rigidity Properties	5
2	Preliminaries	7
2.1	Review of Key Definitions	7
2.2	Components of the Proof	9
2.3	Summary of the Proof of Proposition 2.2	15
2.4	Notation: Genus-Zero Maps	18
2.5	Notation: Genus-One Maps	19
3	On Genus-One Gromov-Witten Invariants	24
3.1	Setup	24
3.2	Proof of Proposition 2.5	25
3.3	Proof of Proposition 2.6	31

*Partially supported by NSF grant DMS-0200477

†Partially supported by an NSF Postdoctoral Fellowship

4	On the Euler Class of the Cone $\mathcal{V}_1^d \rightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$	38
4.1	The Structure of the Moduli Spaces $\overline{\mathfrak{M}}_1^0(\mathbb{P}^n, d; J)$	38
4.2	The Structure of the Cone $\mathcal{V}_1^d \rightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$	40
4.3	Proof of Proposition 2.3	44
4.4	A Genus-One Gluing Procedure	50

1 Introduction

1.1 Summary

Suppose $\gamma \rightarrow \mathbb{P}^n$ is the tautological line bundle, $a \in \mathbb{Z}^+$, and

$$\mathcal{L} = \gamma^* \otimes a.$$

If $s \in H^0(\mathbb{P}^n; \mathcal{L})$ is a generic holomorphic section,

$$Y \equiv s^{-1}(0)$$

is a smooth hypersurface in \mathbb{P}^n . It has long been known how to express the genus-zero Gromov-Witten invariants of Y in terms of the genus-zero GW-invariants of \mathbb{P}^n ; see (1.1) below for a special case. The latter can be computed using the classical localization theorem of [ABo]. In [LZ], we prove a genus-one analogue of (1.1) for an arbitrary hypersurface Y . The proof itself is rather simple. However, it relies on the constructions of **reduced** genus-one GW-invariants in [Z6] and of euler classes of certain natural cones in a setting more general than in [Z5]. The latter in fact constitutes most of [LZ].

In this paper, we rederive a genus-one analogue of (1.1) for a quintic threefold Y in \mathbb{P}^4 , i.e. for $a=5$ in the above notation, in a direct, albeit more laborious, way from a certain rigidity property for genus-zero and genus-one J -holomorphic curves in a quintic threefold; see the next subsection. While it is not known whether the rigidity property is satisfied, it is widely believed to be the case for curves of low genus (at least 0 and 1 and likely up to genus 2 or 3). Our derivation generalizes to arbitrary Calabi-Yau complete-intersection threefolds in projective spaces. It can be used for Fano complete-intersection threefolds as well, but in such cases it can be obtained by taking $\nu=0$ in Subsection 2.2 of [LZ]. In the Calabi-Yau cases, this cannot be done and this paper's derivation is different from that in Subsection 2.2 of [LZ].

Quintic threefolds, as well as other Calabi-Yau manifolds, play a prominent role in theoretical physics. As a result physicists have made a number of important predictions concerning CY-manifolds. Some of these predictions have been verified mathematically; others have not. This paper indicates that one of them fits in nicely with known mathematical facts.

If X is a Kahler manifold, g and k are nonnegative integers, and $A \in H_2(X; \mathbb{Z})$, we denote by $\overline{\mathfrak{M}}_{g,k}(X, A)$ the moduli space of (equivalence classes of) stable holomorphic maps from genus- g curves with k marked points in the homology class A . Let

$$\overline{\mathfrak{M}}_g(X, A) = \overline{\mathfrak{M}}_{g,0}(X, A).$$

If $\iota: Y \rightarrow \mathbb{P}^n$ is an inclusion and ℓ is the homology class of a line in \mathbb{P}^n , let

$$\overline{\mathfrak{M}}_{g,k}(Y, d) = \bigcup_{\iota_* A = d\ell} \overline{\mathfrak{M}}_{g,k}(Y, A).$$

If Y is a Calabi-Yau threefold, the virtual, or expected dimension, of $\overline{\mathfrak{M}}_g(Y, d)$ is zero. The virtual degree of $\overline{\mathfrak{M}}_g(Y, d)$ is the genus- g degree- d GW-invariant of Y . If Y is a quintic threefold, we denote this invariant by $N_g(d)$.

Let

$$\pi_g^d: \mathfrak{U}_g(\mathbb{P}^n, d) \rightarrow \overline{\mathfrak{M}}_g(\mathbb{P}^n, d) \quad \text{and} \quad \text{ev}_g^d: \mathfrak{U}_g(\mathbb{P}^n, d) \rightarrow \mathbb{P}^n$$

be the semi-universal family and the natural evaluation map. In other words, the fiber of π_g^d over $[\mathcal{C}, u]$ is the curve \mathcal{C} , while

$$\text{ev}_g^d([\mathcal{C}, u; z]) = u(z) \quad \text{if } z \in \mathcal{C}.$$

We define a section s_g^d of the sheaf $\pi_{g*}^d \text{ev}_g^{d*} \mathfrak{L} \rightarrow \overline{\mathfrak{M}}_g(\mathbb{P}^n, d)$ by

$$s_g^d([\mathcal{C}, u]) = [s \circ u].$$

If $Y = s^{-1}(0)$, $\overline{\mathfrak{M}}_g(Y, d)$ is the zero set of this section.

If $a=5$, it has long been known that

$$N_0(d) = \langle e(\pi_{0*}^d \text{ev}_0^{d*} \mathfrak{L}), [\overline{\mathfrak{M}}_0(\mathbb{P}^4, d)] \rangle. \quad (1.1)$$

The moduli space $\overline{\mathfrak{M}}_0(\mathbb{P}^4, d)$ is a smooth orbivariety and

$$\pi_{0*}^d \text{ev}_0^{d*} \mathfrak{L} \rightarrow \overline{\mathfrak{M}}_0(\mathbb{P}^4, d) \quad (1.2)$$

is a locally free sheaf, i.e. a vector bundle. Furthermore,

$$\dim_{\mathbb{C}} \overline{\mathfrak{M}}_0(\mathbb{P}^4, d) = 5d + 1 \quad \text{and} \quad \text{rk}_{\mathbb{C}} \pi_{0*}^d \text{ev}_0^{d*} \mathfrak{L} = 5d + 1.$$

Thus, the right-hand side of (1.1) is well-defined. It can be computed via the classical localization theorem of [ABo]. The complexity of this computation increases quickly with the degree d , but a closed formula has been obtained in [Ber], [Ga], [Gi], [Le], and [LLY].

If $g > 0$, the sheaf $\pi_{g*}^d \text{ev}_g^{d*} \mathfrak{L} \rightarrow \overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ is not locally free and does not define an euler class. Thus, the right-hand side of (1.1) does not even make sense if 0 is replaced by $g > 0$. Instead one might try to generalize (1.1) as

$$N_g(d) \stackrel{?}{=} \langle e(R^0 \pi_{g*}^d \text{ev}_g^{d*} \mathfrak{L} - R^1 \pi_{g*}^d \text{ev}_g^{d*} \mathfrak{L}), [\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)]^{vir} \rangle, \quad (1.3)$$

where $R^i \pi_{g*}^d \text{ev}_g^{d*} \mathfrak{L} \rightarrow \overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ is the i th direct image sheaf. The right-hand side of (1.3) can be computed via the virtual localization theorem of [GrP1]. However,

$$N_1(d) \neq \langle e(R^0 \pi_{1*}^d \text{ev}_1^{d*} \mathfrak{L} - R^1 \pi_{1*}^d \text{ev}_1^{d*} \mathfrak{L}), [\overline{\mathfrak{M}}_1(\mathbb{P}^4, d)]^{vir} \rangle,$$

according to a low-degree check of [GrP2] and [K].

Let

$$\mathfrak{M}_g^0(\mathbb{P}^4, d) = \{[\mathcal{C}, u] \in \overline{\mathfrak{M}}_g(\mathbb{P}^4, d) : \mathcal{C} \text{ is smooth}\}.$$

We denote by $\overline{\mathfrak{M}}_g^0(\mathbb{P}^4, d)$ the closure of $\mathfrak{M}_g^0(\mathbb{P}^4, d)$ in $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$. If $g > 0$, $\overline{\mathfrak{M}}_g^0(\mathbb{P}^4, d)$ is one of the many irreducible components of the moduli space $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$.

Theorem 1.1 *If d is a positive integer, $\mathfrak{L} = \gamma^{*\otimes 5} \longrightarrow \mathbb{P}^4$,*

$$\pi_1^d : \mathfrak{U}_1(\mathbb{P}^4, d) \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d) \quad \text{and} \quad \text{ev}_1^d : \mathfrak{U}_1(\mathbb{P}^4, d) \longrightarrow \mathbb{P}^4$$

are the semi-universal family and the natural evaluation map, respectively, then the euler class of the sheaf

$$\pi_{1*}^d \text{ev}_1^{d*} \mathfrak{L} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)$$

is well-defined. Furthermore, if Y satisfies the Rigidity Assumption of Subsection 1.2, then

$$N_1(d) = \frac{1}{12} N_0(d) + \langle e(\pi_{1*}^d \text{ev}_1^{d*} \mathfrak{L}), [\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)] \rangle. \quad (1.4)$$

The moduli space $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)$ is not a smooth orbifold. Nevertheless, it determines a fundamental class in $H_{10d}(\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d); \mathbb{Q})$, as its singularities are fairly simple. The sheaf

$$\pi_{1*}^d \text{ev}_1^{d*} \mathfrak{L} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d) \quad (1.5)$$

is not locally free. Nevertheless, its euler class is well-defined. In other words, the euler class of every desingularization of this sheaf is the same, in the sense described in Subsection 1.2 of [Z5]. The last expression in (1.4) can be computed via the classical localization theorem. Of course, the singularities of the space $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)$ cause additional complications. However, since these singularities can be understood, these complications can be handled. A desingularization of $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)$, i.e. a smooth orbifold $\widetilde{\mathfrak{M}}_1^0(\mathbb{P}^4, d)$ and a map

$$\tilde{\pi} : \widetilde{\mathfrak{M}}_1^0(\mathbb{P}^4, d) \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d),$$

which is biholomorphic onto $\mathfrak{M}_1^0(\mathbb{P}^4, d)$, is constructed in [VaZ]. This desingularization of $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)$ comes with a desingularization of the sheaf (1.5), i.e. a vector bundle

$$\tilde{\mathcal{V}} \longrightarrow \widetilde{\mathfrak{M}}_1^0(\mathbb{P}^4, d) \quad \text{s.t.} \quad \tilde{\pi}_* \tilde{\mathcal{V}} = \pi_{1*}^d \text{ev}_1^{d*} \mathfrak{L}.$$

In particular,

$$\langle e(\pi_{1*}^d \text{ev}_1^{d*} \mathfrak{L}), [\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)] \rangle = \langle e(\tilde{\mathcal{V}}), [\widetilde{\mathfrak{M}}_1^0(\mathbb{P}^4, d)] \rangle.$$

The localization theorem of [ABo] is directly applicable to the right-hand side of this equality.

Using Theorem 1.1 and the desingularization constructed in [VaZ], we have computed the numbers $N_1(d)$ for $d = 1, 2, 3, 4$. The results agree with those predicted in [BCOV]; see Subsection 1.3 in [LZ] for more details.

From the point of view of symplectic topology as described in [FuO] and [LT], the numbers $N_g(d)$ can be interpreted as the euler class of a vector bundle, albeit of an infinite-rank vector bundle over a space of the “same” dimension. As in the finite-dimensional case, this euler class is the number of zeros, counted with appropriate multiplicities, of a transverse (multivalued, admissible) section.

In brief, we prove (1.4) by slightly perturbing the complex structure J_0 on \mathbb{P}^4 , then expressing each of the three terms appearing in (1.4) as the number of zeros of a transverse section of a vector bundle and comparing the results for the two sides of (1.4). There are a vector bundle $\mathfrak{F} \rightarrow \mathfrak{X}$, possibly of infinite rank, and a section φ of \mathfrak{F} associated to each of three terms. The zero set of φ is easy to describe. However, φ is not transverse to the zero set. We determine the number $\mathcal{C}_{\mathcal{Z}}(\varphi)$ of zeros of $\varphi + \varepsilon$, for a small generic multisection ε , that lie near each stratum \mathcal{Z} of $\varphi^{-1}(0)$. These numbers in turn determine the contribution of each J -holomorphic curve in Y to the three numbers in (1.4). We will see that every such curve contributes equally to the two sides of (1.4).

Theorem 1.1 follows immediately from Propositions 1.3-1.5 and separately from Propositions 2.1-2.3. The first three propositions are easier to state and can be deduced from the last three propositions. While the statements of Propositions 2.1-2.3 are more technical, they are easier to prove.

The formula (1.4) also follows from Theorem 1.3 in [LZ] and Theorem 1.1 in [Z6], whether or not the Rigidity Assumption holds. Thus, Theorem (1.1) provides additional evidence for the genus-zero and -one cases of the Rigidity Conjecture; see the next subsection. On the other hand, the methods of [LZ] and [Z6] should extend to give relations between higher-genera GW-invariants of complete intersections and of projective spaces without any assumptions, while the methods of this paper should extend to deduce such relations directly from the Rigidity Conjecture. If the two formulas disagree in some genus g , the Rigidity Conjecture must be false in genus g and in all higher genera.

1.2 Rigidity Properties

Throughout the rest of the paper, Y will denote a quintic threefold in \mathbb{P}^4 . If J is an almost complex structure on Y , (Σ, j) is a Riemann surface, and $u: \Sigma \rightarrow Y$ is a J -holomorphic map, let

$$D_{J,u}: \Gamma(\Sigma; u^*TY) \rightarrow \Gamma(\Sigma; \Lambda_{j,j}^{0,1} T^* \Sigma \otimes u^*TY)$$

be the linearization of $\bar{\partial}_J$ -operator at u ; see Subsection 2.1.

Definition 1.2 *An almost complex structure J on Y satisfies the genus- g rigidity property if for every smooth connected genus- g Riemann surface (Σ, j) and nonconstant J -holomorphic map $u: \Sigma \rightarrow Y$*

- (J_Y1) $u(\Sigma)$ is a smooth curve;
- (J_Y2) $\ker D_{J,u} \subset \Gamma(\Sigma; u^*Tu(\Sigma))$.

If J satisfies the genus- g rigidity property, all genus- g J -holomorphic curves in Y are smooth and isolated. We denote by $\mathcal{J}(Y)$ the space of all C^1 -smooth almost complex structures on Y , with the C^1 -topology, and by $\mathcal{J}_{\text{rig}}^g(Y) \subset \mathcal{J}(Y)$ the subspace of almost complex structures that satisfy the genus- g rigidity property.

Rigidity Conjecture For all g and all Calabi-Yau threefolds Y , $\mathcal{J}_{\text{rig}}^g(Y)$ is dense in $\mathcal{J}(Y)$.

Rigidity Assumption If Y is a quintic threefold, the closure of $\mathcal{J}_{\text{rig}}^0(Y) \cap \mathcal{J}_{\text{rig}}^1(Y)$ in $\mathcal{J}(Y)$ contains J_0 .

We note that $\mathcal{J}_{\text{rig}}^g(Y)$ is open in $\mathcal{J}(Y)$. Thus, the $g=0,1$ cases of the Rigidity Conjecture imply our Rigidity Assumption.

Since the expected dimension of the moduli space $\overline{\mathfrak{M}}_g(Y, d; J)$ of genus- g degree- d J -holomorphic maps into Y is zero, it is easy to show that the property (J_Y1) of Definition 1.2 is satisfied by a generic almost complex structure J . However, despite years of attempts, this has not been shown to be the case for (J_Y2) , even for $g=0$. Nevertheless, this is believed to be case, though with some hesitation for g above 2 or 3.¹

For each $J \in \mathcal{J}(Y)$, let $\mathcal{S}_g^d(Y; J)$ be the set of J -holomorphic genus- g degree- d (simple) curves in Y . If $J \in \mathcal{J}_{\text{rig}}^g(Y)$, this set is finite. By Propositions 1.3 and 1.4 below, the number of elements in $\mathcal{S}_g^d(Y; J)$, counted with appropriate signs, is independent of $J \in \mathcal{J}_{\text{rig}}^g(Y)$ for $g=0,1$. We denote this number by $n_g(d)$. If $\sigma \in \mathbb{Z}^+$, let $m(\sigma)$ denote the number of degree σ covers of an elliptic curve by elliptic curves.

Proposition 1.3 For all $d \in \mathbb{Z}^+$,

$$N_0(d) = \sum_{\sigma|d} \frac{n_0(d/\sigma)}{\sigma^3}.$$

Proposition 1.4 For all $d \in \mathbb{Z}^+$,

$$N_1(d) = \frac{1}{12} \sum_{\sigma|d} \frac{n_0(d/\sigma)}{\sigma} + \sum_{\sigma|d} m(\sigma) \frac{n_1(d/\sigma)}{\sigma}.$$

Proposition 1.5 If d , \mathcal{L} , π_1^d , and ev_1^d are as in the statement of Theorem 1.1, then

$$\langle e(\pi_{1*}^d \text{ev}_1^{d*} \mathcal{L}), [\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)] \rangle = \frac{1}{12} \sum_{\sigma|d} \frac{\sigma^2 - 1}{\sigma^3} n_0(d/\sigma) + \sum_{\sigma|d} m(\sigma) \frac{n_1(d/\sigma)}{\sigma}.$$

We do not prove these three propositions as stated, since this is not necessary for the proof of Theorem 1.1. Instead, we prove the less elegant and more notationally involved Propositions 2.1-2.3, that also imply Theorem 1.1. Propositions 1.3, 1.4, and 1.5 can be derived from Propositions 2.1, 2.2, and 2.3, respectively; see the end of Subsection 2.2.

¹If $u: \Sigma \rightarrow Y$ is an embedding, denote by $N_u \rightarrow \Sigma$ the normal bundle to u . In order to prove the rigidity conjecture, it is sufficient to show that for a generic almost complex structure J on Y , for every J -holomorphic embedding $u: \Sigma \rightarrow Y$ of a Riemann surface, and for every branch cover $f: \tilde{\Sigma} \rightarrow \Sigma$, the operator $D_{J, u \circ f}^\perp$ on $f^* N_u$ induced by $D_{J, u \circ f}$ is injective. If (J_Y2) is to be proved only up to genus g , it is sufficient to consider covers $\tilde{\Sigma}$ of genus up to g . If J is a holomorphic and $N_u \rightarrow \Sigma$ is generic (of degree $2g(\Sigma) - 2$), then $D_{J, u \circ f}^\perp$ is injective for every f if and only if the genus of Σ is 0 or 1. The expectation is that this kind of generic situation can be achieved using a non-integrable J .

2 Preliminaries

2.1 Review of Key Definitions

In this subsection, we give geometric definitions of the three terms that appear in (1.4). The construction of the Gromov-Witten invariants described below is a slight variation on that of [FuO] and [LT], but it is easy to see the only difference is in the presentation. Below we use the term *multisection*, or *multivalued section*, of a vector orbi-bundle as defined in Section 3 of [FuO].

If X is a smooth submanifold of \mathbb{P}^n , we denote by $\mathfrak{X}_g(X, d)$ the space of equivalence classes of stable degree- d smooth maps from genus- g Riemann surfaces to X . Let $\mathfrak{X}_g^0(X, d)$ be the subset of $\mathfrak{X}_g(X, d)$ consisting of stable maps with smooth domains. The spaces $\mathfrak{X}_g(X, d)$ are topologized using L_1^p -convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes. Here and throughout the rest of the paper, p denotes a real number greater than two. The spaces $\mathfrak{X}_g(X, d)$ can be stratified by the smooth infinite-dimensional orbifolds $\mathfrak{X}_{\mathcal{T}}(X)$ of stable maps from domains of the same geometric type. The closure of the main stratum, $\mathfrak{X}_g^0(X, d)$, is $\mathfrak{X}_g(X, d)$.

If J is an almost complex structure on \mathbb{P}^n , let

$$\Gamma_g^{0,1}(X, d; J) \longrightarrow \mathfrak{X}_g(X, d)$$

be the bundle of (TX, J) -valued $(0, 1)$ -forms. In other words, the fiber of $\Gamma_g^{0,1}(X, d; J)$ over a point $[b] = [\Sigma, j; u]$ in $\mathfrak{X}_g(X, d)$ is the space

$$\Gamma_g^{0,1}(X, d; J)|_{[b]} = \Gamma^{0,1}(b; TX; J) / \text{Aut}(b), \quad \text{where} \quad \Gamma^{0,1}(b; TX; J) = \Gamma(\Sigma; \Lambda_{J,j}^{0,1} T^* \Sigma \otimes u^* TX).$$

Here j is the complex structure on Σ , the domain of the smooth map u . The bundle $\Lambda_{J,j}^{0,1} T^* \Sigma \otimes u^* TX$ over Σ consists of (J, j) -antilinear homomorphisms:

$$\Lambda_{J,j}^{0,1} T^* \Sigma \otimes u^* TX = \{ \alpha \in \text{Hom}(T\Sigma, u^* TX) : \alpha \circ j = -J \circ \alpha \}.$$

The total space of the bundle $\Gamma_g^{0,1}(TX, d; J) \longrightarrow \mathfrak{X}_g(X, d)$ is topologized using L^p -convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes. The restriction of $\Gamma_g^{0,1}(TX, d; J)$ to each stratum $\mathfrak{X}_{\mathcal{T}}(X)$ is a smooth vector orbibundle of infinite rank.

We define a continuous section of the bundle $\Gamma_g^{0,1}(TX, d; J) \longrightarrow \mathfrak{X}_g(X, d)$ by

$$\bar{\partial}_J([\Sigma, j; u]) = \frac{1}{2}(du + J \circ du \circ j).$$

By definition, the zero set of this section is the moduli space $\overline{\mathfrak{M}}_g(X, d; J)$ of equivalence classes of stable J -holomorphic degree- d maps from genus- g curves into X . The restriction of $\bar{\partial}_J$ to each stratum of $\mathfrak{X}_g(X, d)$ is smooth. For each element $[b] = [\Sigma, j, u]$ of $\mathfrak{X}_g(X, d)$, we put

$$D_{J,b} \xi = \frac{1}{2}(\nabla^X \xi + J \circ \nabla^X \xi \circ j) + \frac{1}{2}(\nabla_{\xi}^X J) \circ du \circ j \quad \text{if } \xi \in \Gamma(b; TX) \equiv \Gamma(\Sigma; u^* TX),$$

where ∇^X denotes the Levi-Civita connection of a J -compatible metric on X . The linear operator $D_{J,b}$ describes the restriction of a linearization of $\bar{\partial}_J$ at $[b]$ to a finite-codimensional subspace of the tangent bundle of the stratum $\mathfrak{X}_{\mathcal{T}}(X)$ of $\mathfrak{X}_g(X, d)$ containing $[b]$.

The section $\bar{\partial}_J: \mathfrak{X}_g(X, d) \rightarrow \Gamma_g^{0,1}(X, d; J)$ is Fredholm, i.e. its linearization at every point of $\bar{\partial}_J^{-1}(0)$ has finite-dimensional kernel and cokernel. The index of $\bar{\partial}_J$ at a point of $\mathfrak{X}_g^0(X, d)$ is the expected dimension of the moduli space $\overline{\mathfrak{M}}_g(X, d; J)$. If $X=Y$, this expected dimension is 0. By definition,

$$N_g(d) = \pm |\{\bar{\partial}_{J+\varepsilon}^{-1}(0)\}|, \quad (2.1)$$

where ε is a small multivalued perturbation such that $\bar{\partial}_{J+\varepsilon}$ is transverse to the zero set along each stratum $\mathfrak{X}_{\mathcal{T}}(Y)$ of $\mathfrak{X}_g(Y, d)$ and

$$\pm |\{\bar{\partial}_{J+\varepsilon}^{-1}(0)\}|$$

is the number of elements in the finite set $\{\bar{\partial}_{J+\varepsilon}^{-1}(0)\}$, counted with appropriate multiplicities. By the transversality condition,

$$\{\bar{\partial}_{J+\varepsilon}^{-1}(0)\} \subset \mathfrak{X}_g^0(Y, d).$$

The smallness condition implies in particular that the set $\{\bar{\partial}_{J+\varepsilon}^{-1}(0)\}$ is close to $\bar{\partial}_J^{-1}(0)$. Since the set $\bar{\partial}_J^{-1}(0)$ is compact, it follows that the set $\{\bar{\partial}_{J+\varepsilon}^{-1}(0)\}$ is also compact. Let $\mathcal{A}_g^d(\bar{\partial}_J)$ denote the set of all perturbations ε of $\bar{\partial}_J$ that satisfy the two conditions above. Such perturbations will be called $\bar{\partial}_J$ -admissible. Below we will refer to the number in (2.1) as the euler class of the tuple

$$\mathcal{V}_g^d(\bar{\partial}; J) \equiv (\mathfrak{X}_g(Y, d), \Gamma_g^{0,1}(Y, d; J), \pi; \bar{\partial}_J, \mathcal{A}_g^d(\bar{\partial}_J)).$$

This euler class depends on the Fredholm homotopy class of the section $\bar{\partial}_J$.

We now describe the last term in (1.4) in a similar way. If $\mathfrak{L} \rightarrow \mathbb{P}^4$ is as in Theorem 1.1, let $\Gamma_g(\mathfrak{L}, d) \rightarrow \mathfrak{X}_g(\mathbb{P}^4, d)$ be the cone such that the fiber of $\Gamma_g(\mathfrak{L}, d)$ over $[b] = [\Sigma, j; u]$ in $\mathfrak{X}_g(\mathbb{P}^4, d)$ is the Banach space

$$\Gamma_g(\mathfrak{L}, d)|_{[b]} = \Gamma(b; \mathfrak{L}) / \text{Aut}(b), \quad \text{where} \quad \Gamma(b; \mathfrak{L}) = L_1^p(\Sigma; u^* \mathfrak{L}),$$

and the topology on $\Gamma_g(\mathfrak{L}, d)$ is defined analogously to the topology on $\Gamma_g(\mathbb{P}^4, d)$. Let ∇ denote the hermitian connection in the line bundle $\mathfrak{L} \rightarrow \mathbb{P}^4$ induced from the standard connection on the tautological line bundle over \mathbb{P}^4 . If (Σ, j) is a Riemann surface and $u: \Sigma \rightarrow \mathbb{P}^4$ is a smooth map, let

$$\nabla^u: \Gamma(\Sigma; u^* \mathfrak{L}) \rightarrow \Gamma(\Sigma; T^* \Sigma \otimes u^* \mathfrak{L})$$

be the pull-back of ∇ by u . If $b = (\Sigma, j; u)$, we define the corresponding $\bar{\partial}$ -operator by

$$\bar{\partial}_{\nabla, b}: \Gamma(\Sigma; u^* \mathfrak{L}) \rightarrow \Gamma(\Sigma; \Lambda_{i,j}^{0,1} T^* \Sigma \otimes u^* \mathfrak{L}), \quad \bar{\partial}_{\nabla, b} \xi = \frac{1}{2} (\nabla^u \xi + i \nabla^u \xi \circ j), \quad (2.2)$$

where i is the complex multiplication in the bundle $u^* \mathfrak{L}$. Let

$$\mathcal{V}_g^d = \{[b, \xi] \in \Gamma_g(\mathfrak{L}, d) : [b] \in \mathfrak{X}_g(\mathbb{P}^4, d), \xi \in \ker \bar{\partial}_{\nabla, b} \subset \Gamma_g(b; \mathfrak{L})\} \subset \Gamma_{g,k}(\mathfrak{L}, d).$$

The cone $\mathcal{V}_g^d \rightarrow \mathfrak{X}_{g,k}(\mathbb{P}^4, d)$ inherits its topology from $\Gamma_g(\mathfrak{L}, d)$.

Let $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J) \subset \overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$ denote the closed subset containing the set

$$\mathfrak{M}_1^0(\mathbb{P}^4, d; J) = \{[\mathcal{C}, u] \in \overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J) : \mathcal{C} \text{ is smooth}\},$$

which is defined in [Z4]. If the almost complex structure J is sufficiently close to J_0 , $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ is the closure of $\mathfrak{M}_1^0(\mathbb{P}^4, d; J)$ in $\overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$. Furthermore, in such a case, $\mathfrak{M}_1^0(\mathbb{P}^4, d; J)$ is a smooth orbifold of dimension $10d$, while $\partial\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ is a finite union of smooth orbifolds of dimension at most $10d-2$. On the other hand, $\mathcal{V}_1^d|_{\mathfrak{M}_1^0(\mathbb{P}^4, d; J)}$ is a complex vector orbundle of rank $5d$. The last term in (1.4) is the number of zeros, counted with appropriate multiplicities, of any continuous multisection φ of the cone \mathcal{V}_1^d over $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ such that $\varphi^{-1}(0)$ is contained in $\mathfrak{M}_1^0(\mathbb{P}^4, d; J)$ and $\varphi|_{\mathfrak{M}_1^0(\mathbb{P}^4, d; J)}$ is smooth and transverse to the zero set; see Subsections 1.2 and 1.3 in [Z5]. Proposition 3.1 in [Z5] guarantees that a section φ satisfying the two conditions exists. In our case, it is more convenient to think of φ as $s_1^d + \varepsilon$, where ε is a multivalued perturbation of s_1^d . We denote by $\mathcal{A}_1^d(s; J)$ the set of all perturbations ε of s_1^d such that $s_1^d + \varepsilon$ satisfies the two conditions above. Such perturbations ε will be called s_1^d -admissible. Let

$$\mathcal{V}_1^d(s; J) \equiv (\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J), \mathcal{V}_1^d, \pi; s_1^d, \mathcal{A}_1^d(s)).$$

This tuple will be the focus of Section 4.

Remark: Since Y is a semi-positive symplectic manifold, one can define the numbers $N_g(d)$ without using the infinite-rank orbundles $\Gamma_g^{0,1}(Y, d; J)$; see [RT]. However, there would be no effect on the proofs of Propositions 1.3-1.5 and 2.1-2.3, and the construction described above appears more natural in the present context, even though it involves more complicated objects.

2.2 Components of the Proof

We now set up additional notation that allows us to state more notationally involved, but also easier-to-prove, versions of Propositions 1.3-1.5.

By Theorems 1.6 and 2.3 in [Z4], there exists $\delta(d) \in \mathbb{R}^+$ with the property that if J is an almost complex structure on \mathbb{P}^4 such that $\|J - J_0\|_{C^1} \leq \delta(d)$, then J is genus-one $d\ell$ -regular in the sense of Definition 1.4 in [Z4]. This regularity condition implies that the moduli spaces $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^4, d; J)$ and $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^4, d; J)$ have the same stratification structure as the moduli spaces

$$\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^4, d) \equiv \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^4, d; J_0) \quad \text{and} \quad \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^4, d) \equiv \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^4, d; J_0),$$

respectively. In addition, by Theorem 1.2 in [Z5], $\delta(d) \in \mathbb{R}^+$ can be chosen so that the euler class of the cone

$$\mathcal{V}_1^d \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$$

is well-defined and

$$\langle e(\mathcal{V}_1^d), [\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)] \rangle = \langle e(\mathcal{V}_1^d), [\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)] \rangle, \quad (2.3)$$

if $\|J - J_0\|_{C^1} \leq \delta(d)$.

If J is an almost complex structure on \mathbb{P}^4 and $[\Sigma, u] \in \overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$, we put

$$s_1^d([\Sigma, u]) = [s \circ u] \in \Gamma(\mathcal{L}, d)|_{[\Sigma, u]}.$$

If J is ∇s -equivalent to J_0 , i.e.

$$\nabla s \circ J_0 = \nabla s \circ J \in \Gamma(\mathbb{P}^4; \text{Hom}_{\mathbb{R}}(T\mathbb{P}^4, \mathfrak{L})),$$

then $s_1^d([\Sigma, u]) \in \mathcal{V}_1^d|_{[\Sigma, u]}$. Thus, in such a case, we obtain a continuous section of the cone

$$\mathcal{V}_1^d \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J),$$

which restricts to a smooth section over each stratum of $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$. Note that

$$\{s_1^d|_{\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)}\}^{-1}(0) = \overline{\mathfrak{M}}_1^0(Y, d; J) = \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J) \cap \overline{\mathfrak{M}}_1(Y, d; J). \quad (2.4)$$

Since the (∇, J_0) -holomorphic section s of Subsection 1.1 is transverse to the zero set in \mathfrak{L} , the (i, J_0) -linear map

$$\nabla s: T\mathbb{P}^4 \longrightarrow \mathfrak{L}$$

does not vanish along $Y = s^{-1}(0)$. Let U_s be a small neighborhood of Y in \mathbb{P}^4 such that ∇s does not vanish over U_s . The kernel of ∇s over U_s is then a rank-three complex subbundle of $(T\mathbb{P}^4, J_0)|_{U_s}$, which restricts to TY along Y . We denote this subbundle by $\tilde{T}Y$. If J is an almost complex structure on \mathbb{P}^4 such that

- (J1) $J = J_0$ on $\mathbb{P}^4 - U_s$;
- (J2) $J(\tilde{T}Y) = \tilde{T}Y$ and $J = J_0$ on $T\mathbb{P}^4|_{U_s}/\tilde{T}Y$,

then J_0 and J are ∇s -equivalent. Thus, every almost complex structure J_Y on Y extends to an almost complex structure J on \mathbb{P}^4 which is ∇s -equivalent to J_0 . Furthermore, such an extension can be chosen so that

$$\|J - J_0\|_{C^1} \leq 2\|J_Y - J_0|_{TY}\|_{C^1}. \quad (2.5)$$

We denote by $\mathcal{J}_{\text{rig}}(s)$ the set of almost complex structures J on \mathbb{P}^4 such that J is ∇s -equivalent to J_0 and $J_Y \equiv J|_{TY}$ is an element of $\mathcal{J}_{\text{rig}}^0(Y) \cap \mathcal{J}_{\text{rig}}^1(Y)$. By the above and the Rigidity Assumption in Subsection 1.2, the C^1 -closure of $\mathcal{J}_{\text{rig}}(s)$ in $\mathcal{J}(\mathbb{P}^4)$ contains J_0 .

From now on, we assume that $J \in \mathcal{J}_{\text{rig}}(s)$ is an almost complex structure on \mathbb{P}^4 sufficiently close to J_0 . For $g=0, 1$, we put

$$\mathcal{S}_g^d(Y; J) = \mathcal{S}_g^d(Y; J_Y) \quad \forall d \in \mathbb{Z}^+ \quad \text{and} \quad \mathcal{S}_g(Y; J) = \bigsqcup_{d=1}^{\infty} \mathcal{S}_g^d(Y; J).$$

If $\kappa \in \mathcal{S}_g(Y; J)$, let d_κ denote the degree of κ in \mathbb{P}^4 . If $\kappa \in \mathcal{S}_0(Y; J)$ and q is a positive integer, let $\mathfrak{M}_1^q(\kappa, d)$ be the subset of $\overline{\mathfrak{M}}_1(\kappa, d)$ consisting of stable maps $[\mathcal{C}, u]$ such that \mathcal{C} is an elliptic curve E with q rational components attached directly to E and $u|_E$ is constant. Figure 1 shows the domain of a typical element of $\mathfrak{M}_1^3(\kappa, d)$, from the points of view of symplectic topology and of algebraic geometry. In the first diagram, each shaded disc represents a sphere; the integer next to each rational component \mathcal{C}_i indicates the degree of $u|_{\mathcal{C}_i}$. In the second diagram, the components of \mathcal{C} are represented by curves, and the pair of integers next to each component \mathcal{C}_i shows the genus

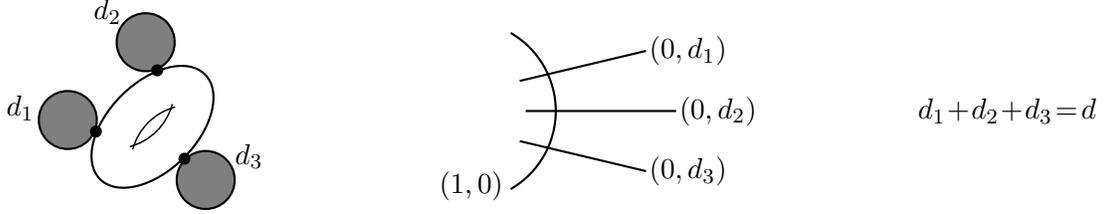


Figure 1: Domain of a typical element of $\mathfrak{M}_1^3(\kappa, d)$

of \mathcal{C}_i and the degree of $u|_{\mathcal{C}_i}$. For stability reasons, the restriction of u to each rational component must be non-constant. We denote by $\overline{\mathfrak{M}}_1^q(\kappa, d)$ the closure of $\mathfrak{M}_1^q(\kappa, d)$ in $\overline{\mathfrak{M}}_1(\kappa, d)$. Note that

$$\dim_{\mathbb{C}} \overline{\mathfrak{M}}_1^q(\kappa, d) = \begin{cases} 2d, & \text{if } q=0; \\ 2d+1-q, & \text{if } q \in \mathbb{Z}^+. \end{cases} \quad (2.6)$$

If $q \in \mathbb{Z}^+$, $\overline{\mathfrak{M}}_1^q(\kappa, d)$ is a smooth orbi-variety. In contrast, $\overline{\mathfrak{M}}_1^0(\kappa, d)$ is a singular orbivariety, if $d > 2$; its structure is described in Subsection 4.2.

For each $q \in \mathbb{Z}^+$, let $[q] = \{1, \dots, q\}$. If $\underline{d} = (d_1, \dots, d_q)$ is a q -tuple of positive integers and $\kappa \in \mathcal{S}_0(Y; J)$, we put

$$\overline{\mathfrak{M}}_0(\kappa, \underline{d}) = \left\{ (b_1, \dots, b_q) \in \prod_{i=1}^{i=q} \overline{\mathfrak{M}}_{0,1}(\kappa, d_i) : \text{ev}_0(b_i) = \text{ev}_0(b_j) \ \forall i, j \in [q] \right\}, \quad (2.7)$$

where $\text{ev}_0: \overline{\mathfrak{M}}_{0,1}(\kappa, d_i) \rightarrow \kappa$ is the evaluation map corresponding to the marked point. Let

$$\overline{\mathfrak{M}}_0^q(\kappa, d) = \bigsqcup_{d_i > 0, \sum d_i = d} \overline{\mathfrak{M}}_0(\kappa, (d_1, \dots, d_q)).$$

The spaces $\overline{\mathfrak{M}}_0^q(\kappa, d)$ are smooth orbi-varieties. We note that

$$\dim_{\mathbb{C}} \overline{\mathfrak{M}}_0^q(\kappa, d) = 2d + 1 - 2q. \quad (2.8)$$

There is a natural node-identifying immersion

$$\iota_q: \overline{\mathcal{M}}_{1,q} \times \overline{\mathfrak{M}}_0^q(\kappa, d) \rightarrow \overline{\mathfrak{M}}_1^q(\kappa, d) \quad (2.9)$$

where $\overline{\mathcal{M}}_{1,q}$ is the moduli space of genus-one curves with q marked points. It descends to an immersion

$$(\overline{\mathcal{M}}_{1,q} \times \overline{\mathfrak{M}}_0^q(\kappa, d)) / S_q \rightarrow \overline{\mathfrak{M}}_1^q(\kappa, d),$$

where S_q is the q th symmetric group; the latter immersion restricts to a diffeomorphism on the preimage of $\mathfrak{M}_1^q(\kappa, d)$. The immersion (2.9) is illustrated in Figure 2. In this figure, we represent an entire space of stable maps by the domain of a typical element of the space. We shade the components of the domain on which the maps are non-constant. The vertical bar indicates that the three marked points are mapped to the same point in κ , as specified by (2.7). Let

$$\pi_P, \pi_B: \overline{\mathcal{M}}_{1,q} \times \overline{\mathfrak{M}}_0^q(\kappa, d) \rightarrow \overline{\mathcal{M}}_{1,q}, \overline{\mathfrak{M}}_0^q(\kappa, d)$$

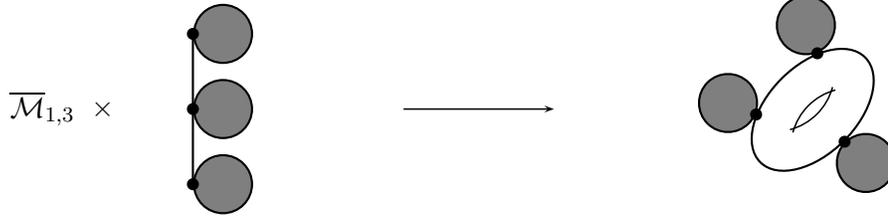


Figure 2: The node-identifying immersion (2.9) for $\overline{\mathfrak{M}}_1^3(\kappa, d)$

be the projection maps.

For each $\kappa \in \mathcal{S}_0(Y; J)$, we denote by $N_Y \kappa$ the normal bundle of κ in Y . If $q \in \mathbb{Z}^+$ and $[b] = ([b_i])_{i \in [q]}$ is an element of $\overline{\mathfrak{M}}_0^q(\kappa, d)$, let

$$\Gamma(b; TY) = \left\{ \xi = (\xi_i)_{i \in [q]} \in \bigoplus_{i \in [q]} \Gamma(b_i; TY) : \xi_i(y_0(b_i)) = \xi_j(y_0(b_j)) \quad \forall i, j \in [q] \right\},$$

where $y_0(b_i)$ is the marked point of the component map b_i . Since $J_Y \in \mathcal{J}_{\text{rig}}^0(Y)$, by the Index Theorem the cokernel $H_J^1(b; TY)$ of the operator

$$\Gamma(b; TY) \longrightarrow \Gamma^{0,1}(b; TY; J) \equiv \bigoplus_{i \in [q]} \Gamma^{0,1}(b_i; TY; J), \quad D_{J,b}((\xi_i)_{i \in [q]}) = (D_{J,b_i} \xi_i)_{i \in [q]}, \quad (2.10)$$

is a vector space of dimension $2d-2$. It is naturally isomorphic to the cokernel $H_J^1(b; N_Y \kappa)$ of the operator

$$\Gamma(b; N_Y \kappa) \longrightarrow \Gamma^{0,1}(b; N_Y \kappa; J), \quad D_{J,b}^\perp((\xi_i)_{i \in [q]}) = (D_{J,b_i}^\perp \xi_i)_{i \in [q]},$$

induced by the operator $D_{J,b}$. These cokernels induce a vector orbibundle over $\overline{\mathfrak{M}}_0^q(\kappa, d)$, which will be denoted by $\mathcal{W}_{\kappa,d}^{0,q}$. If $q=1$, this bundle is the pullback by the forgetful map

$$\tilde{\pi}: \overline{\mathfrak{M}}_0^1(\kappa, d) \equiv \overline{\mathfrak{M}}_{0,1}(\kappa, d) \longrightarrow \overline{\mathfrak{M}}_0(\kappa, d)$$

of the vector bundle defined in a similar way. We denote this last vector bundle by $\mathcal{W}_{\kappa,d}^0$. We have

$$\text{rk } \mathcal{W}_{\kappa,d}^{0,q} = 2d - 2 \quad \forall q \in \mathbb{Z}^+ \quad \text{and} \quad \text{rk } \mathcal{W}_{\kappa,d}^0 = 2d - 2. \quad (2.11)$$

It is straightforward to see that the cokernel bundle for the operators $D_{J,b}$ over $\overline{\mathfrak{M}}_1^q(\kappa, d)$, for $q \in \mathbb{Z}^+$, is given by

$$\iota_q^* \mathcal{W}_{\kappa,d}^{1,q} \approx \pi_P^* \mathbb{E}^* \otimes \pi_B^* \text{ev}_0^* TY \oplus \pi_B^* \mathcal{W}_{\kappa,d}^{0,q}, \quad (2.12)$$

where $\mathbb{E} \longrightarrow \overline{\mathcal{M}}_{1,q}$ is the Hodge line bundle and

$$\text{ev}_0: \overline{\mathfrak{M}}_0^q(\kappa, d) \longrightarrow \kappa$$

is the natural evaluation map, corresponding to the marked point common to all factors. We note that

$$\text{rk } \mathcal{W}_{\kappa,d}^{1,q} = 2d + 1. \quad (2.13)$$

On the other hand, similarly to the genus-zero case, the cokernel $H_J^1(b; TY)$ of the operator $D_{J,b}$ for

$$b \in \mathfrak{M}_1^0(\kappa, d) \subset \overline{\mathfrak{M}}_1(\kappa, d)$$

is naturally isomorphic to the cokernel $H_J^1(b; N_Y \kappa)$ of the operator $D_{J,b}^\perp$ induced by $D_{J,b}$. The cokernels $H_J^1(b; N_Y \kappa)$ have the expected rank for all $b \in \overline{\mathfrak{M}}_1^0(\kappa, d)$ and thus form a vector bundle over $\overline{\mathfrak{M}}_1^0(\kappa, d)$, which we denote by $\mathcal{W}_{\kappa,d}^{1,0}$. We have

$$\mathrm{rk} \mathcal{W}_{\kappa,d}^{1,0} = 2d \quad \text{and} \quad (2.14)$$

$$\iota_q^* \mathcal{W}_{\kappa,d}^{1,0} \approx (\pi_P^* \mathbb{E}^* \otimes \pi_B^* \mathrm{ev}_0^* N_Y \kappa \oplus \pi_B^* \mathcal{W}_{\kappa,d}^{0,q})|_{\iota_q^{-1}(\overline{\mathfrak{M}}_1^0(\kappa,d))} \quad \forall q \in \mathbb{Z}^+. \quad (2.15)$$

We are now ready to reformulate Propositions 1.3-1.5.

Proposition 2.1 *If d and \mathfrak{L} are as in Theorem 1.1, $s \in H^0(\mathbb{P}^4; \mathfrak{L})$ is a transverse section, and $Y = s^{-1}(0)$, there exists $\delta(d) \in \mathbb{R}^+$ with the following property. If $J \in \mathcal{J}_{\mathrm{reg}}(s)$ and $\|J - J_0\|_{C^1} \leq \delta(d)$, then*

$$N_0(d) = \sum_{\kappa \in \mathcal{S}_0(Y; J)} \langle e(\mathcal{W}_{\kappa,d/d_\kappa}^0), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)] \rangle,$$

where $\mathcal{W}_{\kappa,d/d_\kappa}^0 \rightarrow \overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)$ is the cokernel bundle corresponding to the almost complex structure J , as above.

Proposition 2.2 *If d , \mathfrak{L} , s , and Y are as in Proposition 2.1, there exists $\delta(d) \in \mathbb{R}^+$ with the following property. If $J \in \mathcal{J}_{\mathrm{reg}}(s)$ and $\|J - J_0\|_{C^1} \leq \delta(d)$, then*

$$\begin{aligned} N_1(d) = & \sum_{\kappa \in \mathcal{S}_1(Y; J)} \pm |\mathfrak{M}_1^0(\kappa, d/d_\kappa)| \\ & + \sum_{\kappa \in \mathcal{S}_0(Y; J)} \left(\langle e(\mathcal{W}_{\kappa,d/d_\kappa}^{1,0}), [\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)] \rangle + \frac{d/d_\kappa}{12} \langle e(\mathcal{W}_{\kappa,d/d_\kappa}^0), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)] \rangle \right), \end{aligned}$$

where $\mathcal{W}_{\kappa,d/d_\kappa}^0 \rightarrow \overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)$ and $\mathcal{W}_{\kappa,d/d_\kappa}^{1,0} \rightarrow \overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ are the cokernel bundles corresponding to the almost complex structure J , as above.

Proposition 2.3 *If d , \mathfrak{L} , s , and Y are as in Proposition 2.1 and $\mathcal{V}_1^d \rightarrow \mathfrak{X}_1(\mathbb{P}^4, d)$ is the cone corresponding to the line bundle $\mathfrak{L} \rightarrow \mathbb{P}^4$ with its standard connection, there exists $\delta(d) \in \mathbb{R}^+$ with the following properties. If $\|J - J_0\|_{C^1} \leq \delta(d)$, then the moduli space $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ carries a rational fundamental class of dimension $10d$, the euler class of the cone*

$$\mathcal{V}_1^d \rightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$$

is a well-defined element of $H^{10d}(\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J); \mathbb{Q})$, and

$$\langle e(\mathcal{V}_1^d), [\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)] \rangle = \langle e(\mathcal{V}_1^d), [\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)] \rangle.$$

If in addition $J \in \mathcal{J}_{\mathrm{reg}}(s)$,

$$\begin{aligned} \langle e(\mathcal{V}_1^d), [\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)] \rangle = & \sum_{\kappa \in \mathcal{S}_1(Y; J)} \pm |\mathfrak{M}_1^0(\kappa, d/d_\kappa)| \\ & + \sum_{\kappa \in \mathcal{S}_0(Y; J)} \left(\langle e(\mathcal{W}_{\kappa,d/d_\kappa}^{1,0}), [\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)] \rangle + \frac{d/d_\kappa - 1}{12} \langle e(\mathcal{W}_{\kappa,d/d_\kappa}^0), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)] \rangle \right), \end{aligned}$$

where $\mathcal{W}_{\kappa, d/d_\kappa}^0 \longrightarrow \overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)$ and $\mathcal{W}_{\kappa, d/d_\kappa}^{1,0} \longrightarrow \overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ are as in Proposition 2.2.

In the last two propositions, the moduli space $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$, for $\kappa \in \mathcal{S}_1(Y; J)$, contains $m(d/d_\kappa)$ elements: the $m(d/d_\kappa)$ equivalence classes of the degree- d/d_κ covers of the elliptic curve κ by an elliptic curve. Since the order of the automorphism group of such a cover is d/d_κ ,

$$\pm |\mathfrak{M}_1^0(\kappa, d/d_\kappa)| = \pm \frac{m(d/d_\kappa)}{d/d_\kappa}.$$

The sign is determined by viewing the zero-dimensional suborbifold $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$ of $\mathfrak{X}_1(Y, d)$ as a transverse zero of the section $\bar{\delta}_J$. This sign is the same as the sign of κ as an element of the set $\mathcal{S}_1^{d_\kappa}(Y; J)$. In particular,

$$\sum_{\kappa \in \mathcal{S}_1(Y; J)} \pm |\mathfrak{M}_1^0(\kappa, d/d_\kappa)| = \sum_{\sigma|d} m(\sigma) \frac{n_1(d/\sigma)}{\sigma}, \quad (2.16)$$

where $n_1(\cdot)$ is as in Subsection 1.2.

If $\kappa \in \mathcal{S}_0(Y; J)$, the orientations of the vector bundles

$$\mathcal{W}_{\kappa, d/d_\kappa}^0 \longrightarrow \overline{\mathfrak{M}}_0(\kappa, d/d_\kappa) \quad \text{and} \quad \mathcal{W}_{\kappa, d/d_\kappa}^{1,0} \longrightarrow \overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$$

are determined by the linearizations of the sections $\bar{\delta}_J$ over $\mathfrak{X}_0(Y, d)$ and $\mathfrak{X}_1(Y, d)$. According to [IP], by a spectral-flow argument it can be shown that

$$\langle e(\mathcal{W}_{\kappa, \sigma}^0), [\overline{\mathfrak{M}}_0(\kappa, \sigma)] \rangle = \pm \langle e(R^1 \pi_{0*}^\sigma \text{ev}_0^{\sigma*}(\mathcal{O}_\kappa(-1) \oplus \mathcal{O}_\kappa(-1))), [\overline{\mathfrak{M}}_0(\kappa, \sigma)] \rangle, \quad (2.17)$$

$$\langle e(\mathcal{W}_{\kappa, \sigma}^{1,0}), [\overline{\mathfrak{M}}_1^0(\kappa, \sigma)] \rangle = \pm \langle e(R^1 \pi_{1*}^\sigma \text{ev}_1^{\sigma*}(\mathcal{O}_\kappa(-1) \oplus \mathcal{O}_\kappa(-1))), [\overline{\mathfrak{M}}_1^0(\kappa, \sigma)] \rangle, \quad (2.18)$$

where $\sigma = d/d_\kappa$ and the sign agrees with the sign of κ as an element of $\mathcal{S}_0^{d_\kappa}(Y; J)$. By localization,

$$\langle e(R^1 \pi_{0*}^\sigma \text{ev}_0^{\sigma*}(\mathcal{O}_\kappa(-1) \oplus \mathcal{O}_\kappa(-1))), [\overline{\mathfrak{M}}_0(\kappa, \sigma)] \rangle = \frac{1}{\sigma^3}; \quad (2.19)$$

see Section 27.5 of [MirSym]. Using the desingularization of $\overline{\mathfrak{M}}_1^0(\kappa, \sigma)$ constructed in [VaZ], one can show that

$$\langle e(R^1 \pi_{1*}^\sigma \text{ev}_1^{\sigma*}(\mathcal{O}_\kappa(-1) \oplus \mathcal{O}_\kappa(-1))), [\overline{\mathfrak{M}}_1^0(\kappa, \sigma)] \rangle = \frac{1}{12} \frac{\sigma - 1}{\sigma^2}. \quad (2.20)$$

Propositions 1.3-1.5 follow from Propositions 2.1-2.3 via (2.16)-(2.20).

Since Theorem 1.1 follows immediately from Propositions 2.1-2.3, we do not need to deduce Propositions 1.3-1.5 from Propositions 2.1-2.3. We prove Propositions 2.2 and 2.3 in Sections 3 and 4, respectively; see also Propositions 2.5 and 2.6. The proof of Proposition 2.1 is very similar to the proof of Proposition 2.2, but simpler, and we omit it.

2.3 Summary of the Proof of Proposition 2.2

A key notion in our argument, which is also used in the proof of Proposition 2.3, is Definition 2.4 below. For its purposes, we will call either of the two tuples $\mathcal{V}_g^d(\bar{\partial}; J)$ and $\mathcal{V}_1^d(s; J)$, defined in Subsection 2.1, a **generalized vector bundle**. The first tuple involves an infinite-rank bundle over an infinite-dimensional space; the second one involves finite-dimensional objects, albeit non-smooth ones. Nevertheless, both are generalizations of a rank- n vector bundle \mathfrak{F} over an n -dimensional complex compact manifold \mathfrak{X} , with a choice of a section φ and of an appropriate subset $\mathcal{A}(\varphi)$ of $\Gamma(\mathfrak{X}; \mathfrak{F})$ of second category. Such a collection of data can also be considered to be a generalized vector bundle.

Definition 2.4 *Suppose $\mathcal{V} = (\mathfrak{X}, \mathfrak{F}, \pi; \varphi, \mathcal{A}(\varphi))$ is a generalized vector bundle. Subset \mathcal{Z} of $\varphi^{-1}(0)$ is a **regular set** for \mathcal{V} if there exists $\mathcal{C}_{\mathcal{Z}}(\mathcal{V}) \in \mathbb{Q}$ and a dense open subset $\mathcal{A}_{\mathcal{Z}}(\varphi)$ of $\mathcal{A}(\varphi)$ with the following properties. For every $\nu \in \mathcal{A}_{\mathcal{Z}}(\varphi)$,*

- (a) *there exists $\epsilon_{\nu} \in \mathbb{R}^+$ such that $t\nu \in \mathcal{A}(\varphi)$ for all $t \in (0, \epsilon_{\nu})$;*
- (b) *there exist a compact subset $K_{\nu} \subset \mathcal{Z}$, open neighborhood $U_{\nu}(K)$ of K in \mathfrak{X} for each compact subset $K \subset \mathcal{Z}$, and $\epsilon_{\nu}(U) \in (0, \epsilon_{\nu})$ for each open subset U of \mathfrak{X} such that*

$$\#|\{\varphi + t\nu\}^{-1} \cap U| = \mathcal{C}_{\mathcal{Z}}(\mathcal{V}) \quad \text{if } t \in (0, \epsilon_{\nu}(U)) \text{ and } K_{\nu} \subset K \subset U \subset U_{\nu}(K).$$

Every connected component of $\varphi^{-1}(0)$ is regular. However, a regular subset of $\varphi^{-1}(0)$ need not be closed. For example, if φ is a holomorphic section of a rank- k algebraic vector bundle \mathfrak{F} over a k -dimensional compact algebraic variety \mathfrak{X} , every Zariski open subset of $\varphi^{-1}(0)$ is regular. The sections $\bar{\partial}_J$ and s_1^d that play a central role in this paper are in a sense **generalized holomorphic sections**.

If \mathcal{Z} is a regular set for the generalized vector bundle \mathcal{V} , we will call the number $\mathcal{C}_{\mathcal{Z}}(\mathcal{V})$ the φ -**contribution** of \mathcal{Z} to the **euler class** of \mathcal{V} . Note that if $\varphi^{-1}(0) = \sqcup_{i \in I} \mathcal{Z}_i$ is a partition of $\varphi^{-1}(0)$ into regular sets, the euler class of \mathcal{V} , or its Poincare dual, is the sum of φ -contributions:

$$e(\mathcal{V}) = \sum_{i \in I} \mathcal{C}_{\mathcal{Z}_i}(\mathcal{V}). \quad (2.21)$$

We prove Theorem 1.1 by expressing each of the three terms appearing in (1.4) in the form (2.21) and show that we end up with the same terms on the two sides of (1.4).

If d , s , and Y are as in the previous subsection and $J \in \mathcal{J}_{\text{rig}}(s)$,

$$\overline{\mathfrak{M}}_1(Y, d; J) = \bigsqcup_{\kappa \in \mathcal{S}_0(Y; J)} \overline{\mathfrak{M}}_1(\kappa, d/d_{\kappa}) \sqcup \bigsqcup_{\kappa \in \mathcal{S}_1(Y; J)} \mathfrak{M}_1^0(\kappa, d/d_{\kappa}). \quad (2.22)$$

For any $\kappa \in \mathcal{S}_0(Y; J)$, $\sigma \in \mathbb{Z}^+$, and subset ϱ of $\bar{\mathbb{Z}}^+ \equiv \mathbb{Z}^+ \cup \{0\}$, let

$$\mathfrak{M}_1^{\varrho}(\kappa, \sigma) = \bigcap_{q \in \varrho} \overline{\mathfrak{M}}_1^q(\kappa, \sigma) - \bigcup_{q \in \bar{\mathbb{Z}}^+ - \varrho} \overline{\mathfrak{M}}_1^q(\kappa, \sigma).$$

Proposition 2.5 *If d , \mathfrak{L} , s , and Y are as in Proposition 2.1, $J \in \mathcal{J}_{\text{rig}}(s)$ is sufficiently close to J_0 , and $\kappa \in \mathcal{S}_1(Y; J)$, then*

$$\mathcal{C}_{\mathfrak{M}_1^0(\kappa, d/d_\kappa)}(\mathcal{V}_1^d(\bar{\partial}; J)) = \pm |\mathfrak{M}_1^0(\kappa, d/d_\kappa)|.$$

If $\kappa \in \mathcal{S}_0(Y; J)$,

$$\mathcal{C}_{\mathfrak{M}_1^{\{0\}}(\kappa, d/d_\kappa)}(\mathcal{V}_1^d(\bar{\partial}; J)) = \langle e(\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}), [\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)] \rangle.$$

Proposition 2.6 *If d , \mathfrak{L} , s , Y , and J are as in Proposition 2.5, $\kappa \in \mathcal{S}_0(Y; J)$, and ϱ is a subset $\bar{\mathbb{Z}}^+$ different from $\{0\}$, then*

$$\mathcal{C}_{\mathfrak{M}_1^\varrho(\kappa, d/d_\kappa)}(\mathcal{V}_1^d(\bar{\partial}; J)) = \begin{cases} \frac{d/d_\kappa}{12} \langle e(\mathcal{W}_{\kappa, d/d_\kappa}^0), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)] \rangle, & \text{if } \varrho = \{1\}; \\ 0, & \text{if } \varrho \neq \{1\}. \end{cases}$$

One consequence of Propositions 2.5 and 2.6 is that most boundary strata of the moduli space $\overline{\mathfrak{M}}_1(Y, d; J)$ do not contribute to the number $N_1(d)$. In fact, we will show that only the strata $\mathfrak{M}_1^0(Y, d; J)$ and $\mathfrak{M}_1^1(Y, d; J)$ contribute to the number $N_1(d)$.

We now outline the proofs of Propositions 2.5 and 2.6. Let

$$\nu \in \Gamma(\mathfrak{X}_1(\mathbb{P}^4, d); \Gamma_1^{0,1}(\mathbb{P}^4, d; J))$$

be a small generic multisection such that

$$\nu \in \Gamma(\mathfrak{X}_s; \Gamma_1^{0,1}(Y, d; J))$$

for a small neighborhood \mathfrak{X}_s of $\overline{\mathfrak{M}}_1(Y, d; J)$ in $\mathfrak{X}_1(\mathbb{P}^4, d)$ and vanishes outside of \mathcal{U}_s . By definition, $N_1(d)$ is the number of elements $\exp_u \xi \in \mathfrak{X}_1(\mathbb{P}^4, d)$ such that (u, ξ) solves the system

$$\begin{cases} \bar{\partial}_J \exp_u \xi + \nu(\exp_u \xi) = 0; \\ s \circ \exp_u \xi = 0; \end{cases} \quad u \in \overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J), \quad \xi \in T_u \mathfrak{X}_1(\mathbb{P}^4, d). \quad (2.23)$$

Note that

$$\bar{\partial}_{\nabla, \exp_u \xi} s_1^d(\exp_u \xi) = 0$$

if (u, ξ) solves the first equation, due to our assumptions on ν . If $u \in \mathfrak{M}_1^0(\mathbb{P}^4, d; J)$ and ν is sufficiently small, the first equation has a unique small solution $\xi_\nu(u)$ in $\Gamma_+(u)$, the orthogonal complement of $T_u \overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$ in $T_u \mathfrak{X}_1(\mathbb{P}^4, d)$. Plugging this solution into the second equation, we obtain

$$0 = s \circ \exp_u \xi_\nu = s_1^d(u) + \pi_{TY}^\perp \xi_\nu(u) \in \mathcal{V}_1^d, \quad (2.24)$$

where π_{TY}^\perp is the projection map $T\mathbb{P}^4 \rightarrow T\mathbb{P}^4/TY$, defined on a neighborhood of Y in \mathbb{P}^4 . Since all solutions of the system (2.23) are transverse, so are the solutions of (2.24). Thus, the zeros of a generic perturbation ν of the section $\bar{\partial}_J$ that lie close to $\mathfrak{M}_1^0(Y, d; J)$ correspond to the zeros of a perturbation of the section s_1^d that lie close to $\mathfrak{M}_1^0(Y, d; J)$. In Subsection 3.2, we show that the number of these zeros that lie near each component $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$ of $\mathfrak{M}_1^0(Y, d; J)$ is the euler class of the bundle $\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}$ over $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$.

We next look for solutions near $\mathfrak{M}_1^{\{1\}}(Y, d; J)$, i.e. we assume that $u \in \mathfrak{M}_1^{\{1\}}(Y, d; J)$. Note that

$$\overline{\mathfrak{M}}_1^1(Y, d; J) \approx \overline{\mathcal{M}}_{1,1} \times \overline{\mathfrak{M}}_{0,1}(Y, d; J). \quad (2.25)$$

We denote the projection maps onto $\overline{\mathcal{M}}_{1,1}$ and $\overline{\mathfrak{M}}_{0,1}(Y, d; J)$ by π_P and π_B , respectively. Let

$$\tilde{\pi}_B: \overline{\mathfrak{M}}_1^1(Y, d; J) \longrightarrow \overline{\mathfrak{M}}_0(Y, d; J)$$

be the composition of π_B with the forgetful map $\overline{\mathfrak{M}}_{0,1}(Y, d; J) \longrightarrow \overline{\mathfrak{M}}_0(Y, d; J)$. The bundle $T\mathfrak{X}_1(Y, d)$ contains the line subbundle $\mathcal{L} \equiv \pi_P^* L_{P,1} \otimes \pi_B^* L_0$, where

$$L_{P,1} \longrightarrow \overline{\mathcal{M}}_{1,1} \quad \text{and} \quad L_0 \longrightarrow \overline{\mathfrak{M}}_{0,1}(Y, d; J)$$

are the universal tangent line bundles at the marked points. If $u \in \overline{\mathfrak{M}}_1^1(Y, d; J)$ and $v \in \mathcal{L}_u$ is small, we denote by u_v the element $\exp_u v$ of $\mathfrak{X}_1(Y, d)$. Let

$$\text{ev}_P: \overline{\mathfrak{M}}_1^1(Y, d; J) \longrightarrow Y$$

be the composition of π_B with the evaluation map at the marked point. This map sends an element $[\mathcal{C}, u]$ of $\overline{\mathfrak{M}}_1^1(Y, d; J)$ to the value of u on the principal component of \mathcal{C} .

In this case, we work with the analogue of (2.23) intrinsic to Y , i.e. we look for solutions of the equation

$$\bar{\partial}_J \exp_{u_v} \xi + \nu(\exp_{u_v} \xi) = 0 \quad u \in \overline{\mathfrak{M}}_1^{\{1\}}(Y, d; J), \quad \xi \in \Gamma(v; TY) \equiv \Gamma(u_v^* TY). \quad (2.26)$$

This equation usually does not have a small solution in ξ for a fixed u_v , as there is an obstruction bundle

$$\Gamma_-^{0,1}(u; TY; J) = \tilde{\pi}_B^* H^1(u_B^* TY) \oplus \pi_P^* \mathbb{E}^* \otimes \text{ev}_P^* TY \subset \Gamma_1^{0,1}(Y, d; J),$$

where u_B is the restriction of u to the bubble components. Taking the projections $(\pi_{-,B}^{0,1} \oplus \pi_{-,P}^{0,1})$ and $\pi_+^{0,1}$ of (2.26) onto $\Gamma_-^{0,1}(u; TY; J)$ and its complement $\Gamma_+^{0,1}(u; TY; J)$ in $\Gamma_1^{0,1}(Y, d; J)$, respectively, we obtain

$$\begin{cases} \pi_+^{0,1} \bar{\partial} \exp_{u_v} \xi + \pi_+^{0,1} \nu(\exp_{u_v} \xi) = 0 \in \Gamma_+^{0,1}(u; TY; J); \\ \pi_{-,B}^{0,1} \bar{\partial} \exp_{u_v} \xi + \pi_{-,B}^{0,1} \nu(\exp_{u_v} \xi) = 0 \in \tilde{\pi}_B^* H^1(u_B^* TY); \\ \pi_{-,P}^{0,1} \bar{\partial} \exp_{u_v} \xi + \pi_{-,P}^{0,1} \nu(\exp_{u_v} \xi) = 0 \in \pi_P^* \mathbb{E}^* \otimes \text{ev}_P^* TY. \end{cases} \quad (2.27)$$

If ν and v are sufficiently small, the first equation has a unique small solution $\xi_\nu(u, v)$ in $\Gamma_+(u; TY)$. With appropriate choices of neighborhood charts and of the perturbation ν , the value of the left-hand side of the middle equation in (2.27) at $\xi = \xi_\nu(u, v)$ depends only on u_B , and the system (2.27) is equivalent to

$$\pi_{-,P}^{0,1} \bar{\partial} \exp_{u_v} \xi_\nu(u, v) + \pi_{-,P}^{0,1} \nu(\exp_{u_v} \xi_\nu(u, v)) = 0 \in \pi_P^* \mathbb{E}^* \otimes \text{ev}_P^* TY, \quad u_B \in \mathcal{Z}_0, \quad v \in \mathcal{L}, \quad (2.28)$$

where \mathcal{Z}_0 is the zero set of a section of the first component of the bundle $\Gamma_-^{0,1}(\cdot, TY)$ over $\overline{\mathfrak{M}}_0(Y, d; J)$. In particular, $\sharp|\mathcal{Z}_0| = N_0(d)$.

Equation (2.28) is equivalent to

$$\mathcal{D}_u v + \pi_{-,P}^{0,1} \nu(u) = 0 \in \pi_P^* \mathbb{E}^* \otimes T_{\text{ev}_P(u)} Y, \quad u_B \in \mathcal{Z}_0, \quad v \in \mathcal{L}, \quad (2.29)$$

where $\mathcal{D}_u \in \text{Hom}(L_0, T_{\text{ev}_0(u_B)} Y)$. The image of \mathcal{D}_u in $T_{\text{ev}_0(u_B)} Y$ is precisely the tangent line at $\text{ev}_0(u_B)$ to the rational curve $\text{Im } u_B$, as long as the differential of the map u_B does not vanish at the marked point. Thus, for each $u_B \in \mathcal{Z}_0$, the number of solutions of (2.29) is the number of times $\pi_{-,P}^{0,1} \nu(u)$ lies in $\mathbb{E}^* \otimes T_{\text{ev}_P(u)} \text{Im } u_B$. We conclude that

$$\begin{aligned} \mathcal{C}_{\mathfrak{M}_1^{\{1\}}(Y,d;J)}(\mathcal{V}_1^d(\bar{\partial}; J)) &= \sum_{u_B \in \mathcal{Z}_0} \langle c(\pi_P^* \mathbb{E}^* \otimes \text{ev}_P^* TY) c(\pi_P^* \mathbb{E}^* \otimes \text{ev}_P^* T \text{Im } u_B)^{-1}, [\tilde{\pi}_B^{-1}(u_B)] \rangle \\ &= \sum_{u_B \in \mathcal{Z}_0} \langle c_1(\mathbb{E}^*) (c_1(TY) - c_1(T \text{Im } u_B)), [\overline{\mathcal{M}}_{1,1}] \times [\mathbb{P}^1] \rangle \\ &= -\frac{1}{24} \cdot (0-2) \cdot \sum_{\kappa \in \mathcal{S}_0(Y;J)} (d/d_\kappa) \cdot \# |\mathcal{Z}_0 \cap \overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)| \\ &= \sum_{\kappa \in \mathcal{S}_0(Y;J)} \frac{d/d_\kappa}{12} \langle e(\mathcal{W}_{\kappa, d/d_\kappa}^0), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)] \rangle, \end{aligned} \quad (2.30)$$

as claimed in Proposition 2.6.

We analyze the contribution to the number $N_1(d)$ from the complement of $\mathfrak{M}_1^{\{0\}}(\mathbb{P}^4, d; J)$ and $\mathfrak{M}_1^{\{1\}}(\mathbb{P}^4, d; J)$ in $\overline{\mathfrak{M}}_1(Y, d; J)$ in a similar way, but we encounter one of two key differences. If $\varrho = \{0, 1\}$ and $u \in \mathfrak{M}_1^\varrho(Y, d; J)$, $\mathcal{D}_u = 0$. Thus, equation (2.29) has no solutions near $\mathfrak{M}_1^\varrho(Y, d; J)$ if ν is generic. On the other hand, if ϱ is any other subset of $\bar{\mathbb{Z}}^+$ containing 0 and at least one other element, the analogue of the set \mathcal{Z}_0 is empty for dimensional reasons. Thus,

$$\mathcal{C}_{\mathfrak{M}_1^\varrho(Y,d;J)}(\mathcal{V}_1^d(\bar{\partial}; J)) = 0 \quad \text{if } \{0\} \subsetneq \varrho \subset \bar{\mathbb{Z}}^+,$$

as claimed.

The computation of the contribution from $\mathfrak{M}_1^0(Y, d; J)$ to the number $N_1(d)$ can also be carried out in Y , instead of \mathbb{P}^4 . However, the presented version of the computation is meant to indicate why the cone \mathcal{V}_1^d should enter into the Gromov-Witten theory of Y .

We supply more details of the proof of Propositions 2.5 and 2.6 in Section 3. In particular, in order to use the gluing and obstruction-bundle setup described in [Z2], we stratify the moduli spaces that appear in the statements of Propositions 2.5 and 2.6 according to the bubble type, or the dual graph, of stable maps. The notion of contribution to the euler class used in this paper is a direct adaptation, to the orbifold and multisection setting of [FuO] and [LT], of the analogous notion used in [Z1] and [Z3]. However, in the present case, we can get by with far less detailed understanding of the behavior of the bundle sections involved.

2.4 Notation: Genus-Zero Maps

We now summarize our notation for bubble maps from genus-zero Riemann surfaces, with one marked point, and for related objects. For more details on the notation described below, the

reader is referred to Section 2 in [Z2].

In general, moduli spaces of stable maps can be stratified by the dual graph. However, in the present situation, it is more convenient to make use of *linearly ordered sets*:

Definition 2.7 (1) A finite nonempty partially ordered set I is a **linearly ordered set** if for all $i_1, i_2, h \in I$ such that $i_1, i_2 < h$, either $i_1 \leq i_2$ or $i_2 \leq i_1$.

(2) A linearly ordered set I is a **rooted tree** if I has a unique minimal element, i.e. there exists $\hat{0} \in I$ such that $\hat{0} \leq i$ for all $i \in I$.

We use rooted trees to stratify the moduli space $\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^4, d; J)$ of degree- d J -holomorphic maps from genus-zero Riemann surfaces with one marked point to \mathbb{P}^4 .

If I is a linearly ordered set, let \hat{I} be the subset of the non-minimal elements of I . For every $h \in \hat{I}$, denote by $\iota_h \in I$ the largest element of I which is smaller than h , i.e.

$$\iota_h = \max \{i \in I : i < h\}.$$

A **genus-zero \mathbb{P}^4 -valued bubble map** is a tuple $b = (I; x, u)$, where I is a rooted tree, and

$$x: \hat{I} \longrightarrow \mathbb{C} = S^2 - \{\infty\} \quad \text{and} \quad u: I \longrightarrow C^\infty(S^2; \mathbb{P}^4)$$

are maps such that $u_h(\infty) = u_{\iota_h}(x_h)$ for all $h \in \hat{I}$. Such a tuple describes a Riemann surface Σ_b and a continuous map $u_b: \Sigma_b \longrightarrow \mathbb{P}^4$. The irreducible components $\Sigma_{b,i}$ of Σ_b are indexed by the set I and $u_b|_{\Sigma_{b,i}} = u_i$. The Riemann surface Σ_b carries a marked point, i.e. the point $(\hat{0}, \infty) \in \Sigma_{b,\hat{0}}$, if $\hat{0}$ is the minimal element of I . The general structure of genus-zero bubble maps is described by tuples $\mathcal{T} = (I; \underline{d})$, where $\underline{d}: I \longrightarrow \mathbb{Z}$ is a map specifying the degree of $u_b|_{\Sigma_{b,i}}$, if b is a bubble map of type \mathcal{T} . We call such tuples **bubble types**.

If \mathcal{T} is a bubble type as above, let $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ be the subset of $\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^4, d; J)$ consisting of stable maps $[\mathcal{C}, y_1, u]$ such that

$$[\mathcal{C}, y_1, u] = [(\Sigma_b, (\hat{0}, \infty)), u_b],$$

for some bubble map b of type \mathcal{T} . Subsection 2.5 of [Z2] describes a space $\mathcal{U}_{\mathcal{T}}^{(0)}(X; J)$ of *balanced* stable maps, not of equivalence classes of such maps, such that

$$\mathcal{U}_{\mathcal{T}}(X; J) = \mathcal{U}_{\mathcal{T}}^{(0)}(X; J) / \text{Aut}(\mathcal{T}) \times (S^1)^I,$$

for a natural action of $\text{Aut}(\mathcal{T})$ on $(S^1)^I$. This space is convenient to use in gluing constructions.

2.5 Notation: Genus-One Maps

We next set up analogous notation for genus-one stable maps; see Subsection 2.2 in [Z4] for more details. In this case, we also need to specify the structure of the principal component. Thus, we index the strata of $\overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$ by *enhanced linearly ordered sets*:

Definition 2.8 An **enhanced linearly ordered set** is a pair (I, \aleph) , where I is a linearly ordered set, \aleph is a subset of $I_0 \times I_0$, and I_0 is the subset of minimal elements of I , such that if $|I_0| > 1$,

$$\aleph = \{(i_1, i_2), (i_2, i_3), \dots, (i_{n-1}, i_n), (i_n, i_1)\}$$

for some bijection $i: \{1, \dots, n\} \longrightarrow I_0$.

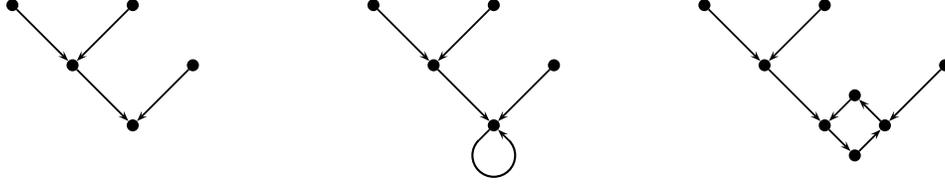


Figure 3: Some enhanced linearly ordered sets

An enhanced linearly ordered set can be represented by an oriented connected graph. In Figure 3, the dots denote the elements of I . The arrows outside the loop, if there are any, specify the partial ordering of the linearly ordered set I . In fact, every directed edge outside of the loop connects a non-minimal element h of I with ι_h . Inside of the loop, there is a directed edge from i_1 to i_2 if and only if $(i_1, i_2) \in \aleph$.

The subset \aleph of $I_0 \times I_0$ will be used to describe the structure of the principal curve of the domain of stable maps in a stratum of the moduli space $\overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$. If $\aleph = \emptyset$, and thus $|I_0| = 1$, the corresponding principal curve Σ_{\aleph} is a smooth torus, with some complex structure. If $\aleph \neq \emptyset$, the principal components form a circle of spheres:

$$\Sigma_{\aleph} = \left(\bigsqcup_{i \in I_0} \{i\} \times S^2 \right) / \sim, \quad \text{where} \quad (i_1, \infty) \sim (i_2, 0) \text{ if } (i_1, i_2) \in \aleph.$$

A **genus-one \mathbb{P}^4 -valued bubble map** is a tuple $b = (I, \aleph; S, x, u)$, where S is a smooth Riemann surface of genus one if $\aleph = \emptyset$ and the circle of spheres Σ_{\aleph} otherwise. The objects x , u , and (Σ_b, u_b) are as in the genus-zero case, except the sphere $\Sigma_{b, \hat{0}}$ is replaced by the genus-one curve $\Sigma_{b, \aleph} \equiv S$. Furthermore, if $\aleph = \emptyset$, and thus $I_0 = \{\hat{0}\}$ is a single-element set, $u_{\hat{0}} \in C^\infty(S; \mathbb{P}^4)$. In the genus-one case, the general structure of bubble maps is encoded by the tuples of the form $\mathcal{T} = (I, \aleph; \underline{d})$. Similarly to the genus-zero case, we denote by $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ the subset of $\overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$ consisting of stable maps $[\mathcal{C}, u]$ such that $[\mathcal{C}, u] = [\Sigma_b, u_b]$, for some bubble map b of type \mathcal{T} as above.

If $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type as above, let

$$I_1 = \{h \in \hat{I} : \iota_h \in I_0\}, \quad \mathcal{T}_0 = (I_1, I_0, \aleph; \iota|_{I_1}, \underline{d}|_{I_0}),$$

and $\text{Aut}^*(\mathcal{T}) = \text{Aut}(\mathcal{T}) / \{g \in \text{Aut}(\mathcal{T}) : g \cdot h = h \ \forall h \in I_1\},$

where I_0 is the subset of minimal elements of I . For each $h \in I_1$, we put

$$I_h = \{i \in I : h \leq i\} \quad \text{and} \quad \mathcal{T}_h = (I_h; \underline{d}|_{I_h}).$$

The tuple \mathcal{T}_0 describes bubble maps from genus-one Riemann surfaces with the marked points indexed by the set I_1 ; see Subsection 2.2 in [Z4]. We have a natural isomorphism

$$\begin{aligned} \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) \approx & \left(\{ (b_0, (b_h)_{h \in I_1}) \in \mathcal{U}_{\mathcal{T}_0}(\mathbb{P}^4; J) \times \prod_{h \in I_1} \mathcal{U}_{\mathcal{T}_h}(\mathbb{P}^4; J) : \right. \\ & \left. \text{ev}_0(b_h) = \text{ev}_{\iota_h}(b_0) \ \forall h \in I_1 \right) / \text{Aut}^*(\mathcal{T}). \end{aligned} \quad (2.31)$$

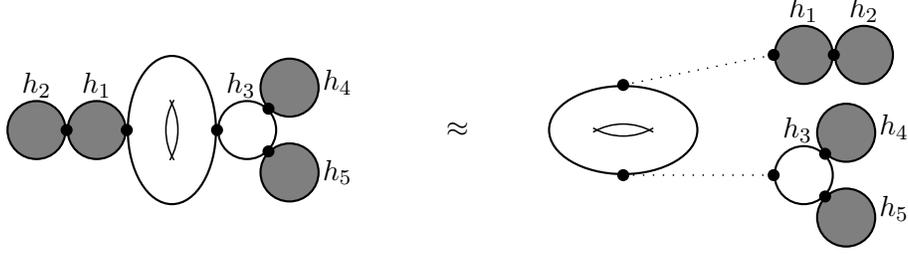


Figure 4: An example of the decomposition (2.31)

This decomposition is illustrated in Figure 4. In this figure, we represent an entire stratum of bubble maps by the domain of the stable maps in that stratum. The right-hand side of Figure 4 represents the subset of the cartesian product of the three spaces of bubble maps, corresponding to the three drawings, on which the appropriate evaluation maps agree pairwise, as indicated by the dotted lines and defined in (2.31).

Let $\mathcal{FT} \rightarrow \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ be the bundle of gluing parameters, or of smoothings at the nodes. This orbi-bundle has the form

$$\mathcal{FT} = \left(\bigoplus_{(h,i) \in \aleph} L_{h,0} \otimes L_{i,1} \oplus \bigoplus_{h \in \hat{I}} L_{h,0} \otimes L_{h,1} \right) / \text{Aut}(\mathcal{T}),$$

for certain line orbi-bundles $L_{h,0}$ and $L_{h,1}$. Similarly to the genus-zero case,

$$\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) = \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J) / \text{Aut}(\mathcal{T}) \times (S^1)^{\hat{I}}, \quad \text{where} \quad (2.32)$$

$$\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J) = \{ (b_0, (b_h)_{h \in I_1}) \in \mathcal{U}_{\mathcal{T}_0}(\mathbb{P}^4; J) \times \prod_{h \in I_1} \mathcal{U}_{\mathcal{T}_h}^{(0)}(\mathbb{P}^4; J) : \text{ev}_0(b_h) = \text{ev}_{\iota_h}(b_0) \ \forall h \in I_1 \}. \quad (2.33)$$

The line bundles $L_{h,0}$ and $L_{h,1}$ arise from the quotient (2.32), and

$$\mathcal{FT} = \tilde{\mathcal{F}}\mathcal{T} / \text{Aut}(\mathcal{T}) \times (S^1)^{\hat{I}}, \quad \text{where} \quad \tilde{\mathcal{F}}\mathcal{T} = \tilde{\mathcal{F}}_{\aleph}\mathcal{T} \oplus \bigoplus_{h \in \hat{I}} \tilde{\mathcal{F}}_h\mathcal{T},$$

$\tilde{\mathcal{F}}_{\aleph}\mathcal{T} \rightarrow \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$ is the bundle of smoothings for the \aleph nodes of the circle of spheres Σ_{\aleph} and $\tilde{\mathcal{F}}_h\mathcal{T} \rightarrow \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$ is the line bundle of smoothings of the attaching node of the bubble indexed by h .

Suppose $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $d_i = 0$ for all $i \in I_0$, i.e. every element in $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ is constant on the principal components. In this case, the decomposition (2.31) is equivalent to

$$\begin{aligned} \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) &\approx \left(\mathcal{U}_{\mathcal{T}_0}(pt) \times \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) \right) / \text{Aut}^*(\mathcal{T}) \\ &\subset \left(\overline{\mathcal{M}}_{1,k} \times \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) \right) / \text{Aut}^*(\mathcal{T}), \end{aligned} \quad (2.34)$$

where $k = |I_1|$ and

$$\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) = \{ (b_h)_{h \in I_1} \in \prod_{h \in I_1} \mathcal{U}_{\mathcal{T}_h}(\mathbb{P}^4; J) : \text{ev}_0(b_{h_1}) = \text{ev}_0(b_{h_2}) \ \forall h_1, h_2 \in I_1 \}.$$

Similarly, (2.33) is equivalent to

$$\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J) \approx \mathcal{U}_{\mathcal{T}_0}(pt) \times \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J) \subset \overline{\mathcal{M}}_{1,k} \times \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J), \quad \text{where} \quad (2.35)$$

$$\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J) = \left\{ (b_h)_{h \in I_1} \in \prod_{h \in I_1} \mathcal{U}_{\mathcal{T}_h}^{(0)}(\mathbb{P}^4; J) : \text{ev}_0(b_{h_1}) = \text{ev}_0(b_{h_2}) \ \forall h_1, h_2 \in I_1 \right\}. \quad (2.36)$$

We denote by

$$\pi_P: \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) \longrightarrow \overline{\mathcal{M}}_{1,k}/\text{Aut}^*(\mathcal{T}) \quad \text{and} \quad \pi_P: \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J) \longrightarrow \overline{\mathcal{M}}_{1,k}$$

the projections onto the first component in the decompositions (2.34) and (2.35). Let

$$\text{ev}_P: \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J), \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J) \longrightarrow \mathbb{P}^4$$

be the map sending each stable map (Σ, u) to its value on the principal component Σ_P of Σ , i.e. the point $u(\Sigma_P)$.

If $\mathcal{T} = (I, \aleph; \underline{d})$ is as in the previous paragraph, let

$$\begin{aligned} \chi(\mathcal{T}) &= \{i \in \hat{I} : d_i \neq 0; \ d_h = 0 \ \forall h < i\}; \\ \tilde{\mathfrak{F}}\mathcal{T} &= \bigoplus_{i \in \chi(\mathcal{T})} \tilde{\mathcal{F}}_{h(i)}\mathcal{T} \longrightarrow \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J), \quad \text{where} \quad h(i) = \min\{h \in \hat{I} : h \leq i\} \in I_1. \end{aligned}$$

The subset $\chi(\mathcal{T})$ of I indexes the first-level effective bubbles of every element of $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$. For each element $b = (\Sigma_b, u_b)$ of $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$ and $i \in \chi(\mathcal{T})$, let

$$\mathcal{D}_i b = \{du_b|_{\Sigma_{b,i}}\}|_{\infty} e_{\infty} \in T_{\text{ev}_P(b)}\mathbb{P}^4, \quad \text{where} \quad e_{\infty} = (1, 0, 0) \in T_{\infty}S^2.$$

The complex span of $\mathcal{D}_i b$ in $T_{\text{ev}_P(b)}\mathbb{P}^4$ is the tangent line to the rational component $\Sigma_{b,i}$ at the node of $\Sigma_{b,i}$ closest to a principal component of Σ_b . If the branch corresponding to $\Sigma_{b,i}$ has a cusp at this node, then $\mathcal{D}_i b = 0$.

Let $\mathbb{E} \longrightarrow \overline{\mathcal{M}}_{1,k}$ denote the Hodge line bundle, i.e. the line bundle of holomorphic differentials. For each $i \in \chi(\mathcal{T})$, we define the bundle map

$$\mathcal{D}_{J,i}: \tilde{\mathcal{F}}_{h(i)}\mathcal{T} \longrightarrow \pi_P^* \mathbb{E}^* \otimes_J \text{ev}_P^* T\mathbb{P}^4,$$

over $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$ by

$$\{\mathcal{D}_{J,i}(\tilde{v})\}(\psi) = \psi_{x_{h(i)}(b)}(\tilde{v}) \cdot_J \mathcal{D}_i b \in T_{\text{ev}_P(b)}\mathbb{P}^4 \quad \text{if} \quad \psi \in \pi_P^* \mathbb{E}, \ \tilde{v} = (b, \tilde{v}) \in \tilde{\mathcal{F}}_{h(i)}\mathcal{T}, \ b \in \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J),$$

and $x_{h(i)}(b) \in \Sigma_{b,\aleph}$ is the node joining the bubble $\Sigma_{b,h(i)}$ of b to the principal component $\Sigma_{b,\aleph}$ of Σ_b . For each $v \in \tilde{\mathcal{F}}\mathcal{T}$, we put

$$\rho(v) = (b; \rho_i(v))_{i \in \chi(\mathcal{T})} \in \tilde{\mathfrak{F}}\mathcal{T}, \quad \text{where} \quad \rho_i(v) = \prod_{h \in \hat{I}, h \leq i} v_h \in \tilde{\mathcal{F}}_{h(i)}\mathcal{T}, \quad \text{if}$$

$$v = (b; v_{\aleph}, (v_i)_{i \in \hat{I}}), \quad b \in \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J), \quad (b, v_{\aleph}) \in \tilde{\mathcal{F}}_{\aleph}\mathcal{T}, \quad (b, v_h) \in \tilde{\mathcal{F}}_h\mathcal{T} \text{ if } h \in I_1, \quad v_i \in \mathbb{C} \text{ if } i \in \hat{I} - I_1.$$

These definitions are illustrated in Figure 5 on page 39. While the bundle maps $D_{J,i}$ and ρ do not necessarily descend to the vector bundle \mathcal{FT} over $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, the map

$$\mathcal{D}_{\mathcal{T}}: \mathcal{FT} \longrightarrow \pi_P^* \mathbb{E}^* \otimes \text{ev}_P^* T\mathbb{P}^4 / \text{Aut}^*(T), \quad \mathcal{D}_{\mathcal{T}}(v) = \sum_{i \in \chi(\mathcal{T})} \mathcal{D}_{J,i} \rho_i(v),$$

is well-defined.

Let $\tilde{\mathcal{V}}_1^d \longrightarrow \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$ be the vector bundle such that the fiber of $\tilde{\mathcal{V}}_1^d$ over a point $b = (\Sigma_b, u_b)$ in $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$ is $\ker \bar{\partial}_{\nabla, b}$, where ∇ is the standard connection in line bundle $\mathcal{L} = \gamma^{*\otimes 5}$ over \mathbb{P}^4 ; see Subsection 2.1, as well as Subsection 3.3 in [Z5]. If $b = (\Sigma_b, u_b) \in \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$, $\xi = (\xi_h)_{h \in I} \in \Gamma(b; \mathcal{L})$, and $i \in \chi(\mathcal{T})$, let

$$\tilde{\mathcal{D}}_{\mathcal{T}, i} \xi = \nabla_{e_\infty} \xi_i \in \mathcal{L}_{\text{ev}_0(b)},$$

as in Subsection 2.2 in [Z5]. We next define the bundle map

$$\mathcal{D}_{\mathcal{T}}: \tilde{\mathcal{V}}_1^d \otimes \tilde{\mathcal{F}}\mathcal{T} \longrightarrow \pi_P^* \mathbb{E}^* \otimes \text{ev}_P^* \mathcal{L}$$

over $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$ by

$$\begin{aligned} \{\tilde{\mathcal{D}}_{\mathcal{T}}(\xi \otimes \tilde{v})\}(\psi) &= \sum_{i \in \chi(\mathcal{T})} \psi_{x_{h(i)}(b)}(\tilde{v}_i) \cdot \tilde{\mathcal{D}}_{\mathcal{T}, i} \xi \in \mathcal{L}_{\text{ev}_P(b)} && \text{if} \\ \xi \in \tilde{\mathcal{V}}_1^d|_b \subset \Gamma(b; \mathcal{L}), \quad \tilde{v} &= (b; \tilde{v}_i)_{i \in \chi(\mathcal{T})} \in \tilde{\mathcal{F}}\mathcal{T}|_b, \quad \text{and} \quad \psi \in \mathbb{E}_{\pi_P(b)}. \end{aligned}$$

The bundle map $\tilde{\mathcal{D}}_{\mathcal{T}}$ induces a linear bundle map over $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$:

$$\mathcal{D}_{\mathcal{T}}: \mathcal{V}_1^d \otimes \mathcal{F}\mathcal{T} \longrightarrow \pi_P^* \mathbb{E}^* \otimes \text{ev}_P^* \mathcal{L} / \text{Aut}^*(T), \quad \text{where} \quad \mathcal{F}\mathcal{T} = \left(\bigoplus_{i \in \chi(\mathcal{T})} \pi_P^* L_{P, h(i)} \otimes \pi_i^* L_0 \right) / \text{Aut}^*(T),$$

$L_{P, h} \longrightarrow \overline{\mathfrak{M}}_{1, k}$ is the universal tangent line bundle at the marked point x_h , $L_0 \longrightarrow \mathcal{U}_{\mathcal{T}'}(\mathbb{P}^4; J)$ is the universal tangent line bundle at the special marked point (i, ∞) for any bubble type \mathcal{T}' of rational stable maps, and

$$\pi_i: \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) \longrightarrow \mathcal{U}_{\mathcal{T}'_i}(\mathbb{P}^4; J)$$

is the projection map sending each bubble map $b = (\Sigma_b, u_b)$ to its restriction to the component $\Sigma_{b, i}$.

Finally, if \mathcal{T} is any bubble type, for genus-zero or genus-one maps, and K is a subset of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, we denote by $K^{(0)}$ the preimage of K under the quotient projection map $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J) \longrightarrow \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$. All vector orbi-bundles we encounter will be assumed to be normed. Some will come with natural norms; for others, we implicitly choose a norm once and for all. If $\pi_{\mathfrak{F}}: \mathfrak{F} \longrightarrow \mathfrak{X}$ is a normed vector bundle and $\delta: \mathfrak{X} \longrightarrow \mathbb{R}$ is any function, possibly constant, let

$$\mathfrak{F}_\delta = \{v \in \mathfrak{F}: |v| < \delta(\pi_{\mathfrak{F}}(v))\}.$$

If Ω is any subset of \mathfrak{F} , we take $\Omega_\delta = \Omega \cap \mathfrak{F}_\delta$.

3 On Genus-One Gromov-Witten Invariants

3.1 Setup

In this section, we prove Propositions 2.5 and 2.6. We start by clarifying the setup described after Proposition 2.6. We also specify the open subsets of admissible perturbations of the $\bar{\partial}_J$ -operator to be used in proving Propositions 2.5 and 2.6; see Definition 2.4.

Let U_s be the neighborhood of Y in \mathbb{P}^4 and $\tilde{T}Y$ the subbundle of $T\mathbb{P}^4|_{U_s}$ as in Subsection 2.2. We set

$$\mathfrak{X}_s = \{[\Sigma, j; u] \in \mathfrak{X}_1(\mathbb{P}^4, d) : u(\Sigma) \subset U_s\}.$$

Let ν be a multisection of the bundle $\Gamma_1^{0,1}(\mathbb{P}^4, d)$ such that

- ($\nu 1$) for every open neighborhood \mathcal{U} of $\overline{\mathfrak{M}}_1(\mathbb{P}^4, d; J)$ in $\mathfrak{X}_1(\mathbb{P}^4, d)$, there exists $\epsilon_\nu(\mathcal{U}) > 0$ such that $\{\bar{\partial}_J + t\nu\}^{-1}(0)$ is contained in \mathcal{U} for all $t \in (0, \epsilon_\nu(\mathcal{U}))$;
- ($\nu 2$) $\nu(b) \in \Gamma(\Sigma; \Lambda_{J,j}^{0,1} T^* \Sigma \otimes u^* \tilde{T}Y) / \text{Aut}(b)$ if $b = [\Sigma, j, u] \in \mathfrak{X}_s$, and $\nu(b) = 0$ if $b \notin \mathfrak{X}_s$;
- ($\nu 3$) for some $\epsilon_\nu > 0$ and for all $t \in (0, \epsilon_\nu)$, the multisection $\bar{\partial}_J + t\nu$ does not vanish on $\mathfrak{X}_1(Y, d) - \mathfrak{X}_1^0(Y, d)$ and is transversal to the zero set in $\Gamma_1^{0,1}(Y, d; J)$ along $\mathfrak{X}_1^0(Y, d)$.

The middle condition implies that $\bar{\partial}_{\nabla, u}\{s \circ u\} = 0$ if $[\Sigma, j, u] \in \{\bar{\partial}_J + t\nu\}^{-1}(0)$. It can be shown, by slightly modifying the proof of Corollary 3.11, that the finite-dimensional conditions ($\nu 3a$)-($\nu 3c$) stated below imply ($\nu 3$).

If ν is a section of the bundle $\Gamma_1^{0,1}(\mathbb{P}^4; d)$ over $\mathfrak{X}_1(\mathbb{P}^4, d)$ as in ($\nu 1$) and ($\nu 2$) above, for every $\kappa \in \mathcal{S}_0(Y; J)$, we define a section of the bundle

$$\mathcal{W}_{\kappa, d/d_\kappa}^{1,1} \longrightarrow \overline{\mathfrak{M}}_1^1(\kappa, d/d_\kappa) \quad \text{by} \quad \pi_{\nu, \kappa}^1(b) = [\nu(b)],$$

where $[\nu(b)]$ is the $(0, 1)$ -cohomology class of $\nu(b)$ and $\mathcal{W}_{\kappa, d/d_\kappa}^{1,1}$ is as in Subsection 2.2. For each $q \in \mathbb{Z}^+$, we define a section of the bundle

$$\begin{aligned} \iota_q^* \widetilde{\mathcal{W}}_{\kappa, d/d_\kappa}^{1,q} &\equiv \pi_P^* \mathbb{E}^* \otimes \pi_B^* \text{ev}_0^* N_Y \kappa \oplus \pi_B^* \mathcal{W}_{\kappa, d/d_\kappa}^{0,q} \longrightarrow \overline{\mathcal{M}}_{1,q} \times \overline{\mathfrak{M}}_0^q(\kappa, d/d_\kappa) \\ &\text{by} \quad \tilde{\pi}_{\nu, \kappa}^q(b) = \pi_\kappa^\perp[\nu(\iota_q(b))], \end{aligned} \quad (3.1)$$

where

$$\pi_\kappa^\perp : \mathcal{W}_{\kappa, d/d_\kappa}^{1,q} \longrightarrow \widetilde{\mathcal{W}}_{\kappa, d/d_\kappa}^{1,q}$$

is the projection map corresponding to the quotient of $\iota_q^* \mathcal{W}_{\kappa, d/d_\kappa}^{1,q}$ by $\pi_P^* \mathbb{E}^* \otimes \pi_B^* \text{ev}_0^* T\kappa$; see (2.12). Finally, we define a section of the bundle

$$\mathcal{W}_{\kappa, d/d_\kappa}^{1,0} \longrightarrow \overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa) \quad \text{by} \quad \pi_{\nu, \kappa}^0(b) = \begin{cases} \tilde{\pi}_{\nu, \kappa}^q(\iota_q^{-1}(b)), & \text{if } b \in \overline{\mathfrak{M}}_1^q(\kappa, d/d_\kappa), \quad q \in \mathbb{Z}^+; \\ [\nu(b)], & \text{otherwise;} \end{cases}$$

see (2.15). This section is well-defined on $\overline{\mathfrak{M}}_1^{q_1}(\kappa, d/d_\kappa) \cap \overline{\mathfrak{M}}_1^{q_2}(\kappa, d/d_\kappa)$.

We denote by $\tilde{\mathcal{A}}_1^d(\bar{\partial}, J)$ the space of multisections ν as in ($\nu 1$) and ($\nu 2$) such that for all $\kappa \in \mathcal{S}_0(Y; J)$:

- ($\nu 3a$) the section $\pi_{\nu, \kappa}^0$ does not vanish on $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa) - \mathfrak{M}_1^0(\kappa, d/d_\kappa)$ and is transversal to the zero set on $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$;

($\nu 3b$) the section $\pi_{\nu, \kappa}^1$ does not vanish on $\overline{\mathfrak{M}}_1^1(\kappa, d/d_\kappa)$;

($\nu 3c$) the section $\tilde{\pi}_{\nu, \kappa}^1$ does not vanish on $\overline{\mathfrak{M}}_1^1(\kappa, d/d_\kappa) - \mathfrak{M}_1^1(\kappa, d/d_\kappa)$ and is transversal to the zero set on $\mathfrak{M}_1^1(\kappa, d/d_\kappa)$.

By (2.6), (2.8), (2.13), and Lemmas 4.1 and 4.2, these conditions are satisfied by a dense open path-connected subset of sections ν .

3.2 Proof of Proposition 2.5

We will focus on the last case of Proposition 2.5, which follows from Proposition 3.1. The claim in the first case is clear, since the finite set $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$ consists of transverse zeros of the section $\bar{\partial}_J$ over $\mathfrak{X}_1(Y, d)$. The proof of Proposition 3.1 applies to this case as well, except there is no gluing to be done.

Let s and Y be as in Proposition 2.5. For every bubble type \mathcal{T} and every rational J -holomorphic curve κ in Y , we put

$$\mathcal{U}_{\mathcal{T}; \kappa} = \{[\mathcal{C}, u] \in \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) : u(\mathcal{C}) = \kappa\}.$$

Proposition 3.1 *Suppose d , Y , and J are as in Proposition 2.5, $\nu \in \tilde{\mathcal{A}}_1^d(\bar{\partial}; J)$ is a generic perturbation of the $\bar{\partial}_J$ -operator on $\mathfrak{X}_1(\mathbb{P}^4, d)$, $\kappa \in \mathcal{S}_0(Y; J)$, and $\mathcal{T} = (I, \mathfrak{N}; \underline{d})$ is a bubble type such that $\sum_{i \in I} d_i = d$ and $d_i \neq 0$ for some minimal element i of I . If $|I| > 1$ or $\mathfrak{N} \neq \emptyset$, for every compact subset K of $\mathcal{U}_{\mathcal{T}; \kappa}$, there exist $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\mathfrak{X}_1(Y, d)$ such that*

$$\{\bar{\partial}_J + t\nu\}^{-1}(0) \cap U(K) = \emptyset \quad \forall t \in (0, \epsilon_\nu(K)).$$

If $|I| = 1$ and $\mathfrak{N} = \emptyset$, for every compact subset K of $\mathcal{U}_{\mathcal{T}; \kappa}$, there exist $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\mathfrak{X}_1(Y, d)$ with the following properties:

- (a) the section $\bar{\partial}_J + t\nu$ is transverse to the zero set in $\Gamma_1^{0,1}(Y, d; J)$ over $U(K)$ for all $t \in (0, \epsilon_\nu(K))$;
- (b) for every open subset U of $\mathfrak{X}_1(Y, d)$, there exists $\epsilon(U) \in (0, \epsilon_\nu(K))$ such that

$$\pm |\{\bar{\partial}_J + t\nu\}^{-1} \cap U| = \langle e(\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}), [\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)] \rangle \quad \text{if } \pi_{\nu, \kappa}^{-1}(0) \subset K \subset U \subset U(K), \quad t \in (0, \epsilon(U)).$$

In other words, the contribution from the main stratum $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$ of $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ to the number $N_1(d)$, as computed via the section $\bar{\partial}_J$, is the euler class of the vector bundle $\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}$ over $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$. None of the boundary strata of $\mathfrak{M}_1^{\{0\}}(\kappa, d/d_\kappa)$ contributes to $N_1(d)$.

We fix a J -compatible metric $g_{\mathbb{P}^4}$ on \mathbb{P}^4 and proceed as in Subsection 4.1 of [Z4]. For each sufficiently small element $v = (b, v)$ of $\tilde{\mathcal{F}}\mathcal{T}^\emptyset$, let

$$b(v) = (\Sigma_v, j_v; u_v), \quad \text{where } u_v = u_b \circ q_v,$$

be the corresponding approximately holomorphic stable map. Here

$$q_v: \Sigma_v \longrightarrow \Sigma_b$$

is the basic gluing map constructed in Subsection 4.1 of [Z4]. Since $d_i \neq 0$ for some minimal element i of I , i.e. the stable map b is non-constant on the principal curve of the domain Σ_b of b ,

the linearization $D_{J,b}$ of the $\bar{\partial}_J$ -operator at b is surjective, since $\|J - J_0\|_{C^1} \leq \delta(b)$. Thus, if v is sufficiently small, the linearization

$$D_{J,v}: \Gamma(v; T\mathbb{P}^4) \cong L_1^p(\Sigma_v; u_v^* T\mathbb{P}^4) \longrightarrow \Gamma^{0,1}(v; T\mathbb{P}^4; J) \cong L^p(\Sigma_v; \Lambda_{J,j}^{0,1} T^* \Sigma_v \otimes u_v^* T\mathbb{P}^4),$$

of the $\bar{\partial}_J$ -operator at $b(v)$, defined via the J -compatible connection ∇^J in $T\mathbb{P}^4$ corresponding to the Levi-Civita connection of the metric $g_{\mathbb{P}^4}$, is also surjective. In particular, we can obtain an orthogonal decomposition

$$\Gamma(v; T\mathbb{P}^4) = \Gamma_-(v; T\mathbb{P}^4) \oplus \Gamma_+(v; T\mathbb{P}^4) \quad (3.2)$$

such that the linear operator

$$D_{J,v}: \Gamma_+(v; T\mathbb{P}^4) \longrightarrow \Gamma^{0,1}(v; T\mathbb{P}^4; J)$$

is an isomorphism, while

$$\Gamma_-(v; T\mathbb{P}^4) = \{\xi \circ q_v: \xi \in \Gamma_-(b; T\mathbb{P}^4)\}, \quad \text{where} \quad \Gamma_-(b; T\mathbb{P}^4) = \ker D_{J,b}.$$

The L^2 -inner product on $\Gamma(v; T\mathbb{P}^4)$ used in the orthogonal decomposition is defined via the metric $g_{\mathbb{P}^4}$ on \mathbb{P}^4 and the metric g_v on Σ_v induced by the pregluing construction. The Banach spaces $\Gamma(v; T\mathbb{P}^4)$ and $\Gamma^{0,1}(v; T\mathbb{P}^4; J)$ carry the norms $\|\cdot\|_{v,p,1}$ and $\|\cdot\|_{v,p}$, respectively, which are also defined by the pregluing construction. These norms are equivalent to the ones used in [LT]. In particular, the norms of $D_{J,v}$ and of the inverse of its restriction to $\Gamma_+(v; T\mathbb{P}^4)$ have fiberwise uniform upper bounds, i.e. dependent only on $[b] \in \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, and not on $v \in \tilde{\mathcal{F}}\mathcal{T}^\emptyset$.

Lemma 3.2 *If \mathcal{T} is a bubble type and ν is an admissible perturbation of the $\bar{\partial}_J$ -operator on $\mathfrak{X}_1(\mathbb{P}^4, d)$ as in Proposition 3.1, for every precompact open subset K of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, there exist $\delta_K, \epsilon_K, C_K \in \mathbb{R}^+$ and an open neighborhood U_K of K in $\mathfrak{X}_1(\mathbb{P}^4, d)$ with the following properties:*

(1) *for all $v = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}^\emptyset|_{K^{(0)}}$,*

$$\begin{aligned} \|D_{J,v}\xi\|_{v,p} &\leq C_K |v|^{1/p} \|\xi\|_{v,p,1} \quad \forall \xi \in \Gamma_-(v; T\mathbb{P}^4) \quad \text{and} \\ C_K^{-1} \|\xi\|_{v,p,1} &\leq \|D_{J,v}\xi\|_{v,p} \leq C_K \|\xi\|_{v,p,1} \quad \forall \xi \in \Gamma_+(v; T\mathbb{P}^4); \end{aligned}$$

(2) *for all $v = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}^\emptyset|_{K^{(0)}}$ and $t \in [0, \delta_K]$, the equation*

$$\bar{\partial}_J \exp_{u_v} \xi + t\nu(\exp_{u_v} \xi) = 0, \quad \xi \in \Gamma_+(v; T\mathbb{P}^4), \quad \|\xi\|_{v,p,1} \leq \epsilon_K,$$

has a unique solution $\xi_{t\nu}(v)$, and $\|\xi_{t\nu}(v)\|_{C^0} \leq C_K(t + |v|^{1/p})$;

(3) *there exist a smooth bundle map $\zeta_\nu: \tilde{\mathcal{F}}\mathcal{T}^\emptyset \longrightarrow \tilde{\Gamma}(T\mathbb{P}^4, d)$ over $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4; J)$ and a continuous function $\varepsilon_\nu: \mathcal{F}\mathcal{T}^\emptyset \longrightarrow \mathbb{R}$ such that for all $v = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}^\emptyset|_{K^{(0)}}$ and $t \in [0, \delta_K]$,*

$$\|\xi_{t\nu}(v) - \xi_0(v) - tq_v^* \zeta_\nu(b)\|_{C^0} \leq C_K(t + \varepsilon_\nu(v))t \quad \text{and} \quad \lim_{v \longrightarrow b} \varepsilon_\nu(v) = 0;$$

(4) *the map*

$$\phi_{\mathcal{T}, t\nu}: \mathcal{F}\mathcal{T}_{\delta_K}^\emptyset|_K \longrightarrow \mathfrak{X}_1^0(\mathbb{P}^4, d), \quad [v] \longrightarrow [\tilde{b}_{t\nu}(v)], \quad \text{where} \quad \tilde{b}_{t\nu}(v) = (\Sigma_v, j_v; \exp_{u_v} \xi_{t\nu}(v)),$$

is an orientation-preserving diffeomorphism onto $\{\bar{\partial}_J + t\nu\}^{-1}(0) \cap \mathfrak{X}_1^0(\mathbb{P}^4, d) \cap U_K$.

The first claim of the lemma is a special case of Lemma 4.1 in [Z4]. The second statement is obtained by expanding the equation at u_v and applying the Contraction Principle; see Subsection 3.6 in [Z2]. The uniqueness part means that there is a unique solution for each branch of the multisection ν . In (3), $\tilde{\Gamma}(T\mathbb{P}^4, d)$ denotes the Banach bundle over the space $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^4, d; J)$ such that

$$\tilde{\Gamma}(T\mathbb{P}^4, d)|_{(\Sigma_b, u_b)} = \Gamma(\Sigma_b; u_b^* T\mathbb{P}^4).$$

Let P_v and P_b denote the inverses of $D_{J,v}$ on $\Gamma_+(v; T\mathbb{P}^4)$ and of $D_{J,b}$ on $\Gamma_+(b; T\mathbb{P}^4)$, respectively. The Banach space $\Gamma_+(b; T\mathbb{P}^4)$ is the orthogonal complement of $\Gamma_-(b; T\mathbb{P}^4)$ in

$$\Gamma(b; T\mathbb{P}^4) \equiv L_1^p(\Sigma_b; u_b^* T\mathbb{P}^4);$$

see Subsection 3.1 in [Z2]. Taking the difference of the expansions for the equations in (2) describing $\xi_{t\nu}(v)$ and $\xi_0(v)$ and applying P_v , one finds that

$$\|\xi_{t\nu}(v) - \xi_0(v) - tP_v\nu(u_v)\|_{C^0} \leq C_K(t+|v|^{1/p})t.$$

On the other hand, a direct computation shows that

$$\begin{aligned} \|P_v\nu(u_v) - q_v^*P_b\nu(u_b)\|_{C^0} &\leq C(b)\|\nu(u_v) - D_{J,v}q_v^*P_b\nu(u_b)\|_{v,p} + \tilde{\varepsilon}_\nu(v) \\ &\leq C(b)\|\nu(u_v) - q_v^*\nu(u_b)\|_{v,p} + \tilde{\varepsilon}'_\nu(v) \leq \varepsilon_\nu(v); \end{aligned}$$

see Subsection 4.1 in [Z2] for a similar computation. These two bounds imply (3) of Lemma 3.2, with $\zeta_\nu(b) = P_b\nu(u_b)$. Finally, the proof of (4) is similar to Subsections 3.8 and 4.3-4.5 of [Z2].

Lemma 3.3 *Suppose \mathcal{T} and ν are as in Lemma 3.2. For every precompact open subset K of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, there exist $\delta_K, \varepsilon_K, C_K \in \mathbb{R}^+$, an open neighborhood U_K of K in $\mathfrak{X}_1(\mathbb{P}^4, d)$, and injective vector-bundle homomorphisms*

$$\tilde{\phi}_{\mathcal{T}, t\nu}: \pi_{\mathcal{F}\mathcal{T}}^* \mathcal{V}_1^d|_{\mathcal{F}^0 \mathcal{T}_{\delta_K}} \longrightarrow \Gamma(\mathfrak{L}; d),$$

covering the maps $\phi_{\mathcal{T}, t\nu}$ of Lemma 3.2, with the following properties:

- (1) requirements (1)-(4) of Lemma 3.2 are satisfied;
- (2) $\lim_{(v,w) \rightarrow (b,w^*)} \tilde{\phi}_{\mathcal{T}, t\nu}(v; w) = w^*$ for all $b \in K$ and $w^* \in \mathcal{V}_1^d$;
- (3) $s_1^d(\phi_{\mathcal{T}, t\nu}(v)) \equiv [s \circ \exp_{u_v} \xi_{t\nu}] \in \text{Im } \tilde{\phi}_{\mathcal{T}, t\nu}$, and for all $[v] = [b, v] \in \mathcal{F}\mathcal{T}_{\delta_K}^0|_K$

$$|\tilde{\phi}_{\mathcal{T}, t\nu}^{-1} s_1^d(\phi_{\mathcal{T}, t\nu}(v)) - \tilde{\phi}_{\mathcal{T}, 0}^{-1} s_1^d(\phi_{\mathcal{T}, 0}(v)) - t\{\nabla s\}\zeta_\nu(b)| \leq C_K(t + \varepsilon_\nu(v))t,$$

where $\varepsilon_\nu: \mathcal{F}\mathcal{T}^0 \rightarrow \mathbb{R}$ is a continuous function such that $\lim_{v \rightarrow b} \varepsilon_\nu(v) = 0$ for all $b \in \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$.

Proof: (1) We need to construct a lift $\tilde{\phi}_{\mathcal{T}, t\nu}$ that has the desired properties. For each element $v = (b, v)$ of $\tilde{\mathcal{F}}\mathcal{T}_{\delta_K}^0|_{K^{(0)}}$, $t \in [0, \delta_K)$, and $\xi \in \Gamma(b; \mathfrak{L})$, let

$$R_v \xi \in \Gamma(v; \mathfrak{L}) \equiv L_1^p(\Sigma_v; u_v^* \mathfrak{L}) \quad \text{and} \quad R_{v, t\nu} \xi \in \Gamma(\tilde{b}_{t\nu}(v); \mathfrak{L}) \equiv L_1^p(\Sigma_v; \{\exp_{u_v} \xi_{t\nu}(v)\}^* \mathfrak{L})$$

be defined by

$$\{R_v \xi\}(z) = \xi(q_v(z)) \quad \text{and} \quad \{R_{v, t\nu} \xi\}(z) = \Pi_{\{\xi_{t\nu}(v)\}(z)} \{R_v \xi\}(z) \quad \forall z \in \Sigma_v,$$

where $\Pi_{\{\xi_{t\nu}(v)\}(z)}\{R_\nu\xi\}(z)$ is the ∇ -parallel transport of $\{R_\nu\xi\}(z)$ along the ∇^J -geodesic

$$\gamma_{\{\xi_{t\nu}(v)\}(z)}: [0, 1] \longrightarrow \mathbb{P}^4, \quad \tau \longrightarrow \exp_{u_\nu(z)} \tau \{\xi_{t\nu}(v)\}(z).$$

We denote the image of

$$\Gamma_-(b; \mathfrak{L}) \equiv \ker \bar{\partial}_{\nabla, b}$$

under the linear map $R_{\nu, t\nu}$ by $\tilde{\Gamma}_-(\tilde{b}_{t\nu}(v); \mathfrak{L})$. If δ_K is sufficiently small, the L^2 -orthogonal projection

$$\tilde{\pi}_{\nu, t\nu}: \Gamma(\tilde{b}_{t\nu}(v); \mathfrak{L}) \longrightarrow \tilde{\Gamma}_-(\tilde{b}_{t\nu}(v); \mathfrak{L}),$$

defined with respect to the metric g_ν on Σ_ν , restricts to an isomorphism on

$$\Gamma_-(\tilde{b}_{t\nu}(v); \mathfrak{L}) \equiv \ker \bar{\partial}_{\nabla, \tilde{b}_{t\nu}(v)};$$

see Subsection 3.2 in [Z5]. Let $\tilde{\pi}_{\nu, t\nu}^{-1}$ be the inverse of this isomorphism. We set

$$\tilde{\phi}_{\mathcal{T}, t\nu}([v; \xi]) = [\tilde{\pi}_{\nu, t\nu}^{-1} R_{\nu, t\nu} \xi] \quad \forall \xi \in \Gamma_-(b; \mathfrak{L}).$$

(2) By our assumptions on ν ,

$$\bar{\partial}_{\nabla, \tilde{b}_{t\nu}(v)}(s \circ \exp_{u_\nu} \xi_{t\nu}(v)) = 0 \quad \implies \quad s_1^d(\phi_{\mathcal{T}, t\nu}(v)) \in \text{Im } \tilde{\phi}_{\mathcal{T}, t\nu}.$$

It remains to prove the estimate in part (3) of the lemma. If ϵ_K is sufficiently small, $v \in \tilde{\mathcal{F}}\mathcal{T}_{\delta_K}^\emptyset|_{K^{(0)}}$, $\xi \in \Gamma(v; T\mathbb{P}^4)$, and $\|\xi_\nu\|_{v, p, 1} < \epsilon_K$, we define

$$N_\nu^s \xi \in \Gamma(v; \mathfrak{L}) \quad \text{by} \quad \Pi_{\xi(z)}^{-1} s(\exp_{u_\nu(z)} \xi(z)) = s(u_\nu(z)) + \nabla s|_{u_\nu(z)} \xi(z) + \{N_\nu^s \xi\}(z) \quad \forall z \in \Sigma_\nu.$$

The quadratic term N_ν^s varies smoothly with ν , $N_\nu^s 0 = 0$, and

$$\|N_\nu^s \xi_1 - N_\nu^s \xi_2\|_{C^0} \leq C_s (\|\xi_1\|_{C^0} + \|\xi_2\|_{C^0}) \|\xi_1 - \xi_2\|_{C^0} \quad (3.3)$$

for some $C_s \in \mathbb{R}^+$ and for all $\xi_1, \xi_2 \in \Gamma(v)$ such that $\|\xi_1\|_{v, p, 1}, \|\xi_2\|_{v, p, 1} < \epsilon_{\mathcal{T}, \nu}(K)$. If $\xi \in \Gamma_-(b; \mathfrak{L})$,

$$\langle\langle s_1^d(\phi_{\mathcal{T}, t\nu}(v)), R_{\nu, t\nu} \xi \rangle\rangle = \langle\langle \Pi_{\xi_{t\nu}(v)}^{-1} s_1^d(\phi_{\mathcal{T}, t\nu}(v)), \xi \circ q_\nu \rangle\rangle.$$

Thus, the estimate in (3) of Lemma 3.3 follows from (3.3) and the estimate in (3) of Lemma 3.2.

Corollary 3.4 *Suppose \mathcal{T} and ν are as in Lemma 3.2. For every precompact open subset K of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, there exist $\delta_K, \epsilon_K, C_K \in \mathbb{R}^+$, an open neighborhood U_K of K in $\mathfrak{X}_1(\mathbb{P}^4, d)$, and for each $t \in (0, \epsilon(K))$ a sign-preserving bijection*

$$\{\bar{\partial}_J + t\nu\}^{-1}(0) \cap \mathfrak{X}_1(Y, d) \cap U_K \longrightarrow \{u \in \mathfrak{M}_1^0(\mathbb{P}^4, d; J) \cap U_K : \{s_1^d + t\vartheta_t\}(u) = 0\},$$

where $\vartheta_t \in \Gamma(\mathfrak{M}_1^0(\mathbb{P}^4, d; J) \cap U_K; \mathcal{V}_1^d)$ is a family of smooth sections such that

$$\begin{aligned} \lim_{v \rightarrow b, t \rightarrow 0} \vartheta_t(v) &= [\{\nabla s\} P_b \nu(b)] \quad \forall b \in K \quad \text{and} \\ |\nabla_X \tilde{\phi}_{\mathcal{T}, 0}^{-1} s_1^d \phi_{\mathcal{T}, 0}(v)| &\leq C_K |X| \quad \forall b \in K, \quad v \in \mathcal{F}\mathcal{T}_{\delta_K}^\emptyset|_b, \quad X \in \ker D_{J, b}, \end{aligned}$$

where $\phi_{\mathcal{T}, 0}$ and $\tilde{\phi}_{\mathcal{T}, 0}$ are as in Lemma 3.3.

Proof: The section ϑ_t is given by

$$t\vartheta_t(\phi_{\mathcal{T},0}(v)) = \tilde{\phi}_{\mathcal{T},0}\tilde{\phi}_{\mathcal{T},t\nu}^{-1}(s_1^d(\phi_{\mathcal{T},t\nu}(v))) - s_1^d(\phi_{\mathcal{T},0}(v)) \quad \forall [v] \in \mathcal{FT}_{\delta_K}^\emptyset|_{K^{(0)}}.$$

This corollary is immediate from Lemma 3.3, with the exception of the last estimate. This estimate follows from the behavior of the various terms involved in defining ϑ_t ; see Subsections 3.4 and 4.2 in [Z2].

For each $\kappa \in \mathcal{S}_0(J; Y)$, $\mathcal{U}_{\mathcal{T};\kappa}$ is a smooth suborbifold of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$. We denote its normal bundle by $\mathcal{N}^\kappa\mathcal{T}$. Its fiber at $[b] \in \mathcal{U}_{\mathcal{T};\kappa}$ is the quotient $\Gamma_-(b; T\mathbb{P}^4)/\Gamma_-(b; TY)$, where

$$\Gamma_-(b; TY) \equiv \Gamma_-(b; T\mathbb{P}^4) \cap \Gamma(b; TY) = \Gamma_-(b; T\kappa),$$

by the assumption (J_Y2) on J_Y . We identify $\mathcal{N}^\kappa\mathcal{T}$ with the L^2 -orthogonal complement of $\Gamma_-(b; TY)$ in $\Gamma_-(b; T\mathbb{P}^4)$. Let

$$\varphi_{\mathcal{T};\kappa} : \mathcal{N}^\kappa\mathcal{T}_{\delta_\kappa} \longrightarrow \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$$

be an orientation-preserving identification of neighborhoods of $\mathcal{U}_{\mathcal{T};\kappa}$ in $\mathcal{N}^\kappa\mathcal{T}$ and in $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ and let

$$\tilde{\varphi}_{\mathcal{T};\kappa} : \pi_{\mathcal{N}^\kappa\mathcal{T}}^* \mathcal{FT}|_{\mathcal{N}^\kappa\mathcal{T}_{\delta_\kappa}} \longrightarrow \mathcal{FT} \quad \text{and} \quad \tilde{\varphi}_{\mathcal{T};\kappa} : \pi_{\mathcal{N}^\kappa\mathcal{T}}^* \mathcal{V}_1^d|_{\mathcal{N}^\kappa\mathcal{T}_{\delta_\kappa}} \longrightarrow \mathcal{V}_1^d$$

be lifts of $\varphi_{\mathcal{T};\kappa}$ to vector-bundle isomorphisms restricting to the identity over $\mathcal{U}_{\mathcal{T};\kappa}$.

The section s_1^d is smooth on $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ and its differential along $\mathcal{U}_{\mathcal{T};\kappa}$, i.e. the homomorphism j_0 in the long exact sequence

$$0 \longrightarrow \Gamma_-(b; TY) \xrightarrow{i_0} \Gamma_-(b; T\mathbb{P}^4) \xrightarrow{j_0} \Gamma_-(b; \mathfrak{L}) \xrightarrow{\partial_0} H_J^1(b; TY) \longrightarrow 0, \quad (3.4)$$

is injective on $\mathcal{N}^\kappa\mathcal{T}$. We denote the image bundle of j_0 by $\mathcal{V}_+ \subset \mathcal{V}_1^d$ and its L^2 -orthogonal complement in \mathcal{V}_1^d by \mathcal{V}_- . Let π_+ and π_- be the corresponding projection maps.

Lemma 3.5 *Suppose \mathcal{T} is a bubble type as in Lemma 3.2 and $\kappa \in \mathcal{S}_0(Y; J)$. For every precompact open subset K of $\mathcal{U}_{\mathcal{T};\kappa}$, there exist*

- (a) $\delta_K, \delta'_K \in \mathbb{R}^+$ and an open neighborhood U_K of K in $\mathfrak{X}_1(\mathbb{P}^4, d)$;
- (b) an orientation-preserving diffeomorphism $\phi_{\mathcal{T},\kappa} : \mathcal{N}^\kappa\mathcal{T}_{\delta'_K} \times_K \mathcal{FT}_{\delta_K}^\emptyset \longrightarrow \mathfrak{M}_1^0(\mathbb{P}^4, d; J) \cap U_K$;
- (c) a lift $\tilde{\phi}_{\mathcal{T},\kappa} : \pi_{\mathcal{N}^\kappa\mathcal{T} \oplus \mathcal{FT}}^* \mathcal{V}_1^d \longrightarrow \mathcal{V}_1^d$ of $\phi_{\mathcal{T},\kappa}$ to a vector-bundle isomorphism;

with the following property. If $\vartheta_t \in \Gamma(\mathfrak{M}_1^0(\mathbb{P}^4, d; J) \cap U_K; \mathcal{V}_1^d)$ is a family of smooth sections such that for some $C \in \mathbb{R}^+$,

$$\begin{aligned} |\vartheta_t(u)| &\leq C \\ |\tilde{\phi}_{\mathcal{T},\kappa}^{-1}\vartheta_t\phi_{\mathcal{T},\kappa}(X, v) - \tilde{\phi}_{\mathcal{T},\kappa}^{-1}\vartheta_t\phi_{\mathcal{T},\kappa}(X', v)| &\leq C|X - X'| \quad \forall X, X' \in \mathcal{N}^\kappa\mathcal{T}_{\delta'_K}, v \in \mathcal{FT}_{\delta_K}^\emptyset|_K, \end{aligned}$$

then there exists $\epsilon \in \mathbb{R}^+$ such that for all $t \in [0, \epsilon]$, $b \in K$, and $v \in \mathcal{FT}_{\delta_K}|_b$, the equation

$$\pi_+\tilde{\phi}_{\mathcal{T},\kappa}^{-1}(\{s_1^d + t\vartheta_t\}\phi_{\mathcal{T},\kappa}(b; X, v)) = 0$$

has a unique solution $X = X_t(v) \in \mathcal{N}^\kappa\mathcal{T}_{\delta'_K}|_b$. Furthermore,

$$\lim_{t \rightarrow 0, v \rightarrow b} t^{-1}\pi_-\tilde{\phi}_{\mathcal{T},\kappa}^{-1}(s_1^d\phi_{\mathcal{T},\kappa}(b; X_t(v), v)) = 0 \quad \forall b \in K.$$

Proof: (1) The desired maps $\phi_{\mathcal{T},\kappa}$ and $\tilde{\phi}_{\mathcal{T},\kappa}$ are simply the compositions $\phi_{\mathcal{T},0} \circ \tilde{\varphi}_{\mathcal{T},\kappa}$ and $\tilde{\phi}_{\mathcal{T},0} \circ \tilde{\varphi}_{\mathcal{T},\kappa}$, respectively. For each $b \in K$, $X \in \mathcal{N}^\kappa \mathcal{T}|_b$, and $v \in \mathcal{FT}^\emptyset|_b$ sufficiently small, we define $\tilde{N}_s(X)$ and $\tilde{N}'_s(X, v)$ in $\mathcal{V}_1^d|_b$ by

$$\tilde{\varphi}_{\mathcal{T},\kappa}^{-1}(s_1^d \phi_{\mathcal{T},\kappa}(b; X)) = s_1^d(b) + j_0 X + \tilde{N}_s(X) = j_0 X + \tilde{N}_s(X); \quad (3.5)$$

$$\tilde{\varphi}_{\mathcal{T},\kappa}^{-1}(s_1^d \phi_{\mathcal{T},\kappa}(b; X, v)) = \tilde{\varphi}_{\mathcal{T},\kappa}^{-1} s_1^d \phi_{\mathcal{T},\kappa}(b; X) + \tilde{N}'_s(X, v). \quad (3.6)$$

Since j_0 is the derivative of s_1^d on $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, for some $C_K \in \mathbb{R}^+$,

$$\tilde{N}_s(0) = 0, \quad |\tilde{N}_s(X) - \tilde{N}_s(X')| < C_K(|X| + |X'|)|X - X'| \quad \forall X, X' \in \mathcal{N}^\kappa \mathcal{T}_{\delta'_K}|_K. \quad (3.7)$$

For $\tilde{N}'_s(\cdot, \cdot)$, we similarly have

$$\begin{aligned} |\tilde{N}'_s(X, v)| &\leq C_K |v|^{1/p}, \\ |\tilde{N}'_s(X, v) - \tilde{N}'_s(X', v)| &< C_K |v|^{1/p} |X - X'| \end{aligned} \quad \forall X, X' \in \mathcal{N}^\kappa \mathcal{T}_{\delta'_K}|_K, \quad v \in \mathcal{FT}^\emptyset_{\delta'_K}|_K. \quad (3.8)$$

The first estimate above is clear from (2) of Lemma 3.2. The second bound follows from the analogous bound on the behavior of the vector field $\xi_0(v)$ of (2) of Lemma 3.2; see Subsection 4.2 in [Z2].

(2) If ϑ_t is a family of smooth sections as in the statement of the lemma, by (3.5) and (3.6),

$$\begin{aligned} \pi_+ \tilde{\phi}_{\mathcal{T},\kappa}^{-1}(\{s_1^d + t\vartheta_t\} \phi_{\mathcal{T},\kappa}(b; X, v)) &= j_0 X + \pi_+ \tilde{N}_s(X) + \pi_+ \tilde{N}'_s(X, v) + t\pi_+ \tilde{\vartheta}_t(X, v), \\ \text{where} \quad \tilde{\vartheta}_t(X, v) &= \tilde{\phi}_{\mathcal{T},\kappa}^{-1}(\vartheta_t \phi_{\mathcal{T},\kappa}(b; X, v)). \end{aligned} \quad (3.9)$$

By (3.7)-(3.9) and the Contraction Principle, there exist $\delta, \delta' \in \mathbb{R}^+$, dependent on j_0 and C_K , and $\epsilon, C' \in \mathbb{R}^+$, dependent on j_0 , C_K , and C , such that for all $t \in [0, \epsilon)$, $b \in K$, and $v \in \mathcal{FT}^\emptyset_{\delta}|_b$, the equation

$$\pi_+ \tilde{\phi}_{\mathcal{T},\kappa}^{-1}(\{s_1^d + t\vartheta_t\} \phi_{\mathcal{T},\kappa}(b; X, v)) = 0$$

has a unique solution $X = X_t(v) \in \mathcal{N}^\kappa \mathcal{T}_{\delta'}|_b$. Furthermore,

$$|X_0(v)| \leq C' |v|^{1/p} \quad \text{and} \quad |X_t(v) - X_0(v)| \leq C' t. \quad (3.10)$$

(3) By (3.5)-(3.8) and (3.10),

$$|\pi_- \tilde{\phi}_{\mathcal{T},\kappa}^{-1}(s_1^d \phi_{\mathcal{T},\kappa}(b; X_t(v), v)) - \pi_- \tilde{\phi}_{\mathcal{T},\kappa}^{-1}(s_1^d \phi_{\mathcal{T},\kappa}(b; X_0(v), v))| \leq C''(t + |v|^{1/p})t. \quad (3.11)$$

On the other hand, as can be seen from Lemma 3.6 below,

$$\pi_- \tilde{\phi}_{\mathcal{T},\kappa}^{-1}(s_1^d \phi_{\mathcal{T},\kappa}(b; X_0(v), v)) = 0 \quad (3.12)$$

for all $v \in \mathcal{FT}^\emptyset|_K$ sufficiently small. The last claim of the lemma follows from (3.11) and (3.12).

Proof of Proposition 3.1: (1) By Corollary 3.4, if $t \in \mathbb{R}^+$ and $U(K)$ are sufficiently small, there is a one-to-one correspondence between $\{\bar{\partial}_J + t\nu\}^{-1}(0) \cap U(K)$ and the set

$$\{u \in \mathfrak{M}_1^0(\mathbb{P}^4, d; J) \cap U(K) : \{s_1^d + t\vartheta_t\}(u) = 0\},$$

where $\vartheta_t \in \Gamma(\mathfrak{M}_1^0(\mathbb{P}^4, d; J) \cap U_K; \mathcal{V}_1^d)$ is a family of smooth sections as in Lemma 3.5. In addition,

$$\lim_{v \rightarrow b, t \rightarrow 0} \vartheta_t(v) = [\{\nabla s\} P_b \nu(b)] \quad \forall b \in K.$$

The homomorphism \mathfrak{d}_0 in the long exact sequence (3.4) restricts to an isomorphism on \mathcal{V}_- and vanishes on \mathcal{V}_+ . By definition of \mathfrak{d}_0 and P_b ,

$$\mathfrak{d}_0([\{\nabla s\} P_b \nu(b)]) = \pi_{\kappa, \nu}^0(b) \quad \forall b \in \mathcal{U}_{\mathcal{T}; \kappa}.$$

Thus, by Lemma 3.5,

$$\{\bar{\partial}_J + t\nu\}^{-1}(0) \cap U(K) = \emptyset \quad \text{if} \quad \pi_{\kappa, \nu}^0{}^{-1}(0) \cap K = \emptyset.$$

The case $|I| > 1$ or $\aleph \neq \emptyset$ of Proposition 3.1 now follows from the assumption $(\nu 3a)$.

(2) If $|I| = 1$ and $\aleph = \emptyset$, by the assumption $(\nu 3a)$ and Lemma 3.5, the section $\bar{\partial}_J + t\nu$ is transverse to the zero set on $U(K)$ and

$$\pm |\{\bar{\partial}_J + t\nu\}^{-1} \cap U(K)| = \pm |\pi_{\kappa, \nu}^0{}^{-1}(0) \cap K|.$$

Since $\pi_{\kappa, \nu}^0{}^{-1}(0) \subset \mathcal{U}_{\mathcal{T}; \kappa} = \mathfrak{M}_1^0(\kappa, d/d_\kappa)$,

$$\pm |\{\bar{\partial}_J + t\nu\}^{-1} \cap U(K)| = \pm |\pi_{\kappa, \nu}^0{}^{-1}(0)| = \langle e(\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}), [\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)] \rangle,$$

provided $\pi_{\kappa, \nu}^0{}^{-1}(0) \subset K$ and t and $U(K)$ are sufficiently small.

We conclude this subsection with Lemma 3.6, which was used in Lemma 3.5.

Lemma 3.6 *Suppose \mathcal{T} is a bubble type as in Lemma 3.2 and $\kappa \in \mathcal{S}_0(Y, J)$. For every precompact open subset K of $\mathcal{U}_{\mathcal{T}; \kappa}$, there exist $\delta \in \mathbb{R}^+$, an open neighborhood U of K in $\mathfrak{X}_1(\mathbb{P}^4, d)$, and an orientation-preserving diffeomorphism*

$$\phi'_{\mathcal{T}, \kappa}: \mathcal{FT}'_\delta|_K \longrightarrow \mathfrak{M}_1^0(\kappa, d/d_\kappa) \cap U \subset \mathfrak{M}_1^0(\mathbb{P}^4, d; J).$$

Proof: If $\mathcal{T} = (I, \aleph; \underline{d})$,

$$\begin{aligned} \mathcal{U}_{\mathcal{T}; \kappa} = \mathcal{U}_{\mathcal{T}'}(\kappa; J_0) &\approx \mathcal{U}_{\mathcal{T}'}(\mathbb{P}^1; J_0) \quad \text{and} \quad \mathcal{FT}|_{\mathcal{U}_{\mathcal{T}; \kappa}} = \mathcal{FT}' \longrightarrow \mathcal{U}_{\mathcal{T}'}(\mathbb{P}^1; J_0), \\ \text{where} \quad \mathcal{T}' &= (I, \aleph; \underline{d}') \quad \text{and} \quad d'_i = d_i/d_\kappa. \end{aligned}$$

Thus, Lemma 3.6 is the \mathbb{P}^1 -analogue of the $t=0$ case of (4) of Lemma 3.2.

3.3 Proof of Proposition 2.6

Proposition 2.6 follows immediately from Proposition 3.7.

Proposition 3.7 *Suppose d , Y , and J are as in Proposition 2.5, $\nu \in \tilde{\mathcal{A}}_1^d(\bar{\partial}; J)$ is a generic perturbation of the $\bar{\partial}_J$ -operator on $\mathfrak{X}_1(\mathbb{P}^4, d)$, $\kappa \in \mathcal{S}_0(Y; J)$, and $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $\sum_{i \in I} d_i = d$ and $d_i = 0$ for all minimal elements i of I . If $|\hat{I}| > 1$ or $\aleph \neq \emptyset$, for every compact subset K of $\mathcal{U}_{\mathcal{T}; \kappa}$, there exist $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U_\nu(K)$ of K in $\mathfrak{X}_1(Y, d)$ such that*

$$\{\bar{\partial}_J + t\nu\}^{-1}(0) \cap U_\nu(K) = \emptyset \quad \forall t \in (0, \epsilon_\nu(K)).$$

If $|\hat{I}| = 1$ and $\aleph = \emptyset$, for every compact subset K of $\mathcal{U}_{T;\kappa}$, there exist $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\mathfrak{X}_1(Y, d)$ with the following properties:

- (a) the section $\bar{\partial}_J + t\nu$ is transverse to the zero set in $\Gamma_1^{0,1}(Y, d; J)$ over $U(K)$ for all $t \in (0, \epsilon_\nu(K))$;
- (b) for every open subset U of $\mathfrak{X}_1(Y, d)$, there exists $\epsilon(U) \in (0, \epsilon_\nu(K))$ such that

$$\begin{aligned} \pm|\{\bar{\partial}_J + t\nu\}^{-1} \cap U| &= \frac{d/d_\kappa}{12} \langle e(\mathcal{W}_{\kappa, d/d_\kappa}^0), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)] \rangle \quad \text{if} \\ \tilde{\pi}_{\nu, \kappa}^1{}^{-1}(0) \cap U &\subset K \subset U \subset U(K), \quad t \in (0, \epsilon(U)). \end{aligned}$$

In simpler words, none of the strata of $\overline{\mathfrak{M}}_1^q(\kappa, d/d_\kappa)$ with $q \geq 2$ contributes to the number $N_1(d)$. Neither does any of the boundary strata of $\overline{\mathfrak{M}}_1^1(\kappa, d/d_\kappa)$. On the other hand, $\mathfrak{M}_1^1(\kappa, d/d_\kappa)$ contributes the euler class of the bundle $\widehat{\mathcal{W}}_{\kappa, d/d_\kappa}^{1,1}$; see Subsection 3.1.

We will proceed similarly to Subsection 3.2, but run the gluing construction in Y , instead of \mathbb{P}^4 , and make use of the assumption $(J_Y 2)$ from the start. We will also use the family of metrics on \mathbb{P}^1 provided by Lemma 2.1 in [Z1], which we now restate:

Lemma 3.8 *There exist $r_{\mathbb{P}^1} > 0$ and a smooth family of Kahler metrics $\{g_{\mathbb{P}^1, q} : q \in \mathbb{P}^1\}$ on \mathbb{P}^1 with the following property. If $B_q(q', r) \subset \mathbb{P}^1$ denotes the $g_{\mathbb{P}^1, q}$ -geodesic ball about q' , the triple $(B_q(q, r_{\mathbb{P}^1}), J_0, g_{\mathbb{P}^1, q})$ is isomorphic to a ball in \mathbb{C}^1 for all $q \in \mathbb{P}^1$.*

In this case, the operators $D_{J,b}|_{\Gamma(\Sigma_b; u_b^*TY)}$ are not surjective for $b \in \mathcal{U}_{T;\kappa}^{(0)}$, where $\mathcal{U}_{T;\kappa}^{(0)}$ is the preimage of $\mathcal{U}_{T;\kappa}$ under the quotient projection map $\mathcal{U}_{T;\kappa}^{(0)}(\mathbb{P}^4; J) \rightarrow \mathcal{U}_{T;\kappa}(\mathbb{P}^4; J)$. Thus, in contrast to the case of Lemma 3.2, we encounter an obstruction bundle in trying to solve the $\bar{\partial}_J$ -equation near $\mathcal{U}_{T;\kappa}$, as in Subsections 3.3-3.5 of [Z2]. Subsections 3.3-3.5 in [Z1] describe a special case of an analogous construction in circumstances similar to the present situation.

First, we describe a convenient ‘‘exponential’’ map for Y defined on a neighborhood of each smooth curve $\kappa \in \mathcal{S}_0(Y; J)$. We identify the rational curve κ with \mathbb{P}^1 . For each $b \in \mathcal{U}_{T;\kappa}$, let $g_{Y,b}$ be a J -compatible extension of the metric $g_{\kappa, b} \equiv g_{\mathbb{P}^1, \text{ev}_P(b)}$ on κ provided by Lemma 3.8 to a Riemannian metric on a neighborhood of κ in Y . We identify the normal bundle $N_Y\kappa$ of κ in Y with the $g_{Y,b}$ -orthogonal complement of $T\kappa$ in $TY|_\kappa$. Let

$$\text{exp}_b : T\kappa \longrightarrow \kappa \quad \text{and} \quad \widetilde{\text{exp}}_b : \pi_{T\kappa}^* N_Y\kappa \longrightarrow N_Y\kappa$$

be the exponential map with respect to the metric $g_{\kappa, b}$ and a lift of exp_b to a vector bundle homomorphism restricting to the identity over κ . For example, $\widetilde{\text{exp}}_b$ can be taken to be the $g_{Y,b}$ -parallel transport along the $g_{\kappa, b}$ -geodesics. For each $q \in \kappa$ and $\xi \in T_q Y$ sufficiently small, let

$$\text{exp}_b \xi = \text{exp}_{g_{Y,b}, \text{exp}_{b,q} \xi_-} (\widetilde{\text{exp}}_{b, \xi_-} \xi_+) \quad \text{if} \quad \xi = \xi_- + \xi_+ \in T\kappa \oplus N_Y\kappa = TY,$$

where $\text{exp}_{g_{Y,b}}$ is the exponential map for the metric $g_{Y,b}$. One useful property of this ‘‘exponential’’ map is that $\text{exp}_b \xi \in \kappa$ if $\xi \in T\kappa \subset TY$.

For each element $b = (\Sigma_b, u_b)$ of $\mathcal{U}_{T;\kappa}^{(0)}$, we identify the cokernel $H_J^1(b; TY)$ of the operator

$$D_{J,b} : \Gamma(b; TY) \longrightarrow \Gamma^{0,1}(b; TY; J)$$

with the space $\Gamma_-^{0,1}(b; TY)$ of (J, j) -antilinear u_b^*TY -valued harmonic forms on Σ_b . The elements of $\Gamma_-^{0,1}(b; TY)$ may have simple poles at the nodes of Σ_b with the residues adding up to zero at each node. If $\mathcal{H}_{b,P}$ denotes the one-dimensional vector space of harmonic antilinear differentials on the principal component(s) $\Sigma_{b,P}$ of Σ_b ,

$$\Gamma_-^{0,1}(b; TY) = \Gamma_-^{0,1}(b; T\kappa) \oplus \Gamma_-^{0,1}(b; N_Y\kappa) = \mathcal{H}_{b,P} \otimes T_{\text{ev}_P(b)}\kappa \oplus \Gamma_-^{0,1}(b; N_Y\kappa).$$

This decomposition is L^2 -orthogonal. Furthermore, $\Gamma_-^{0,1}(b; N_Y\kappa)$ is isomorphic to the cokernel $H_J^1(b; N_Y\kappa)$ of the operator

$$D_{J,b}^\perp: \Gamma(b; N_Y\kappa) \longrightarrow \Gamma^{0,1}(b; N_Y\kappa; J)$$

induced by the operator $D_{J,b}$ via the quotient projection map

$$\pi_\kappa^\perp: TY|_\kappa \longrightarrow N_Y\kappa = TY|_\kappa / T\kappa.$$

We note that if $\mathfrak{N} = \emptyset$ and $|\hat{I}| = 1$, $\Gamma_-^{0,1}(b; TY)$ is a subspace of $\Gamma^{0,1}(b; TY; J)$.

We are now ready to proceed with the pregluing construction. For each sufficiently small element $v = (b, v)$ of $\tilde{\mathcal{F}}T^\emptyset$, let

$$b(v) = (\Sigma_v, j_v; u_v)$$

be the corresponding approximately holomorphic stable map, as in Subsection 3.2. In the present case, the linearization $D_{J,b}$ of the $\bar{\partial}_J$ -operator at b is not surjective. Thus, the linearization $D_{J,v}$ of the $\bar{\partial}_J$ -operator at $b(v)$, defined via the Levi-Civita connection of the metric $\tilde{g}_{Y,b}$, is not uniformly surjective. An approximate cokernel of $D_{J,b}$ is given by

$$\Gamma_-^{0,1}(v; TY) = \Gamma_-^{0,1}(v; T\kappa) \oplus \Gamma_-^{0,1}(v; N_Y\kappa), \quad (3.13)$$

with the vector spaces $\Gamma_-^{0,1}(v; T\kappa)$ and $\Gamma_-^{0,1}(v; N_Y\kappa)$ explicitly describable from $\Gamma_-^{0,1}(b; T\kappa)$ and $\Gamma_-^{0,1}(b; N_Y\kappa)$, respectively, via the basic gluing map $q_v: \Sigma_v \longrightarrow \Sigma_b$. In fact, we can simply take

$$\Gamma_-^{0,1}(v; N_Y\kappa) = \{q_v^*\eta: \eta \in \Gamma_-^{0,1}(b; N_Y\kappa)\}. \quad (3.14)$$

While we can define the space $\Gamma_-^{0,1}(v; T\kappa)$ in the same way from $\Gamma_-^{0,1}(b; T\kappa)$, in the $\mathfrak{N} = \emptyset$, $|\hat{I}| = 1$ case it is more convenient to take

$$\Gamma_-^{0,1}(v; T\kappa) = \{R_v\eta: \eta \in \Gamma_-^{0,1}(b; T\kappa)\},$$

where $R_v\eta$ is a smooth extension of η such that $R_v\eta$ is harmonic on the neck attaching the only bubble $\Sigma_{b,h}$ of Σ_b and below a small collar of the neck and vanishes past a slighter larger collar. For an explicit description of $R_v\eta$, see the construction at the beginning of Subsection 2.2 in [Z1]. We observe that

$$\begin{aligned} \langle\langle \eta_\kappa, \tilde{\eta} \rangle\rangle_{v,2} = 0, \quad \langle\langle \bar{\partial}_J u_v, \tilde{\eta} \rangle\rangle_{v,2} = 0, \quad \langle\langle D_{J,v}\xi, \tilde{\eta} \rangle\rangle_{v,2} = 0, \\ \forall \xi \in \Gamma(v; T\kappa), \quad \eta_\kappa \in \Gamma_-^{0,1}(v; T\kappa), \quad \tilde{\eta} \in \Gamma_-^{0,1}(v; N_Y\kappa), \end{aligned} \quad (3.15)$$

where $\langle\langle \cdot, \cdot \rangle\rangle_{v,2}$ is the L^2 -inner product of the metric $g_{Y,b}$ on Y . This inner-product is independent of the choice of a metric on Σ_v compatible with the complex structure j_v on Σ_v , though we will

always view Σ_v as carrying the metric g_v induced by the pregluing construction. If $\mathfrak{N} = \emptyset$ and $|\hat{I}| = 1$, $\Gamma_-^{0,1}(v; TY)$ is a subspace of $\Gamma^{0,1}(v; TY; J)$, and we denote its L^2 -orthogonal complement by $\tilde{\Gamma}_+(v; TY)$. Let

$$\pi_{v;\kappa}^{0,1}, \tilde{\pi}_v^{0,1}, \pi_{v;+}^{0,1} : \Gamma^{0,1}(v; TY; J) \longrightarrow \Gamma_-^{0,1}(v; T\kappa), \Gamma_-^{0,1}(v; N_Y\kappa), \Gamma_+^{0,1}(v; TY)$$

be the L^2 -projection maps.

As in Subsection 3.2, if v is sufficiently small, we can also obtain a decomposition

$$\Gamma(v; TY) = \Gamma_-(v; TY) \oplus \tilde{\Gamma}_+(v; TY) \quad (3.16)$$

such that the linear operator

$$D_{J,v} : \Gamma_+(v; TY) \longrightarrow \Gamma^{0,1}(v; TY; J)$$

is injective, while

$$\Gamma_-(v; TY) = \{\xi \circ q_v : \xi \in \Gamma_-(b; TY)\}.$$

In this case, $D_{J,v}$ denotes the linearization of the $\bar{\partial}_J$ -operator at $b(v)$ with the respect to the ‘‘exponential’’ map chosen above. In (3.16), we can take the space $\tilde{\Gamma}_+(v; TY)$ to be the L^2 -orthogonal complement of $\Gamma_-(v; TY)$, and we do so unless $\mathfrak{N} = \emptyset$ and $|\hat{I}| = 1$. If $\mathfrak{N} = \emptyset$ and $|\hat{I}| = 1$, we can choose $\tilde{\Gamma}_+(v; TY)$ in such a way that

$$\langle\langle D_{J,v}\xi, \eta \rangle\rangle_{v,2} = 0 \quad \forall \xi \in \tilde{\Gamma}_+(v; TY) \cap \Gamma(v; T\kappa), \quad \eta \in \Gamma_-^{0,1}(v; TY), \quad (3.17)$$

the operator

$$D_{J,v} : \tilde{\Gamma}_+(v; T\kappa) \equiv \tilde{\Gamma}_+(v; TY) \cap \Gamma(v; T\kappa) \longrightarrow \Gamma_+^{0,1}(v; T\kappa) \equiv \Gamma_+^{0,1}(v; TY) \cap \Gamma^{0,1}(v; T\kappa)$$

is an isomorphism, and the intersection of $\tilde{\Gamma}_+(v; TY)$ with the L^2 -orthogonal complement of $\Gamma_-(v; TY)$ has codimension one in both spaces. The subspace $\tilde{\Gamma}_+(v; TY)$ of $\Gamma(v; TY)$ is constructed by restricting the procedure described in Subsection 2.3 of [Z1] to the line $\mathcal{H}_{b,\mathfrak{N}} \otimes T_{\text{ev}_P(b)}\kappa$.

Similarly, let $\Gamma_+(v; N_Y\kappa)$ be the L^2 -orthogonal complement of

$$\Gamma_-(v; N_Y\kappa) = \{\xi \circ q_v : \xi \in \Gamma_-(b; N_Y\kappa)\}$$

in $\Gamma(v; N_Y\kappa) \equiv L_1^p(\Sigma_v; u_v^* N_Y\kappa)$. If v is sufficiently small, the linear operator

$$D_{J,v}^\perp : \Gamma_+(v; N_Y\kappa) \longrightarrow \Gamma^{0,1}(v; N_Y\kappa; J)$$

is injective. The key properties of this setup are described in Lemma 3.9:

Lemma 3.9 *If \mathcal{T} , ν , and κ are as in Proposition 3.7, for every precompact open subset K of $\mathcal{U}_{\mathcal{T};\kappa}$, there exist $\delta_K, C_K \in \mathbb{R}^+$ and an open neighborhood U_K of K in $\mathfrak{X}_1(Y, d)$ with the following properties:*

(1) *for every $[\tilde{b}] \in \mathfrak{X}_1^0(Y, d) \cap U_K$, there exist $v \in \tilde{\mathcal{F}}\mathcal{T}_{\delta_K}^\emptyset|_{K(0)}$ and $\zeta \in \tilde{\Gamma}_+(v; TY)$ such that $\|\zeta\|_{v,p,1} < \delta_K$*

and $[\exp_{b(v)}\zeta] = [\tilde{b}]$, and the pair (b, ζ) is unique up to the action of the group $\text{Aut}(\mathcal{T}) \times (S^1)^{\hat{I}}$;
(2) for all $v = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}_{\delta_K}^0|_{K^{(0)}}$,

$$\begin{aligned} \|\bar{\partial}_J u_v\|_{v,p} &\leq C_K |v|^{1/p}; \\ C_K^{-1} \|\zeta\|_{v,p,1} &\leq \|D_{J,v}\zeta\|_{v,p} \leq C_K \|\zeta\|_{v,p,1} \quad \forall \zeta \in \tilde{\Gamma}_+(v; TY); \\ C_K^{-1} \|\xi\|_{v,p,1} &\leq \|D_{J,v}^\perp \xi\|_{v,p} \leq C_K \|\xi\|_{v,p,1} \quad \forall \xi \in \Gamma_+(v; N_\kappa Y); \end{aligned}$$

(3) for all $v = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}_{\delta_K}^0|_{K^{(0)}}$, $\xi \in \Gamma(v; N_\kappa Y)$, and $\eta \in \Gamma_-^{0,1}(v; N_\kappa Y)$,

$$|\langle\langle D_{J,v}^\perp \xi, \eta \rangle\rangle_{v,2}| \leq C_K |v|^{1/p} \|\xi\|_{v,p,1} \|\eta\|_{v,1}.$$

In the first claim of Lemma 3.9,

$$\exp_{b(v)}\zeta = (\Sigma_v, j_v; \exp_{b, u_v}\zeta).$$

This statement is a variation on (2) of Lemma 4.4 in [Z4] and holds for the same reasons. The first estimate in (2) and (3) of Lemma 3.9 can be obtained by direct computations. The two remaining estimates are proved analogously to the corresponding estimates of Lemma 3.2.

Corollary 3.10 *Suppose ν , \mathcal{T} , and κ are as in Proposition 3.7. If $q \in \mathbb{Z}^+$ and K is a compact subset of $\mathcal{U}_{\mathcal{T}; \kappa} \subset \overline{\mathfrak{M}}_1^q(\kappa, d/d_\kappa)$ such that*

$$\tilde{\pi}_{\kappa, \nu}^q{}^{-1}(0) \cap K = \emptyset,$$

then there exist $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U_\nu(K)$ of K in $\mathfrak{X}_1(Y, d)$ such that

$$\{\bar{\partial}_J + t\nu\}^{-1}(0) \cap U_\nu(K) = \emptyset \quad \forall t \in (0, \epsilon_\nu(K)).$$

Proof: (1) As usually, for all $\zeta \in \Gamma(v; TY)$ sufficiently small,

$$\Pi_\zeta^{-1}\{\bar{\partial}_J + t\nu\} \exp_{b(v)}\zeta = \bar{\partial}_J u_v + D_{J,v}\zeta + N_v\zeta + tN_{\nu,v}\zeta + t\nu|_{u_v},$$

where Π_ζ denotes the parallel transport with respect to the Levi-Civita connection of the metric $g_{Y,b}$ along the geodesics of the map \exp_b . The nonlinear terms satisfy

$$\begin{aligned} \|N_v\zeta - N_v\zeta'\|_{v,p} &\leq C_K (\|\zeta\|_{v,p,1} + \|\zeta'\|_{v,p,1}) \|\zeta - \zeta'\|_{v,p,1} \\ \|N_{\nu,v}\zeta - N_{\nu,v}\zeta'\|_{v,p} &\leq C_K \|\zeta - \zeta'\|_{v,p,1} \end{aligned} \quad \forall \zeta, \zeta' \in \Gamma(v; TY); \quad (3.18)$$

see Subsection 3.6 in [Z2] for example. Our choice of the map \exp_b also implies that

$$N_v\zeta \in \Gamma^{0,1}(v; T\kappa) \quad \forall \zeta \in \Gamma(v; T\kappa) \subset \Gamma(v; TY). \quad (3.19)$$

(2) Suppose $v = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}^0|_{K^{(0)}}$, $\zeta \in \Gamma_+(v; TY)$, and

$$\begin{aligned} \{\bar{\partial}_J + t\nu\} \exp_{b(v)}\zeta &= 0 \quad \implies \\ \bar{\partial}_J u_v + D_{J,v}\zeta + N_v\zeta + tN_{\nu,v}\zeta + t\nu|_{u_v} &= 0. \end{aligned} \quad (3.20)$$

From (2) of Lemma 3.9 and (3.18), we then obtain

$$\|\zeta\|_{v,p,1} \leq C_K (|v|^{1/p} + t). \quad (3.21)$$

On the other hand, applying the projection map π_κ^\perp to both sides of (3.20), we get

$$\begin{aligned} D_{J,\nu}^\perp \zeta^\perp + N_\nu^\perp \zeta + tN_{\nu,\nu}^\perp \zeta + t\nu^\perp|_{u_\nu} &= 0 \in \Gamma^{0,1}(v; N_Y \kappa; J), & \text{if} & \\ \zeta &= \zeta^t + \zeta^\perp \in \Gamma(v; T\kappa) \oplus \Gamma(v; N_Y \kappa), & N_\nu^\perp \zeta &= \pi_\kappa^\perp N_\nu \zeta, \quad N_{\nu,\nu}^\perp \zeta = \pi_\kappa^\perp N_{\nu,\nu} \zeta, \quad \nu^\perp = \pi_\kappa^\perp \nu. \end{aligned} \quad (3.22)$$

By (3.18), (3.19), and (3.21),

$$\|N_\nu^\perp \zeta\|_{v,p} = \|\pi_\kappa^\perp (N_\nu(\zeta^t + \zeta^\perp) - N_\nu \zeta^t)\|_{v,p} \leq C_K (|v|^{1/p} + t) \|\zeta^\perp\|_{v,p,1}.$$

Thus, by (2) of Lemma 3.9, (3.18), and (3.22),

$$\|\zeta^\perp\|_{v,p,1} \leq C_K t, \quad (3.23)$$

provided δ_K is sufficiently small. Combining (3) of Lemma 3.9, (3.18), (3.22), and (3.23), we obtain

$$|\langle \nu|_{u_\nu}, \eta \rangle_{v,2}| \leq C_K (|v|^{1/p} + t) \|\eta\|_{v,1} \quad \forall \eta \in \Gamma_-^{0,1}(v; N_Y \kappa).$$

Since the section $\tilde{\pi}_{\kappa,\nu}^q$ of the bundle $\iota_q^* \widetilde{\mathcal{W}}_{\kappa,d/d_\kappa}^{1,q}$ does not vanish over the compact set K , it follows that

$$\{\bar{\partial}_J + t\nu\}^{-1}(0) \cap U_\nu(K) = \emptyset$$

if t and $U_\nu(K)$ are sufficiently small.

By Lemma 4.2, the spaces $\overline{\mathfrak{M}}_1^q(\kappa, d/d_\kappa)$ with $q \geq 2$ are contained in $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$. In particular, if \mathcal{T} is a bubble type as in the first claim of Proposition 3.7,

$$\mathcal{U}_{\mathcal{T};\kappa} \subset (\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa) - \mathfrak{M}_1^0(\kappa, d/d_\kappa)) \cup (\overline{\mathfrak{M}}_1^1(\kappa, d/d_\kappa) - \mathfrak{M}_1^1(\kappa, d/d_\kappa)).$$

Thus, Corollary 3.10, along with the regularity assumptions ($\nu 3a$) and ($\nu 3c$), implies the first claim of Proposition 3.7.

Corollary 3.11 *Suppose ν , \mathcal{T} , and κ are as in Proposition 3.7. If $\aleph = \emptyset$ and $|\hat{I}| = 1$, for every compact subset K of $\mathcal{U}_{\mathcal{T};\kappa}$ containing $\tilde{\pi}_{\nu,\kappa}^{-1}(0)$, there exist $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\mathfrak{X}_1(Y, d)$ with the following properties:*

- (a) *the section $\bar{\partial}_J + t\nu$ is transverse to the zero set in $\Gamma_1^{0,1}(Y, d; J)$ over $U(K)$ for all $t \in (0, \epsilon_\nu(K))$;*
- (b) *for every open subset U of $\mathfrak{X}_1(Y, d)$, there exists $\epsilon(U) \in (0, \epsilon_\nu(K))$ such that*

$$\sharp |\{\bar{\partial}_J + t\nu\}^{-1} \cap U| = \frac{d/d_\kappa}{12} \langle e(\mathcal{W}_{\kappa,d/d_\kappa}^0), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)] \rangle \quad \text{if } K \subset U \subset U(K), \quad t \in (0, \epsilon(U)).$$

Proof: (1) By Corollary 3.10 and the assumption ($\nu 3a$) on ν , it can be assumed that the compact set K is disjoint from $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$. Thus, if h is the unique element of \hat{I} ,

$$|\mathcal{D}_{J,h}^{(1)} v| \geq C_K |v| \quad \forall [v] = [b, v] \in \mathcal{FT}|_K; \quad (3.24)$$

see Lemma 4.2. By Lemma 3.9 and the proof of Corollary 3.10, we need to determine the number of solutions $[v, \zeta]$ of the equation

$$\bar{\partial}_J u_\nu + D_{J,\nu} \zeta + N_\nu \zeta + tN_{\nu,\nu} \zeta + t\nu = 0, \quad v \in \tilde{\mathcal{F}}_{\delta_K}^\emptyset|_{K^{(0)}}, \quad \zeta \in \tilde{\Gamma}_+(v; TY), \quad \|\zeta\|_{v,p,1} \leq \epsilon_K. \quad (3.25)$$

In this case, $\Gamma_-^{0,1}(v; TY)$ is a subspace of $\Gamma^{0,1}(v; TY; J)$, and the middle estimate in (2) of Lemma 3.9 implies that

$$C_K^{-1} \|\zeta\|_{v,p,1} \leq \|\pi_{v,+}^{0,1} D_{J,v} \zeta\|_{v,p} \leq C_K \|\zeta\|_{v,p,1} \quad \forall \zeta \in \tilde{\Gamma}_+(v; TY).$$

Thus, the linear operator

$$\pi_{v,+}^{0,1} D_{J,v}: \tilde{\Gamma}_+(v; TY) \longrightarrow \Gamma_+^{0,1}(v; TY)$$

is an isomorphism. It then follows from the Contraction Principle, the first estimate in (2) of Lemma 3.9, and (3.18) that the equation

$$\pi_{v,+}^{0,1} (\bar{\partial}_J u_v + D_{J,v} \zeta + N_v \zeta + t N_{\nu,v} \zeta + t \nu) = 0, \quad \zeta \in \tilde{\Gamma}_+(v; TY), \quad \|\zeta\|_{v,p,1} \leq \epsilon_K,$$

has a unique solution $\zeta_v \in \tilde{\Gamma}_+(v; TY)$, provided $v \in \tilde{\mathcal{F}}\mathcal{T}_{\delta_K}^\emptyset|_{K^{(0)}}$ is sufficiently small. Furthermore,

$$\|\zeta_v\|_{v,p,1} \leq C_K (|v|^{1/p} + t). \quad (3.26)$$

Thus, the number of solutions $[v, \zeta]$ of (3.25) is the same as the number of solutions of

$$\Psi_{t\nu}(v) \equiv t^{-1} \cdot \pi_{v,-}^{0,1} (\bar{\partial}_J u_v + D_{J,v} \zeta_v + N_v \zeta_v + t N_{\nu,v} \zeta_v + t \nu) = 0, \quad [v] \in \mathcal{F}\mathcal{T}_{\delta_K}^\emptyset|_{K^{(0)}}, \quad (3.27)$$

where

$$\pi_{v,-}^{0,1} = \pi_{v;\kappa}^{0,1} \oplus \tilde{\pi}_v^{0,1}: \Gamma^{0,1}(v; TY; J) \longrightarrow \Gamma_-^{0,1}(v; T\kappa) \oplus \Gamma_-^{0,1}(v; N_Y \kappa)$$

is the L^2 -projection map.

(2) With our choice of the space $\Gamma_-^{0,1}(v; T\kappa)$,

$$\pi_{v;\kappa}^{0,1} \bar{\partial}_J u_v = R_v \mathcal{D}_{J,h} v \in \Gamma_-^{0,1}(v; T\kappa); \quad (3.28)$$

see Subsection 4.1 in [Z1]. Furthermore,

$$\pi_{v;\kappa}^{0,1} N_v \zeta = 0 \quad \forall \zeta \in \Gamma(v; T\kappa), \quad (3.29)$$

since the supports of all elements of $\eta \in \Gamma_-^{0,1}(v; T\kappa)$ are disjoint from the support of $N_v \zeta$, for $\zeta \in \Gamma(v; T\kappa)$, due to our choice of the ‘‘exponential’’ map. By (3.18), (3.26), (3.28), (3.29), and the same argument as in the proof of Corollary 3.10,

$$\begin{aligned} \|\tilde{\pi}_v^{0,1} \Psi_{t\nu}(v) - R_v \tilde{\pi}_{\kappa,\nu}^1(b)\|_{v,2} &\leq C_K (|v|^{1/p} + t) + \|\nu(u_v) - R_v \nu(b)\|_{v,2} \\ \|\pi_{v;\kappa}^{0,1} \Psi_{t\nu}(v) - R_v (\pi_{\kappa,\nu;\kappa}^1(b) + t^{-1} \mathcal{D}_{J,h} v)\|_{v,2} &\leq C_K (|v|^{1/p} + t) + \|\nu(u_v) - R_v \nu(b)\|_{v,2}, \end{aligned} \quad (3.30)$$

where $R_v \eta = q_v^* \eta$ for $\eta \in \Gamma_-^{0,1}(b; N_Y \kappa)$ and

$$\pi_{\kappa,\nu;\kappa}^1(b) = \pi_\kappa \pi_{\kappa,\nu}^1(b) \equiv \pi_{\kappa,\nu}^1(b) - \pi_\kappa^\perp \pi_{\kappa,\nu}^1(b) \in \Gamma_-^{0,1}(b; T\kappa).$$

Since

$$\lim_{t \rightarrow 0, |v| \rightarrow 0} (|v|^{1/p} + t) + \|\nu(u_v) - R_v \nu(b)\|_{v,2} = 0,$$

by (3.24), (3.30), and the same cobordism argument as in Subsection 3.1 of [Z1], the number of solutions of (3.27) is the same as the number of solutions of the system

$$\begin{cases} \tilde{\pi}_{\kappa,\nu}^1(b) = 0 \in \widetilde{\mathcal{W}}_{\kappa,d/d_\kappa}^{1,1} \\ \pi_{\kappa,\nu;\kappa}^1(b) + \mathcal{D}_{J,h} v = 0 \in \pi_P^* \mathbb{E}^* \otimes \pi_B^* \text{ev}_0^* T\kappa \end{cases} \quad [v] = [b, v] \in \mathcal{F}\mathcal{T} \longrightarrow \overline{\mathfrak{M}}_1^1(k, d/d_\kappa),$$

if the interior of the compact set K contains the finite set $\tilde{\pi}_{\nu,\kappa}^{-1}(0)$. Since $D_{J,h}$ does not vanish on $\tilde{\pi}_{\kappa,\nu}^{-1}(0)$, the number of solution of this system is

$$\begin{aligned} \pm|\tilde{\pi}_{\nu,\kappa}^{-1}(0)| &= \langle e(\widetilde{\mathcal{W}}_{\kappa,d/d_\kappa}^{1,1}), [\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)] \rangle \\ &= \langle e(\pi_P^* \mathbb{E}^* \otimes \pi_B^* \text{ev}_0^* N_Y \kappa) e(\tilde{\pi}_B^* \mathcal{W}_{\kappa,d/d_\kappa}^0), [\overline{\mathcal{M}}_{1,1} \times \overline{\mathfrak{M}}_{0,1}(\kappa, d/d_\kappa)] \rangle \\ &= \left(-\frac{1}{24}\right) (-2(d/d_\kappa)) \langle e(\mathcal{W}_{\kappa,d/d_\kappa}^0), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)] \rangle, \end{aligned}$$

as claimed in Proposition 3.7.

4 On the Euler Class of the Cone $\mathcal{V}_1^d \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$

4.1 The Structure of the Moduli Spaces $\overline{\mathfrak{M}}_1^0(\mathbb{P}^n, d; J)$

In this section, we prove Proposition 2.3 by constructing a perturbation ϑ of the section s_1^d of the cone \mathcal{V}_1^d over $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ and counting the number of zeros of the multisection $s_1^d + t\vartheta$ for a small $t \in \mathbb{R}^+$ that lie near each stratum of

$$s_1^d{}^{-1}(0) \cap \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J) = \overline{\mathfrak{M}}_1^0(Y, d; J) = \bigsqcup_{\kappa \in \mathcal{S}_0(Y; J)} \overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa) \sqcup \bigsqcup_{\kappa \in \mathcal{S}_1(Y; J)} \mathfrak{M}_1^0(\kappa, d/d_\kappa). \quad (4.1)$$

Since the finite set $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$ consists of transverse zeros of s_1^d , for $\kappa \in \mathcal{S}_1(Y; J)$,

$$\mathcal{C}_{\mathfrak{M}_1^0(\kappa, d/d_\kappa)}(s_1^d) = \pm|\{s_1^d + t\vartheta\}^{-1}(0) \cap U_\kappa| = \pm|\mathfrak{M}_1^0(\kappa, d/d_\kappa)|, \quad (4.2)$$

if U_κ is a small neighborhood of $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$ in $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$. The second equality in (4.2) holds for every multisection ϑ of \mathcal{V}_1^d and every $t \in \mathbb{R}$ sufficiently small. Thus, the key to proving Proposition 2.3 is computing the s_1^d -contribution from each stratum of the moduli space $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$. This is achieved by Proposition 4.5 and Corollary 4.7.

In this subsection, we describe the structure of the moduli space $\overline{\mathfrak{M}}_1^0(\mathbb{P}^n, d; J)$, with J sufficiently close to J_0 . Lemmas 4.1 and 4.2 are special cases of Lemmas 2.3 and 2.4, respectively, in [Z5]. In turn, the latter two lemmas follow immediately from Theorems 1.6 and 2.3 in [Z4].

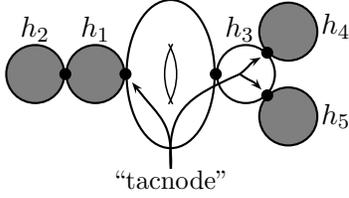
Lemma 4.1 *If $n, d \in \mathbb{Z}^+$, there exists $\delta_n(d) \in \mathbb{R}^+$ with the following property. If J is an almost complex structure on \mathbb{P}^n , such that $\|J - J_0\|_{C^1} < \delta_n(d)$, and $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $\sum_{i \in I} d_i = d$ and $d_i \neq 0$ for some minimal element i of I , then $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$ is a smooth orbifold,*

$$\dim \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J) = 2(d(n+1) - |\aleph| - |\hat{I}|), \quad \text{and} \quad \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J) \subset \overline{\mathfrak{M}}_1^0(\mathbb{P}^n, d; J).$$

Furthermore, there exist $\delta \in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J); \mathbb{R}^+)$, an open neighborhood $U_{\mathcal{T}}$ of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$ in $\mathfrak{X}_1(\mathbb{P}^n, d)$, and an orientation-preserving homeomorphism

$$\phi_{\mathcal{T}}: \mathcal{FT}_{\delta} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^n, d; J) \cap U_{\mathcal{T}},$$

which restricts to a diffeomorphism $\mathcal{FT}_{\delta}^{\emptyset} \longrightarrow \mathfrak{M}_1^0(\mathbb{P}^n, d; J) \cap U_{\mathcal{T}}$.



$$\chi(\mathcal{T}) = \{h_1, h_4, h_5\}, \quad \rho(v) = (v_{h_1}, v_{h_3}v_{h_4}, v_{h_3}v_{h_5})$$

$$\mathcal{F}^1\mathcal{T}^\emptyset = \{[b; v_1, v_2, v_3, v_4, v_5] : v_h \in \mathbb{C}^* \\ \mathcal{D}_{J, h_1}v_{h_1} + \mathcal{D}_{J, h_4}v_{h_3}v_{h_4} + \mathcal{D}_{J, h_5}v_{h_3}v_{h_5} = 0\}$$

Figure 5: An illustration of Lemma 4.2

Lemma 4.2 *If $n, d \in \mathbb{Z}^+$, there exists $\delta_n(d) \in \mathbb{R}^+$ with the following property. If J is an almost complex structure on \mathbb{P}^n , such that $\|J - J_0\|_{C^1} < \delta_n(d)$, and $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $\sum_{i \in I} d_i = d$ and $d_i = 0$ for all minimal elements i of I , then $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$ is a smooth orbifold,*

$$\dim \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J) = 2(d(n+1) - |\aleph| - |\hat{I}| + n), \quad \text{and} \quad \overline{\mathfrak{M}}_1^0(\mathbb{P}^n, d; J) \cap \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J) = \mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n; J),$$

where $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n; J) = \{[b] \in \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J) : \dim_{\mathbb{C}} \text{Span}_{(\mathbb{C}, J)}\{\mathcal{D}_i b : i \in \chi(\mathcal{T})\} < |\chi(\mathcal{T})|\}$.

The space $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n; J)$ admits a stratification by smooth suborbifolds of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$:

$$\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n; J) = \bigsqcup_{m=\max(|\chi(\mathcal{T})|-n, 1)}^{m=|\chi(\mathcal{T})|} \mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n; J), \quad \text{where}$$

$$\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n; J) = \{[b] \in \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J) : \dim_{\mathbb{C}} \text{Span}_{(\mathbb{C}, J)}\{\mathcal{D}_i b : i \in \chi(\mathcal{T})\} = |\chi(\mathcal{T})| - m\},$$

$$\dim \mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n; J) = 2(d(n+1) - |\aleph| - |\hat{I}| + n + (|\chi(\mathcal{T})| - n - m)m) \\ \leq \dim \mathfrak{M}_1^0(\mathbb{P}^n, d; J) - 2.$$

Furthermore, the space

$$\mathcal{F}^1\mathcal{T}^\emptyset \equiv \{[b, v] \in \mathcal{F}\mathcal{T}^\emptyset : \mathcal{D}_{\mathcal{T}}(v) = 0\}$$

is a smooth oriented suborbifold of $\mathcal{F}\mathcal{T}$. Finally, there exist $\delta \in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J); \mathbb{R}^+)$, an open neighborhood $U_{\mathcal{T}}$ of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$ in $\mathfrak{X}_1(\mathbb{P}^n, d)$, and an orientation-preserving diffeomorphism

$$\phi_{\mathcal{T}} : \mathcal{F}^1\mathcal{T}_\delta^\emptyset \longrightarrow \mathfrak{M}_1^0(\mathbb{P}^n, d; J) \cap U_{\mathcal{T}},$$

which extends to a homeomorphism

$$\phi_{\mathcal{T}} : \mathcal{F}^1\mathcal{T}_\delta \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^n, d; J) \cap U_{\mathcal{T}},$$

where $\mathcal{F}^1\mathcal{T}$ is the closure of $\mathcal{F}^1\mathcal{T}^\emptyset$ in $\mathcal{F}\mathcal{T}$.

We illustrate Lemma 4.2 in Figure 5. As before, the shaded discs represent the components of the domain on which every stable map $[b]$ in $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$ is non-constant. The element $[\Sigma_b, u_b]$ of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$ is in the stable-map closure of $\mathfrak{M}_1^0(\mathbb{P}^n, d; J)$ if and only if the branches of $u_b|_{\Sigma_b}$ corresponding to the attaching nodes on the first-level effective bubbles of $[\Sigma_b, u_b]$ form a generalized tacnode. In the case of Figure 5, this means that either

- (a) for some $h \in \{h_1, h_4, h_5\}$, the branch of $u_b|_{\Sigma_{b,h}}$ at the node ∞ has a cusp, or
- (b) for all $h \in \{h_1, h_4, h_5\}$, the branch of $u_b|_{\Sigma_{b,h}}$ at the node ∞ is smooth, but the dimension

of the span of the three lines tangent to these branches is less than three.

If $\kappa \in \mathcal{S}_0(Y; J)$, we put

$$\mathcal{U}_{\mathcal{T};\kappa;1} = \mathcal{U}_{\mathcal{T};\kappa} \cap \mathcal{U}_{\mathcal{T};1}(\mathbb{P}^4; J) \subset \overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa), \quad \mathcal{U}_{\mathcal{T};\kappa;1}^m = \mathcal{U}_{\mathcal{T};\kappa} \cap \mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^4; J) \subset \mathcal{U}_{\mathcal{T};\kappa;1}.$$

By the $n=1$ case of Lemma 4.2,

$$\mathcal{U}_{\mathcal{T};\kappa;1} = \mathcal{U}_{\mathcal{T};\kappa} = \mathcal{U}_{\mathcal{T};\kappa;1}^{|\chi(\mathcal{T})|-1} \cup \mathcal{U}_{\mathcal{T};\kappa;1}^{|\chi(\mathcal{T})|} \quad \text{if } |\chi(\mathcal{T})| \geq 2.$$

The last space may be empty. In particular, $\overline{\mathfrak{M}}_1^q(\kappa, d/d_\kappa) \subset \overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ if $q \geq 2$.

Let

$$\mathfrak{M}_{0,1;1}^0(\mathbb{P}^4, d; J) = \{[\mathbb{P}^1, u] \in \mathfrak{M}_{0,1}^0(\mathbb{P}^4, d; J) : du|_\infty = 0\}.$$

In other words, $\mathfrak{M}_{0,1;1}^0(\mathbb{P}^4, d; J)$ is the subset of $\mathfrak{M}_{0,1}^0(\mathbb{P}^4, d; J)$ consisting of the elements $[\mathbb{P}^1, u]$ such that the differential of u vanishes at the marked point of \mathbb{P}^1 , which we always take to be ∞ . The image of a generic element in $\mathfrak{M}_{0,1;1}^0(\mathbb{P}^4, d; J)$ is a rational curve J -holomorphic curve in \mathbb{P}^4 with a cusp at the image of the marked point. We denote by $\overline{\mathfrak{M}}_{0,1;1}(\mathbb{P}^4, d; J)$ the closure of $\mathfrak{M}_{0,1;1}^0(\mathbb{P}^4, d; J)$ in $\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^4, d; J)$. If $\kappa \in \mathcal{S}_0(Y; J)$, we put

$$\begin{aligned} \mathfrak{M}_{0,1;1}^0(\kappa, d/d_\kappa) &= \mathfrak{M}_{0,1;1}^0(\mathbb{P}^4, d; J) \cap \overline{\mathfrak{M}}_{0,1}(\kappa, d/d_\kappa), \\ \overline{\mathfrak{M}}_{0,1;1}(\kappa, d/d_\kappa) &= \overline{\mathfrak{M}}_{0,1;1}(\mathbb{P}^4, d; J) \cap \overline{\mathfrak{M}}_{0,1}(\kappa, d/d_\kappa), \\ \mathfrak{M}_{1;1}^{1;0}(\kappa, d/d_\kappa) &= \overline{\mathcal{M}}_{1,1} \times \mathfrak{M}_{0,1;1}^0(\kappa, d/d_\kappa), \quad \text{and} \quad \overline{\mathfrak{M}}_{1;1}^1(\kappa, d/d_\kappa) = \overline{\mathcal{M}}_{1,1} \times \overline{\mathfrak{M}}_{0,1;1}(\kappa, d/d_\kappa). \end{aligned}$$

By Lemma 4.2,

$$\overline{\mathfrak{M}}_1^1(\kappa, d) \cap \overline{\mathfrak{M}}_1^0(\kappa, d) = \overline{\mathfrak{M}}_{1;1}^1(\kappa, d) \quad \forall d \in \mathbb{Z}^+.$$

We note that

$$\dim_{\mathbb{C}} \overline{\mathfrak{M}}_{0,1;1}(\kappa, d) = 2d - 2 \quad \text{and} \quad \dim_{\mathbb{C}} \overline{\mathfrak{M}}_{1;1}^1(\kappa, d) = 2d - 1. \quad (4.3)$$

4.2 The Structure of the Cone $\mathcal{V}_1^d \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$

We next describe the structure of the cone \mathcal{V}_1^d near each stratum $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ and $\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^4; J)$ of $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$. We then state several regularity conditions that we will require the perturbation ϑ of s_1^d to satisfy. The first lemma stated is a special case of Lemma 3.2 in [Z5].

Lemma 4.3 *If d , \mathcal{L} , and \mathcal{V}_1^d are as in Proposition 2.3, there exists $\delta(d) \in \mathbb{R}^+$ with the following property. If J is an almost complex structure on \mathbb{P}^4 , such that $\|J - J_0\|_{C^1} < \delta_n(d)$, and $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $\sum_{i \in I} d_i = d$ and $d_i \neq 0$ for some minimal element i of I , then the requirements of Lemma 4.1 are satisfied. Furthermore, the restriction $\mathcal{V}_1^d \longrightarrow \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ is a smooth complex vector orbifold of rank $5d$. Finally, there exists a smooth vector-bundle isomorphism*

$$\tilde{\phi}_{\mathcal{T}} : \pi_{\mathcal{F}\mathcal{T}_\delta}^* (\mathcal{V}_1^d|_{\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)}) \longrightarrow \mathcal{V}_1^d|_{\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J) \cap \mathcal{U}_{\mathcal{T}}},$$

covering the homeomorphism $\phi_{\mathcal{T}}$ of Lemma 4.1, such that $\tilde{\phi}_{\mathcal{T}}$ is the identity over $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ and is smooth over $\mathcal{F}\mathcal{T}_\delta^\emptyset$.

For every $\kappa \in \mathcal{S}_0(Y; J)$, the family of boundary operators \mathfrak{d}_0 in the long exact sequence (3.4), with $b \in \mathcal{U}_{\mathcal{T}; \kappa}^{(0)}$ and \mathcal{T} as in Lemma 4.3, induces a surjective bundle homomorphism

$$\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} : \mathcal{V}_1^d \longrightarrow \mathcal{W}_{\kappa, d/d_\kappa}^{1,0}$$

over $\mathfrak{M}_1^{\{0\}}(\kappa, d/d_\kappa)$. The first two regularity conditions on a perturbation ϑ of the section s_1^d over $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ are that for every $\kappa \in \mathcal{S}_0(Y; J)$

($\vartheta 1a$) the section $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta|_{\mathfrak{M}_1^0(\kappa, d/d_\kappa)}$ is transverse to the zero set in $\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}$;

($\vartheta 1b$) the section $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta$ does not vanish on $\mathfrak{M}_1^{\{0\}}(\kappa, d/d_\kappa) - \mathfrak{M}_1^0(\kappa, d/d_\kappa)$.

By Lemma 4.3, the $n=1$ case of Lemma 4.1, and (2.15), the collection of multisections ϑ of \mathcal{V}_1^d that satisfy ($\vartheta 1a$) and ($\vartheta 1b$) is open and dense in the space of all multisections of \mathcal{V}_1^d .

The next lemma, which is the analogue of Lemma 4.3 for the strata $\mathcal{U}_{\mathcal{T}; 1}^m(\mathbb{P}^4; J)$ of Lemma 4.2, is a special case of Proposition 3.3 and Lemma 3.4 in [Z5]. For any $b \in \mathcal{U}_{\mathcal{T}; 1}^m(\mathbb{P}^4; J)$, we put

$$\mathfrak{F}^1 \mathcal{T}_b = \{ \tilde{v} = (\tilde{v}_i)_{i \in \chi(\mathcal{T})} \in \mathfrak{F} \mathcal{T}_b : \sum_{i \in \chi(\mathcal{T})} \mathcal{D}_{J, i} \tilde{v}_i = 0 \}.$$

Lemma 4.4 *If d , \mathcal{L} , and \mathcal{V}_1^d are as in Proposition 2.3, there exists $\delta(d) \in \mathbb{R}^+$ with the following property. If J is an almost complex structure on \mathbb{P}^4 such that $\|J - J_0\|_{C^1} < \delta(d)$, then the requirements of Lemma 4.2 and of Lemma 4.3 are satisfied for all appropriate bubble types. Furthermore, if $\mathcal{T} = (I, \mathfrak{X}; \underline{d})$ is a bubble type such that $\sum_{i \in I} d_i = d$ and $d_i = 0$ for all minimal elements i of I , then the restriction $\mathcal{V}_1^d \longrightarrow \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$ is a smooth complex vector orbifold of rank $5d+1$. In addition, for every integer*

$$m \in (\max(|\chi(\mathcal{T})| - 4, 1), |\chi(\mathcal{T})|),$$

there exist a neighborhood $U_{\mathcal{T}}^m$ of $\mathcal{U}_{\mathcal{T}; 1}^m(\mathbb{P}^4; J)$ in $\mathfrak{X}_1(\mathbb{P}^4, d)$ and a topological vector orbifold

$$\mathcal{V}_{1; \mathcal{T}}^{d; m} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J) \cap U_{\mathcal{T}}^m$$

such that $\mathcal{V}_{1; \mathcal{T}}^{d; m} \longrightarrow \mathfrak{M}_1^0(\mathbb{P}^4, d; J) \cap U_{\mathcal{T}}^m$ is a smooth complex vector orbifold contained in \mathcal{V}_1^d and

$$\mathcal{V}_{1; \mathcal{T}}^{d; m}|_{\mathcal{U}_{\mathcal{T}; 1}^m(\mathbb{P}^4; J)} = \{ \xi \in \mathcal{V}_1^d|_b : b \in \mathcal{U}_{\mathcal{T}; 1}^m(\mathbb{P}^4; J); \mathfrak{D}_{\mathcal{T}}(\xi \otimes \tilde{v}) = 0 \ \forall \tilde{v} \in \mathfrak{F}^1 \mathcal{T}_b \}.$$

There also exists a continuous vector-bundle isomorphism

$$\tilde{\phi}_{\mathcal{T}}^m : \pi_{\mathcal{F}^1 \mathcal{T}_\delta}^* (\mathcal{V}_{1; \mathcal{T}}^{d; m}|_{\mathcal{U}_{\mathcal{T}; 1}^m(\mathbb{P}^4; J) \cap U_{\mathcal{T}}^m}) \longrightarrow \mathcal{V}_{1; \mathcal{T}}^{d; m}|_{\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J) \cap U_{\mathcal{T}}^m},$$

covering the homeomorphism $\phi_{\mathcal{T}}$ of Lemma 4.1, such that $\tilde{\phi}_{\mathcal{T}}^m$ is the identity over $\mathcal{U}_{\mathcal{T}; 1}^m(\mathbb{P}^4; J)$. Finally, if \mathcal{T} and \mathcal{T}' are two bubble types as above and $m, m' \in \mathbb{Z}^+$, then

$$\mathcal{V}_{1; \mathcal{T}'}^{d; m'}|_{\mathcal{U}_{\mathcal{T}'}^m(\mathbb{P}^4; J) \cap U_{\mathcal{T}'}^{m'}} \subset \mathcal{V}_{1; \mathcal{T}}^{d; m}|_{\mathcal{U}_{\mathcal{T}; 1}^m(\mathbb{P}^4; J) \cap U_{\mathcal{T}}^{m'}} \quad \text{if } m' \geq m.$$

If $[b] \in \mathcal{U}_{\mathcal{T};\kappa;1}^m$, we put

$$\begin{aligned}\Gamma_-(b; \mathfrak{L}; 0) &= \{\xi \in \Gamma_-(b; \mathfrak{L}) : \tilde{\mathfrak{D}}_{\mathcal{T},i}\xi = 0 \ \forall i \in \chi(\mathcal{T})\}; \\ \tilde{\mathcal{V}}_{1;\mathcal{T}}^d|_b &= \{[\xi] \in \mathcal{V}_{1;\mathcal{T}}^d|_b : \mathfrak{D}_{\mathcal{T},i}\xi = 0 \ \forall i \in \chi(\mathcal{T})\} \subset \mathcal{V}_{1;\mathcal{T}}^{d;m}.\end{aligned}$$

In this case, the standard analogue for b of the long exact sequence (3.4) has six terms. However, replacing the fourth term by the kernel of the outgoing map at the fourth term, we get

$$0 \longrightarrow \Gamma_-(b; TY) \xrightarrow{i_0} \Gamma_-(b; T\mathbb{P}^4) \xrightarrow{j_0} \Gamma_-(b; \mathfrak{L}) \xrightarrow{\mathfrak{d}_0} H_J^1(\pi_B(b); TY) \longrightarrow 0. \quad (4.4)$$

By Theorem 1.6 in [Z4], the linear operator

$$\tilde{\mathfrak{D}}_{\mathcal{T},i}^{\mathbb{P}^4} : \{\zeta \in \Gamma_-(b; T\mathbb{P}^4) : \zeta|_{\Sigma_{b,\mathbb{N}}} = 0\} \longrightarrow T_{\text{ev}_P(b)}\mathbb{P}^4, \quad \zeta \longrightarrow \nabla_{e_\infty}^J(\zeta|_{\Sigma_{b,i}}),$$

is surjective for every $[b] \in \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J)$, with \mathcal{T} as in Lemma 4.2 and $i \in \chi(\mathcal{T})$, if J is sufficiently close to J_0 . It follows that the homomorphism

$$\tilde{j}_0 : \Gamma_-(b; T\mathbb{P}^4) \longrightarrow \Gamma_-(b; \mathfrak{L})/\Gamma_-(b; \mathfrak{L}; 0),$$

induced by the map j_0 in (4.4) is surjective for every $[b] \in \mathcal{U}_{\mathcal{T};\kappa}$. Thus, the family of boundary operators \mathfrak{d}_0 in (4.4) with $[b] \in \mathcal{U}_{\mathcal{T};\kappa}$ induces a surjective bundle homomorphism

$$\mathfrak{d}_{\kappa,d/d_\kappa}^{1,q} : \tilde{\mathcal{V}}_{1;\mathcal{T}}^d \longrightarrow \pi_B^* \mathcal{W}_{\kappa,d/d_\kappa}^{0,q} \subset \mathcal{W}_{\kappa,d/d_\kappa}^{1,0} \quad (4.5)$$

over $\mathcal{U}_{\mathcal{T};\kappa;1}$, if $\mathcal{U}_{\mathcal{T};\kappa} \subset \overline{\mathfrak{M}}_1^q(\kappa, d/d_\kappa)$. Furthermore,

$$\mathfrak{d}_{\kappa,d/d_\kappa}^{1,q} = \pi_B^* \mathfrak{d}_{\kappa,d/d_\kappa}^{0,q}, \quad (4.6)$$

where $\mathfrak{d}_{\kappa,d/d_\kappa}^{0,q}$ is the surjective bundle homomorphism over $\mathcal{U}_{\mathcal{T};\kappa} \subset \overline{\mathfrak{M}}_0^q(\kappa, d/d_\kappa)$ defined similarly to $\mathfrak{d}_{\kappa,d/d_\kappa}^{1,q}$.

We now state additional regularity conditions on a perturbation ϑ of s_1^d . We will require that for every $\kappa \in \mathcal{S}_0(Y; J)$:

($\vartheta 2a$) the sections $\mathfrak{d}_{\kappa,d/d_\kappa}^{1,1} \vartheta|_{\mathcal{M}_{1,1} \times \mathfrak{M}_{0,1;1}^0(\kappa, d/d_\kappa)}$ and $\mathfrak{d}_{\kappa,d/d_\kappa}^{1,1} \vartheta|_{\partial \overline{\mathcal{M}}_{1,1} \times \mathfrak{M}_{0,1;1}^0(\kappa, d/d_\kappa)}$ are transverse

to the zero set in $\pi_B^* \mathcal{W}_{\kappa,d/d_\kappa}^{0,1}$;

($\vartheta 2b$) the section $\mathfrak{d}_{\kappa,d/d_\kappa}^{1,1} \vartheta$ does not vanish on $\overline{\mathfrak{M}}_{1,1}^1(\kappa, d/d_\kappa) - \mathfrak{M}_{1,1}^{1,0}(\kappa, d/d_\kappa)$;

($\vartheta 2c$) for $q \geq 2$, the section $\mathfrak{d}_{\kappa,d/d_\kappa}^{1,q} \vartheta$ does not vanish on $\overline{\mathfrak{M}}_1^q(\kappa, d/d_\kappa)$.

By Lemma 4.4, (2.6), and (2.11), the collection of multisections ϑ of \mathcal{V}_1^d that satisfy ($\vartheta 2a$) and ($\vartheta 2c$) with $q \geq 4$ is open and dense in the space of all multisections of \mathcal{V}_1^d . By (2.8), (2.11), (4.3), and (4.6), the collection of multisections ϑ of \mathcal{V}_1^d that satisfy one of the three remaining conditions, i.e. ($\vartheta 2b$), ($\vartheta 2c$) with $q = 2$, or ($\vartheta 2c$) with $q = 3$, is nonempty and open in the space of all multisections of \mathcal{V}_1^d , but not dense. Nevertheless, by considering the decompositions of the intersections of the corresponding subspaces of $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ analogous to (2.9), it is straightforward to see that the intersection of these three open sets is still nonempty. Alternatively, note that

$$\begin{aligned}\dim_{\mathbb{C}} (\overline{\mathfrak{M}}_{1,1}^1(\kappa, d/d_\kappa) - \mathfrak{M}_{1,1}^{1,0}(\kappa, d/d_\kappa)) &= 2(d/d_\kappa) - 2 = \text{rk } \mathcal{W}_{\kappa,d/d_\kappa}^{0,1}; \\ \dim_{\mathbb{C}} \overline{\mathfrak{M}}_1^3(\kappa, d/d_\kappa) &= 2(d/d_\kappa) - 2 = \text{rk } \mathcal{W}_{\kappa,d/d_\kappa}^{0,3}; \\ \dim_{\mathbb{C}} \overline{\mathfrak{M}}_1^2(\kappa, d/d_\kappa) \cap \overline{\mathfrak{M}}_1^3(\kappa, d/d_\kappa) &\leq 2(d/d_\kappa) - 3.\end{aligned}$$

Furthermore, the space $\overline{\mathfrak{M}}_{1;1}^1(\kappa, d/d_\kappa) - \mathfrak{M}_{1;1}^{1;0}(\kappa, d/d_\kappa)$ has two irreducible components. One of them is contained in $\overline{\mathfrak{M}}_1^2(\kappa, d/d_\kappa)$, while the other intersects $\overline{\mathfrak{M}}_1^2(\kappa, d/d_\kappa)$ in subvariety of complex dimension $2(d/d_\kappa) - 3$. Thus, if ϑ is a generic multisection that satisfies $(\vartheta 2c)$ with $q = 2$, its restrictions to

$$\overline{\mathfrak{M}}_{1;1}^1(\kappa, d/d_\kappa) - \mathfrak{M}_{1;1}^{1;0}(\kappa, d/d_\kappa) \quad \text{and} \quad \overline{\mathfrak{M}}_1^3(\kappa, d/d_\kappa) \quad (4.7)$$

have finite zero sets, divided equally between positive and negative zeros. These zeros can be removed in pairs by modifying ϑ outside of the boundary strata of (4.7).

It remains to state one more regularity assumption on ϑ . If $\hat{I} = \chi(\mathcal{T}) = \{h\}$ is a single-element set, for every $[b] \in \mathcal{U}_{\mathcal{T};\kappa;1}$, we put

$$\begin{aligned} \Gamma_-(b; T\mathbb{P}^4; TY) &= \{\zeta \in \Gamma_-(b; T\mathbb{P}^4) : \tilde{\mathfrak{D}}_{\mathcal{T},h}^{\mathbb{P}^4} \zeta \in T_{\text{ev}_P(b)} Y\} \\ \Gamma_-(b; T\mathbb{P}^4; T\kappa) &= \{\zeta \in \Gamma_-(b; T\mathbb{P}^4) : \tilde{\mathfrak{D}}_{\mathcal{T},h}^{\mathbb{P}^4} \zeta \in T_{\text{ev}_P(b)} \kappa\} \subset \Gamma_-(b; T\mathbb{P}^4; TY). \end{aligned}$$

Since $d\{u_b|_{\Sigma_{b,h}}\}|_\infty = 0$ by Lemma 4.2, the subspaces

$$\Gamma_-(b; T\mathbb{P}^4; TY), \Gamma_-(b; T\mathbb{P}^4; T\kappa) \subset \Gamma_-(b; T\mathbb{P}^4)$$

are in fact independent of the choice of connection ∇^J in $T\mathbb{P}^4$. Furthermore, by Theorem 1.6 in [Z4],

$$\Gamma_-(b; T\mathbb{P}^4; TY) / \Gamma_-(b; T\mathbb{P}^4; T\kappa) \approx N_{Y\kappa}|_{\text{ev}_P(b)} \quad \text{via} \quad \zeta \longrightarrow [\tilde{\mathfrak{D}}_{\mathcal{T},h}^{\mathbb{P}^4} \zeta]. \quad (4.8)$$

By the paragraph following Lemma 4.4 and condition $(J_Y 2)$ of Definition 1.2,

$$\text{Im } j_0|_{\Gamma_-(b; T\mathbb{P}^4; TY)} = \ker \mathfrak{d}_0 \cap \Gamma_-(b; \mathfrak{L}; 0) \quad \text{and} \quad \ker j_0 \subset \Gamma_-(b; T\mathbb{P}^4; T\kappa), \quad (4.9)$$

where j_0 and \mathfrak{d}_0 are as in (4.4). Let

$$\tilde{H}_J^1(\pi_B(b); TY) = \Gamma_-(b; \mathfrak{L}; 0) / \text{Im } j_0|_{\Gamma_-(b; T\mathbb{P}^4; T\kappa)}.$$

The vector spaces $\tilde{H}_J^1(\pi_B(b); TY)$ and the quotient projection maps induce a vector bundle over $\mathfrak{M}_{1;1}^{1;0}(\kappa, d/d_\kappa)$, which we denote by $Q_{\kappa, d/d_\kappa}^{1,1}$, and a surjective bundle homomorphism

$$\tilde{\mathfrak{d}}_{\kappa, d/d_\kappa}^{1,1} : \tilde{\mathcal{V}}_{\mathcal{T};1}^d = \mathcal{V}_{\mathcal{T};1}^{d;1} \longrightarrow Q_{\kappa, d/d_\kappa}^{1,1}.$$

On the other hand, the boundary operators \mathfrak{d}_0 in (4.4) induce a surjective bundle homomorphism

$$\pi_\kappa^+ : Q_{\kappa, d/d_\kappa}^{1,1} \longrightarrow \pi_B^* \mathcal{W}_{\kappa, d/d_\kappa}^{0,1}$$

over $\mathfrak{M}_{1;1}^{1;0}(\kappa, d/d_\kappa)$. By (4.8) and (4.9),

$$\ker \pi_\kappa^+ \approx \pi_B^*(L_0^* \otimes \text{ev}_0^* N_{Y\kappa}).$$

We also have a surjective bundle homomorphism

$$\pi_\kappa^- : Q_{\kappa, d/d_\kappa}^{1,1} \longrightarrow \pi_B^*(L_0^* \otimes \text{ev}_0^* N_{Y\kappa}).$$

It is induced by the map

$$j_0\zeta \longrightarrow (-2\pi J)[\nabla_{e_\infty}^J \zeta|_{\Sigma_{b,h}}] \in N_Y \kappa \quad \text{if } \zeta \in \Gamma(b; T\mathbb{P}^4), \quad j_0\zeta \in \Gamma_-(b; \mathfrak{L}; 0). \quad (4.10)$$

Thus, we obtain a splitting of $Q_{\kappa, d/d_\kappa}^{1,1}$:

$$\pi_\kappa^- \oplus \pi_\kappa^+ : Q_{\kappa, d/d_\kappa}^{1,1} \longrightarrow \pi_B^*(L_0^* \otimes \text{ev}_0^* N_Y \kappa) \oplus \pi_B^* \mathcal{W}_{\kappa, d/d_\kappa}^{0,1} \quad (4.11)$$

over $\mathfrak{M}_{1;1}^{1,0}(\kappa, d/d_\kappa)$. We note that

$$\text{rk } Q_{\kappa, d}^{1,1} = 2d \quad (4.12)$$

for all $d \in \mathbb{Z}^+$.

Our final regularity condition on ϑ is that for every $\kappa \in \mathcal{S}_0(Y; J)$:

($\vartheta 3$) the section $\tilde{\mathfrak{d}}_{\kappa, d/d_\kappa}^{1,1} \vartheta$ does not vanish over $\mathfrak{M}_{1;1}^{1,0}(\kappa, d/d_\kappa)$.

By Lemma 4.4, (4.3), and (4.12), the collection of multisections ϑ of \mathcal{V}_1^d that satisfy ($\vartheta 3$) is open and dense in the space of all multisections of \mathcal{V}_1^d . We denote by $\tilde{\mathcal{A}}_1^d(s; J)$ the collection of all multi-valued perturbations of the section s_1^d of \mathcal{V}_1^d over $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ that satisfy the regularity conditions ($\vartheta 1$)-($\vartheta 3$). By the above, $\tilde{\mathcal{A}}_1^d(s; J)$ is a nonempty open, but not dense, subset of the space of all multisections of \mathcal{V}_1^d .

It is possible to use a dense open collection of perturbations in the statement of Proposition 4.5 below. However, using such a collection would needlessly complicate its proof by enlarging the zero set of the sections $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta$ by homologically trivial subspaces of $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$. This would also require stating the analogue of ($\vartheta 3$) for the $q=2, 3$ cases of ($\vartheta 2c$).

4.3 Proof of Proposition 2.3

In this subsection, we finally prove Proposition 2.3. It follows immediately from (4.1), (4.2), Proposition 4.5, and Corollary 4.7.

Proposition 4.5 *Suppose J, d, \mathfrak{L} , and \mathcal{V}_1^d are as in Proposition 2.3, $\vartheta \in \tilde{\mathcal{A}}_1^d(s; J)$ is a regular perturbation of the section s_1^d of \mathcal{V}_1^d on $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$, $\kappa \in \mathcal{S}_0(Y; J)$, and $\mathcal{T} = (I, \mathfrak{N}; \underline{d})$ is a bubble type such that $\sum_{i \in I} d_i = d$. If $|I| > 1$ or $\mathfrak{N} \neq \emptyset$, for every compact subset K of $\mathcal{U}_{\mathcal{T}; \kappa}$, there exist $\epsilon_\vartheta(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ such that*

$$\{s_1^d + t\vartheta\}^{-1}(0) \cap U(K) = \emptyset \quad \forall t \in (0, \epsilon_\vartheta(K)).$$

If $|I| = 1$ and $\mathfrak{N} = \emptyset$, for every compact subset K of $\mathcal{U}_{\mathcal{T}; \kappa}$, there exist $\epsilon_\vartheta(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$ with the following properties:

- (a) *the section $s_1^d + t\vartheta$ is transverse to the zero set in \mathcal{V}_1^d over $U(K)$ for all $t \in (0, \epsilon_\vartheta(K))$;*
- (b) *for every open subset U of $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)$, there exists $\epsilon(U) \in (0, \epsilon_\vartheta(K))$ such that*

$$\begin{aligned} \#|\{s_1^d + t\vartheta\}^{-1} \cap U| &= \langle e(\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}), [\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)] \rangle - \mathcal{C}_{\mathfrak{M}_{1;1}^{1,0}(\kappa, d/d_\kappa)}(\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta) \\ \text{if } \quad \{ \mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta \}^{-1}(0) - \mathfrak{M}_{1;1}^1(\kappa, d/d_\kappa) &\subset K \subset U \subset U(K), \quad t \in (0, \epsilon(U)). \end{aligned}$$

In other words, the s_1^d -contribution from the main stratum $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$ of the space $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ to the number

$$\langle e(\mathcal{V}_1^d), [\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d; J)] \rangle,$$

as computed via a perturbation from the open collection $\tilde{A}_1^d(s; J)$, is the euler class of the vector bundle $\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}$ over $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ minus the $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0}$ ϑ -contribution to the latter euler class from the zeros of $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0}$ that lie in $\partial\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$. Since $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta|_{\mathfrak{M}_1^0(\kappa, d/d_\kappa)}$ is transverse to the zero set in $\mathfrak{M}_1^0(\kappa, d/d_\kappa)$,

$$\pm |\{\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta\}^{-1}(0) \cap \mathfrak{M}_1^0(\kappa, d/d_\kappa)| = \langle e(\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}), [\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)] \rangle - \mathcal{C}_{\partial\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)}(\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta),$$

by Definition 2.4. None of the boundary strata of $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ contributes to the euler class of \mathcal{V}_1^d .

Proof: (1) If \mathcal{T} is a bubble type such that $d_i \neq 0$ for some minimal element $i \in I$, the conclusion of Proposition 4.5 follows by the same argument as in the proof of Lemma 3.5 and at the end of Subsection 3.2. The key difference in the $|I|=1, \aleph=\emptyset$ case is that the section $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta$ may vanish on $\partial\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$. In addition, by the regularity assumptions, $(\vartheta 1b)$, $(\vartheta 2b)$, and $(\vartheta 2c)$,

$$\begin{aligned} \{\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta\}^{-1}(0) - \mathfrak{M}_1^0(\kappa, d/d_\kappa) &\subset \mathfrak{M}_{1;1}^{1;0}(\kappa, d/d_\kappa) \implies \\ \pm |\{\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta\}^{-1}(0) \cap \mathfrak{M}_1^0(\kappa, d/d_\kappa)| &= \langle e(\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}), [\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)] \rangle - \mathcal{C}_{\mathfrak{M}_{1;1}^{1;0}(\kappa, d/d_\kappa)}(\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta). \end{aligned}$$

(2) If \mathcal{T} is a bubble type such that $d_i = 0$ for all minimal elements $i \in I$ and $|\hat{I}| = 1$, or more generally $|\chi(\mathcal{T})| = 1$, nearly the same argument still applies. In this case, $\mathcal{F}^1\mathcal{T} = \mathcal{F}\mathcal{T}$ and the conclusions of Lemmas 3.5 and 3.6 are still valid. The key difference is that the normal bundle of $\mathcal{U}_{\mathcal{T};\kappa;1}$ in $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^4; J)$ is not given by the cokernels of the homomorphisms i_0 in the long exact sequence (3.4). Instead, up to the action of the automorphism group of b , the fiber of $\mathcal{N}^\kappa\mathcal{T}$ at $[b] \in \mathcal{U}_{\mathcal{T};\kappa;1}$ is

$$\begin{aligned} \mathcal{N}^\kappa\mathcal{T} &= \Gamma_-(b; T\mathbb{P}^4; 0) / \Gamma_-(b; TY; 0), \quad \text{where} \\ \Gamma_-(b; T\mathbb{P}^4; 0) &= \{\zeta \in \Gamma_-(b; T\mathbb{P}^4) : \tilde{\mathfrak{D}}_{\mathcal{T}, h}^{\mathbb{P}^4} \zeta = 0\} \\ \Gamma_-(b; TY; 0) &= \{\zeta \in \Gamma_-(b; TY) : \tilde{\mathfrak{D}}_{\mathcal{T}, h}^{\mathbb{P}^4} \zeta = 0\} \subset \Gamma_-(b; T\kappa), \end{aligned}$$

if h is the unique element of $\chi(\mathcal{T})$. The reason for this is that

$$\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^4; J) = \{b \in \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) : d\{u_b|_{\Sigma_{b,h}}\} = 0\},$$

by Lemma 4.2. Since the linear operator

$$\mathfrak{D}_{\mathcal{T}, h}^\kappa : \{\zeta \in \Gamma_-(b; T\kappa) : \zeta|_{\Sigma_{b,\aleph}} = 0\} \longrightarrow T_{\text{ev}_P(b)}\kappa, \quad \zeta \longrightarrow \nabla_{e_\infty}^\kappa(\zeta|_{\Sigma_{b,h}}),$$

is surjective, the image of the homomorphism j_0 in the long exact sequence (4.4) on $\Gamma_-(b; T\mathbb{P}^4; 0)$ is the same as on $\Gamma_-(b; T\mathbb{P}^4; T\kappa)$. Thus, the analogue of the bundle \mathcal{V}_- of Subsection 3.2 in this case is the bundle $Q_{\kappa, d/d_\kappa}^{1,1}$ described in the previous subsection. The section of \mathcal{V}_- induced by ϑ is $\tilde{\mathfrak{d}}_{\kappa, d/d_\kappa}^{1,1} \vartheta$. Its composition with the map π_κ^+ in (4.11) is $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,1} \vartheta$. Thus, Proposition 4.5 in this case follows from the regularity assumptions $(\vartheta 2b)$ and $(\vartheta 3)$, by the same argument as in Subsection 3.2.

(3) Finally, suppose $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $d_i = 0$ for all minimal elements $i \in I$

and $|\hat{I}| \geq 2$. In this case, the dimension of the fibers of $\mathcal{F}^1\mathcal{T}$ may not be constant over $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^4; J)$. Thus, we modify the setup of the second part of Subsection 3.2 by working directly with the normal bundle to the smooth submanifold $\mathcal{F}^1\mathcal{T}^\emptyset|_{\mathcal{U}_{\mathcal{T};\kappa;1}}$ in $\mathcal{F}^1\mathcal{T}^\emptyset|_{\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^4; J)}$. Let $\gamma \longrightarrow \mathbb{P}\mathfrak{F}\mathcal{T}$ be the tautological line bundle and

$$V = \pi_{\mathbb{P}\mathfrak{F}\mathcal{T}}^*(\pi_P^*\mathbb{E}^* \otimes \text{ev}_P^*T\mathbb{P}^4) \longrightarrow \mathbb{P}\mathfrak{F}\mathcal{T}.$$

We define the section $\alpha_{\mathcal{T}}$ of $\gamma^* \otimes V$ over $\mathbb{P}\mathfrak{F}\mathcal{T}$ by

$$\begin{aligned} \{\alpha_{\mathcal{T}}(b, (\tilde{v}_i)_{i \in \chi(\mathcal{T})})\}(b, \psi) &= \sum_{i \in \chi(\mathcal{T})} \mathcal{D}_{\mathcal{T}, i}(b, \psi_{x_{h(i)}} \tilde{v}_i) \in T_{\text{ev}_P(b)}\mathbb{P}^4, \\ \text{if } (b, (\tilde{v}_i)_{i \in \chi(\mathcal{T})}) &\in \gamma, \quad (b, \psi) \in \mathbb{E}_{\pi_P(b)}. \end{aligned}$$

With our assumptions on J , this section is transverse to the zero set and thus

$$\mathcal{U}_1(\mathbb{P}^4; J) \equiv \alpha_{\mathcal{T}}^{-1}(0)$$

is a smooth suborbifold of $\mathbb{P}\mathfrak{F}\mathcal{T}$. For a similar reason, so is

$$\mathcal{U}_{1;\kappa} \equiv \mathcal{U}_1 \cap \mathbb{P}\mathfrak{F}\mathcal{T}|_{\mathcal{U}_{\mathcal{T};\kappa}}.$$

Let $\mathcal{N}^\kappa\mathcal{T}$ denote the normal bundle of $\mathcal{U}_{1;\kappa}$ in $\mathcal{U}_1(\mathbb{P}^4; J)$. Up to the action of the automorphism group of $[b, [\tilde{v}]] \in \mathcal{U}_{1;\kappa}$,

$$\begin{aligned} \mathcal{N}^\kappa\mathcal{T}|_{[b, [\tilde{v}]]} &= \Gamma_-(b; T\mathbb{P}^4; \tilde{v}) / \Gamma_-(b; TY; \tilde{v}), \quad \text{where} \\ \Gamma_-(b; T\mathbb{P}^4; \tilde{v}) &= \left\{ \zeta \in \Gamma_-(b; T\mathbb{P}^4) : \sum_{i \in \chi(\mathcal{T})} (\psi_{x_{h(i)}} \tilde{v}_i) \tilde{\mathfrak{D}}_{\mathcal{T}, i}^{\mathbb{P}^4} \zeta = 0 \ \forall \psi \in \mathbb{E}_{\pi_P(b)} \right\}, \\ \Gamma_-(b; TY; \tilde{v}) &= \left\{ \zeta \in \Gamma_-(b; TY) : \sum_{i \in \chi(\mathcal{T})} (\psi_{x_{h(i)}} \tilde{v}_i) \tilde{\mathfrak{D}}_{\mathcal{T}, i}^{\mathbb{P}^4} \zeta = 0 \ \forall \psi \in \mathbb{E}_{\pi_P(b)} \right\} \subset \Gamma_-(b; T\kappa). \end{aligned}$$

Thus, there is a natural surjective bundle homomorphism

$$\mathcal{V}_- \equiv \pi_{\mathbb{P}\mathfrak{F}\mathcal{T}}^* \mathcal{V}_1^d / j_\kappa(\mathcal{N}^\kappa\mathcal{T}) \longrightarrow \pi_B^* \mathcal{W}_{\kappa, d/d_\kappa}^{0, q} \quad \text{if } \mathcal{U}_{\mathcal{T}}(\mathbb{P}^4; J) \subset \overline{\mathfrak{M}}_1^q(\mathbb{P}^4, d; J),$$

where $j_\kappa : \mathcal{N}^\kappa\mathcal{T} \longrightarrow \pi_{\mathbb{P}\mathfrak{F}\mathcal{T}}^* \mathcal{V}_1^d$ is the injective bundle homomorphism induced by the maps j_0 in (4.4). We put

$$\begin{aligned} \mathcal{F} &= \pi_{\mathbb{P}\mathfrak{F}\mathcal{T}}^* \mathcal{F}\mathcal{T} \longrightarrow \mathbb{P}\mathfrak{F}\mathcal{T}; \\ \mathcal{F}^{1, \emptyset} &= \{(b, [\tilde{v}]; v) \in \mathcal{F}^\emptyset : (b, [\tilde{v}]) \in \mathcal{U}_1(\mathbb{P}^4; J); [\rho(v)] = [\tilde{v}]\}. \end{aligned}$$

The smooth orbifold $\mathcal{F}^{1, \emptyset}$ is diffeomorphic to $\mathcal{F}^1\mathcal{T}^\emptyset$ by the projection map

$$(b, [\tilde{v}]; v) \longrightarrow (b; v).$$

Furthermore, $\mathcal{F}^{1, \emptyset} \longrightarrow \mathcal{U}_1(\mathbb{P}^4; J)$ is a fiber bundle of smooth varieties. We can thus apply the same argument as in the proof of Lemma 3.5 and the end of Subsection 3.2, along with the regularity assumption $(\vartheta 2b)$, to show that

$$\{s_1^d + t\vartheta\}^{-1}(0) \cap U(K) = \emptyset \quad \forall t \in (0, \epsilon_\vartheta(K))$$

if $U(K)$ and t are sufficiently small.

It remains to compute the $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0}$ - ϑ -contribution to the euler class of the bundle $\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}$ over $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ from the set

$$\mathcal{Z}_{\kappa, \vartheta} \equiv \{\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta\}^{-1}(0) - \mathfrak{M}_1^0(\kappa, d/d_\kappa) = \{\mathfrak{d}_{\kappa, d/d_\kappa}^{1,1} \vartheta\}^{-1}(0) \subset \mathfrak{M}_{1;1}^{1,0}(\kappa, d/d_\kappa).$$

This contribution is computed by counting the zeros of the section $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta + t\nu$, for a generic section ν of $\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}$, that lie near $\mathcal{Z}_{\kappa, \vartheta}$. First, let

$$\pi_{-; \kappa}^\perp: \widetilde{\mathcal{W}}_{\kappa, d/d_\kappa}^{1,1} \longrightarrow \pi_P^* \mathbb{E}^* \otimes \pi_B^* \text{ev}_0^* N_Y \kappa$$

denote the (quotient) projection map; see (3.1). Our regularity assumptions on ν will be that the affine map

$$\psi_{\vartheta, \nu}: \pi_P^* L_{P,1} \otimes \pi_B^* L_0 \longrightarrow \pi_P^* \mathbb{E}^* \otimes \pi_B^* \text{ev}_0^* N_Y \kappa, \quad \psi_{\vartheta, \nu}(b; \nu) = \pi_{-; \kappa}^\perp \nu(b) + \{\tilde{\mathfrak{d}}_{\kappa, d/d_\kappa}^{1,1} \vartheta\}|_b \nu, \quad (4.13)$$

over $\mathcal{Z}_{\kappa, \vartheta}$ is transverse to the zero set and all zeros of $\psi_{\vartheta, \nu}$ lie over

$$\mathcal{Z}_{\kappa, \vartheta}^0 \equiv \mathcal{Z}_{\kappa, \vartheta} \cap (\mathcal{M}_{1,1} \times \mathfrak{M}_{0,1;1}^0(\kappa, d/d_\kappa)).$$

Since the set $\psi_{\vartheta, \nu}^{-1}(0)$ is finite, it follows that it lies over a compact subset $K_{\vartheta, \nu}$ of $\mathcal{Z}_{\kappa, \vartheta}^0$. By the regularity assumption ($\vartheta 2a$), these conditions are satisfied by sections ν in a dense open subset of the space of all sections of $\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}$. We put

$$\partial \mathcal{Z}_{\kappa, \vartheta} = \mathcal{Z}_{\kappa, \vartheta} \cap (\partial \overline{\mathcal{M}}_{1,1} \times \mathfrak{M}_{0,1;1}^0(\kappa, d/d_\kappa)).$$

Lemma 4.6 *Suppose $J, d, \mathfrak{L}, \mathcal{V}_1^d, \vartheta \in \tilde{\mathcal{A}}_1^d(s; J)$, and $\kappa \in \mathcal{S}_0(Y; J)$ are as in Proposition 4.5 and in Lemma 4.4. If $\mathcal{T} = (I, \aleph; \underline{d})$ is a bubble type such that $d_i = 0$ for all minimal elements i of I and $\hat{I} = \{h\}$ is a single-element set, then there exist $\delta \in C(\mathcal{U}_{\mathcal{T}; \kappa; 1}; \mathbb{R}^+)$, $U_{\mathcal{T}}^1$, and $\phi_{\mathcal{T}}^1$ as in the $n=1$ case of Lemma 4.2, $\varepsilon \in C(\mathcal{FT}; \mathbb{R})$, and a vector bundle isomorphism*

$$\Phi_{\mathcal{T}}: \pi_{\mathcal{FT}_\delta}^* (\widetilde{\mathcal{W}}_{\kappa, d/d_\kappa}^{1,1} |_{\mathcal{U}_{\mathcal{T}; \kappa; 1}}) \longrightarrow \mathcal{W}_{\kappa, d/d_\kappa}^{1,0} |_{\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa) \cap U_{\mathcal{T}}^1},$$

covering the homeomorphism $\phi_{\mathcal{T}}^1$ and restricting to the identity over $\mathcal{U}_{\mathcal{T}; \kappa; 1}$ such that

$$|\pi_{-; \kappa}^\perp \Phi_{\mathcal{T}}^{-1}(\{\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta\}(\phi_{\mathcal{T}}^1(v)) - \{\pi_{\kappa}^- \tilde{\mathfrak{d}}_{\kappa, d/d_\kappa}^{1,1} \vartheta|_b\} \rho(v))| \leq \varepsilon(v) |\rho(v)| \quad \forall v = (b, \nu) \in \mathcal{FT}_\delta^\emptyset,$$

and $\lim_{|v| \rightarrow 0} \varepsilon(v) = 0$.

In this case, $\aleph = \emptyset$ or \aleph contains one element, and

$$\mathcal{U}_{\mathcal{T}; \kappa; 1} = \begin{cases} \mathcal{M}_{1,1} \times \mathcal{M}_{0,1;1}^0(\kappa, d/d_\kappa), & \text{if } \aleph = \emptyset; \\ \partial \overline{\mathcal{M}}_{1,1} \times \mathfrak{M}_{0,1;1}^0(\kappa, d/d_\kappa), & \text{otherwise.} \end{cases}$$

In either case, by Lemma 4.2, the normal bundle $\mathcal{F}^1 \mathcal{T}$ of $\mathcal{U}_{\mathcal{T}; \kappa}$ in $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ is \mathcal{FT} . If $\aleph = \emptyset$,

$$\mathcal{FT} = \pi_P^* L_{P,1} \otimes \pi_B^* L_0 \quad \text{and} \quad \rho(v) = v.$$

Otherwise, \mathcal{FT} is the direct sum of $\pi_P^* L_{P,1} \otimes \pi_B^* L_0$ with the line of smoothings of the node of $\Sigma_{b,\aleph}$, which in this case is a sphere with two points identified. If $v \in \mathcal{FT}$, $\rho(v)$ is the $\pi_P^* L_{P,1} \otimes \pi_B^* L_0$ -component of v .

Lemma 4.6 follows fairly easily from constructions in [Z4] and [Z5]. However, its proof is notationally involved, and we postpone it until the next subsection.

Corollary 4.7 *Suppose $d, \mathfrak{L}, \mathcal{V}_1^d, J, \vartheta \in \tilde{\mathcal{A}}_1^d(s; J)$, and $\kappa \in \mathcal{S}_0(Y; J)$ are as in Proposition 4.5. If ν is a generic perturbation of the section $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta$ of $\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}$ over $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$, there exist $\epsilon_\nu \in \mathbb{R}^+$ and an open neighborhood U of $\partial \mathcal{Z}_{\kappa, \vartheta}$ in $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ such that*

$$\{\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta + t\nu\}^{-1}(0) \cap U = \emptyset \quad \forall t \in (0, \epsilon_\nu).$$

Furthermore, for every compact subset K of $\mathcal{Z}_{\kappa, \vartheta}^0$, there exist $\epsilon_\nu(K) \in \mathbb{R}^+$ and an open neighborhood $U(K)$ of K in $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ with the following properties:

- (a) the section $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta + t\nu$ is transverse to the zero set in $\mathcal{W}_{\kappa, d/d_\kappa}^{1,0}$ over $U(K)$ for all $t \in (0, \epsilon_\nu(K))$;
- (b) for every open subset U of $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$, there exists $\epsilon(U) \in (0, \epsilon_\nu(K))$ such that

$$\begin{aligned} \pm |\{\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta + t\nu\}^{-1} \cap U| &= -\frac{d/d_\kappa - 1}{12} \langle e(\mathcal{W}_{\kappa, d/d_\kappa}^0), [\overline{\mathfrak{M}}_0(\kappa, d/d_\kappa)] \rangle \\ \text{if } K_{\vartheta, \nu} \subset K \subset U \subset U(K), t &\in (0, \epsilon(U)). \end{aligned}$$

Proof: (1) Let $\mathcal{T} = (I, \aleph; \underline{d})$ be a bubble type such that $\sum_{i \in I} d_i = d$ and $d_i = 0$ for all minimal elements i of I . By the regularity assumption ($\vartheta 2b$), if

$$\mathcal{Z}_{\vartheta, \mathcal{T}} \equiv \{\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta\}^{-1}(0) \cap \mathcal{U}_{\mathcal{T}; \kappa; 1} \neq \emptyset,$$

then $\hat{I} = \{h\}$ is a single-element set, while $|\aleph| \in \{0, 1\}$.

(2) We denote by $\mathcal{N}^\vartheta \mathcal{T}$ the normal bundle of $\mathcal{Z}_{\vartheta, \mathcal{T}}$ in $\mathcal{U}_{\mathcal{T}; \kappa; 1}$. Similarly to the proof of Lemma 3.5, using the homeomorphism $\phi_{\mathcal{T}}^1$ of Lemma 4.2 and the bundle isomorphism $\Phi_{\mathcal{T}}$ of Lemma 4.6, we can obtain an identification of neighborhoods of $\mathcal{Z}_{\vartheta, \mathcal{T}}$ in $\mathcal{N}^\vartheta \mathcal{T} \oplus \mathcal{FT}$ and in $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$,

$$\phi_{\mathcal{T}; \vartheta}: \mathcal{N}^\vartheta \mathcal{T}_\delta \times_{\mathcal{Z}_{\vartheta, \mathcal{T}}} \mathcal{FT}_\delta \longrightarrow \overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa),$$

and a lift of $\phi_{\mathcal{T}; \vartheta}$ to a bundle isomorphism,

$$\Phi_{\mathcal{T}; \vartheta}: \pi_{\mathcal{N}^\vartheta \mathcal{T} \times_{\mathcal{Z}_{\vartheta, \mathcal{T}}} \mathcal{FT}_\delta}^* \left(\widetilde{\mathcal{W}}_{\kappa, d/d_\kappa}^{1,1} |_{\mathcal{Z}_{\vartheta, \mathcal{T}}} \right) \longrightarrow \mathcal{W}_{\kappa, d/d_\kappa}^{0,1}.$$

For $(b; X, v) \in \mathcal{N}^\vartheta \mathcal{T} \times_{\mathcal{Z}_{\vartheta, \mathcal{T}}} \mathcal{FT}_\delta$, we put

$$\begin{aligned} s(b; X, v) &= \Phi_{\mathcal{T}; \vartheta}^{-1}(\{\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta\} \phi_{\mathcal{T}}^1(b; X, v)) \in \widetilde{\mathcal{W}}_{\kappa, d/d_\kappa}^{1,1} |_b; \\ \tilde{\nu}(b; X, v) &= \Phi_{\mathcal{T}; \vartheta}^{-1}(\nu(\phi_{\mathcal{T}}^1(b; X, v))) \in \widetilde{\mathcal{W}}_{\kappa, d/d_\kappa}^{1,1} |_b. \end{aligned}$$

We define $N_s(X)$ and $N'_s(X, v)$ in $\widetilde{\mathcal{W}}_{\kappa, d/d_\kappa}^{1,1}|_b$ by

$$s(b; X, 0) = s(b; 0, 0) + j_b X + N_s X = j|_b X + N_s X; \quad (4.14)$$

$$s(b; X, v) = s(b; X, 0) + N'_s(X, v), \quad (4.15)$$

where $j_b: \mathcal{N}^\vartheta \mathcal{T} \longrightarrow \pi_B^* \mathcal{W}_{\kappa, d/d_\kappa}^{0,1}|_b \subset \widetilde{\mathcal{W}}_{\kappa, d/d_\kappa}^{1,1}|_b$ is the derivative of s . Thus,

$$N_s(0) = 0, \quad |N_s(X) - N_s(X')| < C_K(|X| + |X'|)|X - X'| \quad \forall X, X' \in \mathcal{N}^\vartheta \mathcal{T}_\delta. \quad (4.16)$$

By the continuity ϑ ,

$$|N'_s(X, v)| \leq \varepsilon(v), \quad |N'_s(X, v) - N'_s(X', v)| \leq \varepsilon(v)|X - X'| \quad \forall X, X' \in \mathcal{N}^\vartheta \mathcal{T}_\delta, v \in \mathcal{FT}_\delta, \quad (4.17)$$

for some $\varepsilon \in C(\mathcal{FT}; \mathbb{R}^+)$ such that $\lim_{|v| \rightarrow 0} \varepsilon(v) = 0$. We also have

$$\begin{aligned} |\tilde{\nu}(b; X, v)| &\leq C(b) \\ |\tilde{\nu}(b; X, v) - \tilde{\nu}(b; X', v)| &\leq C(b)|X - X'| \end{aligned} \quad \forall b \in \mathcal{Z}_{\vartheta, \mathcal{T}}, X, X' \in \mathcal{N}^\vartheta \mathcal{T}_\delta, v \in \mathcal{FT}_\delta, \quad (4.18)$$

for some $C \in C(\mathcal{Z}_{\vartheta, \mathcal{T}}; \mathbb{R})$.

(3) Let $\pi_{-,B}: \widetilde{\mathcal{W}}_{\kappa, d/d_\kappa}^{1,1} \longrightarrow \pi_B^* \mathcal{W}_{\kappa, d/d_\kappa}^{0,1}$ be the natural projection map; see (3.1). Let K be a precompact open subset of $\mathcal{Z}_{\vartheta, \mathcal{T}}$. Since the homomorphism

$$j_b: \mathcal{N}^\vartheta \mathcal{T} \longrightarrow \pi_B^* \mathcal{W}_{\kappa, d/d_\kappa}^{0,1}$$

is an isomorphism by the regularity assumption ($\vartheta 2a$), by (4.14)-(4.18) and the Contraction Principle, the equation

$$\pi_{-,B}(s(b; X, v) + t\tilde{\nu}(b; X, v)) = 0$$

has a unique small solution $X = X_t(v) \in \mathcal{N}^\vartheta \mathcal{T}_b$ for all $t \in [0, \delta_K)$, $v \in \mathcal{FT}_{\delta_K}|_b$, and $b \in K$. Furthermore,

$$|X_t(v)| \leq C_K(t + \varepsilon(v)). \quad (4.19)$$

(4) By the above, the number of zeros of $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta + t\nu$, for $t \in (0, \delta_K)$, in a small neighborhood U_K of K in $\overline{\mathfrak{M}}_1^0(\kappa, d/d_\kappa)$ is the number of solutions of the equation

$$\begin{aligned} \Psi_t(b; v) &\equiv t^{-1} \pi_{-, \kappa}^\perp (s(b; X_t(v); v) + t\tilde{\nu}(b; X_t(v); v)) \\ &= t^{-1} \pi_{-, \kappa}^\perp N'_s(v; X_t(v)) + \pi_{-, \kappa}^\perp \tilde{\nu}(b; X_t(v); v), \end{aligned}$$

since $\mathfrak{d}_{\kappa, d/d_\kappa}^{1,0} \vartheta|_{\mathcal{U}_{\mathcal{T}; \kappa; 1}}$ is a section of $\pi_B^* \mathcal{W}_{\kappa, d/d_\kappa}^{0,1}$ and thus $\pi_{-, \kappa}^\perp N_s X = 0$ for all $X \in \mathcal{N}^\vartheta \mathcal{T}_\delta$. By the estimate of Lemma 4.6, (4.19), and the smoothness of ν

$$|\Psi_t(b; v) - (\pi_{-, \kappa}^\perp \nu(b) + t^{-1} \{\pi_{\kappa}^- \tilde{\mathfrak{d}}_{\kappa, d/d_\kappa}^{1,1} \vartheta\}|_b \rho(v))| \leq C_K(t + \varepsilon(v)) |\rho(v)|. \quad (4.20)$$

By the regularity assumption ($\vartheta 3$), the section $\pi_{\kappa}^- \tilde{\mathfrak{d}}_{\kappa, d/d_\kappa}^{1,1} \vartheta$ does not vanish over $\mathcal{Z}_{\vartheta, \mathcal{T}}$. Suppose \mathcal{T} is a bubble type as in the $|\mathfrak{N}| = 1$ case in (1) above. By our assumptions on ν , the affine map

$$\mathcal{FT} \longrightarrow \pi_P^* \mathbb{E}^* \otimes \pi_B^* \text{ev}_0^* N_Y \kappa, \quad v \longrightarrow \pi_{-, \kappa}^\perp \nu(b) + \{\pi_{\kappa}^- \tilde{\mathfrak{d}}_{\kappa, d/d_\kappa}^{1,1} \vartheta\}|_b \rho(v), \quad (4.21)$$

which factors through $\psi_{\vartheta,\nu}$, does not vanish over the compact set $\partial\mathcal{Z}_{\kappa,\vartheta}$. Thus, $\Psi_t(b;v) \neq 0$ for all $t \in \mathbb{R}^+$ and $v \in \mathcal{FT}|_{\partial\mathcal{Z}_{\kappa,\vartheta}}$ sufficiently small. This concludes the proof of the first statement of Corollary 4.7.

(5) Finally, suppose \mathcal{T} is a bubble type as in the $|\aleph|=0$ case in (1). If K is a compact subset of $\mathcal{Z}_{\kappa,\vartheta}^0$ containing $K_{\vartheta,\nu}$, by (4.20), the regularity assumption $(\vartheta 3)$, and the same cobordism argument as in Subsection 3.1 of [Z1], the number of solutions of $\Psi_t(b;v) = 0$ with $t \in \mathbb{R}^+$ and $v \in \mathcal{FT}|_K$ sufficiently small is the number of zeros of the affine map in (4.21), i.e. $\pm|\psi_{\vartheta,\nu}^{-1}(0)|$. Since $\pi_{\kappa}^{-1}\tilde{\mathfrak{d}}_{\kappa,d/d_{\kappa}}^{1,1}\vartheta$ does not vanish over $\mathcal{Z}_{\kappa,\vartheta}$,

$$\begin{aligned} \pm|\psi_{\vartheta,\nu}^{-1}(0)| &= \langle c(\pi_P^*\mathbb{E}^* \otimes \pi_B^*\text{ev}_0^*N_Y\kappa)c(\pi_P^*L_{P,1} \otimes \pi_B^*L_0)^{-1}, [\mathcal{Z}_{\kappa,\vartheta}] \rangle \\ &= \langle \pi_P^*(-2\lambda + \psi_{P,1}) + \pi_B^*(c_1(\text{ev}_0^*N_Y\kappa) + \psi_0), [\mathcal{Z}_{\kappa,\vartheta}] \rangle, \end{aligned}$$

where λ and $\psi_{P,1}$ are the usual tautological classes on $\overline{\mathcal{M}}_{1,1}$. The space $\mathcal{Z}_{\kappa,\vartheta}$ is the zero set of the section $\mathfrak{d}_{\kappa,d/d_{\kappa}}^{1,1}\vartheta$ of the bundle $\pi_B^*\mathcal{W}_{\kappa,d/d_{\kappa}}^{0,1}$ over $\overline{\mathfrak{M}}_{1,1}^1(\kappa, d/d_{\kappa})$. Since this section is transverse to the zero set by $(\vartheta 2a)$,

$$\begin{aligned} \pm|\psi_{\vartheta,\nu}^{-1}(0)| &= \langle \pi_P^*(-2\lambda + \psi_{P,1}) \cdot \pi_B^*e(\mathcal{W}_{\kappa,d/d_{\kappa}}^{0,1}), [\overline{\mathcal{M}}_{1,1} \times \overline{\mathfrak{M}}_{0,1;1}(\kappa, d/d_{\kappa})] \rangle \\ &= -\frac{1}{24} \langle e(\mathcal{W}_{\kappa,d/d_{\kappa}}^{0,1}), [\overline{\mathfrak{M}}_{0,1;1}(\kappa, d/d_{\kappa})] \rangle. \end{aligned}$$

We note that a generic fiber of the forgetful map

$$\tilde{\pi}: \overline{\mathfrak{M}}_{0,1;1}(\kappa, d/d_{\kappa}) \longrightarrow \overline{\mathfrak{M}}_0(\kappa, d/d_{\kappa})$$

consists of $2(d/d_{\kappa})-2$ points, corresponding to the branch points a degree- d/d_{κ} cover $\mathbb{P}^1 \longrightarrow \kappa$. We conclude that for every compact subset K of $\mathcal{Z}_{\kappa,\vartheta}$ containing $K_{\vartheta,\nu}$

$$\begin{aligned} \pm|\{\mathfrak{d}_{\kappa,d/d_{\kappa}}^{1,0}\vartheta + t\nu\}^{-1} \cap U| &= \pm|\psi_{\vartheta,\nu}^{-1}(0)| = -\frac{1}{24} \langle \tilde{\pi}^*e(\mathcal{W}_{\kappa,d/d_{\kappa}}^0), [\overline{\mathfrak{M}}_{0,1;1}(\kappa, d/d_{\kappa})] \rangle \\ &= -\frac{1}{24} (2(d/d_{\kappa})-2) \langle e(\mathcal{W}_{\kappa,d/d_{\kappa}}^0), [\overline{\mathfrak{M}}_0(\kappa, d/d_{\kappa})] \rangle, \end{aligned}$$

provided that U is a sufficiently small neighborhood of K in $\overline{\mathfrak{M}}_1^0(\kappa, d/d_{\kappa})$ and $t \in (0, \delta(U))$.

4.4 A Genus-One Gluing Procedure

In this subsection, we prove Lemma 4.6. We review the genus-one gluing procedure of Subsection 4.2 in [Z4] and its extensions to the spaces $\Gamma_-(b; T\mathbb{P}^4)$ and $\Gamma_-(b; \mathfrak{L})$. As a result, we will be able to describe the behavior of the boundary operator \mathfrak{d}_0 in the long exact sequence (3.4) for $[\tilde{b}(v)] \in \mathfrak{M}_1^0(\kappa, d/d_{\kappa})$ with $v \in \tilde{\mathcal{FT}}^0$ sufficiently small.

Let $\mathcal{T} = (I, \aleph; \underline{d})$ be a bubble type as in the statement of Lemma 4.6. If $v = (b, v) \in \tilde{\mathcal{FT}}_{\delta}^0$ is small gluing parameter, let

$$b(v) = (\Sigma_v, j_v; u_v), \quad \text{where } u_v = u_b \circ \tilde{q}_v,$$

be the (second-stage) approximately J -holomorphic map. Here

$$\tilde{q}_v = \tilde{q}_{v_0;2}: \Sigma_v \longrightarrow \Sigma_b$$

is the basic gluing map constructed in Subsection 4.2 of [Z4]. In the present case, there is no first stage in this usually two-stage gluing construction, as there is only one level of bubbles (in fact, only one bubble) to attach. The key advantage of this gluing construction is that the map \tilde{q}_v is closer to being holomorphic than in the gluing construction used in Section 3. In particular,

$$\|\bar{\partial}_J u_v\|_{v,p} \leq C(b)|\rho(v)|.$$

If $b \in \mathcal{U}_{\mathcal{T};\kappa;1}$, then $du_{b,h}|_\infty = 0$ and this estimate improves to

$$\|\bar{\partial}_J u_v\|_{v,p} \leq C(b)|\rho(v)|^2. \quad (4.22)$$

This is immediate from the definition of the map \tilde{q}_v .

We extend the metric $g_{Y,b}$ described in Subsection 3.3 to a metric $g_{\mathbb{P}^4,b}$ on the bundle $T\mathbb{P}^4$. Let ∇^J be the J -compatible connection corresponding to the Levi-Civita of the metric $g_{\mathbb{P}^4,b}$.

Similarly to Subsection 3.3, let

$$\Gamma_-^{0,1}(b; T\mathbb{P}^4) = \mathcal{H}_{b,P}^{0,1} \otimes T_{\text{ev}_P(b)}\mathbb{P}^4$$

be the space of $u_b^*T\mathbb{P}^4$ -valued harmonic $(0,1)$ -forms on Σ_b . If $v = (b, v)$ and $b \in \mathcal{U}_{\mathcal{T};\kappa;1}$ are as above, we put

$$\Gamma_-^{0,1}(v; T\mathbb{P}^4) = \{R_v\eta : \eta \in \Gamma_-^{0,1}(b; T\mathbb{P}^4)\} \subset \Gamma^{0,1}(v; T\mathbb{P}^4),$$

where $R_v\eta$ is a smooth extension of η such that $R_v\eta$ is nearly harmonic on the neck attaching the only bubble $\Sigma_{b,h}$ of Σ_b and below a small collar of the neck and vanishes past a slighter larger collar; see Subsection 4.2 in [Z4]. Let

$$\pi_{v;-}^{0,1} : \Gamma^{0,1}(v; T\mathbb{P}^4) \longrightarrow \Gamma_-^{0,1}(v; T\mathbb{P}^4)$$

be the L^2 -projection map. We denote its kernel by $\Gamma_+^{0,1}(v; T\mathbb{P}^4)$. By the same argument as in Subsection 2.3 in [Z1], we have a decomposition

$$\begin{aligned} \Gamma(v; T\mathbb{P}^4) &= \Gamma_-(v; T\mathbb{P}^4) \oplus \tilde{\Gamma}_+(v; T\mathbb{P}^4), \\ \text{where } \Gamma_-(v; T\mathbb{P}^4) &= \{R_v\zeta \equiv \xi \circ \tilde{q}_v : \zeta \in \Gamma_-(b; T\mathbb{P}^4)\}, \end{aligned}$$

such that

$$D_{J,v} : \tilde{\Gamma}_+(v; T\mathbb{P}^4) \longrightarrow \Gamma_+^{0,1}(v; T\mathbb{P}^4)$$

is an isomorphism with fiber-uniformly bounded inverse and

$$\langle\langle D_{J,v}\zeta, \eta \rangle\rangle_{v,2} = 0 \quad \forall \zeta \in \tilde{\Gamma}_+(v; T\kappa) \equiv \tilde{\Gamma}_+(v; T\mathbb{P}^4) \cap \Gamma(v; T\kappa), \eta \in \Gamma_-^{0,1}(v; T\mathbb{P}^4).$$

Analogously to (4.22), we also have

$$\|D_{J,v}\zeta\|_{v,p} \leq C(b)|\rho(v)|\|\zeta\|_{v,p,1} \quad \forall \zeta \in \Gamma_-(v; T\mathbb{P}^4). \quad (4.23)$$

Furthermore,

$$|\pi_{v;-}^{0,1} D_{J,v} R_v \zeta + 2\pi\rho(v) J R_v \tilde{\mathfrak{D}}_{\mathcal{T},h}^{\mathbb{P}^4} \zeta| \leq C(b)|\rho(v)|^2 \|\zeta\|_{b,p,1} \quad \forall \zeta \in \Gamma_-(b; T\mathbb{P}^4); \quad (4.24)$$

see (5) of Lemma 4.4 in [Z4]. Due to the assumption that $du_{b,h}|_\infty = 0$, we do not need to require that $\zeta|_{\Sigma_{b,\mathbb{N}}} = 0$. We also get a slightly sharper bound, though this is not essential. The estimate (4.24) is the fundamental fact behind the estimate of Lemma 4.6.

Similarly to Subsection 3.3, the restriction of the homeomorphism $\phi_{\mathcal{T}}^1$ of Lemma 4.2 can be taken to be of the form

$$\begin{aligned} \phi_{\mathcal{T}}^1([v]) &= ([\tilde{b}(v)]), \quad \text{where } \tilde{b}(v) = (\Sigma_v, j_v; \tilde{u}_v), \quad \tilde{u}_v = \exp_{b,u_v} \zeta_v \\ \zeta_v &\in \tilde{\Gamma}_+(v; T\kappa), \quad \|\zeta_v\|_{v,p,1} \leq C(b)|\rho(v)|^2. \end{aligned} \quad (4.25)$$

The last estimate follows from (4.22) by the usual argument.

We denote by

$$\begin{aligned} \Pi_v^J: \Gamma(v; T\mathbb{P}^4) &\longrightarrow \tilde{\Gamma}(v; T\mathbb{P}^4) \equiv L_1^p(\tilde{b}(v); T\mathbb{P}^4) \quad \text{and} \\ \Pi_v: \Gamma(v; \mathcal{L}) &\equiv L_1^p(b(v); \mathcal{L}) \longrightarrow \tilde{\Gamma}(v; \mathcal{L}) \equiv L_1^p(\tilde{b}(v); \mathcal{L}) \end{aligned}$$

the ∇^J -parallel transport in $T\mathbb{P}^4$ and the ∇ -parallel transport in \mathcal{L} along the geodesics γ_{ζ_v} in κ of the metric $g_{\kappa,b}$. By (4.23)-(4.25) and the same argument as in Subsection 4.3 of [Z5], there exists an isomorphism

$$\begin{aligned} \tilde{R}_v: \Gamma_-(b; T\mathbb{P}^4; 0) &\longrightarrow \tilde{\Gamma}_-(v; T\mathbb{P}^4) \equiv \ker D_{J,\tilde{b}(v)} \quad \text{s.t.} \\ \|\tilde{R}_v \zeta - \Pi_v^J R_v \zeta\|_{v,p,1} &\leq C(b)|\rho(v)|^2 \|\zeta\|_{b,p,1} \quad \forall \zeta \in \Gamma_-(b; T\mathbb{P}^4; 0). \end{aligned} \quad (4.26)$$

Similarly, there exists an isomorphism

$$\begin{aligned} \tilde{R}_v: \Gamma_-(b; \mathcal{L}; 0) &\longrightarrow \tilde{\Gamma}_-(v; \mathcal{L}) \equiv \ker \bar{\partial}_{\nabla, \tilde{b}(v)} \quad \text{s.t.} \\ \|\tilde{R}_v \xi - \Pi_v R_v \xi\|_{v,p,1} &\leq C(b)|\rho(v)|^2 \|\xi\|_{b,p,1} \quad \forall \xi \in \Gamma_-(b; \mathcal{L}; 0), \end{aligned} \quad (4.27)$$

where again $R_v \xi = \xi \circ \tilde{q}_v$.

We next describe a convenient family of finite-dimensional spaces

$$\Gamma(b; T\mathbb{P}^4; \mathcal{L}) \subset \Gamma(b; T\mathbb{P}^4),$$

parameterized by $b \in \mathcal{U}_{\mathcal{T}; \kappa; 1}^{(0)}$, such that the homomorphism

$$j_0: \Gamma(b; T\mathbb{P}^4; \mathcal{L}) \longrightarrow \Gamma_-(b; \mathcal{L}; 0)$$

is an isomorphism. For every $b \in \mathcal{U}_{\mathcal{T}; \kappa; 1}^{(0)}$, let

$$\Gamma_{-,+}(b; T\mathbb{P}^4; N_Y \kappa) \approx \text{ev}_P^* N_Y \kappa \quad \text{and} \quad \Gamma_{-,+}(b; T\mathbb{P}^4; \kappa)$$

be the L^2 -orthogonal complements of $\Gamma_-(b; T\mathbb{P}^4; T\kappa)$ in $\Gamma_-(b; T\mathbb{P}^4; TY)$ and of $\Gamma_-(b; T\kappa; 0)$ in $\Gamma_-(b; T\mathbb{P}^4; 0)$, respectively. The map in (4.10) induces a surjective homomorphism

$$\tilde{\pi}_\kappa^-: \Gamma_-(b; \mathcal{L}; 0) \longrightarrow \text{ev}_P^* N_Y \kappa,$$

which restricts to an isomorphism on $j_0(\Gamma_{-,+}(b; T\mathbb{P}^4; N_Y\kappa))$ and vanishes on $j_0(\Gamma_{-,+}(b; T\mathbb{P}^4; \kappa))$, where j_0 is as in (4.4). Let $\Gamma_{-,+}(b; \mathfrak{L}; 0)$ be the L^2 -orthogonal complement of $j_0(\Gamma_{-,+}(b; T\mathbb{P}^4; \kappa))$ in $\ker \tilde{\pi}_\kappa^-$. We set

$$\Gamma_{+,-}(b; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) = \{\zeta \in \Gamma(b; N_{\mathbb{P}^4}Y) : \pi_Y^\perp \circ \zeta \in \Gamma_{-,+}(b; \mathfrak{L}; 0)\},$$

where the normal bundle $N_{\mathbb{P}^4}Y$ of Y in \mathbb{P}^4 is identified with the $g_{\mathbb{P}^4,b}$ -orthogonal complement of TY in $T\mathbb{P}^4$ and

$$\pi_Y^\perp : T\mathbb{P}^4 \longrightarrow \mathfrak{L} \approx N_{\mathbb{P}^4}Y$$

is the quotient projection. Then, the map

$$\pi_Y^\perp : \Gamma(b; T\mathbb{P}^4; \mathfrak{L}) \equiv \Gamma_{-,+}(b; T\mathbb{P}^4; \kappa) \oplus \Gamma_{-,+}(b; T\mathbb{P}^4; N_Y\kappa) \oplus \Gamma_{+,-}(b; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) \longrightarrow \Gamma_-(b; \mathfrak{L}; 0)$$

is an isomorphism. Furthermore, since

$$\Gamma_{-,+}(b; T\mathbb{P}^4; \kappa) \oplus \Gamma_{-,+}(b; T\mathbb{P}^4; N_Y\kappa) \subset \Gamma_-(b; T\mathbb{P}^4),$$

the map

$$\mathfrak{d}_b : \Gamma_{+,-}(b; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) \longrightarrow H_J^1(\pi_B(b); N_Y\kappa), \quad \zeta \longrightarrow [\pi_\kappa^\perp D_{J,b}\zeta],$$

is also an isomorphism, by the definition of the boundary operator \mathfrak{d}_0 in (4.4).

We now use the subspace $\Gamma(b; T\mathbb{P}^4; \mathfrak{L})$ of $\Gamma(b; T\mathbb{P}^4)$ to construct an analogous subspace $\tilde{\Gamma}(v; T\mathbb{P}^4; \mathfrak{L})$ of $\tilde{\Gamma}(v; T\mathbb{P}^4)$ for $v \in \tilde{\mathcal{F}}\mathcal{T}_\delta^0$ sufficiently small. For every

$$\zeta \in \Gamma_{-,+}(b; T\mathbb{P}^4; N_Y\kappa) \oplus \Gamma_{+,-}(b; T\mathbb{P}^4; N_{\mathbb{P}^4}Y),$$

we define $\tilde{R}_v\zeta \in \tilde{\Gamma}(v; T\mathbb{P}^4)$ by

$$\pi_Y^\perp \tilde{R}_v\zeta = \tilde{R}_v\pi_Y^\perp\zeta \in \tilde{\Gamma}_-(v; \mathfrak{L}) \quad \text{and} \quad \tilde{R}_v\zeta - \Pi_v^J R_v\zeta \in \Gamma(\Sigma_v; \tilde{u}_v^* N_{\mathbb{P}^4}Y),$$

where again $R_v\zeta = \zeta \circ \tilde{q}_v$. By (4.26) and (4.27),

$$\|\tilde{R}_v\zeta - \Pi_v R_v\zeta\|_{v,p,1} \leq C(b)|\rho(v)|^2 \|\zeta\|_{b,p,1} \quad \forall \zeta \in \Gamma(b; \mathbb{P}^4; \mathfrak{L}). \quad (4.28)$$

Let $\tilde{\Gamma}_{-,+}(v; T\mathbb{P}^4; \kappa)$, $\tilde{\Gamma}_{-,+}(v; T\mathbb{P}^4; N_Y\kappa)$, $\tilde{\Gamma}_{+,-}(v; T\mathbb{P}^4; N_{\mathbb{P}^4}Y)$, and $\tilde{\Gamma}(v; T\mathbb{P}^4; \mathfrak{L})$ denote the images of $\Gamma_{-,+}(b; T\mathbb{P}^4; \kappa)$, $\Gamma_{-,+}(b; T\mathbb{P}^4; N_Y\kappa)$, $\Gamma_{+,-}(b; T\mathbb{P}^4; N_{\mathbb{P}^4}Y)$, and $\Gamma(b; T\mathbb{P}^4; \mathfrak{L})$ under \tilde{R}_v . By (4.28), the map

$$\pi_Y^\perp : \tilde{\Gamma}(v; T\mathbb{P}^4; \mathfrak{L}) \longrightarrow \tilde{\Gamma}_-(v; \mathfrak{L})$$

is injective and thus an isomorphism. Furthermore, since

$$\tilde{\Gamma}_{-,+}(v; T\mathbb{P}^4; \kappa) \subset \tilde{\Gamma}_-(v; T\mathbb{P}^4),$$

by the definition of the boundary operator \mathfrak{d}_0 in (3.4) the map

$$\mathfrak{d}_v : \tilde{\Gamma}_{-,+}(v; T\mathbb{P}^4; N_Y\kappa) \oplus \tilde{\Gamma}_{+,-}(v; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) \longrightarrow H_J^1(\tilde{b}(v); N_Y\kappa), \quad \zeta \longrightarrow [\pi_\kappa^\perp D_{J,\tilde{b}(v)}\zeta],$$

is surjective and thus an isomorphism. We set

$$\begin{aligned} \tilde{\Gamma}_{-,+}^{0,1}(v; T\mathbb{P}^4; N_Y\kappa) &= \{\pi_\kappa^\perp D_{J,\tilde{b}(v)}\zeta : \zeta \in \tilde{\Gamma}_{-,+}(v; T\mathbb{P}^4; N_Y\kappa)\} \subset \tilde{\Gamma}^{0,1}(v; N_Y\kappa) \equiv L^p(\tilde{b}(v); N_Y\kappa); \\ \tilde{\Gamma}_{+,-}^{0,1}(v; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) &= \{\pi_\kappa^\perp D_{J,\tilde{b}(v)}\zeta : \zeta \in \tilde{\Gamma}_{+,-}(v; T\mathbb{P}^4; N_{\mathbb{P}^4}Y)\} \subset \tilde{\Gamma}^{0,1}(v; N_Y\kappa). \end{aligned}$$

It follows from above that the projection map

$$\tilde{\pi}_v^{0,1} : \tilde{\Gamma}_{-,+}^{0,1}(v; T\mathbb{P}^4; N_Y\kappa) \oplus \tilde{\Gamma}_{+,-}^{0,1}(v; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) \longrightarrow H_J^1(\tilde{b}(v); N_Y\kappa)$$

is an isomorphism.

The space $\Gamma_-^{0,1}(b; N_Y\kappa)$ of $u_b^*N_Y\kappa$ -valued harmonic $(0,1)$ -forms on Σ_b splits as

$$\Gamma_-^{0,1}(b; N_Y\kappa) = \Gamma_{-,P}^{0,1}(b; N_Y\kappa) \oplus \Gamma_{-,B}^{0,1}(b; N_Y\kappa) = \mathcal{H}_{b,P} \otimes \text{ev}_P^*N_Y\kappa \oplus \Gamma_{-,B}^{0,1}(b; N_Y\kappa).$$

Here $\Gamma_{-,P}^{0,1}(b; N_Y\kappa)$ and $\Gamma_{-,B}^{0,1}(b; N_Y\kappa)$ are the subspaces of $\Gamma_-^{0,1}(b; N_Y\kappa)$ consisting of the differentials supported on the main components $\Sigma_{b,\mathbb{N}}$ of Σ_b and on the only bubble component $\Sigma_{b,h}$ of Σ_b , respectively. Let

$$\pi_{b;P}^{0,1}, \pi_{b;B}^{0,1} : \Gamma_-^{0,1}(b; N_Y\kappa) \longrightarrow \Gamma_{-,P}^{0,1}(b; N_Y\kappa), \Gamma_{-,B}^{0,1}(b; N_Y\kappa)$$

denote the projection maps. If $\eta \in \mathcal{H}_{b,P} \otimes \text{ev}_P^*N_Y\kappa$, we define $R_v\eta \in \Gamma_-^{0,1}(v; N_Y\kappa)$ as above by identifying $N_Y\kappa$ with the $g_{Y,b}$ -orthogonal complement of $T\kappa$ in $TY \subset T\mathbb{P}^4$. If $\eta \in \Gamma_{-,B}^{0,1}(b; N_Y\kappa)$, let $R_v\eta = \tilde{q}_v^*\eta$. We denote by

$$\tilde{\Gamma}_{-,P}^{0,1}(v; N_Y\kappa), \tilde{\Gamma}_{-,B}^{0,1}(v; N_Y\kappa) \subset \tilde{\Gamma}^{0,1}(v; N_Y\kappa)$$

the images of $\Gamma_{-,P}^{0,1}(b; N_Y\kappa)$ and $\Gamma_{-,B}^{0,1}(b; N_Y\kappa)$ under the map $\tilde{R}_v \equiv \Pi_v^J R_v$. Let

$$\tilde{\pi}_{v;P}^{0,1} : \tilde{\Gamma}^{0,1}(v; N_Y\kappa) \longrightarrow \tilde{\Gamma}_{-,P}^{0,1}(v; N_Y\kappa) \quad \text{and} \quad \tilde{\pi}_{v;B}^{0,1} : \tilde{\Gamma}^{0,1}(v; N_Y\kappa) \longrightarrow \tilde{\Gamma}_{-,B}^{0,1}(v; N_Y\kappa)$$

be the L^2 -projection maps.

By (4.24), (4.25), and (4.28),

$$\left| \tilde{\pi}_{v;P}^{0,1} D_{J,\tilde{b}(v)} \tilde{R}_v \zeta + 2\pi\rho(v) J \tilde{R}_v \pi_\kappa^\perp \tilde{\mathcal{D}}_{T,h}^{\mathbb{P}^4} \zeta \right| \leq C(b) |\rho(v)|^2 \|\zeta\|_{b,p,1} \quad \forall \zeta \in \Gamma_{-,+}^{0,1}(v; T\mathbb{P}^4; N_Y\kappa). \quad (4.29)$$

In particular, the projection map

$$\tilde{\pi}_{v;P}^{0,1} : \tilde{\Gamma}_{-,+}^{0,1}(v; T\mathbb{P}^4; N_Y\kappa) \longrightarrow \tilde{\Gamma}_{-,P}^{0,1}(v; N_Y\kappa)$$

is an isomorphism. We denote its inverse by $S_{v;P}$. The projection map

$$\tilde{\pi}_{v;B}^{0,1} : \tilde{\Gamma}_{+,-}^{0,1}(v; T\mathbb{P}^4; N_{\mathbb{P}^4}Y) \longrightarrow \tilde{\Gamma}_{-,B}^{0,1}(v; N_Y\kappa).$$

is also an isomorphism, since the map \mathfrak{d}_b is. We denote its inverse by $S_{v;B}$.

Finally, let

$$T_v = \tilde{\pi}_v^{0,1} \circ (S_{v;P} \oplus S_{v;B}) \circ \tilde{R}_v : \Gamma_-^{0,1}(b; N_Y\kappa) \longrightarrow H_J^1(\tilde{b}(v); N_Y\kappa).$$

The maps T_v with $v \in \tilde{\mathcal{F}}\mathcal{T}_\delta^\emptyset$ induce a bundle isomorphism

$$\Phi_T : \pi_{\mathcal{F}\mathcal{T}_\delta^\emptyset}^* (\widetilde{\mathcal{W}}_{\kappa,d/d_\kappa}^{1,1} |_{\mathcal{U}_T; \kappa; 1}) \longrightarrow \mathcal{W}_{\kappa,d/d_\kappa}^{0,1} |_{\mathfrak{M}_1^0(\kappa,d/d_\kappa) \cap U_T^1},$$

covering $\phi_T^1|_{\mathcal{F}T_\delta^\emptyset}$. This isomorphism extends continuously over $\mathcal{F}T_\delta - \mathcal{F}T_\delta^\emptyset$, as can be seen directly from the definition.

If $\vartheta(v) \in \tilde{\Gamma}_-(v; \mathfrak{L})$, we can find a unique

$$\begin{aligned} \zeta_\vartheta(v) &= \zeta_\vartheta^-(v) \oplus \zeta_\vartheta^0(v) \oplus \zeta_\vartheta^+(v) \\ &\in \tilde{\Gamma}_{-,+}(v; T\mathbb{P}^4; \kappa) \oplus \tilde{\Gamma}_{-,+}(v; T\mathbb{P}^4; N_Y \kappa) \oplus \tilde{\Gamma}_{+,-}(v; T\mathbb{P}^4; N_{\mathbb{P}^4} Y) \end{aligned}$$

such that $\pi_Y^\perp \zeta_\vartheta(v) = \vartheta(v)$. By (4.29),

$$|\pi_{b;P}^{0,1} T_v^{-1} \mathfrak{d}_0 \vartheta(v) + 2\pi \rho(v) J \pi_\kappa^\perp \tilde{\mathfrak{D}}_{T,h}^{\mathbb{P}^4} \tilde{R}_v^{-1} \zeta_\vartheta^0(v)| \leq C_\vartheta(b) |\rho(v)|^2. \quad (4.30)$$

On the other hand, by the definition of the map π_k^- in Subsection 4.2,

$$\pi_k^- \tilde{\mathfrak{d}}_{\kappa,d/d_\kappa}^{1,1} \vartheta(b) = -2\pi J \pi_\kappa^\perp \tilde{\mathfrak{D}}_{T,h}^{\mathbb{P}^4} \zeta_\vartheta^0(b). \quad (4.31)$$

The estimate of Lemma 4.6 follows from (4.28), (4.30), (4.31), and the continuity of the section ϑ .

Department of Mathematics, Stanford University, Stanford, CA 94305-2125
jli@math.stanford.edu

Department of Mathematics, Stanford University, Stanford, CA 94305-2125
Current Address: Department of Mathematics, SUNY, Stony Brook, NY 11794-3651
azinger@math.sunysb.edu

References

- [ABo] M. Atiyah and R. Bott, *The Moment Map and Equivariant Cohomology*, *Topology* 23 (1984), 1–28.
- [Ber] A. Bertram, *Another Way to Enumerate Rational Curves with Torus Actions*, *Invent. Math.* 142 (2000), no. 3, 487–512.
- [BehFa] K. Behrend and B. Fantechi, *The Intrinsic Normal Cone*, *Invent. Math.* 128 (1997), no. 1, 45–88.
- [BCOV] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Holomorphic Anomalies in Topological Field Theories*, *Nucl. Phys. B* 405 (1993), 279–304.
- [FuO] K. Fukaya and K. Ono, *Arnold Conjecture and Gromov-Witten Invariant*, *Topology* 38 (1999), no. 5, 933–1048.
- [Ga] A. Gathmann, *Absolute and Relative Gromov-Witten Invariants of Very Ample Hypersurfaces*, *Duke Math. J.* 115 (2002), no. 2, 171–203.

- [Gi] A. Givental, *The Mirror Formula for Quintic Threefolds*, Amer. Math. Soc. Transl. Ser. 2, 196 (1999).
- [GrP1] T. Graber and R. Pandharipande, *Localization of Virtual Classes*, Invent. Math. 135 (1999), no. 2, 487–518.
- [GrP2] T. Graber and R. Pandharipande, personal communication.
- [IP] E. Ionel and T. Parker, seminar talk.
- [K] S. Katz, personal communication.
- [Le] Y. P. Lee, *Quantum Lefschetz Hyperplane Theorem*, Invent. Math. 145 (2001), no. 1, 121–149.
- [LLY] B. Lian, K. Liu, and S.T. Yau, *Mirror Principle I*, Asian J. of Math. 1, No. 4 (1997), 729–763.
- [LT] J. Li and G. Tian, *Virtual Moduli Cycles and Gromov-Witten Invariants of General Symplectic Manifolds*, Topics in Symplectic 4-Manifolds, 47-83, First Int. Press Lect. Ser., I, Internat. Press, 1998.
- [LZ] J. Li and A. Zinger, *On Genus-One Gromov-Witten Invariants of Complete Intersections*, math/0507104.
- [MirSym] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror Symmetry*, Clay Math. Inst., Amer. Math. Soc., August 2003.
- [RT] Y. Ruan and G. Tian, *Higher Genus Symplectic Invariants and Sigma Models Coupled with Gravity*, Invent. Math. 130 (1997), no. 2, 455–516.
- [VaZ] R. Vakil and A. Zinger, *A Desingularization of the Main Component of the Moduli Space of Genus-One Stable Maps into \mathbb{P}^n* , math/0603353.
- [Z1] A. Zinger, *Enumeration of Genus-Two Curves with a Fixed Complex Structure in \mathbb{P}^2 and \mathbb{P}^3* , J. Diff. Geom. 65 (2003), no. 3, 341-467.
- [Z2] A. Zinger, *Enumerative vs. Symplectic Invariants and Obstruction Bundles*, math.SG/0201255, J. Symplectic Geom. 2 (2004), no. 4, 445–543.
- [Z3] A. Zinger, *Counting Rational Curves of Arbitrary Shape in Projective Spaces*, math.AG/0210146, Geom.&Top. 9 (2005), 571-697.
- [Z4] A. Zinger, *A Sharp Compactness Theorem for Genus-One Pseudo-Holomorphic Maps*, math.SG/0406103.
- [Z5] A. Zinger, *On the Structure of Certain Natural Cones over Moduli Spaces of Genus-One Holomorphic Maps*, Adv. Math. 214 (2007), 878–933.
- [Z6] A. Zinger, *Reduced Genus-One Gromov-Witten Invariants*, math.SG/0507103.