On the Structure of Certain Natural Cones over Moduli Spaces of Genus-One Holomorphic Maps

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Abstract

We show that certain naturally arising cones over the main component of a moduli space of J_0-holomorphic maps into $\mathbb{P}^n$ have a well-defined euler class. We also prove that this is the case if the standard complex structure J_0 on $\mathbb{P}^n$ is replaced by a nearby almost complex structure J. The genus-zero analogue of the cone considered in this paper is a vector bundle. The genus-zero Gromov-Witten invariant of a projective complete intersection can be viewed as the euler class of such a vector bundle. As shown in a separate paper, this is also the case for the "genus-one part" of the genus-one GW-invariant. The remaining part is a multiple of the genus-zero GW-invariant.

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1 Introduction

1.1 Motivation

The GW-invariants of symplectic manifolds have been an area of much research in the past decade. These invariants are however often hard to compute.

If Y is a compact Kahler submanifold of the complex projective space $\mathbb{P}^n$, one could try to compute the GW-invariants of Y by relating them to the GW-invariants of $\mathbb{P}^n$. For example, suppose Y is a hypersurface in $\mathbb{P}^n$ of degree a. In other words, if $\gamma \longrightarrow \mathbb{P}^n$ is the tautological line bundle[1] and $\mathfrak{L} = \gamma^{*\otimes a} \longrightarrow \mathbb{P}^n$, then

$$Y = s^{-1}(0),$$

for some $s \in H^0(\mathbb{P}^n; \mathfrak{L})$ such that s is transverse to the zero set. If J_0 is the standard complex structure on $\mathbb{P}^n$ and g, k, and d are nonnegative integers, denote by $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n, d)$ and $\overline{\mathfrak{M}}_{g,k}(Y, d)$ the moduli spaces of stable J_0-holomorphic degree-d maps from genus-g Riemann surfaces with k marked points to $\mathbb{P}^n$ and Y, respectively. These moduli spaces determine the genus-g degree-d GW-invariants of $\mathbb{P}^n$ and Y.

By definition, the moduli space $\overline{\mathfrak{M}}_{g,k}(Y, d)$ is a subset of the moduli space $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n, d)$. In fact,

$$\overline{\mathfrak{M}}_{g,k}(Y, d) = \big\{[\mathcal{C}, u] \in \overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n, d) \colon s \circ u = 0 \in H^0\big(\mathcal{C}; u^*\mathfrak{L}\big)\big\}. \tag{1.1}$$

Here $[\mathcal{C}, u]$ denotes the equivalence class of the holomorphic map $u \colon \mathcal{C} \longrightarrow \mathbb{P}^n$ from a genus-g curve $\mathcal{C}$ with k marked points. The relationship (1.1) can be restated more globally as follows. Suppose $\mathcal{U}$ is the universal curve over $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n, d)$, with structure map π and evaluation map ev:

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{\ \mathrm{ev}\ } & \mathbb{P}^n \\ \big\downarrow{\scriptstyle \pi} & & \\ \overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n, d). & & \end{array}$$

In other words, the fiber of π over $[\mathcal{C}, u]$ is the curve $\mathcal{C}$ with k marked points, while

$$\mathrm{ev}\big([\mathcal{C}, u; z]\big) = u(z) \qquad \text{if} \quad z \in \mathcal{C}.$$[2]

We define a section s of the sheaf $\pi_*\mathrm{ev}^*\mathfrak{L} \longrightarrow \overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n, d)$ by

$$s\big([\mathcal{C}, u]\big) = [s \circ u].$$

By (1.1), $\overline{\mathfrak{M}}_{g,k}(Y, d)$ is the zero set of this section.

The previous paragraph suggests that it should be possible to relate the genus-g degree-d GW-invariants of the hypersurface Y to the moduli space $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n, d)$ in general and to the sheaf

$$\pi_*\mathrm{ev}^*\mathfrak{L} \longrightarrow \overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n, d)$$

[1] the line bundle corresponding to the locally free sheaf $\mathcal{O}_{\mathbb{P}^n}(-1)$

[2] $\mathcal{U}$ can be viewed as $\overline{\mathfrak{M}}_{g,k+1}(\mathbb{P}^n, d)$, in which case ev is the evaluation map ev_{k+1} at the last marked point

in particular. In fact, it can be shown that

$$\mathrm{GW}^Y_{0,k}(d;\psi)\equiv\big\langle\psi,\big[\overline{\mathfrak{M}}_{0,k}(Y,d)\big]^{vir}\big\rangle=\big\langle\psi\cdot e\big(\pi_*\mathrm{ev}^*\mathfrak{L}\big),\big[\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)\big]\big\rangle \tag{1.2}$$

for all $\psi\in H^*(\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d);\mathbb{Q})$. The moduli space $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$ is a smooth orbifold[3] and

$$\pi_*\mathrm{ev}^*\mathfrak{L}\longrightarrow\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d) \tag{1.3}$$

is a locally free sheaf, i.e. a vector bundle.[4] Furthermore,

$$\begin{gathered}\dim_{\mathbb{C}}\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)=d(n+1)+(n-3)+k,\qquad \mathrm{rk}_{\mathbb{C}}\,\pi_*\mathrm{ev}^*\mathfrak{L}=da+1,\\ \text{and}\qquad \dim^{vir}_{\mathbb{C}}\overline{\mathfrak{M}}_{0,k}(Y,d)=d(n+1-a)+(n-1-3)+k.\end{gathered}$$

Thus, the right-hand side of (1.2) is well-defined and vanishes for dimensional reasons precisely when the left-hand side of (1.2) does. In other cases, the right-hand side of (1.2) can be computed via the classical localization theorem of [AB], though the complexity of this computation increases rapidly with the degree d.

If $g>0$, the sheaf $\pi_*\mathrm{ev}^*\mathfrak{L}\longrightarrow\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$ is not locally free and does not define an euler class. Thus, the right-hand side of (1.2) does not even make sense if 0 is replaced by $g>0$. Instead one might try to generalize (1.2) as

$$\begin{aligned}\mathrm{GW}^Y_{g,k}(d;\psi)&\equiv\big\langle\psi,\big[\overline{\mathfrak{M}}_{g,k}(Y,d)\big]^{vir}\big\rangle\\ &\overset{?}{=}\big\langle\psi\cdot e\big(R^0\pi_*\mathrm{ev}^*\mathfrak{L}-R^1\pi_*\mathrm{ev}^*\mathfrak{L}\big),\big[\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)\big]^{vir}\big\rangle,\end{aligned} \tag{1.4}$$

where $R^i\pi_*\mathrm{ev}^*\mathfrak{L}\longrightarrow\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$ is the ith direct image sheaf. The right-hand side of (1.4) can be computed via the virtual localization theorem of [GP1]. However,

$$N_1(d)\equiv\mathrm{GW}^Y_{g,0}(d;1)\neq\big\langle e\big(R^0\pi_*\mathrm{ev}^*\mathfrak{L}-R^1\pi_*\mathrm{ev}^*\mathfrak{L}\big),\big[\overline{\mathfrak{M}}_1(\mathbb{P}^4,d)\big]^{vir}\big\rangle,$$

according to a low-degree check of [GP2] and [K] for the quintic threefold $Y\subset\mathbb{P}^4$.

It turns out that a $g=1$ analogue of the role played by the euler class of sheaf (1.3) is played by the euler class of the sheaf

$$\pi_*\mathrm{ev}^*\mathfrak{L}\longrightarrow\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d), \tag{1.5}$$

where $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ is the primary, algebraically irreducible, component of $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$. In other words, $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ is the closure in $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$, either in the Zariski or stable-map topology[5], of the subspace

$$\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d)=\big\{[\mathcal{C},u]\in\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)\colon\mathcal{C}\text{ is smooth}\big\}.$$

One of the results of this paper is that the euler class of the sheaf (1.5) is in fact well-defined.

[3]It is a smooth algebraic stack by [FlP].

[4]Strictly speaking, $\pi_*\mathrm{ev}^*\mathfrak{L}\longrightarrow\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$ is the orbi-sheaf of holomorphic multisections of a vector orbi-bundle. We occasionally drop "orbi" to streamline the presentation. The reader is referred to Sections 2-4 for a detailed discussion of the orbifold category.

[5]also known as Gromov's convergence topology

Theorem 1.1 *If n, d, and a are positive integers, k is a nonnegative integer, $\mathfrak{L}=\gamma^{*\otimes a}\longrightarrow\mathbb{P}^n$,*

$$\pi\colon\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$$

is the universal curve, and

$$\mathrm{ev}\colon\mathfrak{U}\longrightarrow\mathbb{P}^n$$

is the natural evaluation map, the sheaf

$$\pi_*\mathrm{ev}^*\mathfrak{L}\longrightarrow\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$$

determines a homology class and a cohomology class on $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$:

$$\mathrm{PD}_{\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)}\big(e(\pi_*\mathrm{ev}^*\mathfrak{L})\big)\in H_{2(d(n+1-a)+k)}\big(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d);\mathbb{Q}\big)$$
$$\textit{and}\qquad e(\pi_*\mathrm{ev}^*\mathfrak{L})\in H^{2da}\big(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d);\mathbb{Q}\big).$$

Remark 1: If $a_1,\ldots,a_m\in\mathbb{Z}^+$ and

$$\mathfrak{L}=\gamma^{*\otimes a_1}\oplus\ldots\oplus\gamma^{*\otimes a_m}\longrightarrow\mathbb{P}^n,$$

then the sheaf $\pi_*\mathrm{ev}^*\mathfrak{L}$ is the direct sum of the sheaves corresponding to the line bundles $\gamma^{*\otimes a_l}$. Thus, Theorem 1.1 applies to any split vector bundle over $\mathbb{P}^n$.

Remark 2: The moduli space $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ is an orbivariety, which *not* smooth if $d\geq 3$ and $n\geq 1$. Thus, the Poincare dual of a cohomology element on $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ may not exist. As explained in the next subsection, we will define a homology element, which will be called $\mathrm{PD}_{\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)}\big(e(\pi_*\mathrm{ev}^*\mathfrak{L})\big)$, first and then use it to construct a cohomology element, which we call $e(\pi_*\mathrm{ev}^*\mathfrak{L})$.

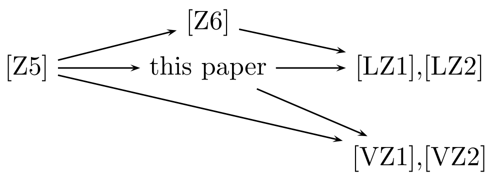
Remark 3: In the genus-zero case, the space of maps from smooth domains is dense in the entire moduli space (i.e. the domain of a generic element of $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$ is $\mathbb{P}^1$). Thus, if $\overline{\mathfrak{M}}_{0,k}^0(\mathbb{P}^n,d)$ is defined analogously to $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$, then

$$\overline{\mathfrak{M}}_{0,k}^0(\mathbb{P}^n,d)=\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$$

and the equality (1.2) remains valid if we replace $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$ with $\overline{\mathfrak{M}}_{0,k}^0(\mathbb{P}^n,d)$. By Theorem (1.1), the analogue of the right-hand side of (1.2) makes sense in the genus-one case for $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$, but not for

$$\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)\supsetneq\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d).$$

This paper is continued in [LZ1] to show that the resulting right-hand side computes the *reduced* genus-one invariants of Y defined in [Z6]. Since these invariants differ from the standard genus-one invariants by a combination of genus-zero invariants, Theorem 1.1 of [LZ1] and Theorem 1.1 above, along with the original equation and [AB], open a way for computing the (standard) Gromov-Witten invariants of complete intersections.



[Z5]: Refined Compactness Theorem
[Z6]: Reduced Genus-1 Invariants
[LZ1],[LZ2]: Genus-1 GWs of CIs
[VZ1],[VZ2]: Desingularization of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ and (1.5)

Figure 1: Connections between Papers

One way to view the statement of Theorem 1.1 is that the sheaf (1.5) admits a desingularization, and the euler class of every desingularization of (1.5) is the same, in the appropriate sense. This is not the point of view taken in this paper. However, one approach to computing the number

$$\big\langle \psi, \mathrm{PD}_{\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)}\big(e(\pi_*\mathrm{ev}^*\mathfrak{L})\big)\big\rangle = \big\langle \psi\cdot e\big(\pi_*\mathrm{ev}^*\mathfrak{L}\big), \big[\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)\big]\big\rangle \tag{1.6}$$

for a natural cohomology class $\psi\in H^*(\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d);\mathbb{Q})$ is to apply the localization theorem of [AB] to a desingularization of (1.5). In [VZ1] (outlined in [VZ2]), we construct a desingularization of the space $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$, i.e. a smooth orbivariety $\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ and a map

$$\tilde{\pi}\colon \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d),$$

which is biholomorphic onto $\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d)$. This desingularization of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ comes with a desingularization of the sheaf (1.5), i.e. a vector bundle

$$\tilde{\mathcal{V}}^d_{1,k} \longrightarrow \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \qquad \text{s.t.} \qquad \tilde{\pi}_*\tilde{\mathcal{V}}^d_{1,k} = \pi_*\mathrm{ev}^*\mathfrak{L}.$$

In particular,

$$\big\langle \psi\cdot e\big(\pi_*\mathrm{ev}^*\mathfrak{L}\big), \big[\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)\big]\big\rangle = \big\langle \tilde{\pi}^*\psi\cdot e(\tilde{\mathcal{V}}^d_{1,k}), \big[\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)\big]\big\rangle. \tag{1.7}$$

Since a group action on $\mathbb{P}^n$ induces actions on $\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ and on $\tilde{\mathcal{V}}^d_{1,k}$, the localization theorem of [AB] is directly applicable to the right-hand side of (1.7), for a natural cohomology class ψ.

Before the results of [LZ1] were announced, no positive-genus analogue of (1.2) had been even conjectured. On the other hand, Theorem 1.1 suggests a natural genus-one analogue of (1.2), which is proved in [LZ1], and a conjectural extension of (1.2) to higher genera, which is stated in [LZ1].

Theorem 1.1 is the $J=J_0$ case of Theorem 1.2, which is stated in Subsection 1.3. In the next subsection, we describe the main topological arguments that lie behind the proof of Theorems 1.1 and 1.2.

Remark: This paper is part of a series that studies limiting properties of pseudoholomorphic maps from genus-one Riemann surfaces and their applications in the Gromov-Witten theory and enumerative geometry. The primary relations between the papers in the series so far are illustrated in Figure 1.

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1.2 General Approach

In this paper we approach Theorem 1.1 from the point of view of differential topology, rather than of algebraic geometry. As a motivation, we recall the following standard fact. Suppose $\overline{\mathfrak{M}}$ is a compact oriented manifold of dimension m and $\mathcal{V}\longrightarrow\overline{\mathfrak{M}}$ is a complex vector bundle of rank k. If φ is a smooth section of $\overline{\mathfrak{M}}$ which is transverse to the zero set, then $\varphi^{-1}(0)$ is a smooth oriented submanifold of $\overline{\mathfrak{M}}$ and the homology class it determines in M is Poincare dual to the euler class of V:

$$\big[\varphi^{-1}(0)\big] = \mathrm{PD}_{\overline{\mathfrak{M}}}\big(e(\mathcal{V})\big) \in H_{m-2k}\big(\overline{\mathfrak{M}};\mathbb{Z}\big). \tag{1.8}$$

In the orbifold category, this identity holds with rational coefficients for any transverse multisection φ.[6] We will define $\mathrm{PD}_{\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)}\big(e(\pi_*\mathrm{ev}^*\mathfrak{L})\big)$ by using equation (1.8) in the opposite direction.

There are two complications here. First, $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ is not a smooth manifold. However, it is stratified by smooth oriented orbifolds of even dimensions; see Subsection 2.3. For such spaces, pseudocycles provide a convenient replacement for the usual singular homology.[7] An m-`pseudocycle`

$$f\colon M \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$$

is a continuous map from a compact topological space M which is stratified by smooth orbifolds such that

(PS1) the main stratum M^0 of M is an oriented orbifold of real dimension m,

(PS2) the complement of M^0 in M is a union of orbifolds of real dimension of at most $m-2$, and

(PS3) the restriction of f to each stratum of M is smooth.

In particular, every stratum of M is mapped into a stratum of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$. Each m-pseudocycle determines an mth homology class and vice versa.[8]

The second complication concerns the sheaf (1.5). It is *not* locally free if $d\ge 3$ and does not correspond to a vector bundle over $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$. Instead it is the orbi-sheaf of sections of the (orbi-)cone

$$\pi\colon \mathcal{V}^d_{1,k}\equiv\overline{\mathfrak{M}}^0_{1,k}(\mathfrak{L},d) \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d), \qquad \pi\big([\mathcal{C},\xi]\big) = p\circ\xi, \tag{1.9}$$

where $p\colon\mathfrak{L}\longrightarrow\mathbb{P}^n$ is the bundle projection map. Every fiber of π is a vector space, up to a quotient by a finite group, and the vector space operations are continuous. More precisely,

$$\mathcal{V}^d_{1,k}\big|_{[\mathcal{C},u]}\equiv\pi^{-1}\big([\mathcal{C},u]\big) = H^0(\mathcal{C};u^*\mathfrak{L})/\mathrm{Aut}(\mathcal{C},u).$$

The fibers of π do not have constant rank. The restriction of $\mathcal{V}^d_{1,k}$ to the dense open subspace $\mathfrak{M}^{\mathrm{eff}}_{1,k}(\mathbb{P}^n,d)$ of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ consisting of stable maps u that are effective (not constant) on the

[6] Our notion of multisection agrees with the one commonly used in symplectic topology and corresponds to the notion of locally liftable multisection of Section 3 in [FkO].

[7] See the beginning of Chapter 7 in [MS] for an overview of pseudocycle constructions in the basic manifold case and [Z7] for a more thorough treatment.

[8] In [Z7] this statement is proved for smooth manifolds. However, the proof goes through for any space $\overline{\mathfrak{M}}$ as long as the conclusion of Proposition 2.2 in [Z7] remains valid, i.e. the image of every smooth map from a smooth m-manifold has an arbitrarily small neighborhood U with $H_l(U)=0$ for all $l>k$. Lemmas 2.3 and 2.4 imply that $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ satisfies this property.

principal (genus-carrying) component(s) of the domain of u is indeed a vector bundle and of the expected rank, i.e. da; see Lemma 3.2. However, the rank of $\mathcal{V}_{1,k}^d$ jumps to $da+1$ over

$$\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d) - \mathfrak{M}_{1,k}^{\mathrm{eff}}(\mathbb{P}^n,d);$$

see Subsection 3.3.

While the cone (1.9) is not a vector bundle, it turns out to be not too degenerate. In particular, we will show that it admits a continuous multisection φ such that

($\mathcal{V}1$) $\varphi|_{\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)}$ is smooth and transverse to the zero set in $\mathcal{V}_{1,k}^d|_{\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)}$, and

($\mathcal{V}2$) the intersection of $\varphi^{-1}(0)$ with a boundary stratum of $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ is a smooth suborbifold of the stratum of real dimension of at most $2(d(n+1-a)+k)-2$.

These two properties imply that $\varphi^{-1}(0)$ is a pseudocycle in $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ under the inclusion map and thus determines an element

$$[\varphi_0^{-1}(0)] \in H_{2d(n+1-a)+2k}(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d);\mathbb{Q}).$$

We will also show that for any two continuous sections φ_0 and φ_1 of (1.9) satisfying ($\mathcal{V}1$) and ($\mathcal{V}2$), there exists a continuous homotopy

$$\Phi\colon [0,1]\times\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d) \longrightarrow \mathcal{V}_{1,k}^d$$

such that $\Phi|_{\{t\}\times\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)}=\varphi_t$ for $t=0,1$,

($\mathcal{V}1'$) $\Phi|_{[0,1]\times\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)}$ is smooth and transverse to the zero set in $[0,1]\times\mathcal{V}_{1,k}^d|_{\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)}$, and

($\mathcal{V}2'$) the intersection of $\Phi^{-1}(0)$ with a stratum of $[0,1]\times\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ is a smooth suborbifold of the stratum of real dimension of at most $2(d(n+1-a)+k)-1$.

The existence of such a homotopy is called implies that

$$[\varphi_0^{-1}(0)] = [\varphi_1^{-1}(0)] \in H_{2(d(n+1-a)+k)}(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d);\mathbb{Q}).^9$$

We call this homology class *the Poincare dual of the euler class* of the cone (1.9) and of the sheaf (1.5).

If $X\in H_{2da}(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d);\mathbb{Q})$, let

$$f_X\colon M_X \longrightarrow \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$$

be a pseudocycle representing X. If φ is a section of (1.9) satisfying ($\mathcal{V}1$) and ($\mathcal{V}2$), we can also require that

($\varphi X1$) $f_X(M_X)\cap\varphi^{-1}(0)\subset\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)$, $f_X^{-1}\big(\varphi^{-1}(0)\big)\subset M_X^0$;

($\varphi X2$) $f_X|_{M_X^0}$ intersects $\varphi^{-1}(0)$ transversally in $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)$.

These assumptions imply that $\varphi^{-1}(0)\cap f_X(M_X^0)$ is a compact oriented zero-dimensional suborbifold of $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)$. We then set

$$\big\langle e(\pi_*\mathrm{ev}^*\mathfrak{L}),X\big\rangle = {}^{\pm}\big|\varphi^{-1}(0)\cap f_X(M_X^0)\big|, \tag{1.10}$$

[9] The projection map $\Phi^{-1}(0)\longrightarrow\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ is a pseudocycle equivalence from $\varphi_0^{-1}(0)$ to $\varphi_1^{-1}(0)$.

where ${}^{\pm}|\mathcal{Z}|$ denotes the cardinality of a compact oriented zero-dimensional orbifold $\mathcal{Z}$, i.e. the number of elements in the finite set $\mathcal{Z}$ counted with the appropriate multiplicities.

If $f_{X,0}\colon M_{X,0}\longrightarrow\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ and $f_{X,1}\colon M_{X,1}\longrightarrow\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ are two pseudocycles satisfying $(\varphi X1)$ and $(\varphi X2)$, we can choose a pseudocycle equivalence

$$F\colon \tilde{M}\longrightarrow\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$$

between $f_{X,0}$ and $f_{X,1}$ such that

$(\varphi X1')$ $F(\tilde{M})\cap\varphi^{-1}(0)\subset\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d)$, $F^{-1}\big(\varphi^{-1}(0)\big)\subset\tilde{M}^0$;

$(\varphi X2')$ $F|_{\tilde{M}^0}$ intersects $\varphi^{-1}(0)$ transversally in $\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d)$.

These two assumptions imply that $\varphi^{-1}(0)\cap F(\tilde{M}^0)$ is a compact oriented one-dimensional suborbifold of $\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d)$ and

$$\begin{aligned}\partial\big(\varphi^{-1}(0)\cap F(\tilde{M}^0)\big)&=\varphi^{-1}(0)\cap f_{X,1}(M^0_{X,1})-\varphi^{-1}(0)\cap f_{X,0}(M^0_{X,0})\\ \Longrightarrow\qquad {}^{\pm}\big|\varphi^{-1}(0)\cap f_{X,0}(M^0_{X,0})\big|&={}^{\pm}\big|\varphi^{-1}(0)\cap f_{X,1}(M^0_{X,1})\big|.\end{aligned}$$

Thus, the number in (1.10) is independent of the choice of pseudocycle representative f_X for X satisfying $(\varphi X1)$ and $(\varphi X2)$.

Similarly, if φ_0 and φ_1 are two multisections satisfying $(\mathcal{V}1)$ and $(\mathcal{V}2)$, let Φ be a homotopy between φ_0 and φ_1 satisfying $(\mathcal{V}1')$ and $(\mathcal{V}2')$. We can then choose a pseudocycle representative

$$f_X\colon M_X\longrightarrow\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$$

for X such that

$(\Phi X1)$ $f_X(M_X)\cap\Phi^{-1}(0)\subset\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d)$, $f_X^{-1}\big(\Phi^{-1}(0)\big)\subset M^0_X$;

$(\Phi X2)$ $f_X|_{M^0_X}$ intersects $\Phi^{-1}(0)$ transversally in $\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d)$,

and f_X satisfies $(\varphi X2)$ with $\varphi=\varphi_0$ and $\varphi=\varphi_1$. These assumptions imply that $\Phi^{-1}(0)\cap f_X(M^0_X)$ is a compact oriented one-dimensional suborbifold of $\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d)$ and

$$\begin{aligned}\partial\big(\Phi^{-1}(0)\cap f_X(M^0_X)\big)&=\varphi_1^{-1}(0)\cap f_X(M^0_X)-\varphi_0^{-1}(0)\cap f_X(M^0_X)\\ \Longrightarrow\qquad {}^{\pm}\big|\varphi_0^{-1}(0)\cap f_X(M^0_X)\big|&={}^{\pm}\big|\varphi_1^{-1}(0)\cap f_X(M^0_X)\big|.\end{aligned}$$

Thus, the number in (1.10) is independent of the choice of section φ satisfying $(\mathcal{V}1)$ and $(\mathcal{V}2)$. We conclude that (1.10) defines an element of

$$\mathrm{Hom}\big(H_{2da}(\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d);\mathbb{Q});\mathbb{Q}\big)=H^{2da}(\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d);\mathbb{Q}).$$

We call this cohomology class `the euler class` of the cone (1.9) and of the sheaf (1.5).

We note that the existence of a continuous section φ of (1.9) satisfying $(\mathcal{V}1)$ and $(\mathcal{V}2)$ implies that the euler class of every desingularization of (1.9), or of (1.5), is the same, in the appropriate sense, for the following reason. If

$$\begin{array}{ccc}\tilde{\mathcal{V}}^d_{1,k} & \xrightarrow{\ \tilde{\pi}_*\ } & \mathcal{V}^d_{1,k}\\ \big\downarrow & & \big\downarrow\\ \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) & \xrightarrow{\ \tilde{\pi}\ } & \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)\end{array}\qquad(1.11)$$

is a desingularization of the cone (1.9), or of the sheaf (1.5), the section φ induces a section $\tilde{\varphi}$ of the vector bundle

$$\tilde{\mathcal{V}}_{1,k}^d \longrightarrow \widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$$

such that $\tilde{\varphi}=\varphi$ on $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)$ and $\tilde{\varphi}^{-1}(0)-\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)$ is a finite union of smooth orbifolds of real dimension of at most $2(d(n+1-a)+k)-2$. Suppose $X\in H_{2da}(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d);\mathbb{Q})$ is represented by a pseudocycle

$$f_X\colon M_X \longrightarrow \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d),$$

and

$$\begin{aligned}\psi_X \equiv \mathrm{PD}_{\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)}X \in & H^{2(d(n+1-a)+k)}(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d);\mathbb{Q})\\ &= \mathrm{Hom}\big(H_{2(d(n+1-a)+k)}(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d);\mathbb{Q});\mathbb{Q}\big)\end{aligned}$$

is the `Poincare dual of` X, i.e. the element constructed by intersecting $2(d(n+1-a)+k)$-pseudocycles with $f_X(M_X)$. The Poincare dual of the cohomology class $\tilde{\pi}^*\psi_X$ in $\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ can then be represented by a pseudocycle

$$\begin{aligned} f_{\tilde{X}}\colon M_{\tilde{X}} &\longrightarrow \widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d) \qquad \text{s.t.}\\ M_X^0 \subset M_{\tilde{X}}^0, &\quad f_{\tilde{X}}(M_{\tilde{X}}-M_X^0)\subset \tilde{\pi}^{-1}\big(f_X(M_X-M_X^0)\big),\\ \text{and} \qquad f_X\big|_{M_X^0} = f_{\tilde{X}}\big|_{M_X^0} &: M_X^0 \longrightarrow \mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)\subset \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d), \widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d).\end{aligned}$$

Our assumptions on φ and f_X then imply that all intersections of $f_{\tilde{X}}(M_{\tilde{X}})$ with $\tilde{\varphi}^{-1}(0)$ are contained in $f_{\tilde{X}}(M_X^0)\cap\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)$, are transverse, and correspond to the intersections of $f_X(M_X)$ with $\varphi^{-1}(0)$. Thus,

$$\begin{aligned}\big\langle \tilde{\pi}^*\psi_X\cdot e(\tilde{\mathcal{V}}_{1,k}^d), \big[\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)\big]\big\rangle &= {}^{\pm}\big|\tilde{\varphi}^{-1}(0)\cap f_{\tilde{X}}(M_{\tilde{X}})\big|\\ &= {}^{\pm}\big|\varphi^{-1}(0)\cap f_X(M_X)\big|.\end{aligned} \tag{1.12}$$

In particular, the left-hand side of (1.12) depends only on the homology class X used in constructing the cohomology class ψ_X and is independent of the desingularization (1.11).

The above argument also shows that if the cone (1.9) admits a multisection φ satisfying $(\mathcal{V}1)$ and $(\mathcal{V}2)$ and admits a desingularization as in (1.11), then the number

$$\big\langle e(\mathcal{V}_{1,k}^d), X\big\rangle \equiv \big\langle \tilde{\pi}^*\psi_X\cdot e(\tilde{\mathcal{V}}_{1,k}^d), \big[\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)\big]\big\rangle$$

is well-defined for every homology class X on $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$. Thus, the euler class $e(\mathcal{V}_{1,k}^d)$ of the cone (1.9) and the sheaf (1.5) is also well-defined. In particular, the existence of homotopies satisfying $(\mathcal{V}1')$ and $(\mathcal{V}2')$ is not absolutely necessary for showing that the euler class of (1.9) is well-defined.

The construction of a section φ satisfying $(\mathcal{V}1)$ and $(\mathcal{V}2)$ is the subject of Section 3. Since

$$\mathcal{V}_{1,k}^d \longrightarrow \mathfrak{M}_{1,k}^{\mathrm{eff}}(\mathbb{P}^n,d)$$

is a vector bundle, it is simple to construct a section φ over $\mathfrak{M}^{\mathrm{eff}}_{1,k}(\mathbb{P}^n,d)$ that satisfies $(\mathcal{V}1)$ and $(\mathcal{V}2)$ for the strata of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ that are contained in $\mathfrak{M}^{\mathrm{eff}}_{1,k}(\mathbb{P}^n,d)$. Thus, the key is to show that such a section can be constructed over a neighborhood of

$$\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)-\mathfrak{M}^{\mathrm{eff}}_{1,k}(\mathbb{P}^n,d).$$

In order to do this, we have to describe the structure of the cone $\mathcal{V}^d_{1,k}$ on a neighborhood of each stratum $\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^n;J_0)$ contained in the complement of $\mathfrak{M}^{\mathrm{eff}}_{1,k}(\mathbb{P}^n,d)$. For each such stratum $\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^n;J_0)$, there is a vector subbundle (not a cone)

$$\mathcal{V}^{d;m}_{1,k;\mathcal{T}}\longrightarrow\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^n;J_0)$$

spanned by the sections of $\mathcal{V}^d_{1,k}$ over $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$, i.e.

$$\mathcal{V}^{d;m}_{1,k;\mathcal{T}}=\big\{s(b)\colon b\in\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^n;J_0);\, s\in\Gamma(\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d);\mathcal{V}^d_{1,k})\big\}.$$ [10]

The subbundles $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}$ of $\mathcal{V}^d_{1,k}$ are described explicitly by Lemma 3.4. It turns out that the corank of $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}$ is sufficiently small relatively to the codimension of $\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^n;J_0)$ so that a generic section of $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}$ satisfies $(\mathcal{V}2)$; see (3.3). By Proposition 3.3, the bundles $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}$ over the various strata $\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^n;J_0)$ match up sufficiently well so that one can build a section of $\mathcal{V}^d_{1,k}$ over $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ by extending generic sections of $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}$ over $\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^n;J_0)$ starting from lowest strata. This construction is carried out in Subsection 3.1.

In the next subsection we give a more analytic description of the cone $\mathcal{V}^d_{1,k}$ and extend Theorem 1.1 to deformations of the standard complex structure J_0 on $\mathbb{P}^n$. We introduce the notation needed to describe the strata $\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^n;J_0)$ and the bundles $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}$ accurately in Subsections 2.1 and 2.2. As $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ and $\mathcal{V}^d_{1,k}$ are singular along $\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^n;J_0)$, this notation is unfortunately rather involved. The structure of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$, including the strata $\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^n;J_0)$ and their neighborhoods, is described in Subsection 2.3. The structure of $V^d_{1,k}$ along the strata of $\mathfrak{M}^{\mathrm{eff}}_{1,k}(\mathbb{P}^n,d)$ and the strata $\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^n;J_0)$ is described in Subsections 3.2 and 3.4, respectively. The technical portion of the analysis needed to justify parts of Proposition 3.3 and Lemma 3.4 has been relegated to Section 4. The construction in Section 4 is the lifting of the gluing construction of Section 6 in [Z5] for stable maps into $\mathbb{P}^n$ to bundle sections, i.e. maps into $\mathfrak{L}$. To a certain extent, it can be viewed as the construction of [Z5] applied to the complex manifold $\mathfrak{L}$. However, some care has to be exercised so that the lifting of the gluing procedure for maps into $\mathbb{P}^n$ to maps into $\mathfrak{L}$ is $\mathbb{C}$-linear on the fibers.

1.3 Main Theorem

While the standard complex structure J_0 on $\mathbb{P}^n$ is ideal for many purposes, such as computing obstruction bundles in the Gromov-Witten theory and applying the localization theorems of [AB] and [GP1], it is sometimes more convenient to work with an almost complex structure J on $\mathbb{P}^n$

[10] In Sections 3 and 4, $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}$ denotes an extension the bundle just defined to a neighborhood of $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}$.

obtained by perturbing J_0.[11] For this reason, we generalize Theorem 1.1 to almost complex structures J that are close to J_0.

We denote by $\mathfrak{X}_{g,k}(\mathbb{P}^n,d)$ the space of equivalence classes of stable degree-d smooth maps from genus-g Riemann surfaces with k marked points to $\mathbb{P}^n$. Let $\mathfrak{X}^0_{g,k}(\mathbb{P}^n,d)$ be the subset of $\mathfrak{X}_{g,k}(\mathbb{P}^n,d)$ consisting of stable maps with smooth domains. The spaces $\mathfrak{X}_{g,k}(\mathbb{P}^n,d)$ are topologized using L^p_1-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes; see Section 3 in [LT] for more details. Here and throughout the rest of the paper, p denotes a real number greater than two. The spaces $\mathfrak{X}_{g,k}(\mathbb{P}^n,d)$ are stratified by the smooth infinite-dimensional orbifolds $\mathfrak{X}_{\mathcal{T}}(\mathbb{P}^n)$ of stable maps from domains of the same geometric type and with the same degree distribution between the components. The closure of the main stratum, $\mathfrak{X}^0_{g,k}(\mathbb{P}^n,d)$, is $\mathfrak{X}_{g,k}(\mathbb{P}^n,d)$.

Using modified Sobolev norms, [LT] also defines a cone $\Gamma_{g,k}(T\mathbb{P}^n,d)\longrightarrow\mathfrak{X}_{g,k}(\mathbb{P}^n,d)$ such that the fiber of $\Gamma_{g,k}(T\mathbb{P}^n,d)$ over a point $[b]=[\Sigma,j;u]$ in $\mathfrak{X}_{g,k}(\mathbb{P}^n,d)$ is the Banach space

$$\Gamma_{g,k}(T\mathbb{P}^n,d)\big|_b=\Gamma(b;T\mathbb{P}^n)\big/\mathrm{Aut}(b),\qquad\text{where}\qquad\Gamma(b;T\mathbb{P}^n)=L^p_1(\Sigma;u^*T\mathbb{P}^n).$$

The topology on $\Gamma_{g,k}(T\mathbb{P}^n,d)$ is defined similarly to the convergence topology on $\mathfrak{X}_{g,k}(\mathbb{P}^n,d)$. If $\mathfrak{L}$ is the line bundle $\gamma^{*\otimes a}\longrightarrow\mathbb{P}^n$, let $\Gamma_{g,k}(\mathfrak{L},d)\longrightarrow\mathfrak{X}_{g,k}(\mathbb{P}^n,d)$ be the cone such that the fiber of $\Gamma_{g,k}(\mathfrak{L},d)$ over $[b]=[\Sigma,j;u]$ in $\mathfrak{X}_{g,k}(\mathbb{P}^n,d)$ is the Banach space

$$\Gamma_{g,k}(\mathfrak{L},d)\big|_b=\Gamma(b;\mathfrak{L})\big/\mathrm{Aut}(b),\qquad\text{where}\qquad\Gamma(b;\mathfrak{L})=L^p_1(\Sigma;u^*\mathfrak{L}),$$

and the topology on $\Gamma_{g,k}(\mathfrak{L},d)$ is defined analogously to the topology on $\Gamma_{g,k}(\mathbb{P}^n,d)$.

Let ∇ denote the hermitian connection in the line bundle $\mathfrak{L}\longrightarrow\mathbb{P}^n$ induced from the standard connection on the tautological line bundle over $\mathbb{P}^n$. If (Σ,j) is a Riemann surface and $u\colon\Sigma\longrightarrow\mathbb{P}^n$ is a smooth map, let

$$\nabla^u\colon\Gamma(\Sigma;u^*\mathfrak{L})\longrightarrow\Gamma\big(\Sigma;T^*\Sigma\otimes u^*\mathfrak{L}\big)$$

be the pull-back of ∇ by u. If $b=(\Sigma,j;u)$, we define the corresponding $\bar\partial$-operator by

$$\bar\partial_{\nabla,b}\colon\Gamma(\Sigma;u^*\mathfrak{L})\longrightarrow\Gamma\big(\Sigma;\Lambda^{0,1}_{\mathfrak{i},j}T^*\Sigma\otimes u^*\mathfrak{L}\big),\quad\bar\partial_{\nabla,b}\xi=\frac{1}{2}\big(\nabla^u\xi+\mathfrak{i}\nabla^u\xi\circ j\big),\tag{1.13}$$

where $\mathfrak{i}$ is the complex multiplication in the bundle $u^*\mathfrak{L}$ and

$$\Lambda^{0,1}_{\mathfrak{i},j}T^*\Sigma\otimes u^*\mathfrak{L}=\big\{\eta\in\mathrm{Hom}(T\Sigma,u^*\mathfrak{L})\colon\eta\circ j=-\mathfrak{i}\eta\big\}.$$

The kernel of $\bar\partial_{\nabla,b}$ is necessarily a finite-dimensional complex vector space. If $u:\Sigma\longrightarrow\mathbb{P}^n$ is a (J_0,j)-holomorphic map, then

$$\ker\bar\partial_{\nabla,b}=H^0\big((\Sigma,j);u^*\mathfrak{L}\big)$$

is the space of holomorphic sections of the line bundle $u^*\mathfrak{L}\longrightarrow(\Sigma,j)$. Let

$$\mathcal{V}^d_{g,k}=\big\{[b,\xi]\in\Gamma_{g,k}(\mathfrak{L},d)\colon[b]\in\mathfrak{X}_{g,k}(\mathbb{P}^n,d),\ \xi\in\ker\bar\partial_{\nabla,b}\subset\Gamma_{g,k}(b;\mathfrak{L})\big\}\subset\Gamma_{g,k}(\mathfrak{L},d).$$

[11] A perturbed almost complex structure may possess certain regularity properties that J_0 does not have; see [LZ2], for example.

The cone $\mathcal{V}_{g,k}^d \longrightarrow \mathfrak{X}_{g,k}(\mathbb{P}^n,d)$ inherits its topology from $\Gamma_{g,k}(\mathfrak{L},d)$.

If J is an almost complex structure on $\mathbb{P}^n$, let $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d;J)$ denote the moduli spaces of stable J-holomorphic degree-d maps from genus-g Riemann surfaces with k marked points to $\mathbb{P}^n$. Let

$$\mathfrak{M}_{g,k}^0(\mathbb{P}^n,d;J) = \big\{[\mathcal{C},u]\in\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d;J) \colon \mathcal{C} \text{ is smooth}\big\}.$$

We denote by $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ the closed subset of $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d;J)$ containing $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)$ defined in Subsection 1.2 of [Z5]. If J is sufficiently close to J_0, $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ is the closure of $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)$ in $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d;J)$.[12] We describe the structure of $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ in this case in Lemma 2.4 below. Finally, let $\bar{\mathbb{Z}}^+$ denote the set of nonnegative integers.

Remark: The spaces $\mathfrak{X}_{g,k}$ considered in [LT] consist of L_1^p-maps. In our case, it is sufficient to restrict to the subspace consisting of smooth maps (which we call $\mathfrak{X}_{g,k}(\mathbb{P}^n,d)$) as the *base* of the bundle $\Gamma_{g,k}(\mathfrak{L},d)$. However, for the purposes of the analysis of Section 4, we have to consider L_1^p-spaces of bundle sections of $\mathfrak{L}$ as the *fibers* of $\Gamma_{g,k}(\mathfrak{L},d)$. On the other hand, the entire infinite-dimensional setting for the base is not necessary for the purposes of this paper and is introduced primarily for convenience, while the topology on the total space of $\Gamma_{g,k}(\mathfrak{L},d)$ defined in [LT] is not necessary for the statement of Theorem 1.2, Propositions 3.1 or 3.3, or Lemmas 3.2 or 3.4. The only bases we work with are $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ and $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;\underline{J})$, where J is an almost complex structure close to J_0 and $\underline{J}$ is a smooth one-dimensional family of such structures. Furthermore, in the topology of [LT], the cone

$$\mathcal{V}_{1,k}^d \longrightarrow \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$$

is simply the preimage of $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ under the projection map

$$\overline{\mathfrak{M}}_{1,k}^0(\mathfrak{L},d;\tilde{J}) \longrightarrow \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d;J), \qquad [\mathcal{C},\tilde{u}] \longrightarrow [\mathcal{C},\pi\circ\tilde{u}],$$

where $\pi\colon \mathfrak{L}\longrightarrow\mathbb{P}^n$ is the bundle projection map and $\tilde{J}$ is the lift of J to $\mathfrak{L}$ via the connection ∇.

Theorem 1.2 *If $n,d,a\in\mathbb{Z}^+$ and $k\in\bar{\mathbb{Z}}^+$, there exists $\delta_n(d,a)\in\mathbb{R}^+$ with the following property. If J is an almost complex structure on $\mathbb{P}^n$ such that*

$$\|J-J_0\|_{C^1} < \delta_n(d,a),$$

the moduli space $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ carries a fundamental class

$$\big[\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)\big] \in H_{2(d(n+1)+k)}\big(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J);\mathbb{Q}\big).$$

Furthermore, the cone $\mathcal{V}_{1,k}^d\longrightarrow\mathfrak{X}_{1,k}(\mathbb{P}^n,d)$ corresponding to the line bundle $\mathfrak{L}=\gamma^{*\otimes a}\longrightarrow\mathbb{P}^n$ determines a homology class and a cohomology class on $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$:

$$PD_{\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)}\big(e(\mathcal{V}_{1,k}^d)\big) \in H_{2(d(n+1-a)+k)}\big(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J);\mathbb{Q}\big)$$
$$\text{and} \qquad e(\mathcal{V}_{1,k}^d) \in H^{2da}\big(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J);\mathbb{Q}\big).$$

[12] Since in this paper we work only with almost complex structures J sufficiently close to J_0, $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ can be taken to be the closure of $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)$ in $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d;J)$ by definition.

Finally, if $\mathcal{W}\longrightarrow\mathfrak{X}_{1,k}(\mathbb{P}^n,d)$ is a vector orbi-bundle such that the restriction of $\mathcal{W}$ to each stratum $\mathfrak{X}_{\mathcal{T}}(\mathbb{P}^n)$ of $\mathfrak{X}_{1,k}(\mathbb{P}^n)$ is smooth, then

$$\big\langle e(\mathcal{W})\cdot e(\mathcal{V}_{1,k}^d),\big[\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)\big]\big\rangle=\big\langle e(\mathcal{W})\cdot e(\mathcal{V}_{1,k}^d),\big[\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)\big]\big\rangle. \tag{1.14}$$

Remark: This theorem remains valid if the compact Kahler manifold $(\mathbb{P}^n,\omega_0,J_0)$, positive integer d, the holomorphic line bundle $\mathfrak{L}=\gamma^{*\otimes a}\longrightarrow\mathbb{P}^n$, and the connection ∇ in $\mathfrak{L}$ are replaced by a compact almost Kahler manifold (X,ω,J_0), a homology class $A\in H_2(X;\mathbb{Z})$, and a split positive vector bundle with connection $(\mathfrak{L},\nabla)\longrightarrow X$ such that the almost complex structure J_0 on X is genus-one A-regular in the sense of Definition 1.4 in [Z5].

It is well-known that the standard complex structure is genus-one $d\ell$-regular, where $\ell\in H_2(\mathbb{P}^n;\mathbb{Z})$ is the homology class of a line. Thus, if J is an almost complex structure on $\mathbb{P}^n$ which is close to J_0, Corollary 1.5 and Theorem 1.6 in [Z5] imply that $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ is the closure of $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)$ in $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ and is contained in a small neighborhood of $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ in $\mathfrak{X}_{1,k}(\mathbb{P}^n,d)$. In addition, the stratification structure of the moduli space $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ is the same as that of $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$; see Lemmas 2.3 and 2.4 below. Thus, $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ carries a rational fundamental class; see the paragraph at the end of Subsection 2.3 in [Z5].

The two remaining claims of Theorem 1.2 are the subject of Proposition 3.1. The restriction of the cone $\mathcal{V}_{1,k}^d$ to $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)$ is a complex vector bundle of the expected rank, i.e. da. The cone $\mathcal{V}_{1,k}^d$ admits a multisection φ that satisfies the analogues of $(\mathcal{V}1)$ and $(\mathcal{V}2)$ for $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$. As in the previous subsection, the zero set of this section determines a homology class in real codimension $2da$ and a cohomology class of real dimension $2da$. On the other hand, if $\underline{J}=(J_t)_{t\in[0,1]}$ is a smooth family of almost complex structures on $\mathbb{P}^n$ such that J_t is close to J_0 for all $t\in[0,1]$, the moduli space

$$\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;\underline{J})\equiv\bigcup_{t\in[0,1]}\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J_t)$$

is compact, by Theorem 1.2 in [Z5]. We can construct a multisection Φ of the cone $\mathcal{V}_{1,k}^d$ over $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;\underline{J})$ with properties analogous to $(\mathcal{V}1)$ and $(\mathcal{V}2)$. If $\mathcal{W}\longrightarrow\mathfrak{X}_{1,k}(\mathbb{P}^n,d)$ is a complex vector bundle of rank $d(n+1-a)+k$ as in Theorem 1.2, we can choose a section F of $\mathcal{W}$ over $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;\underline{J})$ such that $\Phi^{-1}(0)\cap F^{-1}(0)$ is a compact oriented one-dimensional suborbifold of $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,A;\underline{J})$ and

$$\begin{aligned}\partial\big(\Phi^{-1}(0)\cap F^{-1}(0)\big)&=\Phi^{-1}(0)\cap F^{-1}(0)\cap\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J_1)-\Phi^{-1}(0)\cap F^{-1}(0)\cap\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J_0)\\ \Longrightarrow\qquad {}^{\pm}\big|\Phi^{-1}(0)\cap F^{-1}(0)\cap\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J_1)\big|&={}^{\pm}\big|\Phi^{-1}(0)\cap F^{-1}(0)\cap\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)\big|.\end{aligned}$$

This equality implies (1.14).

2 Preliminaries

2.1 Notation: Genus-Zero Maps

We now summarize our notation for bubble maps from genus-zero Riemann surfaces with at least one marked point, for the spaces of such bubble maps that form the standard stratifications of moduli spaces of stable maps, and for important vector bundles over them. For more details on the notation described below, the reader is referred to Subsections 2.1 and 2.2 in [Z5].

In general, moduli spaces of stable maps can be stratified by the dual graph. However, in the present situation, it is more convenient to make use of *linearly ordered sets*:

Definition 2.1 *(1) A finite nonempty partially ordered set I is a* `linearly ordered set` *if for all $i_1, i_2, h \in I$ such that $i_1, i_2 < h$, either $i_1 \le i_2$ or $i_2 \le i_1$.*
(2) A linearly ordered set I is a `rooted tree` *if I has a unique minimal element, i.e. there exists $\hat{0} \in I$ such that $\hat{0} \le i$ for all $i \in I$.*

If I is a linearly ordered set, let $\hat{I}$ be the subset of the non-minimal elements of I. For every $h \in \hat{I}$, denote by $\iota_h \in I$ the largest element of I which is smaller than h, i.e. $\iota_h = \max\{i \in I : i < h\}$.

If M is a finite set, a `genus-zero` $\mathbb{P}^n$`-valued bubble map with` M`-marked points` is a tuple

$$b = \big(M, I; x, (j, y), u\big),$$

where I is a rooted tree, and

$$x\colon \hat{I} \longrightarrow \mathbb{C} = S^2 - \{\infty\}, \quad j\colon M \longrightarrow I, \quad y\colon M \longrightarrow \mathbb{C}, \quad \text{and} \quad u\colon I \longrightarrow C^\infty(S^2; \mathbb{P}^n) \tag{2.1}$$

are maps such that $u_h(\infty) = u_{\iota_h}(x_h)$ for all $h \in \hat{I}$. We associate such a tuple with Riemann surface

$$\Sigma_b = \Big(\bigsqcup_{i \in I} \Sigma_{b,i}\Big)\Big/\sim, \quad \text{where} \qquad \Sigma_{b,i} = \{i\} \times S^2 \quad \text{and} \quad (h, \infty) \sim (\iota_h, x_h) \ \ \forall h \in \hat{I},$$

with marked points,

$$y_l(b) \equiv (j_l, y_l) \in \Sigma_{b,j_l} \qquad \text{and} \qquad y_0(b) \equiv (\hat{0}, \infty) \in \Sigma_{b,\hat{0}},$$

and with the continuous map $u_b\colon \Sigma_b \longrightarrow X$, given by $u_b|_{\Sigma_{b,i}} = u_i$ for all $i \in I$.

The general structure of genus-zero bubble maps is described by tuples

$$\mathcal{T} = (M, I; j, \underline{d}),$$

where $\underline{d}\colon I \longrightarrow \mathbb{Z}$ is a map specifying the degree of $u_b|_{\Sigma_{b,i}}$, if b is a bubble map of type $\mathcal{T}$. We call such tuples *bubble types*. Let $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n; J)$ denote the subset of $\overline{\mathfrak{M}}_{0,\{0\}\sqcup M}(\mathbb{P}^n, d; J)$ consisting of stable maps $[\mathcal{C}; u]$ such that

$$[\mathcal{C}; u] = \big[(\Sigma_b, (\hat{0}, \infty), (j_l, y_l)_{l \in M}); u_b\big],$$

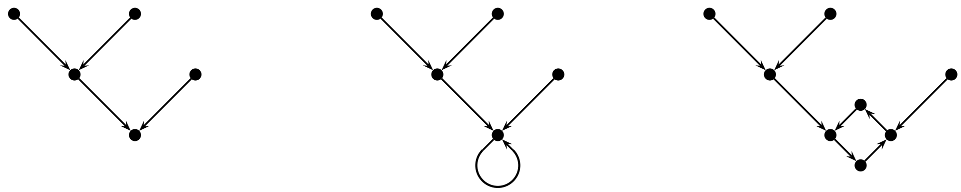


Figure 2: Some Enhanced Linearly Ordered Sets

for some bubble map b of type $\mathcal{T}$. We recall that

$$\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)=\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)\big/\mathrm{Aut}(\mathcal{T})\propto(S^1)^I,$$

for a certain submanifold $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ of the space $\mathcal{H}_{\mathcal{T}}(\mathbb{P}^n;J)$ of J-holomorphic maps into $\mathbb{P}^n$ of type $\mathcal{T}$, not of equivalence classes of such maps; see Subsection 2.5 in [Z3]. For $l\in\{0\}\sqcup M$, let

$$\mathrm{ev}_l\colon \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J),\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)\longrightarrow\mathbb{P}^n$$

be the evaluation maps corresponding to the marked point y_l.

2.2 Notation: Genus-One Maps

We next set up analogous notation for maps from genus-one Riemann surfaces. In this case, we also need to specify the structure of the principal component. Thus, we index the strata of the moduli space $\overline{\mathfrak{M}}_{1,M}(\mathbb{P}^n,d;J)$ by *enhanced linearly ordered sets*:

Definition 2.2 *An* `enhanced linearly ordered set` *is a pair $(I,\aleph)$, where I is a linearly ordered set, $\aleph$ is a subset of $I_0\times I_0$, and I_0 is the subset of minimal elements of I, such that if $|I_0|>1$,*

$$\aleph=\big\{(i_1,i_2),(i_2,i_3),\ldots,(i_{n-1},i_n),(i_n,i_1)\big\}$$

for some bijection $i\colon\{1,\ldots,n\}\longrightarrow I_0$.

An enhanced linearly ordered set can be represented by an oriented connected graph. In Figure 2, the dots denote the elements of I. The arrows outside the loop, if there are any, specify the partial ordering of the linearly ordered set I. In fact, every directed edge outside of the loop connects a non-minimal element h of I with ι_h. Inside of the loop, there is a directed edge from i_1 to i_2 if and only if $(i_1,i_2)\in\aleph$.

The subset $\aleph$ of $I_0\times I_0$ will be used to describe the structure of the principal curve of the domain of stable maps in a stratum of the moduli space $\overline{\mathfrak{M}}_{1,M}(\mathbb{P}^n,d;J)$. If $\aleph=\emptyset$, and thus $|I_0|=1$, the corresponding principal curve $\Sigma_\aleph$ is a smooth torus, with some complex structure. If $\aleph\neq\emptyset$, the principal components form a circle of spheres:

$$\Sigma_\aleph=\Big(\bigsqcup_{i\in I_0}\{i\}\times S^2\Big)\Big/\sim,\qquad\text{where}\qquad(i_1,\infty)\sim(i_2,0)\ \ \text{if}\ \ (i_1,i_2)\in\aleph.$$

For example, the principal components $\Sigma_\aleph$ described by the three diagrams in Figure 2 are a smooth torus, a sphere with two points identified, and a circle of spheres (a smooth torus with four disjoint circles, that are not null-homotopic, collapsed).

A `genus-one` $\mathbb{P}^n$`-valued bubble map with` M`-marked points` is a tuple

$$b = \big(M, I, \aleph; S, x, (j, y), u\big),$$

where S is a smooth Riemann surface of genus one if $\aleph=\emptyset$ and the circle of spheres $\Sigma_\aleph$ otherwise. The objects x, j, y, u, and (Σ_b, u_b) are as in the genus-zero case above, except the sphere $\Sigma_{b,\hat{0}}$ is replaced by the genus-one curve $\Sigma_{b,\aleph} \equiv S$. Furthermore, if $\aleph = \emptyset$, and thus $I_0 = \{\hat{0}\}$ is a single-element set, $u_{\hat{0}} \in C^\infty(S; \mathbb{P}^n)$. In the genus-one case, the general structure of bubble maps is encoded by the tuples of the form

$$\mathcal{T} = (M, I, \aleph; j, \underline{d}).$$

Similarly to the genus-zero case, we denote by $\mathcal{U}_\mathcal{T}(\mathbb{P}^n; J)$ the subset of $\overline{\mathfrak{M}}_{1,M}(\mathbb{P}^n, d; J)$ consisting of stable maps $[\mathcal{C}; u]$ such that

$$[\mathcal{C}; u] = \big[(\Sigma_b, (j_l, y_l)_{l\in M}); u_b\big],$$

for some bubble map b of type $\mathcal{T}$ as above.

If $\mathcal{T}=(M, I, \aleph; j, \underline{d})$ is a bubble type as above, let

$$I_1 = \big\{h\in\hat{I}\colon \iota_h\in I_0\big\}, \qquad M_0 = \big\{l\in M\colon j_l\in I_0\big\}, \qquad \text{and}$$
$$\mathcal{T}_0 = \big(M_0\sqcup I_1, I_0, \aleph; j|_{M_0}\sqcup\iota|_{I_1}, \underline{d}|_{I_0}\big),$$

where I_0 is the subset of minimal elements of I as before. For each $h\in I_1$, we put

$$I_h = \big\{i\in I\colon h\le i\big\}, \qquad M_h = \big\{l\in M\colon j_l\in I_h\big\}, \quad \text{and} \quad \mathcal{T}_h = \big(M_h, I_h; j|_{M_h}, \underline{d}|_{I_h}\big).$$

The tuple $\mathcal{T}_0$ describes bubble maps from genus-one Riemann surfaces with the marked points indexed by the set $M_0\sqcup I_1$. By definition, we have a natural isomorphism

$$\begin{split}\mathcal{U}_\mathcal{T}(\mathbb{P}^n; J) \approx \Big(\big\{\big(b_0, (b_h)_{h\in I_1}\big)\in\mathcal{U}_{\mathcal{T}_0}(\mathbb{P}^n; J)\times\prod_{h\in I_1}\mathcal{U}_{\mathcal{T}_h}(\mathbb{P}^n; J)\colon \\ \mathrm{ev}_0(b_h)=\mathrm{ev}_{\iota_h}(b_0)\ \forall h\in I_1\big\}\Big)\big/\mathrm{Aut}^*(\mathcal{T}),\end{split} \tag{2.2}$$

where the group $\mathrm{Aut}^*(\mathcal{T})$ is defined by

$$\mathrm{Aut}^*(\mathcal{T}) = \mathrm{Aut}(\mathcal{T})/\{g\in\mathrm{Aut}(\mathcal{T})\colon g\cdot h=h\ \forall h\in I_1\}.$$

This decomposition is illustrated in Figure 3. In this figure, we represent an entire stratum of bubble maps by the domain of the stable maps in that stratum. We shade the components of the domain on which every (or any) stable map in $\mathcal{U}_\mathcal{T}(\mathbb{P}^n; J)$ is nonconstant. The right-hand side of Figure 3 represents the subset of the cartesian product of the three spaces of bubble maps, corresponding to the three drawings, on which the appropriate evaluation maps agree pairwise, as indicated by the dotted lines and defined in (2.2).

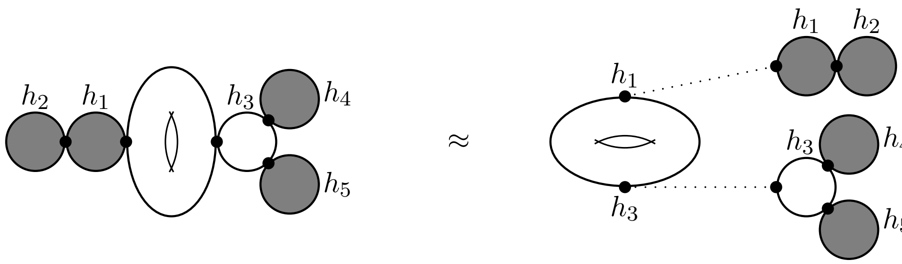


Figure 3: An Example of the Decomposition (2.2)

Let $\mathcal{F}\mathcal{T} \longrightarrow \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ be the bundle of gluing parameters, or of smoothings at the nodes. This orbi-bundle has the form

$$\mathcal{F}\mathcal{T} = \Big(\bigoplus_{(h,i)\in\aleph} L_{h,0}\otimes L_{i,1} \oplus \bigoplus_{h\in\hat{I}} L_{h,0}\otimes L_{h,1}\Big)/\mathrm{Aut}(\mathcal{T}),$$

for certain line orbi-bundles $L_{h,0}$ and $L_{h,1}$. Similarly to the genus-zero case,

$$\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J) = \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)/\mathrm{Aut}(\mathcal{T}) \propto (S^1)^{\hat{I}}, \qquad \text{where} \tag{2.3}$$

$$\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J) = \big\{\big(b_0,(b_h)_{h\in I_1}\big)\in\mathcal{U}_{\mathcal{T}_0}(\mathbb{P}^n;J)\times\prod_{h\in I_1}\mathcal{U}_{\mathcal{T}_h}^{(0)}(\mathbb{P}^n;J)\colon \mathrm{ev}_0(b_h)=\mathrm{ev}_{\iota_h}(b_0)\ \forall h\in I_1\big\}. \tag{2.4}$$

The line bundles $L_{h,0}$ and $L_{h,1}$ arise from the quotient (2.3), and

$$\mathcal{F}\mathcal{T} = \tilde{\mathcal{F}}\mathcal{T}/\mathrm{Aut}(\mathcal{T})\propto (S^1)^{\hat{I}}, \qquad \text{where} \qquad \tilde{\mathcal{F}}\mathcal{T} = \tilde{\mathcal{F}}_{\aleph}\mathcal{T} \oplus \bigoplus_{h\in\hat{I}} \tilde{\mathcal{F}}_h\mathcal{T},$$

$\tilde{\mathcal{F}}_{\aleph}\mathcal{T} \longrightarrow \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ is the bundle of smoothings for the $|\aleph|$ nodes of the circle of spheres $\Sigma_{\aleph}$ and $\tilde{\mathcal{F}}_h\mathcal{T} \longrightarrow \mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ is the line bundle of smoothings of the attaching node of the bubble indexed by h. We denote by $\mathcal{F}\mathcal{T}^{\emptyset}$ and $\tilde{\mathcal{F}}\mathcal{T}^{\emptyset}$ the subsets of $\mathcal{F}\mathcal{T}$ and $\tilde{\mathcal{F}}\mathcal{T}$, respectively, consisting of the elements with all components nonzero.

For the rest of this subsection, we restrict to the case when $\mathcal{T}$ describes stable maps that are of degree zero on the principal components. Thus, let

$$\mathcal{T} \equiv (M, I, \aleph; j, \underline{d})$$

be a bubble type such that $d_i=0$ for all $i\in I_0$. Since a degree-zero pseudoholomorphic map is constant, the decomposition (2.2) becomes

$$\begin{aligned}\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J) &\approx \Big(\mathcal{U}_{\mathcal{T}_0}(pt)\times\mathcal{U}_{\bar{\mathcal{T}}}(\mathbb{P}^n;J)\Big)/\mathrm{Aut}^*(\mathcal{T})\\ &\subset \Big(\overline{\mathcal{M}}_{1,k_0}\times\mathcal{U}_{\bar{\mathcal{T}}}(\mathbb{P}^n;J)\Big)/\mathrm{Aut}^*(\mathcal{T}),\end{aligned} \tag{2.5}$$

where $k_0=|I_1|+|M_0|$, $\overline{\mathcal{M}}_{1,k_0}$ is the moduli space of genus-one curves with k_0 marked points, and

$$\mathcal{U}_{\bar{\mathcal{T}}}(\mathbb{P}^n;J) = \big\{(b_h)_{h\in I_1}\in\prod_{h\in I_1}\mathcal{U}_{\mathcal{T}_h}(\mathbb{P}^n;J)\colon \mathrm{ev}_0(b_{h_1})=\mathrm{ev}_0(b_{h_2})\ \forall h_1,h_2\in I_1\big\}.$$

Similarly, (2.3) and (2.4) are equivalent to

$$\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J) \approx \mathcal{U}_{\mathcal{T}_0}(pt)\times\mathcal{U}_{\bar{\mathcal{T}}}^{(0)}(\mathbb{P}^n;J)\subset\overline{\mathcal{M}}_{1,k_0}\times\mathcal{U}_{\bar{\mathcal{T}}}^{(0)}(\mathbb{P}^n;J), \qquad \text{where} \tag{2.6}$$

$$\mathcal{U}_{\bar{\mathcal{T}}}^{(0)}(\mathbb{P}^n;J) = \big\{(b_h)_{h\in I_1}\in\prod_{h\in I_1}\mathcal{U}_{\mathcal{T}_h}^{(0)}(\mathbb{P}^n;J)\colon \mathrm{ev}_0(b_{h_1})=\mathrm{ev}_0(b_{h_2})\ \forall h_1,h_2\in I_1\big\}. \tag{2.7}$$

We denote by

$$\pi_P\colon \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J),\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)\longrightarrow\overline{\mathcal{M}}_{1,k_0}$$

the projections onto the first component in the decompositions (2.5) and (2.6). Let

$$\mathrm{ev}_P\colon \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J),\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)\longrightarrow\mathbb{P}^n$$

be the maps sending every element $b=(\Sigma_b,u_b)$ of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ and $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ to the image of the principal component $\Sigma_{b;P}$ of Σ_b under u_b.

If $\mathcal{T}=(M,I,\aleph;j,\underline{d})$ is as in the previous paragraph, let

$$\chi(\mathcal{T}) = \big\{i\in\hat{I}\colon d_i\neq0;\ d_h=0\ \forall h<i\big\}.$$

The subset $\chi(\mathcal{T})$ of I indexes the first-level effective bubbles of every element of $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$. For each element $b=(\Sigma_b,u_b)$ of $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ and $i\in\chi(\mathcal{T})$, let

$$\mathcal{D}_ib = \big\{du_b|_{\Sigma_{b,i}}\big\}\big|_\infty e_\infty\in T_{\mathrm{ev}_P(b)}\mathbb{P}^n, \qquad \text{where} \qquad e_\infty=(1,0,0)\in T_\infty S^2.$$

In geometric terms, the complex span of $\mathcal{D}_ib$ in $T_{\mathrm{ev}_P(b)}\mathbb{P}^n$ is the line tangent to the rational component $\Sigma_{b,i}$ at the node of $\Sigma_{b,i}$ closest to a principal component of Σ_b. If the branch corresponding to $\Sigma_{b,i}$ has a cusp at this node, then $\mathcal{D}_ib=0$. Let

$$\tilde{\mathfrak{F}}\mathcal{T} = \bigoplus_{i\in\chi(\mathcal{T})}\tilde{\mathcal{F}}_{h(i)}\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J), \quad \text{where} \quad h(i)=\min\{h\in\hat{I}\colon h\le i\}\in I_1.$$

We define the bundle map

$$\rho\colon \tilde{\mathcal{F}}\mathcal{T}\longrightarrow\tilde{\mathfrak{F}}\mathcal{T}$$

over $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ by

$$\rho(\upsilon)=\big(b;(\rho_i(\upsilon))_{i\in\chi(\mathcal{T})}\big)\in\tilde{\mathfrak{F}}\mathcal{T}, \quad \text{where} \quad \rho_i(\upsilon)=\prod_{h\in\hat{I},h\le i}v_h\in\tilde{\mathcal{F}}_{h(i)}\mathcal{T}, \qquad \text{if}$$

$$\upsilon=\big(b;v_\aleph,(v_h)_{h\in\hat{I}}\big), \quad b\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J), \quad v_\aleph\in\tilde{\mathcal{F}}_\aleph\mathcal{T}|_b, \quad v_h\in\tilde{\mathcal{F}}_h\mathcal{T}|_b\approx\begin{cases}T_{x_h(b)}\Sigma_{b;P}, & \text{if } h\in I_1,\\ \mathbb{C}, & \text{if } h\in\hat{I}-I_1,\end{cases}$$

where $x_h(b)\in\Sigma_{b;P}$ is the node joining the bubble $\Sigma_{b,h}$ of b to the principal component $\Sigma_{b;P}$ of Σ_b. This definition is illustrated in Figure 4 on page 22.

Let $\mathbb{E}\longrightarrow\overline{\mathcal{M}}_{1,k_0}$ be the Hodge line bundle, i.e. the line bundle of holomorphic differentials. For each $i\in\chi(\mathcal{T})$, we define the bundle map

$$\mathcal{D}_{J,i}\colon\tilde{\mathcal{F}}_{h(i)}\mathcal{T}\longrightarrow\pi_P^*\mathbb{E}^*\otimes_J\mathrm{ev}_P^*T\mathbb{P}^n$$

over $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ by

$$\big\{\mathcal{D}_{J,i}(b,w_i)\big\}(\psi)=\psi_{x_{h(i)}(b)}(w_i)\cdot_J\mathcal{D}_i b\in T_{\mathrm{ev}_P(b)}\mathbb{P}^n$$
$$\text{if}\qquad b\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J),\ w_i\in\tilde{\mathcal{F}}_{h(i)}\mathcal{T}|_b,\ \psi\in\pi_P^*\mathbb{E}|_b,$$

where $\cdot_J$ is the complex multiplication in the vector bundle $(T\mathbb{P}^n,J)$.[13] Let

$$\mathcal{D}_{\mathcal{T}}\colon\tilde{\mathfrak{F}}\mathcal{T}\longrightarrow\pi_P^*\mathbb{E}^*\otimes\mathrm{ev}_P^*T\mathbb{P}^n$$

be the bundle map over $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ given by

$$\mathcal{D}_{\mathcal{T}}\big(b,(w_i)_{i\in\chi(\mathcal{T})}\big)=\sum_{i\in\chi(\mathcal{T})}\mathcal{D}_{J,i}(b,w_i).$$

It descends to a bundle map

$$\mathcal{D}_{\mathcal{T}}\colon\mathfrak{F}\mathcal{T}\longrightarrow\pi_P^*\mathbb{E}^*\otimes\mathrm{ev}_P^*T\mathbb{P}^n/\mathrm{Aut}^*(\mathcal{T})$$

over $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$, for a bundle $\mathfrak{F}\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$.

Let $\tilde{\mathcal{V}}_{1,k}^d\longrightarrow\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ be the cone such that the fiber of $\tilde{\mathcal{V}}_{1,k}^d$ over a point $b=(\Sigma_b,u_b)$ in $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ is $\ker\bar\partial_{\nabla,b}$; see Subsection 1.3. If $b=(\Sigma_b,u_b)\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$, $\xi=(\xi_h)_{h\in I}\in\Gamma(b;\mathfrak{L})$, and $i\in\chi(\mathcal{T})$, let

$$\mathfrak{D}_{\mathcal{T},i}\xi=\nabla_{e_\infty}\xi_i\in\mathfrak{L}_{\mathrm{ev}_P(b)}.$$

The element $\nabla_{e_\infty}\xi_i$ of $u_{b,i}^*\mathfrak{L}|_\infty$ is the covariant derivative of the section $\xi_i\in\Gamma(\Sigma_{b,i};u_{b,i}^*\mathfrak{L})$ at $\infty\in\Sigma_{b,i}$ with respect to the connection ∇ in $\mathfrak{L}$ along e_∞; see Subsection 1.3. Note that if $\xi\in\ker\bar\partial_{\nabla,b}$, then

$$\nabla_{c\cdot e_\infty}\xi_i=c\cdot\mathfrak{D}_{\mathcal{T},i}\xi\qquad\forall\,c\in\mathbb{C}.\tag{2.8}$$

We next define the bundle map

$$\mathfrak{D}_{\mathcal{T}}\colon\tilde{\mathfrak{F}}\mathcal{T}\longrightarrow\mathrm{Hom}\big(\tilde{\mathcal{V}}_{1,k}^d,\pi_P^*\mathbb{E}^*\otimes\mathrm{ev}_P^*\mathfrak{L}\big)$$

over $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ by

$$\big\{\mathfrak{D}_{\mathcal{T}}(b,\xi\otimes w)\big\}(\psi)=\sum_{i\in\chi(\mathcal{T})}\psi_{x_{h(i)}(b)}(w_i)\cdot\mathfrak{D}_{\mathcal{T},i}\xi\in\mathfrak{L}_{\mathrm{ev}_P(b)}\qquad\text{if}$$
$$\xi\in\tilde{\mathcal{V}}_{1,k}^d|_b\subset\Gamma(b;\mathfrak{L}),\qquad w=(w_i)_{i\in\chi(\mathcal{T})}\in\tilde{\mathfrak{F}}\mathcal{T}|_b,\quad\text{and}\quad\psi\in\mathbb{E}_{\pi_P(b)}.$$

By (2.8), the bundle map $\mathfrak{D}_{\mathcal{T}}$ induces a linear bundle map

$$\mathfrak{F}\mathcal{T}\longrightarrow\mathrm{Hom}(\mathcal{V}_{1,k}^d,\pi_P^*\mathbb{E}^*\otimes\mathrm{ev}_P^*\mathfrak{L}/\mathrm{Aut}^*(\mathcal{T})\big)$$

over $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$. The maps $\mathfrak{D}_{\mathcal{T},i}$ and $\mathfrak{D}_{\mathcal{T}}$ are the analogues of $\mathcal{D}_{\mathcal{T},i}$ and $\mathcal{D}_{\mathcal{T}}$ for the target space $\mathfrak{L}$, in place of $\mathbb{P}^n$.

Finally, all vector orbi-bundles we encounter will be assumed to be normed. Some will come with natural norms; for others, we implicitly choose a norm once and for all. If $\pi_{\mathfrak{F}}\colon\mathfrak{F}\longrightarrow\mathfrak{X}$ is a normed vector bundle and $\delta\colon\mathfrak{X}\longrightarrow\mathbb{R}$ is any function, possibly constant, let

$$\mathfrak{F}_\delta=\big\{\upsilon\in\mathfrak{F}\colon|\upsilon|<\delta(\pi_{\mathfrak{F}}(\upsilon))\big\}.$$

If Ω is any subset of $\mathfrak{F}$, we take $\Omega_\delta=\Omega\cap\mathfrak{F}_\delta$.

[13] The complex number $\psi_{x_{h(i)}(b)}(w_i)$ is simply the evaluation of $\psi_{x_{h(i)}(b)}\in T^*_{x_{h(i)}}\Sigma_\aleph$ on $w\in T_{x_{h(i)}}\Sigma_\aleph$.

2.3 The Structure of the Moduli Space $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$

We now describe the structure of the moduli space $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$ near each of its strata. The first part of Theorem 1.2 follows from the first claims of Lemmas 2.3 and 2.4 below. If $k\in\mathbb{Z}$, we denote by $[k]$ the set of positive integers that do not exceed k.

Lemma 2.3 *If n, k, and d are as in Theorem 1.2, there exists $\delta_n(d)\in\mathbb{R}^+$ with the following property. If J is an almost complex structure on $\mathbb{P}^n$, such that $\|J-J_0\|_{C^1}<\delta_n(d)$, and*

$$\mathcal{T}=([k],I,\aleph;j,\underline{d})$$

is a bubble type such that $\sum_{i\in I}d_i=d$ and $d_i\neq 0$ for some minimal element i of I, then $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ is a smooth orbifold,

$$\dim\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)=2\big(d(n+1)+k-|\aleph|-|\hat{I}|\big), \qquad \textit{and} \qquad \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)\subset\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J).$$

Furthermore, there exist $\delta\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$, an open neighborhood $U_{\mathcal{T}}$ of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ in $\mathfrak{X}_{1,k}(\mathbb{P}^n,d)$, and an orientation-preserving homeomorphism

$$\phi_{\mathcal{T}}:\mathcal{F}\mathcal{T}_\delta\longrightarrow\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)\cap U_{\mathcal{T}},$$

which restricts to a diffeomorphism $\mathcal{F}\mathcal{T}^\emptyset_\delta\longrightarrow\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;J)\cap U_{\mathcal{T}}$.

By Theorem 1.6 in [Z5], there exists $\delta_n(d)\in\mathbb{R}^+$ with the following property. If J is an almost complex structure on $\mathbb{P}^n$, such that $\|J-J_0\|_{C^1}<\delta_n(d)$, Σ is a genus-one prestable Riemann surface, and $u:\Sigma\longrightarrow\mathbb{P}^n$ is a J-holomorphic map, such that the restriction of u to the principal component(s) of Σ is not constant, then the linearization $D_{J,u}$ of the $\bar\partial_J$-operator at u is surjective. From standard arguments, such as in Chapter 3 of [MS], it then follows that the stratum $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$, where $\mathcal{T}$ is a bubble type as in Lemma 2.3, is a smooth orbifold of the expected dimension. Furthermore, there is no obstruction to gluing the maps in $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$, in the sense of the following paragraph.

We fix a metric g_n and a connection ∇^n on $(T\mathbb{P}^n,J)$. For each sufficiently small element $\upsilon=(b,v)$ of $\tilde{\mathcal{F}}\mathcal{T}^\emptyset$ and $b=(\Sigma_b,u_b)\in\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^n;J)$, let

$$q_\upsilon:\Sigma_\upsilon\longrightarrow\Sigma_b$$

be the basic gluing map constructed in Subsection 4.1 of [Z5]. In this case, Σ_υ is a smooth elliptic curve and q_υ collapses certain disjoint circles in Σ_υ onto the nodes of Σ_b. Let

$$b(\upsilon)=\big(\Sigma_\upsilon,j_\upsilon,u_\upsilon\big), \qquad \text{where} \qquad u_\upsilon=u_b\circ q_\upsilon,$$

be the corresponding approximately J-holomorphic stable map. By the previous paragraph, the linearization $D_{J,b}$ of the $\bar\partial_J$-operator at b is surjective. Thus, if υ is sufficiently small, the linearization

$$D_{J,\upsilon}:\Gamma(\upsilon)\equiv L^p_1(\Sigma_\upsilon;u_\upsilon^*T\mathbb{P}^n)\longrightarrow\Gamma^{0,1}(\upsilon)\equiv L^p(\Sigma_\upsilon;\Lambda^{0,1}_{J,j}T^*\Sigma_\upsilon\otimes u_\upsilon^*T\mathbb{P}^n),$$

of the $\bar\partial_J$-operator at $b(\upsilon)$, defined via ∇^n, is also surjective. In particular, we can obtain an orthogonal decomposition

$$\Gamma(\upsilon)=\Gamma_-(\upsilon)\oplus\Gamma_+(\upsilon) \tag{2.9}$$

such that the linear operator $D_{J,\upsilon}:\Gamma_+(\upsilon)\longrightarrow\Gamma^{0,1}(\upsilon)$ is an isomorphism, while $\Gamma_-(\upsilon)$ is close to $\ker D_{J,b}$. The L^2-inner product on $\Gamma(\upsilon)$ used in the orthogonal decomposition is defined via the metric g_n on $\mathbb{P}^n$ and the metric g_υ on Σ_υ induced by the pregluing construction. The Banach spaces $\Gamma(\upsilon)$ and $\Gamma^{0,1}(\upsilon)$ carry the norms $\|\cdot\|_{\upsilon,p,1}$ and $\|\cdot\|_{\upsilon,p}$, respectively, which are also defined by the pregluing construction. These norms are equivalent to the ones used in Section 3 of [LT]. In particular, the norms of $D_{J,\upsilon}$ and of the inverse of its restriction to $\Gamma_+(\upsilon)$ have fiberwise uniform upper bounds, i.e. dependent only on $[b]\in\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$, and not on $\upsilon\in\tilde{\mathcal{F}}\mathcal{T}^\emptyset$. It then follows that the equation

$$\bar\partial_J\exp_{u_\upsilon}\zeta=0\qquad\Longleftrightarrow\qquad[\Sigma_\upsilon,\exp_{u_\upsilon}\zeta]\in\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;J)$$

has a unique small solution $\zeta_\upsilon\in\Gamma_+(\upsilon)$. Furthermore,

$$\|\zeta_\upsilon\|_{\upsilon,p,1}\le C(b)|\upsilon|^{1/p},$$

for some $C\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$. The diffeomorphism on $\mathcal{F}\mathcal{T}^\emptyset_\delta$ is given by

$$\phi_{\mathcal{T}}:\mathcal{F}\mathcal{T}^\emptyset_\delta\longrightarrow\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;J),\qquad\phi_{\mathcal{T}}([\upsilon])=\big[\tilde b(\upsilon)\big],\qquad\text{where}\qquad\tilde b(\upsilon)=(\Sigma_\upsilon,\exp_{u_\upsilon}\zeta_\upsilon);$$

see the paragraph following Lemma 3.1 in [Z5]. This map extends to a homeomorphism

$$\phi_{\mathcal{T}}:\mathcal{F}\mathcal{T}_\delta\longrightarrow\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J),$$

as can be seen by an argument similar to Subsections 3.9 and 4.1 in [Z3].

We denote by $\mathfrak{M}^{\text{eff}}_{1,k}(\mathbb{P}^n,d;J)$ the union of the strata $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ with $\mathcal{T}$ as in Lemma 2.3. In other words,

$$\mathfrak{M}^{\text{eff}}_{1,k}(\mathbb{P}^n,d;J)=\big\{[\mathcal{C},u]\in\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d;J)\colon u|_{\mathcal{C}_P}\text{ is not constant}\big\},$$

where $\mathcal{C}_P$ is the principal component of the domain $\mathcal{C}$ of u.

Lemma 2.4 *If n, k, and d are as in Theorem 1.2, there exists $\delta_n(d)\in\mathbb{R}^+$ with the following property. If J is an almost complex structure on $\mathbb{P}^n$, such that $\|J-J_0\|_{C^1}<\delta_n(d)$, and*

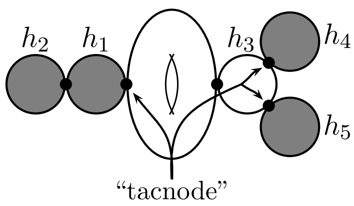
$$\mathcal{T}=([k],I,\aleph;j,\underline{d})$$

is a bubble type such that $\sum_{i\in I}d_i=d$ and $d_i=0$ for all minimal elements i of I, then $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ is a smooth orbifold,

$$\dim\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)=2\big(d(n+1)+k-|\aleph|-|\hat I|+n\big),\quad\textit{and}\quad\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)\cap\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)=\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J),$$
$$\textit{where}\qquad\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)=\big\{[b]\in\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)\colon\dim_{\mathbb{C}}\mathit{Span}_{(\mathbb{C},J)}\{\mathcal{D}_ib\colon i\in\chi(\mathcal{T})\}<|\chi(\mathcal{T})|\big\}.$$

The space $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)$ admits a stratification by smooth suborbifolds of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$:

$$\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)=\bigsqcup_{m=\max(|\chi(\mathcal{T})|-n,1)}^{m=|\chi(\mathcal{T})|}\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)\qquad\textit{such that}$$
$$\begin{aligned}\dim\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)&=2\big(d(n+1)+k-|\aleph|-|\hat I|+n+(|\chi(\mathcal{T})|-n-m)m\big)\\&\le\dim\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;J)-2.\end{aligned}$$



$$\chi(\mathcal{T})=\{h_1,h_4,h_5\},\ \ \rho(\upsilon)=(\upsilon_{h_1},\upsilon_{h_3}\upsilon_{h_4},\upsilon_{h_3}\upsilon_{h_5})$$

$$\begin{aligned}\mathcal{F}^1\mathcal{T}^\emptyset = \big\{[b;\upsilon_{h_1},\upsilon_{h_2},\upsilon_{h_3},\upsilon_{h_4},\upsilon_{h_5}]\colon \upsilon_{h_2},\upsilon_{h_4},\upsilon_{h_5}\in\mathbb{C}^*;\\ \upsilon_{h_1}\in T_{x_{h_1}}\Sigma_\aleph-\{0\},\ \upsilon_{h_3}\in T_{x_{h_3}}\Sigma_\aleph-\{0\};\\ \upsilon_{h_1}\mathcal{D}_{J,h_1}b+\upsilon_{h_3}\upsilon_{h_4}\mathcal{D}_{J,h_4}b+\upsilon_{h_3}\upsilon_{h_5}\mathcal{D}_{J,h_5}b=0\big\}\end{aligned}$$

Figure 4: An Illustration of Lemma 2.4

Furthermore, the space

$$\mathcal{F}^1\mathcal{T}^\emptyset \equiv \big\{[b,\upsilon]\in\mathcal{F}\mathcal{T}^\emptyset\colon \mathcal{D}_\mathcal{T}\big(\rho(\upsilon)\big)=0\big\}$$

is a smooth oriented suborbifold of $\mathcal{F}\mathcal{T}$. Finally, there exist $\delta\in C(\mathcal{U}_\mathcal{T}(\mathbb{P}^n;J);\mathbb{R}^+)$, an open neighborhood $U_\mathcal{T}$ of $\mathcal{U}_\mathcal{T}(\mathbb{P}^n;J)$ in $\mathfrak{X}_{1,k}(\mathbb{P}^n,d)$, and an orientation-preserving diffeomorphism

$$\phi_\mathcal{T}\colon \mathcal{F}^1\mathcal{T}_\delta^\emptyset \longrightarrow \mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)\cap U_\mathcal{T},$$

which extends to a homeomorphism

$$\phi_\mathcal{T}\colon \mathcal{F}^1\mathcal{T}_\delta \longrightarrow \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)\cap U_\mathcal{T},$$

where $\mathcal{F}^1\mathcal{T}$ is the closure of $\mathcal{F}^1\mathcal{T}^\emptyset$ in $\mathcal{F}\mathcal{T}$.

We now clarify the statement of Lemma 2.4 and illustrate it using Figure 4. As before, the shaded discs represent the components of the domain on which every stable map $[b]$ in $\mathcal{U}_\mathcal{T}(\mathbb{P}^n;J)$ is non-constant. The element $[\Sigma_b,u_b]$ of $\mathcal{U}_\mathcal{T}(\mathbb{P}^n;J)$ is in the stable-map closure of $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)$ if and only if the branches of $u_b(\Sigma_b)$ corresponding to the attaching nodes of the first-level effective (shaded) bubbles of $[\Sigma_b,u_b]$ form a generalized tacnode. In the case of Figure 4, this means that either

(a) for some $h\in\{h_1,h_4,h_5\}$, the branch of $u_b|_{\Sigma_{b,h}}$ at the node ∞ has a cusp, or

(b) for all $h\in\{h_1,h_4,h_5\}$, the branch of $u_b|_{\Sigma_{b,h}}$ at the node ∞ is smooth, but the dimension of the span of the three lines tangent to these branches is less than three.

The last statement of Lemma 2.4 identifies a normal neighborhood of $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)$ in $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ with a small neighborhood of $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)$ in the bundle $\mathcal{F}^1\mathcal{T}$ over $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)$. Each fiber of the projection map $\mathcal{F}^1\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)$ is an algebraic variety. See Figure 4 for an example.

The first statement of Lemma 2.4 follows immediately from Theorems 1.6 and 2.3 in [Z5] and the decomposition (2.2). The other two statements of Lemma 2.4 are a special case of the last two statements of the latter theorem.

If $\mathcal{T}$ is a bubble type as in Lemma 2.4 and m is a positive integer, let

$$\begin{aligned}\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J) &=\big\{[b]\in\mathcal{U}_\mathcal{T}(\mathbb{P}^n;J)\colon \dim_\mathbb{C}\mathrm{Span}_{(\mathbb{C},J)}\{\mathcal{D}_ib\colon i\in\chi(\mathcal{T})\} = |\chi(\mathcal{T})|-m\big\}\\ &\subset \mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J).\end{aligned}$$

By definition, the subspaces $\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)$ of $\mathcal{U}_\mathcal{T}(\mathbb{P}^n;J)$ partition $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)$. On the other hand,

$$\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)\neq\emptyset \qquad\Longrightarrow\qquad \max\big(|\chi(\mathcal{T})|-n,1\big)\le m\le|\chi(\mathcal{T})|.$$

In order to show that the space $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$ is a smooth suborbifold of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ of the claimed dimension, below we describe $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$ in a different way.

For each $i\in\hat{I}$, let

$$M_i=\big\{l\in M\colon j_l=i\big\}\sqcup\big\{h\in\hat{I}\colon \iota_h=i\big\}.$$

We denote by

$$\pi_i\colon \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)\longrightarrow \mathfrak{M}^0_{0,\{0\}\sqcup M_i}(\mathbb{P}^n,d_i;J)$$

the map sending each bubble map (Σ_b,u_b) to its restriction to the component $\Sigma_{b,i}\subset\Sigma$. Let

$$L_0\longrightarrow \mathfrak{M}^0_{0,\{0\}\sqcup M_i}(\mathbb{P}^n,d_i;J)\subset\overline{\mathfrak{M}}_{0,\{0\}\sqcup M_i}(\mathbb{P}^n,d_i;J)$$

be the universal tangent line bundle for the special point labeled by 0, i.e. (i,∞) in the notation of Subsection 2.1. We put

$$\mathcal{F}=\bigoplus_{i\in\chi(\mathcal{T})}\pi_i^*L_0\longrightarrow \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J).$$

While each line bundle $\pi_i^*L_0$ may not be well-defined,[14] the orbibundle $\mathcal{F}$ is always well-defined. We denote by

$$\pi_m\colon \mathrm{Gr}_m\mathcal{F}\longrightarrow \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)\qquad\text{and}\qquad \gamma_m\longrightarrow \mathrm{Gr}_m\mathcal{F}$$

the Grassmannian bundle of m-dimensional linear subspaces and the tautological m-plane bundle, respectively. Let

$$\mathcal{S}_m=\mathcal{D}_m^{-1}(0)\subset \mathrm{Gr}_m\mathcal{F},\qquad\text{where}$$

$$\mathcal{D}_m\in\Gamma(\mathrm{Gr}_m\mathcal{F};\gamma_m^*\otimes\pi_m^*\mathrm{ev}_P^*T\mathbb{P}^n),\qquad \mathcal{D}_m([\upsilon])=\sum_{i\in\chi(\mathcal{T})}\mathcal{D}_{J,i}\upsilon_i\in\mathrm{ev}_P^*T\mathbb{P}^n\quad\text{if}\quad[\upsilon]=\big[(\upsilon_i)_{i\in\chi(\mathcal{T})}\big].$$

By Theorem 1.6 in [Z5], the section $\mathcal{D}_m$ is transverse to the zero set if $\delta_n(d)$ is sufficiently small. Thus, $\mathcal{S}_m$ is a smooth suborbifold of $\mathrm{Gr}_m\mathcal{F}$ of dimension

$$\begin{aligned}\dim\mathcal{S}_m&=\dim\mathrm{Gr}_m\mathcal{F}-2\,\mathrm{rk}\,\gamma_m^*\otimes\pi_m^*\mathrm{ev}_P^*T\mathbb{P}^n\\&=2\big(d(n+1)+k-|\aleph|-|\hat{I}|+n\big)+2m\big(|\chi(\mathcal{T})|-m\big)-2\cdot n\cdot m\\&=2\big(d(n+1)+k-|\aleph|-|\hat{I}|+n+m\big(|\chi(\mathcal{T})|-n-m\big)\big).\end{aligned}$$

The image of $\mathcal{S}_m$ under the bundle projection map π_m is the union of the spaces $\mathcal{U}^{m'}_{\mathcal{T};1}(\mathbb{P}^n;J)$ with $m'\geq m$. The map $\pi_m|_{\mathcal{S}_m}$ is an immersion at $[\upsilon]\in\mathcal{S}_m$ if

$$\pi_m^{-1}(\pi_m([\upsilon]))=[\upsilon].$$

The latter is the case if and only $\pi_m([\upsilon])\in\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$. Thus, the subspace $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$ is a smooth suborbifold of $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ of $\dim\mathcal{S}_m$.

[14] If $\mathcal{T}$ has an automorphism that does not fix an element i of $\chi(\mathcal{T})\subset I$, then the projection map π_i is not well-defined on $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$. It is however well-defined on $\mathcal{U}^0_{\mathcal{T}}(\mathbb{P}^n;J)$, since the components of the elements of $\mathcal{U}^0_{\mathcal{T}}(\mathbb{P}^n;J)$ are indeed indexed by the set I.

3 Proof of Theorem 1.2

3.1 The Global Structure of the Cone $\mathcal{V}_{1,k}^d \longrightarrow \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$

In this subsection we deduce Proposition 3.1, which contains the last two statements of Theorem 1.2, from Lemma 3.2 and Proposition 3.3. We suggest the reader review Subsection 1.2 at this point. The argument following Proposition 3.1 makes use of the small neighborhoods $U_{\mathcal{T}}^m$ of the strata $\mathcal{U}_{\mathcal{T}}^m(\mathbb{P}^n;J)$ and vector subbundles

$$\mathcal{V}_{1,k;\mathcal{T}}^{d;m} \longrightarrow \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)\cap U_{\mathcal{T}}^m$$

of the cone $\mathcal{V}_{1,k}^d$ described in Proposition 3.3. It might thus be helpful to study the statements of Lemma 3.2 and Proposition 3.3 before going through the argument for Proposition 3.1. However, with the help of the last two paragraphs of Subsection 1.2, it should be possible to get a rough idea of the argument even without looking at Lemma 3.2 and Proposition 3.3.

Proposition 3.1 *If n, k, d, a, $\mathfrak{L}$, and $\mathcal{V}_{1,k}^d$ are as in Theorem 1.2, there exists $\delta_n(d,a)\in\mathbb{R}^+$ with the following property. If J is an almost complex structure on $\mathbb{P}^n$, such that $\|J-J_0\|_{C^1}<\delta_n(d,a)$, the requirements of Lemmas 2.3 and 2.4 are satisfied. Furthermore, $\mathcal{V}_{1,k}^d \longrightarrow \mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)$ is a smooth complex vector orbibundle of rank da. In addition, there exists a continuous multisection $\varphi\colon \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)\longrightarrow\mathcal{V}_{1,k}^d$ such that*

($\mathcal{V}1$) $\varphi|_{\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)}$ is smooth and transverse to the zero set in $\mathcal{V}_{1,k}^d|_{\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)}$;

($\mathcal{V}2$) the intersection of $\varphi^{-1}(0)$ with each boundary stratum $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ and $\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)$ of $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ is a smooth suborbifold of the stratum of real dimension of at most $2(d(n+1-a)+k)-2$.

If φ_0 and φ_1 are two such multisections, there exists a continuous homotopy

$$\Phi\colon [0,1]\times\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J) \longrightarrow [0,1]\times\mathcal{V}_{1,k}^d$$

such that $\Phi|_{\{t\}\times\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)}=\varphi_t$ for $t=0,1$, and

($\mathcal{V}1'$) $\Phi|_{[0,1]\times\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)}$ is smooth and transverse to the zero set in $[0,1]\times\mathcal{V}_{1,k}^d|_{\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)}$;

($\mathcal{V}2'$) the intersection of $\Phi^{-1}(0)$ with each boundary stratum $[0,1]\times\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ and $[0,1]\times\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)$ of $[0,1]\times\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$ is a smooth suborbifold of the stratum of real dimension of at most $2(d(n+1-a)+k)-1$.

Thus, the cone $\mathcal{V}_{1,k}^d$ determines a homology class and a cohomology class on $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$:

$$PD_{\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)}\big(e(\mathcal{V}_{1,k}^d)\big) \in H_{2(d(n+1-a)+k)}\big(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J);\mathbb{Q}\big)$$
$$\text{and}\qquad e(\mathcal{V}_{1,k}^d) \in H^{2da}\big(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J);\mathbb{Q}\big).$$

Finally, if $\mathcal{W}\longrightarrow\mathfrak{X}_{1,k}(\mathbb{P}^n,d)$ is a vector orbi-bundle such that the restriction of $\mathcal{W}$ to each stratum $\mathfrak{X}_{\mathcal{T}}(\mathbb{P}^n)$ of $\mathfrak{X}_{1,k}(\mathbb{P}^n,d)$ is smooth, then

$$\big\langle e(\mathcal{W})\cdot e(\mathcal{V}_{1,k}^d), \big[\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)\big]\big\rangle = \big\langle e(\mathcal{W})\cdot e(\mathcal{V}_{1,k}^d), \big[\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)\big]\big\rangle.$$

The second statement of this proposition is a special case of Lemma 3.2. We use Lemma 3.2 and Proposition 3.3 to construct a multisection φ satisfying $(\mathcal{V}1)$ and $(\mathcal{V}2)$, starting from the lowest-dimensional strata of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$. If $\mathcal{T}$ and m are as in Lemma 2.4, let

$$\partial\bar{\mathcal{U}}^m_{\mathcal{T};1}(\mathbb{P}^n;J) \equiv \bar{\mathcal{U}}^m_{\mathcal{T};1}(\mathbb{P}^n;J) - \mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$$

be the boundary of the stratum $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$.

Suppose $\mathcal{T}$ and m are as in Lemma 2.4 and we have constructed

(i) a neighborhood U of $\partial\bar{\mathcal{U}}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$ in $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$ and

(ii) a continuous multisection φ of the cone $\mathcal{V}^d_{1,k}$ over U

such that for all $\mathcal{T}'$ and m' as in Lemma 2.4 the restriction of φ to $\mathcal{U}^{m'}_{\mathcal{T}';1}(\mathbb{P}^n;J)\cap U$

(a) is a smooth multisection of the vector bundle $\mathcal{V}^{d;m'}_{1,k;\mathcal{T}'}$ of Proposition 3.3 and

(b) is transverse to the zero set in $\mathcal{V}^{d;m'}_{1,k;\mathcal{T}'}$.

We then extend the restriction of φ to $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)\cap U$ to a smooth section of $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}$ over $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$ and to a continuous section $\varphi^m_{\mathcal{T}}$ of $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}$ over $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)\cap U^m_{\mathcal{T}}$, using the bundle isomorphism $\tilde{\phi}^m_{\mathcal{T}}$ of Proposition 3.3.[15] By the definition of the bundles $\mathcal{V}^{k;m}_{1,k;\mathcal{T}}$ in Subsection 3.3, the restriction of $\varphi^m_{\mathcal{T}}$ to each space $\mathcal{U}^{m'}_{\mathcal{T}';1}(\mathbb{P}^n;J)\cap U^m_{\mathcal{T}}$ is a section of $\mathcal{V}^{k;m'}_{1,k;\mathcal{T}'}$, for all $\mathcal{T}'$ and m' as in Lemma 2.4. We can also insure that the restriction of $\varphi^m_{\mathcal{T}}$ to $\mathcal{U}^{m'}_{\mathcal{T}';1}(\mathbb{P}^n;J)\cap U^m_{\mathcal{T}}$ is smooth and transverse to the zero set in $\mathcal{V}^{d;m'}_{1,k;\mathcal{T}'}$. Finally, by using a partition of unity and the newly constructed section $\varphi^m_{\mathcal{T}}$, we can extend the section φ to a neighborhood of $\bar{\mathcal{U}}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$ in $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$, without changing it on $\bar{\mathcal{U}}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$ or on a neighborhood of $\partial\bar{\mathcal{U}}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$ in $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$. After finitely many steps, we end up with

(1) a neighborhood U of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J) - \mathfrak{M}^{\mathrm{eff}}_{1,k}(\mathbb{P}^n,d;J)$ in $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$ and

(2) a continuous multisection φ of the cone $\mathcal{V}^d_{1,k}$ over U

such that the properties (a) and (b) hold for all $\mathcal{T}'$ and m' as in Lemma 2.4.

We then extend φ in the same stratum-by-stratum way to a section over all of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$, using Lemma 3.2. In fact, the existence of such an extension is immediate from the fact that

$$\mathcal{V}^d_{1,k} \longrightarrow \mathfrak{M}^{\mathrm{eff}}_{1,k}(\mathbb{P}^n,d;J)$$

is a vector bundle. Since the real dimension of a boundary stratum $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$, with $\mathcal{T}$ as in Lemma 2.3, is at least two less than the dimension of $\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;J)$, the transversality of $\varphi|_{\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)}$ to the zero set in $\mathcal{V}^d_{1,k}$ implies $(\mathcal{V}2)$ for this stratum. Similarly, the transversality of $\varphi|_{\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)}$ to the zero set in $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}$ and equation (3.3) imply $(\mathcal{V}2)$ for each stratum $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$ of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$, with $\mathcal{T}$ and m as in Lemma 2.4. The homotopy statement of Proposition 3.1 is proved by a nearly identical construction.

The second-to-last statement of Proposition 3.1 follows from the preceding claims by the same argument as in Subsection 1.2. The final statement of Proposition 3.1 follows from the proof of the

[15] $\tilde{\phi}^m_{\mathcal{T}}$ is a trivialization of $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}$ in the normal directions to $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)\cap U$

first part of Proposition 3.1 and from the last statement of Theorem 1.6 in [Z5]. The latter states that there exists $\delta_n(d)\in\mathbb{R}^+$ with the following property. If $\underline{J}=(J_t)_{t\in[0,1]}$ is a C^1-smooth family of almost complex structures on $\mathbb{P}^n$ such that $\|J_t-J_0\|_{C^1}\le\delta_n(d)$ for all $t\in[0,1]$, then the compact moduli space

$$\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;\underline{J})\equiv\bigcup_{t\in[0,1]}t\times\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J_t)\ \subset[0,1]\times\mathfrak{X}_{1,k}(\mathbb{P}^n,d)$$

has the general topological structure of a variety with boundary. It is stratified by the smooth orbifolds with boundary,

$$\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;\underline{J})\equiv\bigcup_{t\in[0,1]}t\times\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J_t)\quad\text{and}\quad\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;\underline{J})\equiv\bigcup_{t\in[0,1]}t\times\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J_t),$$

each of dimension one greater than the corresponding dimension given by Lemmas 2.3 or 2.4. By the same argument as above, we can construct a multisection Φ of the cone $\mathcal{V}^d_{1,k}$ over $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;\underline{J})$ such that

($\mathcal{V}1''$) $\Phi|_{\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;\underline{J})}$ is smooth and transverse to the zero set in $\mathcal{V}^d_{1,k}|_{\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;\underline{J})}$;

($\mathcal{V}2''$) the intersection of $\Phi^{-1}(0)$ with each boundary stratum $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;\underline{J})$ and $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;\underline{J})$ of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;\underline{J})$ is a smooth suborbifold of the stratum of real dimension of at most $2(d(n+1-a)+k)-1$,

and the restrictions $\varphi_0\equiv\Phi|_{\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J_0)}$ and $\varphi_1\equiv\Phi|_{\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J_1)}$ satisfy conditions ($\mathcal{V}1$) and ($\mathcal{V}2$).

If $\mathcal{W}\longrightarrow\mathfrak{X}_{1,k}(\mathbb{P}^n,d)$ is a complex vector bundle of rank $d(n+1-a)+k$ as in Proposition 3.1, we can choose a continuous section F of $\mathcal{W}$ over $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;\underline{J})$ such that

($\Phi\mathcal{W}1$) $\Phi^{-1}(0)\cap F^{-1}(0)\subset\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;\underline{J})$;

($\Phi\mathcal{W}2$) $F|_{\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;\underline{J})}$ is smooth and transverse to the zero set in $\mathcal{W}|_{\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;\underline{J})}$;

($\Phi\mathcal{W}3$) $F^{-1}(0)$ intersects $\Phi^{-1}(0)$ transversely in $\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;\underline{J})$,

($\Phi\mathcal{W}4$) $f_t^{-1}(0)$ intersects $\varphi_t^{-1}(0)$ transversely in $\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;J_t)$ for $t=0,1$, where $f_t\equiv F|_{\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;J_t)}$.

It follows that $\Phi^{-1}(0)\cap F^{-1}(0)$ is a compact oriented one-dimensional suborbifold of $\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d;\underline{J})$ and

$$\begin{aligned}\partial\big(\Phi^{-1}(0)\cap F^{-1}(0)\big)&=\varphi_1^{-1}(0)\cap f_1^{-1}(0)-\varphi_0^{-1}(0)\cap f_0^{-1}(0)\\ \Longrightarrow\qquad {}^{\pm}\big|\varphi_1^{-1}(0)\cap f_1^{-1}(0)\big|&={}^{\pm}\big|\varphi_0^{-1}(0)\cap f_0^{-1}(0)\big|.\end{aligned}$$

This equality implies the last claim of Proposition 3.1.

3.2 The Local Structure of the Cone $\mathcal{V}^d_{1,k}\longrightarrow\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$, I

In this subsection we describe the structure of the cone $\mathcal{V}^d_{1,k}\longrightarrow\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)$ over a neighborhood of each stratum $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ of Lemma 2.3. Lemma 3.2 is essentially equivalent to the statement that

$$\mathcal{V}^d_{1,k}\longrightarrow\mathfrak{M}^{\text{eff}}_{1,k}(\mathbb{P}^n,d;J)$$

is a vector bundle. The proof of this lemma begins to introduce the setup needed to carry out the delicate analysis of Section 4 for the case of the boundary strata $\mathcal{U}^m_{\mathcal{T}}(\mathbb{P}^n;J)$ of Lemma 2.4.

Lemma 3.2 *If n, k, d, a, $\mathfrak{L}$, and $\mathcal{V}_{1,k}^d$ are as in Theorem 1.2, there exists $\delta_n(d)\in\mathbb{R}^+$ with the following property. If J is an almost complex structure on $\mathbb{P}^n$, such that $\|J-J_0\|_{C^1}<\delta_n(d)$, and*

$$\mathcal{T}=([k],I,\aleph;j,\underline{d})$$

is a bubble type such that $\sum_{i\in I}d_i=d$ and $d_i\neq 0$ for some minimal element i of I, then the requirements of Lemma 2.3 are satisfied. Furthermore, the restriction of $\mathcal{V}_{1,k}^d$ to $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ is a smooth complex vector orbibundle of rank da. Finally, there exists a continuous vector-bundle isomorphism

$$\tilde{\phi}_{\mathcal{T}}\colon \pi_{\mathcal{FT}_\delta}^*\big(\mathcal{V}_{1,k}^d\big|_{\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)}\big)\longrightarrow \mathcal{V}_{1,k}^d\big|_{\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)\cap U_{\mathcal{T}}},$$

covering the homeomorphism $\phi_{\mathcal{T}}$ of Lemma 2.3, such that $\tilde{\phi}_{\mathcal{T}}$ is the identity over $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ and is smooth over $\mathcal{FT}_\delta^\emptyset$.

The restriction $\mathcal{V}_{1,k}^d\longrightarrow\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ is the quotient of the cone $\tilde{\mathcal{V}}_{1,k}^d\longrightarrow\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ by the group $\mathrm{Aut}(\mathcal{T})\propto(S^1)^{\hat{I}}$; see Subsection 2.2 for notation. The fiber of $\tilde{\mathcal{V}}_{1,k}^d$ at a point $b=(\Sigma_b,u_b)$ of $\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ is the Dolbeault cohomology group $H^0_{\bar\partial}(\Sigma_b;u_b^*\mathfrak{L})$, for a holomorphic structure in the bundle $u_b^*\mathfrak{L}$. Since $d_i\neq0$ for some minimal element $i\in I$, the degree of the restriction of $u_b^*\mathfrak{L}$ to the principal curve of Σ_b is positive. Thus, by an argument similar to Subsections 6.2 and 6.3 in [Z2],

$$H^1_{\bar\partial}(\Sigma_b;u_b^*\mathfrak{L})=\{0\}\qquad\Longrightarrow\qquad \dim\tilde{\mathcal{V}}_{1,k}^d|_b=\dim H^0_{\bar\partial}(\Sigma_b;u_b^*\mathfrak{L})=\mathrm{ind}\,\bar\partial_{\nabla,b}=da.$$

Since the holomorphic structure in the line bundles $u_b^*\mathfrak{L}$ varies smoothly with $b\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$, it follows that $\tilde{\mathcal{V}}_{1,k}^d\longrightarrow\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ is a smooth complex vector bundle of rank da and $\mathcal{V}_{1,k}^d\longrightarrow\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ is a smooth complex vector orbibundle of rank da.

We construct a lift $\tilde{\phi}_{\mathcal{T}}$ of $\phi_{\mathcal{T}}$ to the cone $\mathcal{V}_{1,k}^d\longrightarrow\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ as follows. For each sufficiently small element $\upsilon=(b,v)$ of $\tilde{\mathcal{F}}\mathcal{T}^\emptyset$, we define the maps

$$\begin{aligned} R_\upsilon\colon \Gamma(b;\mathfrak{L})\equiv L_1^p\big(\Sigma_b;u_b^*\mathfrak{L}\big)&\longrightarrow\Gamma(\upsilon;\mathfrak{L})\equiv L_1^p\big(\Sigma_\upsilon;u_\upsilon^*\mathfrak{L}\big) &&\text{by}\quad \big\{R_\upsilon\xi\big\}(z)=\xi\big(q_\upsilon(z)\big)\quad\text{if } z\in\Sigma_\upsilon,\\ \Pi_\upsilon\colon \Gamma\big(\upsilon;\mathfrak{L}\big)&\longrightarrow\tilde\Gamma(\upsilon;\mathfrak{L})\equiv L_1^p\big(\Sigma_\upsilon;\tilde{u}_\upsilon^*\mathfrak{L}\big) &&\text{by}\quad \big\{\Pi_\upsilon\xi\big\}(z)=\Pi_{\zeta_\upsilon(z)}\xi(z)\quad\text{if } z\in\Sigma_\upsilon, \end{aligned}$$

where $\Pi_{\zeta_\upsilon(z)}\xi(z)$ is the ∇-parallel transport of $\xi(z)$ along the g_n-geodesic

$$\gamma_{\zeta_\upsilon(z)}\colon[0,1]\longrightarrow\mathbb{P}^n,\qquad \tau\longrightarrow\exp_{u_\upsilon(z)}\tau\zeta_\upsilon(z),$$

and $\zeta_\upsilon\in\Gamma(\upsilon)$ is as in Subsection 2.3. As in Subsection 1.3, we use the [LT]-modified L_1^p- and L^p-Sobolev norms, defined in the present setting as in Subsection 3.3 of [Z3]. By a direct computation, for some $C\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$,

$$\big\|\bar\partial_{\nabla,b(\upsilon)}R_\upsilon\xi\big\|_{\upsilon,p}\le C(b)|\upsilon|^{1/p}\|\xi\|_{b,p,1}\quad\forall\,\xi\in\Gamma_-(b;\mathfrak{L})\equiv\ker\bar\partial_{\nabla,b}\qquad\text{and}\tag{3.1}$$

$$\begin{aligned}\big\|\Pi_\upsilon^{-1}\circ\bar\partial_{\nabla,\tilde b(\upsilon)}\circ\Pi_\upsilon\xi-\bar\partial_{\nabla,b(\upsilon)}\xi\big\|_{\upsilon,p}&\le C'(b)\|\zeta\|_{\upsilon,p,1}^2\|\xi\|_{\upsilon,p,1}\\ &\le C(b)|\upsilon|^{2/p}\|\xi\|_{b,p,1}\qquad\forall\,\xi\in\Gamma(\upsilon;\mathfrak{L});\end{aligned}\tag{3.2}$$

see the proof of Corollary 2.3 in [Z1] for the first inequality in (3.2). We denote by $\Gamma_-(\upsilon;\mathfrak{L})$ the image of $\Gamma_-(b;\mathfrak{L})$ under the map R_υ and by $\Gamma_+(\upsilon;\mathfrak{L})$ its L^2-orthogonal complement in $\Gamma(\upsilon;\mathfrak{L})$. Since the operator

$$\bar\partial_{\nabla,b}\colon\Gamma(b;\mathfrak{L})\longrightarrow\Gamma^{0,1}(b;\mathfrak{L})\equiv L^p(\Sigma_b;\Lambda^{0,1}_{\mathfrak{i},j_b}T^*\Sigma_b\otimes u_b^*\mathfrak{L})$$

is surjective for all $b\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$, similarly to Subsection 2.3 the operator

$$\bar{\partial}_{\nabla,b(\upsilon)}:\Gamma_+(\upsilon;\mathfrak{L})\longrightarrow\Gamma^{0,1}(\upsilon;\mathfrak{L})\equiv L^p(\Sigma_\upsilon;\Lambda^{0,1}_{\mathfrak{i},j_\upsilon}T^*\Sigma_\upsilon\otimes u_\upsilon^*\mathfrak{L})$$

is an isomorphism if υ is sufficiently small. Its norm and the norm of its inverse depend only on $[b]\in\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$. Thus, by (3.1) and (3.2), for every $\xi\in\Gamma_-(b;\mathfrak{L})$ there exists a unique $\xi_+(\upsilon)\in\Gamma_+(\upsilon;\mathfrak{L})$ such that

$$\Pi_\upsilon^{-1}\circ\bar{\partial}_{\nabla,\tilde{b}(\upsilon)}\circ\Pi_\upsilon\big(R_\upsilon\xi+\xi_+(\upsilon)\big)=0\qquad\Longleftrightarrow\qquad\Pi_\upsilon\big(R_\upsilon\xi+\xi_+(\upsilon)\big)\in\ker\bar{\partial}_{\nabla,\tilde{b}(\upsilon)}.$$

Furthermore,

$$\|\xi_+(\upsilon)\|_{\upsilon,p,1}\le C(b)|\upsilon|^{2/p}\|\xi\|_{b,p,1},$$

for some $C\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$. We can thus define a smooth lift $\tilde{\phi}_{\mathcal{T}}$ of the diffeomorphism on $\phi_{\mathcal{T}}|_{\mathcal{FT}_\delta^\emptyset}$ by

$$\tilde{\phi}_{\mathcal{T}}:\pi^*_{\mathcal{FT}_\delta^\emptyset}\big(\mathcal{V}_{1,k}^d|_{\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)}\big)\longrightarrow\mathcal{V}_{1,k}^d|_{\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)\cap U_{\mathcal{T}}},\qquad\tilde{\phi}_{\mathcal{T}}([\upsilon;\xi])=[\tilde{R}_\upsilon\xi],$$
$$\text{where}\qquad\tilde{R}_\upsilon\xi=\Pi_\upsilon\big(R_\upsilon\xi+\xi_+(\upsilon)\big).$$

This map extends to a continuous bundle homomorphism

$$\tilde{\phi}_{\mathcal{T}}:\pi^*_{\mathcal{FT}_\delta}\big(\mathcal{V}_{1,k}^d|_{\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)}\big)\longrightarrow\mathcal{V}_{1,k}^d|_{\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)\cap U_{\mathcal{T}}},$$

as can be seen by an argument similar to Subsections 3.9 and 4.1 in [Z3].

3.3 The Local Structure of the Cone $\mathcal{V}_{1,k}^d\longrightarrow\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$, II

This subsection presents the central statements of the paper concerning the structure of the cone $\mathcal{V}_{1,k}^d$ along the complement of the dense open subset

$$\mathfrak{M}_{1,k}^{\mathrm{eff}}(\mathbb{P}^n,d;J)\subset\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J).$$

This is precisely where the singularities (failure to be a vector bundle) of the cone

$$\mathcal{V}_{1,k}^d\longrightarrow\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)$$

arise, as the rank of $\mathcal{V}_{1,k}^d$ on the complement of $\mathfrak{M}_{1,k}^{\mathrm{eff}}(\mathbb{P}^n,d;J)$ is one higher than on $\mathfrak{M}_{1,k}^{\mathrm{eff}}(\mathbb{P}^n,d;J)$. Proposition 3.3 is the analogue of Lemma 2.4 for the cone $\mathcal{V}_{1,k}^d$. Lemma 3.4 can be viewed as a condensed version of Proposition 3.3. The technical parts of the proof of these two results are the subject of Section 4.

Proposition 3.3 *If n, k, d, a, $\mathfrak{L}$, and $\mathcal{V}_{1,k}^d$ are as in Theorem 1.2, there exists $\delta_n(d)\in\mathbb{R}^+$ with the following property. If J is an almost complex structure on $\mathbb{P}^n$ such that $\|J-J_0\|_{C^1}<\delta_n(d)$, then the requirements of Lemma 2.4 and of Lemma 3.2 are satisfied for all appropriate bubble types. Furthermore, if*

$$\mathcal{T}=([k],I,\aleph;j,\underline{d})$$

is a bubble type such that $\sum_{i\in I}d_i=d$ *and* $d_i=0$ *for all minimal elements* i *of* I*, then the restriction of* $\mathcal{V}_{1,k}^d$ *to* $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ *is a smooth complex vector orbibundle of rank* $da+1$*. In addition, for every integer*

$$m\in\big(\max(|\chi(\mathcal{T})|-n,1),|\chi(\mathcal{T}|\big),$$

there exist a neighborhood $U_{\mathcal{T}}^m$ *of* $\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)$ *in* $\mathfrak{X}_{1,k}(\mathbb{P}^n,d)$ *and a topological vector orbibundle*

$$\mathcal{V}_{1,k;\mathcal{T}}^{d;m}\longrightarrow\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)\cap U_{\mathcal{T}}^m$$

contained in the cone $\mathcal{V}_{1,k}^d$ *such that*

$$\mathcal{V}_{1,k;\mathcal{T}}^{d;m}\longrightarrow\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)\cap U_{\mathcal{T}}^m$$

is a smooth vector orbibundle and

$$\text{rk }\mathcal{V}_{1,k;\mathcal{T}}^{d;m}=da+1-m>\frac{1}{2}\dim\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)-\big(d(n+1-a)+k\big).\tag{3.3}$$

There also exists a continuous vector-bundle isomorphism

$$\tilde{\phi}_{\mathcal{T}}^m:\pi_{\mathcal{F}^1\mathcal{T}_\delta}^*\big(\mathcal{V}_{1,k;\mathcal{T}}^{d;m}\big|_{\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)\cap U_{\mathcal{T}}^m}\big)\longrightarrow\mathcal{V}_{1,k;\mathcal{T}}^{d;m}\big|_{\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)\cap U_{\mathcal{T}}^m},$$

covering the homeomorphism $\phi_{\mathcal{T}}$ *of Lemma 2.4, such that* $\tilde{\phi}_{\mathcal{T}}^m$ *is the identity over* $\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)$*. Finally,*

$$\mathcal{U}_{\mathcal{T}';1}^{m'}(\mathbb{P}^n;J)\cap\overline{\mathcal{U}}_{\mathcal{T};1}^m(\mathbb{P}^n;J)\neq\emptyset\quad\Longrightarrow\quad\mathcal{V}_{1,k;\mathcal{T}'}^{d;m'}\big|_{\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)\cap U_{\mathcal{T}'}^{m'}}\subset\mathcal{V}_{1,k;\mathcal{T}}^{d;m}\big|_{\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)\cap U_{\mathcal{T}'}^{m'}}.$$

The restriction of every element of $\mathcal{V}_{1,k}^d|_b$ to the domain of the image of b under the projection onto the first component in the decomposition (2.5) is a constant function. Thus, every element of $\mathcal{V}_{1,k}^d|_b$ is determined by its restriction to the domain of the image of b under the projection onto the second component in (2.5). The statement concerning the restriction $\mathcal{V}_{1,k}^d$ to $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)$ in Proposition 3.3 now follows by the same argument as for the corresponding statement in Lemma 3.2, but applied to the second component in the decomposition (2.5). The index in this case is $da+1$.

The bundle $\mathcal{V}_{1,k;\mathcal{T}}^{d;m}\longrightarrow\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)\cap U_{\mathcal{T}}^m$ is not unique. However, its restriction to $\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)$ is:

$$\mathcal{V}_{1,k;\mathcal{T}}^{d;m}|_{\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)}\equiv\big\{\xi\in\mathcal{V}_{1,k}^d|_b\colon b\in\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J);\ \text{if } b_r\in\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d;J)\text{ and }\lim_{r\longrightarrow\infty}b_r=b\in\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J),\\ \text{then }\exists\,\xi_r\in\mathcal{V}_{1,k}^d|_{b_r}\text{ s.t. }\lim_{r\longrightarrow\infty}\xi_r=\xi\big\}.$$

In other words, $\mathcal{V}_{1,k;\mathcal{T}}^{k;m}|_{\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)}$ is the largest subspace of $\mathcal{V}_{1,k}^d|_{\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)}$ with the property that a continuous lift

$$\tilde{\phi}_{\mathcal{T}}:\pi_{\mathcal{F}^1\mathcal{T}_\delta}^*\big(\mathcal{V}_{1,k;\mathcal{T}}^{d;m}\big|_{\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)\cap U_{\mathcal{T}}^m}\big)\longrightarrow\mathcal{V}_{1,k;\mathcal{T}}^{d;m}\big|_{\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d;J)\cap U_{\mathcal{T}}^m}$$

of $\phi_{\mathcal{T}}$ that restricts to the identity over $\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)$ can possibly exist for a vector-bundle extension for the subspace $\mathcal{V}_{1,k;\mathcal{T}}^{d;m}|_{\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)}$ to a neighborhood of $\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)$ in $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)$. The next lemma describes the subspace $\mathcal{V}_{1,k;\mathcal{T}}^{d;m}|_{\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)}$ of $\mathcal{V}_{1,k}^d|_{\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)}$ explicitly. Let

$$\tilde{\mathcal{F}}^1\mathcal{T}=\big\{\upsilon\in\tilde{\mathcal{F}}\mathcal{T}:[\upsilon]\in\mathcal{F}^1\mathcal{T}\big\}.$$

Lemma 3.4 *Suppose n, k, d, a, $\mathfrak{L}$, $\mathcal{V}_{1,k}^d$, J, and $\mathcal{T}$ are as in the first and third sentences of Proposition 3.3. If $b\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$, $\xi\in\tilde{\mathcal{V}}_{1,k}^d|_b$, and $\upsilon_r\in\tilde{\mathcal{F}}^1\mathcal{T}^\emptyset$ is a sequence of gluing parameters such that*

$$\lim_{r\longrightarrow\infty}\upsilon_r=b \quad \textit{and} \quad \lim_{r\longrightarrow\infty}\big[\big(\rho_i(\upsilon_r)\big)_{i\in\chi(\mathcal{T})}\big]=[w]\in\mathbb{P}\tilde{\mathfrak{F}}\mathcal{T}|_b,$$

then

$$\exists\ [\xi_r]\in\tilde{\mathcal{V}}_{1,k}^k|_{\phi_{\mathcal{T}}([\upsilon_r])}\ \text{s.t.}\ \lim_{r\longrightarrow\infty}[\xi_r]=[\xi] \qquad\Longleftrightarrow\qquad \mathfrak{D}_{\mathcal{T}}(\xi\otimes w)=0.$$

Therefore,

$$\begin{gathered}\mathcal{V}_{1,k;\mathcal{T}}^{d;m}|_{\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)}=\big\{\xi\in\mathcal{V}_{1,k}^d|_{[b]}:[b]\in\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J);\ \mathfrak{D}_{\mathcal{T}}(\xi\otimes w)=0\ \forall w\in\tilde{\mathfrak{F}}^1\mathcal{T}_b\big\}, \\ \textit{where}\qquad \tilde{\mathfrak{F}}^1\mathcal{T}_b=\big\{w\in\tilde{\mathfrak{F}}\mathcal{T}_b:\mathcal{D}_{\mathcal{T}}w=0\big\}.\end{gathered}\tag{3.4}$$

Thus, $\mathcal{V}_{1,k;\mathcal{T}}^{d;m}|_{\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)}\longrightarrow\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)$ is a smooth complex vector orbibundle of rank $da+1-m$.

The bundle map $\mathfrak{D}_{\mathcal{T}}$ constructed at the end of Subsection 2.2 depends on the choice of connection in the bundle $\mathfrak{L}\longrightarrow\mathbb{P}^n$. It may appear that so do the first two statements of Lemma 3.4. This is however not the case for the following reason. Suppose

$$b\equiv(\Sigma_b,u_b)\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J), \qquad \xi\in\tilde{\mathcal{V}}_{1,k}^d|_b, \qquad \upsilon_r\in\tilde{\mathcal{F}}^1\mathcal{T}^\emptyset, \quad \text{and} \quad w\equiv(w_i)_{i\in\chi(\mathcal{T})}\in\tilde{\mathfrak{F}}\mathcal{T}|_b$$

are as in Lemma 3.4. Then, by the definition of $\tilde{\mathcal{F}}^1\mathcal{T}^\emptyset$ in Lemma 2.4,

$$\mathcal{D}_{\mathcal{T}}(b,w)\equiv\sum_{i\in\chi(\mathcal{T})}\psi_{x_{h(i)}(b)}(w_i)\cdot_J du_{b,i}|_\infty e_\infty=0\in T_{\mathrm{ev}_P(b)}\mathbb{P}^n \qquad \forall\,\psi\in\mathbb{E}_{\pi_P(b)}.\tag{3.5}$$

On the other hand, since the map u_b is constant on every component $\Sigma_{b,h}$ of the domain Σ_b of b with $h<i$ for some $i\in\chi(\mathcal{T})$, ξ is a holomorphic function on $\Sigma_{b,h}$ and thus must be a constant $\xi_P\in\mathfrak{L}_{\mathrm{ev}_P(b)}$. It follows that

$$\xi_{i_1}(\infty)=\xi_{i_2}(\infty)=\xi_P \qquad \forall\ i_1,i_2\in\chi(\mathcal{T}).\tag{3.6}$$

Suppose that ∇' is a connection in the line bundle $\mathfrak{L}\longrightarrow\mathbb{P}^n$ that induces the same $\bar{\partial}$-operator in the line bundle $u_b^*\mathfrak{L}\longrightarrow\Sigma_b$ as the connection ∇; see Subsection 1.3. Then, there exists a complex-valued one-form θ on $\mathbb{P}^n$ such that

$$\nabla_v\zeta-\nabla'_v\zeta=(\theta_q v)\cdot\zeta(z) \quad \forall\,q\in\mathbb{P}^n,\ v\in T_q\mathbb{P}^n,\ \zeta\in\Gamma(\mathbb{P}^n;\mathfrak{L}), \quad \text{and} \quad u_b^*\theta\circ j_b=\mathfrak{i}\cdot u_b^*\theta.\tag{3.7}$$

Thus, if $\mathfrak{D}_{\mathcal{T}}$ and $\mathfrak{D}'_{\mathcal{T}}$ are the bundle maps corresponding to the connections ∇ and ∇' as at the end of Subsection 2.2,

$$\begin{split}\big\{\mathfrak{D}_{\mathcal{T}}(\xi\otimes w)-\mathfrak{D}'_{\mathcal{T}}(\xi\otimes w)\big\}(\psi)&=\sum_{i\in\chi(\mathcal{T})}\psi_{x_{h(i)}(b)}(w_i)\cdot\big(\theta_{\mathrm{ev}_P(b)}(du_{b,i}|_\infty e_\infty)\big)\cdot\xi_i(\infty)\\ &=\theta_{\mathrm{ev}_P(b)}\Big(\sum_{i\in\chi(\mathcal{T})}\psi_{x_{h(i)}(b)}(w_i)\cdot_J(du_{b,i}|_\infty e_\infty)\Big)\cdot\xi_P=0.\end{split}\tag{3.8}$$

The middle equality above follows from (3.6), the second condition in (3.7), and the assumption that u_b is a J-holomorphic map. The last equality above is an immediate consequence of (3.5).

More generally, the middle equality in (3.8) implies that the expression $\mathfrak{D}_{\mathcal{T}}(\xi\otimes w)$ is intrinsically defined whenever $w\in\tilde{\mathfrak{F}}^1\mathcal{T}$.

The second statement of Lemma 3.4 follows immediately from the definition of $\mathcal{V}_{1,k;\mathcal{T}}^{d;m}|_{\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)}$, the first statement of Lemma 3.4, and the last statement of Lemma 2.4. For the final statement of Lemma 3.4, let

$$\tilde{\mathcal{U}}_{\mathcal{T};1}^m(\mathbb{P}^n;J)=\big\{b\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)\colon [b]\in\mathcal{U}_{\mathcal{T};1}^m(\mathbb{P}^n;J)\big\}.$$

By the proof of Lemma 2.4,

$$\tilde{\mathfrak{F}}^1\mathcal{T}\longrightarrow\tilde{\mathcal{U}}_{\mathcal{T};1}^m(\mathbb{P}^n;J)$$

is a vector bundle of rank m. On the other hand, by the same argument as in Subsection 6.2 of [Z2], for every $b\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$ and $i\in\chi(\mathcal{T})$, the linear map

$$\big\{\xi=(\xi_h)_{h\in I}\in\tilde{\mathcal{V}}_{1,k}^d|_b\colon \xi_i(\infty)=0\big\}\longrightarrow\mathfrak{L}_{\mathrm{ev}_P(b)},\qquad \xi\longrightarrow\nabla_{e_\infty}\xi_i,$$

is surjective. It follows that the linear bundle map

$$\tilde{\mathcal{V}}_{1,k}^d\longrightarrow\mathrm{Hom}(\mathfrak{F}^1\mathcal{T},\pi_P^*\mathbb{E}^*\otimes\mathrm{ev}_P^*\mathfrak{L})$$

over $\tilde{\mathcal{U}}_{\mathcal{T};1}^m(\mathbb{P}^n;J)$ induced by $\mathfrak{D}_{\mathcal{T}}$ is surjective on every fiber. Thus, its kernel is a smooth vector bundle of rank

$$\mathrm{rk}\ \mathcal{V}_{1,k;\mathcal{T}}^{d;m}=\mathrm{rk}\ \mathcal{V}_{1,k}^d-\mathrm{rk}\ \mathrm{Hom}(\mathfrak{F}^1\mathcal{T},\pi_P^*\mathbb{E}^*\otimes\mathrm{ev}_P^*\mathfrak{L})=da+1-m,$$

as claimed in the last statement of Lemma 3.4.

We prove the remaining claims of Proposition 3.3 and Lemma 3.4 at the end of Section 4. An element $\xi\in\mathcal{V}_{1,k}^d|_{[\Sigma,u]}$ can be viewed as a map $\tilde{u}\colon\Sigma\longrightarrow\mathfrak{L}$. We will show in particular that the obstruction to smoothing out $\tilde{u}$ in the direction $\upsilon\in\mathcal{F}^1\mathcal{T}^{\emptyset}$ is precisely $\mathfrak{D}_{\mathcal{T}}(\xi\otimes\rho(\upsilon))$.

4 A Gluing Construction

4.1 Smoothing Stable Maps

We begin by reviewing the gluing construction of Section 6 in [Z5]. If $b=(\Sigma_b,u_b)$ is any genus-one bubble map such that $u_b|_{\Sigma_{b;P}}$ is constant, let $\Sigma_b^0\subset\Sigma_b$ be the maximum connected union of the irreducible components of Σ_b such that $\Sigma_{b;P}\subset\Sigma_b^0$ and $u_b|_{\Sigma_b^0}$ is constant. If $u_b|_{\Sigma_{b;P}}$ is not constant, let $\Sigma_b^0=\emptyset$. We put

$$\begin{aligned}\Gamma_B(b)&=\big\{\zeta\in\Gamma(\Sigma_b;u_b^*T\mathbb{P}^n)\colon \zeta|_{\Sigma_b^0}=0\big\},\\ \Gamma_B(b;\mathfrak{L})&=\big\{\xi\in\Gamma(\Sigma_b;u_b^*\mathfrak{L})\colon \xi|_{\Sigma_b^0}=0\big\},\qquad\text{and}\\ \Gamma_B^{0,1}(b;\mathfrak{L})&=\big\{\eta\in\Gamma(\Sigma_b;\Lambda_{\mathfrak{i},j}^{0,1}T^*\Sigma_b\otimes u_b^*\mathfrak{L})\colon \eta|_{\Sigma_b^0}=0\big\}.\end{aligned}$$

Suppose $\mathcal{T} = ([k], I, \aleph; j, \underline{d})$ is a bubble type as in Proposition 3.3, i.e. $d_i = 0$ for all $i \in I_0$, where $I_0 \subset I$ is the subset of minimal elements. We put

$$\chi^0(\mathcal{T}) = \{h \in I : d_i = 0 \ \forall\, i \le h\}, \qquad \chi^-(\mathcal{T}) = \bigcup_{i \in \chi(\mathcal{T})} \{h \in \hat{I} : h < i\} \subset \chi^0(\mathcal{T}),$$

$$\langle \mathcal{T} \rangle = \max\big\{ \big|\{h \in \hat{I} : h \le i\}\big| : i \in \chi(\mathcal{T})\big\} \ge 1, \qquad \mathcal{I}^*_{\langle \mathcal{T} \rangle} = \chi(\mathcal{T}), \qquad \mathcal{I}_{\langle \mathcal{T} \rangle} = \hat{I} - \chi(\mathcal{T}) - \chi^-(\mathcal{T}) - I_1,$$

where $I_1 \subset I$ is as in Subsection 2.2. For each $s \in \{0\} \cup [\langle \mathcal{T} \rangle - 1]$, let

$$\mathcal{I}_s = \big\{ i \in \chi(\mathcal{T}) \cup \chi^-(\mathcal{T}) : \big|\{h \in \hat{I} : h < i\}\big| = s \big\}, \qquad \mathcal{I}^*_s = \mathcal{I}_s \cup \bigcup_{t=0}^{s-1} \big(\mathcal{I}_t \cap \chi(\mathcal{T})\big).$$

In the case of Figure 4 on page 22,

$$\langle \mathcal{T} \rangle = 2, \qquad \mathcal{I}_0 = \{h_1, h_3\}, \qquad \mathcal{I}_1 = \{h_4, h_5\}, \qquad \mathcal{I}_2 = \{h_2\}.$$

In general, the set $\mathcal{I}_{\langle \mathcal{T} \rangle}$ could be empty, but the sets $\mathcal{I}_s$ with $s < \langle \mathcal{T} \rangle$ never are.

If b is a bubble map of type $\mathcal{T}$ as in Subsection 2.2 and $s \in [\langle \mathcal{T} \rangle]$, we put

$$\Sigma_b^{(s)} = \bigcup_{i \in \chi^0(\mathcal{T}) - \chi^-(\mathcal{T})} \Sigma_{b,i} \ \cup \bigcup_{h \in \mathcal{I}^*_{s-1}} \bigcup_{i < h} \Sigma_{b,i} \ \subset \Sigma_b.$$

If $h \in \mathcal{I}^*_{s-1}$, let

$$\Sigma_b^h = \bigcup_{h \le i} \Sigma_{b,i} \subset \Sigma_b, \qquad \chi_h(\mathcal{T}) = \big\{ i \in \chi(\mathcal{T}) : h \le i \big\}, \qquad \tilde{\mathfrak{F}}_h \mathcal{T} = \mathcal{U}^{(0)}_{\mathcal{T}}(X; J) \times \mathbb{C}^{\chi_h(\mathcal{T})}.$$

If in addition $\upsilon = (b, v) \in \widetilde{\mathcal{F}}\mathcal{T}$, we put

$$\rho_{s;h}(\upsilon) = \big(b, (\rho_{h;i}(\upsilon))_{i \in \chi_h(\mathcal{T})}\big) \in \tilde{\mathfrak{F}}_h \mathcal{T}, \qquad \text{where} \quad \rho_{h;i}(\upsilon) = \prod_{h < h' \le i} v_{h'} \in \mathbb{C};$$

$$\mathcal{I}^0_{s-1}(\upsilon) = \big\{ h \in \mathcal{I}^*_{s-1} : \rho_{s;h}(\upsilon) = 0 \big\};$$

see Subsection 2.2 for notation.

If $\upsilon = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}$, let

$$\upsilon_0 = \big(b, v_\aleph, (v_h)_{h \in I_1}\big) \qquad \text{if} \quad \upsilon = \big(b, v_\aleph, (v_h)_{h \in \hat{I}}\big).$$

Let $\upsilon_{\langle 0 \rangle} = \upsilon$ and $\upsilon_{\langle \langle \mathcal{T} \rangle + 1 \rangle} = b$. If $s \in [\langle \mathcal{T} \rangle]$, we put

$$\upsilon_s = \big(b, (v_h)_{h \in \mathcal{I}_s}\big) \qquad \text{and} \qquad \upsilon_{\langle s \rangle} = \big(b, (v_h)_{h \in \mathcal{I}_t, t \ge s}\big).$$

The component $\upsilon_{\langle \mathcal{T} \rangle}$ of υ consists of smoothings at the nodes of Σ_b that do not lie on the principal component $\Sigma_{b;P}$ of Σ_b and do not lie between $\Sigma_{b;P}$ and the bubble components indexed by the set $\chi(\mathcal{T})$. In Section 6 of [Z5], these nodes are smoothed out at the first step of the gluing construction, as specified by $\upsilon_{\langle \mathcal{T} \rangle}$. After that, the nodes indexed by the set $\mathcal{I}_{\langle \mathcal{T} \rangle - 1}$ are smoothed out, and so on. At the last step, the nodes that lie on the principle component are smoothed according to υ_0, provided $\upsilon\in\tilde{\mathcal{F}}^1\mathcal{T}^{\emptyset}$ is sufficiently small.

If $\upsilon\in\tilde{\mathcal{F}}^1\mathcal{T}^{\emptyset}$ is sufficiently small and $s\in\{0\}\cup[\langle\mathcal{T}\rangle]$, let

$$q_{\upsilon_{\langle s\rangle}}\colon\Sigma_{\upsilon_{\langle s\rangle}}\longrightarrow\Sigma_b$$

be the basic gluing map constructed in Subsection 2.2 of [Z3]. Via the construction of Subsection 3.3 in [Z3], the map $q_{\upsilon_{\langle s\rangle}}$ induces a metric $g_{\upsilon_{\langle s\rangle}}$ and a weight function $\rho_{\upsilon_{\langle s\rangle}}$ that define weighted L^p_1-Sobolev norms $\|\cdot\|_{\upsilon,p,1}$ on the spaces $\Gamma_B(b')$ and $\Gamma_B(b';\mathfrak{L})$ and weighted L^p-Sobolev norms $\|\cdot\|_{\upsilon,p}$ on the corresponding spaces of differentials, for any bubble map $b'=(\Sigma_{\upsilon_{\langle s\rangle}},u)$ such that u is constant on $q^{-1}_{\upsilon_{\langle s\rangle}}(\Sigma^{(s)}_b)$ if $s>0$. In this case, $(\Sigma_{\upsilon_{\langle s\rangle}},g_{\upsilon_{\langle s\rangle}})$ is obtained from Σ_b with its metric g_b by replacing the nodes of Σ_b indexed by the sets $\mathcal{I}_t$ with $t\geq s$ by thin necks. The norms $\|\cdot\|_{\upsilon,p,1}$ and $\|\cdot\|_{\upsilon,p}$ are analogous to the ones used in Section 3 of [LT]. Let

$$q_{\upsilon_s;\langle\mathcal{T}\rangle+1-s}\colon\Sigma_{\upsilon_{\langle s\rangle}}\longrightarrow\Sigma_{\upsilon_{\langle s+1\rangle}}$$

be the basic gluing map of Subsection 2.2 in [Z3] corresponding to the gluing parameter υ_s. We recall that

$$q_{\upsilon_{\langle s\rangle}}=q_{\upsilon_{\langle s+1\rangle}}\circ q_{\upsilon_s;\langle\mathcal{T}\rangle+1-s}$$

for all $s\in\{0\}\cup[\langle\mathcal{T}\rangle-1]$. If $s\in[\langle\mathcal{T}\rangle]$ and $h\in\mathcal{I}^*_{s-1}$, let

$$\Sigma^h_{\upsilon_{\langle s\rangle}}=q^{-1}_{\upsilon_{\langle s\rangle}}\big(\Sigma^h_b\big)\subset\Sigma_{\upsilon_{\langle s\rangle}}.$$

We note that $\Sigma^h_{\upsilon_{\langle s\rangle}}$ is a union of components of $\Sigma_{\upsilon_{\langle s\rangle}}$.

For any $\upsilon=(b,v)\in\tilde{\mathcal{F}}\mathcal{T}$, we put

$$\tilde{b}_{\langle\mathcal{T}\rangle+1}(\upsilon)\equiv\big(\Sigma_b,\tilde{u}_{\upsilon,\langle\mathcal{T}\rangle+1}\big)=(\Sigma_b,u_b).$$

In Section 6 of [Z5], for J sufficiently close to J_0, $\delta\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$ sufficiently small and all $\upsilon\in\widetilde{\mathcal{F}}^1\mathcal{T}^{\emptyset}_{\delta}$, we construct J-holomorphic bubble maps

$$\tilde{b}_s(\upsilon)=\big(\Sigma_{\upsilon_{\langle s\rangle}},\tilde{u}_{\upsilon,s}\big)\qquad\forall\ s=0,\ldots,\langle\mathcal{T}\rangle$$

such that the following properties are satisfied. First, for all $s\in[\langle\mathcal{T}\rangle]$,

$$\Sigma^0_{\tilde{b}_s(\upsilon)}=q^{-1}_{\upsilon_{\langle s\rangle}}\big(\Sigma^{(s)}_b\big)\qquad\text{and}\qquad\tilde{u}_{\upsilon,s}\big(\Sigma^0_{\tilde{b}_s(\upsilon)}\big)=u_b(\Sigma^0_b)\equiv\mathrm{ev}_P(b).\tag{4.1}$$

Second, for all $s\in[\langle\mathcal{T}\rangle]$,

$$\begin{aligned}&\tilde{u}_{\upsilon,s}=\exp_{u_{\upsilon,s}}\zeta_{\upsilon,s}\\ \text{for some}\qquad&\zeta_{\upsilon,s}\in\Gamma_B\big(b_s(\upsilon)\big)\ \text{ s.t. }\ \big\|\zeta_{\upsilon,s}\big\|_{\upsilon_{\langle s\rangle},p,1}\leq C(b)|\upsilon|^{1/p},\end{aligned}\tag{4.2}$$

where

$$b_s(\upsilon)=\big(\Sigma_{\upsilon_{\langle s\rangle}},u_{\upsilon,s}\big),\qquad u_{\upsilon,s}=\tilde{u}_{s+1}\circ q_{\upsilon_s;\langle\mathcal{T}\rangle+1-s}.$$

Third,

$$\begin{aligned}&\tilde{u}_{\upsilon,0}=\exp_{u_{\upsilon,0}}\zeta_{\upsilon,0}\\ \text{for some}\qquad&\zeta_{\upsilon,0}\in\Gamma_B\big(b_0(\upsilon)\big)\ \text{ s.t. }\ \big\|\zeta_{\upsilon,0}\big\|_{\upsilon,p,1}\leq C(b)\big|\rho(\upsilon)\big|,\end{aligned}\tag{4.3}$$

where

$$b_0(\upsilon) = \big(\Sigma_\upsilon, u_{\upsilon,0}\big), \qquad u_{\upsilon,0} = \tilde{u}_1 \circ \tilde{q}_{\upsilon_s;\langle\mathcal{T}\rangle+1},$$

and

$$\tilde{q}_{\upsilon_s;\langle\mathcal{T}\rangle+1} : \Sigma_\upsilon \longrightarrow \Sigma_{\upsilon_{\langle 1\rangle}}$$

is the modified gluing map corresponding to the parameter of $\delta(b)^{1/2}$ constructed in Subsection 4.2 of [Z5]. Finally, the maps $\upsilon \longrightarrow \zeta_{\upsilon,s}$ are smooth over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset$ and extend continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$. These extensions satisfy

$$\zeta_{b,s} = 0 \quad \forall\, b\in\mathcal{U}_{\mathcal{T}}^{(0)}(X;J),\ s\in\{0\}\cup[\langle\mathcal{T}\rangle] \qquad \text{and} \tag{4.4}$$

$$\zeta_{\upsilon,s}|_{\Sigma^h_{\upsilon_{\langle s\rangle}}} = 0 \quad \forall\ \upsilon\in\tilde{\mathcal{F}}^1\mathcal{T},\ s\in[\langle\mathcal{T}\rangle],\ h\in\mathcal{I}^0_{s-1}(\upsilon). \tag{4.5}$$

The homeomorphism of Lemma 2.4 is given by

$$\phi_{\mathcal{T}} : \mathcal{F}^1\mathcal{T}_\delta \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)\cap U_{\mathcal{T}}, \qquad \phi_{\mathcal{T}}\big([\upsilon]\big) = \big[\tilde{b}_0(\upsilon)\big].$$

Remark: The bubble maps $b_s(\upsilon)$ and $\tilde{b}_s(\upsilon)$ above correspond to the bubble maps $b_s(\tilde{\mu}_0(\upsilon,\zeta_{\upsilon,0}))$ and $\tilde{b}_s(\tilde{\mu}_0(\upsilon,\zeta_{\upsilon,0}))$ in Section 6 of [Z5], where $\tilde{\mu}_0(\upsilon,\zeta_{\upsilon,0})$ is the perturbation of υ constructed in Subsection 6.2 in [Z5].

4.2 Smoothing Bundle Sections, I

In this subsection we extend all but the last step of the gluing construction summarized above to the cone $\mathcal{V}^d_{1,k}$ over $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d;J)$. In contrast to the last step, these steps are unobstructed.

We will use a convenient family of connections in the line bundles $u^*\mathfrak{L}\longrightarrow\Sigma$, which is chosen in Lemma 4.1 below. First, if $b=(\Sigma_b,u_b)$ is a stable genus-one bubble map such that $u_b|_{\Sigma_{b;P}}$ is constant, g_b is a Hermitian metric in the line bundle $u_b^*\mathfrak{L}\longrightarrow\Sigma_b$, and ∇^b is a connection in $u_b^*\mathfrak{L}$, we will call the pair (g,∇)-`admissible` if

$(g\nabla 1)$ ∇^b is g_b-compatible and $\bar\partial_{\nabla,b}$-compatible;

$(g\nabla 2)$ $g_b = g_{u_b}$ and $\nabla^b = \nabla^{u_b}$ on Σ_b^0,

where g_{u_b} is the Hermitian metric in $u_b^*\mathfrak{L}$ induced from the standard metric in $\mathfrak{L}$. The second condition in $(g\nabla 1)$ means that

$$\bar\partial_{\nabla,b} \equiv \frac{1}{2}\big(\nabla^{u_b} + \mathfrak{i}\nabla^{u_b}\circ j\big) = \frac{1}{2}\big(\nabla^b + \mathfrak{i}\nabla^b\circ j\big),$$

with notation as in (1.13). If the pair (g,∇) satisfies $(g\nabla 1)$, the connection ∇^b is uniquely determined by the metric g_b. The second conditions in $(g\nabla 1)$ and in $(g\nabla 2)$ imply that the bundle map $\mathfrak{D}_{\mathcal{T}}$ does not change if it is defined using the connection ∇^b instead of ∇^{u_b}; see Subsection 2.2 and the paragraph following Lemma 3.4.

If $b\in\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^n;J)$, $\delta\in\mathbb{R}^+$, $i\in\hat{I}$, let

$$\begin{aligned}
A^-_{b,i}(\delta) &= \big\{(i,z)\in\Sigma_{b,i}=\{i\}\times S^2\colon |z|\geq\delta^{-1/2}/2\big\}\subset\Sigma_b;\\
\partial^-A^-_{b,i}(\delta) &= \big\{(i,z)\in\Sigma_{b,i}=\{i\}\times S^2\colon |z|=\delta^{-1/2}/2\big\}\subset\Sigma_b;\\
\Sigma^0_b(\delta) &= \bigcup_{i\in\chi(\mathcal{T})}A^-_{b,i}(\delta)\cup\bigcup_{h\in\chi^0(\mathcal{T})}\Sigma_{b,h}.
\end{aligned}$$

If $\upsilon\in\tilde{\mathcal{F}}\mathcal{T}$ is sufficiently small, we put

$$A^-_{\upsilon,i}(\delta)=q_\upsilon^{-1}(A^-_{b,i}(\delta))\subset\Sigma_\upsilon,\quad \partial^-A^-_{\upsilon,i}(\delta)=q_\upsilon^{-1}\big(\partial^-A^-_{b,i}(\delta)\big),\quad \Sigma^0_\upsilon(\delta)=q_\upsilon^{-1}\big(\Sigma^0_b(\delta)\big).$$

If $s\in[\langle\mathcal{T}\rangle+1]$ and $h\in\mathcal{I}^*_{s-1}$, let

$$\Sigma^{h;0}_{\upsilon_{\langle s\rangle}}(\delta)=\Sigma^0_{\upsilon_{\langle s\rangle}}(\delta)\cap\Sigma^h_{\upsilon_{\langle s\rangle}}.$$

Lemma 4.1 *If n, d, k, a, and $\mathfrak{L}$ are as in Proposition 3.3, there exists $\delta_n(d)\in\mathbb{R}^+$ such that for every almost complex structure J on $\mathbb{P}^n$, such that $\|J-J_0\|_{C^1}\leq\delta_n(d)$, and a bubble type $\mathcal{T}$ as above, there exist $\delta,C\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$ with the following property. For every*

$$\upsilon\equiv(b,v)\in\tilde{\mathcal{F}}^1\mathcal{T}^{\emptyset}_\delta\qquad\text{and}\qquad s\in[\langle\mathcal{T}\rangle+1],$$

there exist a metric $g_{\upsilon,s}$ and a connection $\nabla^{\upsilon,s}$ in the line bundle $\tilde{u}^*_{\upsilon,s}\mathfrak{L}\longrightarrow\Sigma_\upsilon$ such that
(1) all pairs $(g_{\upsilon,s},\nabla^{\upsilon,s})$ are admissible;
(2) the curvature of $\nabla^{\upsilon,s}$ vanishes on $\Sigma^0_{\upsilon_{\langle s\rangle}}(2\delta(b))$.
Furthermore, the maps $\upsilon\longrightarrow(g_{\upsilon,s},\nabla^{\upsilon,s})$ are $\mathrm{Aut}(\mathcal{T})\propto(S^1)^I$-invariant and smooth over $\tilde{\mathcal{F}}^1\mathcal{T}^{\emptyset}_\delta$. They extend continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$. The extension satisfies (1) and (2). In addition,

$$(g_{b,s},\nabla^{b,s})=(g_{b,\langle\mathcal{T}\rangle+1},\nabla^{b,\langle\mathcal{T}\rangle+1})\qquad\forall\,b\in\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^n;J),\ s\in[\langle\mathcal{T}\rangle];\tag{4.6}$$

$$(g_{\upsilon,s},\nabla^{\upsilon,s})\big|_{\Sigma^h_{\upsilon_{\langle s\rangle}}}=q^*_{\upsilon_s;\langle\mathcal{T}\rangle+1-s}(g_{\upsilon,s+1},\nabla^{\upsilon,s+1})\big|_{\Sigma^h_{\upsilon_{\langle s\rangle}}}\qquad\forall\,s\in[\langle\mathcal{T}\rangle],\ h\in\mathcal{I}^0_{s-1}(\upsilon).\tag{4.7}$$

This lemma is an analogue of Lemma 3.4 in [Z5] for the bundle $\mathfrak{L}$ and is proved in a similar way as follows. Let $\beta\colon\mathbb{R}^+\longrightarrow[0,1]$ be a smooth function such that

$$\beta(t)\in\begin{cases}0, & \text{if } t\leq1;\\ 1, & \text{if } t\geq2.\end{cases}$$

If $r\in\mathbb{R}^+$, let $\beta_r(t)=\beta(t/\sqrt{r})$. We define $\beta_b\in C^\infty(\Sigma_b;\mathbb{R})$ by

$$\beta_b(z)=\begin{cases}1, & \text{if } z\in\Sigma_{b,i},\ i\in\chi^0(\mathcal{T});\\ 1-\beta_{\delta(b)}(r(z)/2), & \text{if } z\in\Sigma_{b,i},\ i\in\chi(\mathcal{T});\\ 0, & \text{otherwise},\end{cases}\tag{4.8}$$

where $r(z)=|q_S^{-1}(z)|$ if $q_S:\mathbb{C}\longrightarrow S^2$ is the stereographic projection mapping the origin to the south pole of S^2. In other words, $\beta_b=1$ on $\Sigma^0_b(2\delta(b))$ and vanishes outside of $\Sigma^0_b(8\delta(b))\subset\Sigma_b$. Let

$\beta_\upsilon = \beta_b \circ q_\upsilon$.

For $s \in [\langle \mathcal{T} \rangle + 1]$, $h \in \mathcal{I}^*_{s-1}$, and $\upsilon \in \tilde{\mathcal{F}}^1 \mathcal{T}^\emptyset_\delta$, we use parallel transport with respect to the connection $\nabla^{\tilde{u}_{\upsilon,s}}$ along the meridians to the south pole of the sphere $\Sigma^h_{\upsilon_{\langle s \rangle}}$ to identify $\tilde{u}^*_{\upsilon,s}\mathfrak{L}$ over $\Sigma^{h;0}_{\upsilon_{\langle s \rangle}}(8\delta(b))$ with the trivial holomorphic line bundle

$$\Sigma^{h;0}_{\upsilon_{\langle s \rangle}}(8\delta(b)) \times \mathfrak{L}_{\mathrm{ev}_P(b)}.$$

A connection $\nabla^{\upsilon,s}$ with the desired properties can then be found by solving an equation of the form

$$\bar{\partial}\theta = \beta_\upsilon \Omega_{\upsilon,h}, \qquad \theta(\infty) = 0, \qquad \theta \in C^\infty(\Sigma^{h;0}_{\upsilon_{\langle s \rangle}}(8\delta(b)); \mathbb{C}), \tag{4.9}$$

where $\Omega_{\upsilon,h} \in C^\infty(\Sigma^{h;0}_\upsilon(8\delta(b)); \mathbb{C})$ is determined by υ and satisfies

$$\|\Omega_{\upsilon,h}\|_{\upsilon_{\langle s \rangle},p} \le C(b)\delta(b)^{1/p}.$$

This bound follows immediately from the definition of the set $\chi(\mathcal{T})$ and (4.2). The equation (4.9) can be viewed as an equation on $\Sigma^h_{\upsilon_{\langle s \rangle}}$, which is a two-sphere with the metric $g_{\upsilon_{\langle s \rangle}}$ arising in the pregluing construction. If $\delta(b) \in \mathbb{R}^+$ is sufficiently small, (4.9) has a unique solution $\theta_{\upsilon,h} \in C^\infty(\Sigma^h_{\upsilon_{\langle s \rangle}}; \mathbb{C})$. The curvature of the connection

$$\tilde{\nabla}^{\upsilon,h} \equiv \nabla^{\tilde{u}_{\upsilon,s}} + \beta_\upsilon \theta_{\upsilon,h}$$

then vanishes on $\Sigma^{h;0}_{\upsilon_{\langle s \rangle}}(2\delta(b))$.

Let $g_{\upsilon,h}$ be the metric in $\tilde{u}^*_{\upsilon,s}\mathfrak{L}|_{\Sigma^h_{\upsilon_{\langle s \rangle}}}$ obtained by patching the flat metric in $\tilde{u}^*_{\upsilon,s}\mathfrak{L}|_{\Sigma^{h;0}_{\upsilon_{\langle s \rangle}}(8\delta(b))}$ induced via parallel transport from $\infty \in \Sigma^h_{\upsilon_{\langle s \rangle}}$ with respect to $\tilde{\nabla}^{\upsilon,h}$ with the metric $g_{\tilde{u}_{\upsilon,s}}$ over

$$\Sigma^{h;0}_{\upsilon_{\langle s \rangle}}(8\delta(b)) - \Sigma^{h;0}_{\upsilon_{\langle s \rangle}}(4\delta(b)).$$

We put

$$g_{\upsilon,s}|_z = \begin{cases} g_{\upsilon,h}|_z, & \text{if } z \in \Sigma^h_{\upsilon_{\langle s \rangle}},\ h \in \mathcal{I}^*_{s-1}; \\ g_{\tilde{u}_{\upsilon,s}}|_z, & \text{if } z \in \Sigma^0_{\tilde{b}_s(\upsilon)}. \end{cases}$$

Since $\Sigma^0_{\tilde{b}_s(\upsilon)}$ is the union of the components of $\Sigma_{\upsilon_{\langle s \rangle}}$ that are not in $\Sigma^h_{\upsilon_{\langle s \rangle}}$ for any $h \in \mathcal{I}^*_{s-1}$ by (4.1), the metric $g_{\upsilon,s}$ on $\tilde{u}^*_{\upsilon,s}\mathfrak{L}$ is well-defined. In particular, the two definitions agree at the node of $\Sigma^h_{\upsilon_{\langle s \rangle}}$. Let $\nabla^{\upsilon,s}$ be the $\bar{\partial}_{\nabla,\tilde{u}_{\upsilon,s}}$-compatible and $g_{\upsilon,s}$-compatible connection. By construction, $\nabla^{\upsilon,s} = \tilde{\nabla}^{\upsilon,h}$ on $\Sigma^{h;0}_{\upsilon_{\langle s \rangle}}(2\delta(b))$. Thus, the pair $(g_{\upsilon,s}, \nabla^{\upsilon,s})$ satisfies the requirements (1) and (2) of Lemma 4.1. By construction, the map $\upsilon \longrightarrow (g_{\upsilon,s}, \nabla^{\upsilon,s})$ is $\mathrm{Aut}(\mathcal{T}) \propto (S^1)^I$-invariant and smooth. Since the maps $\upsilon \longrightarrow \zeta_{\upsilon,s}$ extend continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$, so does the map $\upsilon \longrightarrow (g_{\upsilon,s}, \nabla^{\upsilon,s})$, as can be seen by an argument analogous to Subsections 3.9 and 4.1 in [Z3]. It is immediate from the construction that (4.6) is satisfied, while (4.7) follows from (4.5).

For each $s \in [\langle \mathcal{T} \rangle]$, we will next choose a family of identifications

$$\Pi_{\upsilon,s}\big|_z : u^*_{\upsilon,s}\mathfrak{L}|_z \longrightarrow \tilde{u}^*_{\upsilon,s}\mathfrak{L}_z, \qquad z \in \Sigma_{\upsilon_{\langle s \rangle}},$$

which is smooth in υ on $\tilde{\mathcal{F}}^1\mathcal{T}^\emptyset$ and in z. If $z\in\Sigma^{h;0}_{\upsilon_{\langle s\rangle}}(2\delta(b))$ for some $h\in\mathcal{I}^*_{s-1}$, let $\Pi^{\upsilon,s}_{\infty,z}$ and $\tilde{\Pi}^{\upsilon,s}_{\infty,z}$ be the parallel transports in the line bundles $u^*_{\upsilon,s}\mathfrak{L}$ and $\tilde{u}^*_{\upsilon,s}\mathfrak{L}$, respectively, along a path from ∞ to z in $\Sigma^{h;0}_{\upsilon_{\langle s\rangle}}(2\delta(b))$ with respect to the connections $q^*_{\upsilon_s;\langle\mathcal{T}\rangle+1-s}\nabla^{\upsilon,s+1}$ and $\nabla^{\upsilon,s}$. Due to the requirement (2) of Lemma 4.1, these parallel transports are path-independent. If $z\in\Sigma_{\upsilon_{\langle s\rangle}}$ and $\xi\in u^*_{\upsilon,s}\mathfrak{L}|_z$, we require that

$$\Pi_{\upsilon,s}|_z\xi=\begin{cases}\xi, & \text{if } z\in\Sigma^0_{b_s(\upsilon)};\\ \tilde{\Pi}^{\upsilon,s}_{\infty,z}\big\{\Pi^{\upsilon,s}_{\infty,z}\big\}^{-1}\xi, & \text{if } z\in\Sigma^{h;0}_{\upsilon_{\langle s\rangle}}(\delta(b)),\ h\in\mathcal{I}^*_{s-1};\\ \Pi_{\zeta_{\upsilon,s}(z)}\xi, & \text{if } z\notin\Sigma^{h;0}_{\upsilon_{\langle s\rangle}}(2\delta(b))\ \forall\, h\in\mathcal{I}^*_{s-1}.\end{cases} \tag{4.10}$$

We patch the last two identifications in (4.10) over $\Sigma^{h;0}_{\upsilon_{\langle s\rangle}}(2\delta(b))-\Sigma^{h;0}_{\upsilon_{\langle s\rangle}}(\delta(b))$, using a cutoff function constructed from β. Let

$$\Pi_{\upsilon,s}\colon\Gamma\big(\Sigma_{\upsilon_{\langle s\rangle}};u^*_{\upsilon,s}\mathfrak{L}\big)\longrightarrow\Gamma\big(\Sigma_{\upsilon_{\langle s\rangle}};\tilde{u}^*_{\upsilon,s}\mathfrak{L}\big)$$

be the operator induced by the maps $\Pi_{\upsilon,s}|_z$. We note that if

$$\xi\in\tilde{\Gamma}_-\big(\upsilon_{\langle s\rangle};\mathfrak{L}\big)\equiv\ker\bar{\partial}_{\nabla,\tilde{b}(\upsilon)},$$

then $\{\tilde{\Pi}^{\upsilon,s}_{\infty,\cdot}\}^{-1}\xi$ is a holomorphic function on $\Sigma^{h;0}_{\upsilon_{\langle s\rangle}}(2\delta(b))$, since covariant differentiation commutes with parallel transport due to (2) of Lemma 4.1.

For $b\in\mathcal{U}^{(0)}_{\mathcal{T}}(X;J)$, $s\in[\langle\mathcal{T}\rangle]$, and $h\in\mathcal{I}^*_{s-1}$, let

$$\Gamma_h(b;\mathfrak{L})=\big\{\xi\in\Gamma_B(b;\mathfrak{L})\colon\xi|_{\Sigma_b-\Sigma^h_b}=0\big\},\qquad\Gamma_{h;-}(b;\mathfrak{L})=\Gamma_h(b;\mathfrak{L})\cap\Gamma_-(b;\mathfrak{L}),$$
$$\Gamma^{0,1}_h(b;\mathfrak{L})=\big\{\eta\in\Gamma^{0,1}_B(b;\mathfrak{L})\colon\eta|_{\Sigma_b-\Sigma^h_b}=0\big\}.$$

If $\upsilon\in\tilde{\mathcal{F}}^1\mathcal{T}^\emptyset_\delta$, $s\in[\langle\mathcal{T}\rangle+1]$, and $h\in\mathcal{I}^*_{s-1}$, we put

$$\tilde{\Gamma}_{h;-}\big(\upsilon_{\langle s\rangle};\mathfrak{L}\big)=\big\{\xi\in\tilde{\Gamma}_-\big(\upsilon_{\langle s\rangle};\mathfrak{L}\big)\colon\xi|_{\Sigma_{\upsilon_{\langle s\rangle}}-\Sigma^h_{\upsilon_{\langle s\rangle}}}=0\big\}.$$

For each $m\in\mathbb{Z}^+$, we define

$$\mathfrak{D}^{(m)}_{s;h}\colon\tilde{\Gamma}_-\big(\upsilon_{\langle s\rangle};\mathfrak{L}\big)\longrightarrow\mathfrak{L}_{\mathrm{ev}_P(b)}\qquad\text{by}\qquad\mathfrak{D}^{(m)}_{s;h}\xi=\frac{d}{dw_h}\{\tilde{\Pi}^{\upsilon,s}_{\infty,\cdot}\}^{-1}\xi\Big|_{w_h=0}=\big\{\nabla^{\upsilon,s}_{e_\infty}\big\}^m\xi|_{\Sigma^h_{\upsilon_{\langle s\rangle}}},$$

where w_h is the standard holomorphic coordinate around ∞ in $\Sigma^h_{\upsilon_{\langle s\rangle}}$. We will construct isomorphisms

$$\tilde{R}_{\upsilon,s}\colon\Gamma_-(b;\mathfrak{L})\equiv\ker\bar{\partial}_{\nabla,b}\longrightarrow\tilde{\Gamma}_-\big(\upsilon_{\langle s\rangle};\mathfrak{L}\big)\qquad\forall\ s\in\big[\langle\mathcal{T}\rangle\big]$$

such that the following properties are satisfied. First, for all $h\in\mathcal{I}^*_{s-1}$,

$$\tilde{R}_{\upsilon,s}\xi\in\tilde{\Gamma}_{h;-}\big(\upsilon_{\langle s\rangle};\mathfrak{L}\big)\qquad\forall\,\xi\in\Gamma_{h;-}(b;\mathfrak{L}). \tag{4.11}$$

Second, for all $h\in\mathcal{I}^*_{s-1}$,

$$\mathfrak{D}^{(1)}_{s;h}\tilde{R}_{\upsilon,s}\xi=\alpha_{s;h}\big(\rho_{s;h}(\upsilon);\xi\big)\equiv\sum_{i\in\chi_h(\mathcal{T})}\rho_{h;i}(\upsilon)\,\mathfrak{D}_{\mathcal{T},i}\xi\qquad\forall\,\xi\in\Gamma_-(b;\mathfrak{L}). \tag{4.12}$$

Finally, the maps $\upsilon \longrightarrow \tilde{R}_{\upsilon,s}$ are smooth over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset$ and extend continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$. These extensions satisfy

$$\tilde{R}_{\upsilon,s}|_b = \mathrm{id}\colon \Gamma_-(b;\mathfrak{L}) \longrightarrow \Gamma_-(b;\mathfrak{L}). \tag{4.13}$$

In order to construct isomorphisms $\tilde{R}_{\upsilon,s}$, we observe that certain operators are surjective. If $b\in\mathcal{U}_{\mathcal{T}}^{(0)}(X;J)$, $s\in[\langle\mathcal{T}\rangle]$, $h\in\mathcal{I}_{s-1}^*$, and $[w_h]\in\mathbb{P}\tilde{\mathfrak{F}}_h\mathcal{T}|_b$, let

$$\Gamma_{h;-}\big(b;\mathfrak{L};[w_h]\big) = \big\{\xi\in\Gamma_{h;-}(b;\mathfrak{L})\colon \alpha_{s;h}(w_h;\xi)=0\big\}.$$

We denote the L^2-orthogonal complement of $\Gamma_{h;-}(b;\mathfrak{L};[w_h])$ in $\Gamma_{h;-}(b;\mathfrak{L})$ by $\Gamma_{h;-}^{\perp}(b;\mathfrak{L};[w_h])$. Since Σ_b^h is a genus-zero Riemann surface and the degree of $u_b^*\mathfrak{L}$ over every component of Σ_b^h is nonnegative,

$$H^1\big(\Sigma_b^h;\{u_b|_{\Sigma_b^h}\}^*\mathfrak{L}\otimes\mathcal{O}(-z)\big) = \{0\} \qquad \forall\ z\in\Sigma_b^{h*},$$

where $\Sigma_b^{h*}\subset\Sigma_b^h$ is the subset of smooth points. Thus, the operator

$$\bar{\partial}_{\nabla,b}^h\colon \Gamma_h(b;\mathfrak{L}) \longrightarrow \Gamma_h^{0,1}(b;\mathfrak{L})$$

induced by $\bar{\partial}_{\nabla,b}$ is surjective. Similarly, since the degree of $u_b^*\mathfrak{L}|_{\Sigma_{b,i}}$ is positive for all $i\in\chi(\mathcal{T})$,

$$H^1\big(\Sigma_b^h;\{u_b|_{\Sigma_b^i}\}^*\mathfrak{L}\otimes\mathcal{O}(-2z)\big) = \{0\} \qquad \forall\ z\in\Sigma_b^{i*}\cap\Sigma_b^h.$$

Thus, for every element $w_h\in\tilde{\mathfrak{F}}_h\mathcal{T}|_b$, the linear map

$$\alpha_{s;h}(w_h;\cdot)\colon \Gamma_{h;-}(b;\mathfrak{L}) \longrightarrow \mathfrak{L}_{\mathrm{ev}_P(b)}$$

is surjective. In particular,

$$\alpha_{s;h}(w_h;\cdot)\colon \Gamma_{h;-}^{\perp}(b;\mathfrak{L};[w_h]) \longrightarrow \mathfrak{L}_{\mathrm{ev}_P(b)}$$

is an isomorphism and

$$C(b)^{-1}|w_h||\xi| \le \big|\alpha_{s;h}(w_h;\xi)\big| \le C(b)|w_h||\xi| \qquad \forall\ \xi\in\Gamma_{h;-}^{\perp}(b;\mathfrak{L};[w_h]), \tag{4.14}$$

for some $C\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$.

If $\upsilon\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset$, $s\in[\langle\mathcal{T}\rangle]$, and $h\in\mathcal{I}_{s-1}^*$, we denote by $\Gamma_h(\upsilon_{\langle s\rangle};\mathfrak{L})$ and $\Gamma_h^{0,1}(\upsilon_{\langle s\rangle};\mathfrak{L})$ the completions of the spaces

$$\big\{\xi\in\Gamma_B(b_s(\upsilon);\mathfrak{L})\colon \xi|_{\Sigma_{\upsilon_{\langle s\rangle}}-\Sigma^h_{\upsilon_{\langle s\rangle}}}=0\big\} \qquad \text{and} \qquad \big\{\eta\in\Gamma_B^{0,1}(b_s(\upsilon);\mathfrak{L})\colon \eta|_{\Sigma_{\upsilon_{\langle s\rangle}}-\Sigma^h_{\upsilon_{\langle s\rangle}}}=0\big\}$$

with respect to the norms $\|\cdot\|_{\upsilon_{\langle s\rangle},p,1}$ and $\|\cdot\|_{\upsilon_{\langle s\rangle},p}$, respectively. Let

$$\Gamma_{h;-}(\upsilon_{\langle s\rangle};\mathfrak{L}) = \Gamma_-(\upsilon_{\langle s\rangle};\mathfrak{L})\cap\Gamma_h(\upsilon_{\langle s\rangle};\mathfrak{L}), \qquad \text{where}$$
$$\Gamma_-(\upsilon_{\langle s\rangle};\mathfrak{L}) = \big\{\xi\circ q_{\upsilon_s;\langle\mathcal{T}\rangle+1-s}\colon \xi\in\tilde{\Gamma}_-(\upsilon_{\langle s+1\rangle};\mathfrak{L})\big\}.$$

We denote the L^2-orthogonal complement of $\Gamma_{h;-}(\upsilon_{\langle s\rangle};\mathfrak{L})$ in $\Gamma_h(\upsilon_{\langle s\rangle};\mathfrak{L})$ by $\Gamma_{h;+}(\upsilon_{\langle s\rangle};\mathfrak{L})$. By (4.2) and the same argument as in Subsection 3.5 in [Z3],

$$C(b)^{-1}\|\xi\|_{\upsilon_{\langle s\rangle},p,1}\le \big\|\bar\partial^h_{\nabla,b_s(\upsilon)}\xi\big\|_{\upsilon_{\langle s\rangle},p}\le C(b)\|\xi\|_{\upsilon_{\langle s\rangle},p,1}\qquad \forall\,\xi\in\Gamma_{h;+}(\upsilon_{\langle s\rangle};\mathfrak{L}) \tag{4.15}$$

for some $C\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$, provided $\delta\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$ is sufficiently small. In particular, the operator

$$\bar\partial^h_{\nabla,b_s(\upsilon)}\colon\Gamma_{h;+}(\upsilon_{\langle s\rangle};\mathfrak{L})\longrightarrow\Gamma^{0,1}_h(\upsilon_{\langle s\rangle};\mathfrak{L})$$

is an isomorphism. On the other hand, by the construction of the map $q_{\upsilon_s;\langle\mathcal{T}\rangle+1-s}$ in Subsection 2.2 of [Z3],

$$\begin{aligned}\big\|\bar\partial_{\nabla,b_s(\upsilon)}\big(\xi\circ q_{\upsilon_s;\langle\mathcal{T}\rangle+1-s}\big)\big\|_{\upsilon_{\langle s\rangle},p}&\le C(b)|\upsilon|^{1/p}\|\xi\|_{\upsilon_{\langle s+1\rangle},p,1}\\ \text{and}\qquad \bar\partial_{\nabla,b_s(\upsilon)}\big(\xi\circ q_{\upsilon_s;\langle\mathcal{T}\rangle+1-s}\big)\big|_{\Sigma^0_{b_s(\upsilon)}}&=0\qquad\forall\,\xi\in\tilde\Gamma_-(\upsilon_{\langle s+1\rangle};\mathfrak{L}).\end{aligned}\tag{4.16}$$

Thus, by the analogue of (3.2) for $\zeta_{\upsilon,s}$, there exist unique linear maps

$$\varepsilon_{\upsilon,s;h}\colon\tilde\Gamma_-(\upsilon_{\langle s+1\rangle};\mathfrak{L})\longrightarrow\Gamma_{h;+}(\upsilon_{\langle s\rangle};\mathfrak{L}),\qquad h\in\mathcal{I}^*_{s-1},$$

such that

$$\tilde R'_{\upsilon,s}\xi\equiv\Pi_{\upsilon,s}\Big(\xi\circ q_{\upsilon_s;\langle\mathcal{T}\rangle+1-s}+\sum_{h\in\mathcal{I}^*_{s-1}}\varepsilon_{\upsilon,s;h}(\xi)\Big)\in\tilde\Gamma_-(\upsilon_{\langle s\rangle};\mathfrak{L})\qquad\forall\,\xi\in\tilde\Gamma_-(\upsilon_{\langle s+1\rangle};\mathfrak{L}).$$

Furthermore, for all $\xi\in\tilde\Gamma_-(\upsilon_{\langle s+1\rangle};\mathfrak{L})$ and $h\in\mathcal{I}^*_{s-1}$,

$$\begin{aligned}\big\|\varepsilon_{\upsilon,s;h}(\xi)\big\|_{\upsilon_{\langle s\rangle},p,1}\le C(b)\big(&\big\|\zeta_{\upsilon,s}|_{\Sigma^h_{\upsilon_{\langle s\rangle}}}\big\|^2_{\upsilon_{\langle s\rangle},p,1}\|\xi\|_{\upsilon_{\langle s+1\rangle},p,1}\\ &+\big\|\bar\partial_{\nabla,b_s(\upsilon)}\big(\xi\circ q_{\upsilon_s;\langle\mathcal{T}\rangle+1-s}\big)|_{\Sigma^h_{\upsilon_{\langle s\rangle}}}\big\|_{\upsilon_{\langle s\rangle},p}\big).\end{aligned}\tag{4.17}$$

In addition, for all $h,h'\in\mathcal{I}^*_{s-1}$ such that $h'\neq h$,

$$\varepsilon_{\upsilon,s;h}(\xi)=0\qquad\forall\,\xi\in\tilde\Gamma_{h';-}(\upsilon_{\langle s+1\rangle};\mathfrak{L}).$$

The expansion in Lemma 4.2 below is a key step in constructing the homomorphisms $\tilde R_{\upsilon,s}$ with the desired properties. For every $h\in\mathcal{I}^*_{s-1}-\chi(\mathcal{T})$, let

$$\chi'_h(\mathcal{T})=\big\{h'\in\hat I\colon \iota_{h'}=h\big\}.$$

If $b\in\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^n;J)$ and $h'\in\chi'_h(\mathcal{T})$, we denote by

$$x_{h'}(b)\in\mathbb{C}=\Sigma_{b,h}-\{\infty\}$$

the node shared by $\Sigma_{b,h}$ and $\Sigma_{b,h'}$.

Lemma 4.2 *If n, d, k, a, and $\mathfrak{L}$ are as in Proposition 3.3, there exists $\delta_n(d)\in\mathbb{R}^+$ such that for every almost complex structure J on $\mathbb{P}^n$, such that $\|J-J_0\|_{C^1}\le\delta_n(d)$, and a bubble type $\mathcal{T}$ as above, there exist $\delta, C\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$ such that the requirement of Lemma 4.1 is satisfied. Furthermore, for every $\upsilon=(b,v)\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset$ and $s\in[\langle\mathcal{T}\rangle]$, there exists an isomorphism*

$$\tilde{R}'_{\upsilon,s}\colon \tilde{\Gamma}_-(\upsilon_{\langle s+1\rangle};\mathfrak{L})\longrightarrow\tilde{\Gamma}_-(\upsilon_{\langle s\rangle};\mathfrak{L})$$

such that

$$\big\|\tilde{R}'_{\upsilon,s}\xi-\Pi_{\zeta_{\upsilon,s}}(\xi\circ q_{\langle\mathcal{T}\rangle+1-s})\big\|_{\upsilon_{\langle s\rangle},p,1}\le C(b)|\upsilon|^{1/p}\|\xi\|_{\upsilon_{\langle s+1\rangle},p,1}\quad\forall\,\xi\in\tilde{\Gamma}_-(\upsilon_{\langle s+1\rangle};\mathfrak{L});\tag{4.18}$$

$$\tilde{R}'_{\upsilon,s}\xi\in\tilde{\Gamma}_{h;-}(\upsilon_{\langle s\rangle};\mathfrak{L})\qquad\forall\,\xi\in\tilde{\Gamma}_{h;-}(\upsilon_{\langle s+1\rangle};\mathfrak{L}).\tag{4.19}$$

In addition, there exist homomorphisms

$$\varepsilon_{\upsilon,h;i}\colon\tilde{\Gamma}_-(\upsilon_{\langle s+1\rangle};\mathfrak{L})\longrightarrow\mathfrak{L}_{\mathrm{ev}_P(b)},\qquad h\in\mathcal{I}^*_{s-1}\cap\chi^-(\mathcal{T}),\ i\in\chi_h(\mathcal{T}),$$

such that for all $\xi\in\tilde{\Gamma}_-(\upsilon_{\langle s+1\rangle};\mathfrak{L})$, $h\in\mathcal{I}^*_{s-1}\cap\chi^-(\mathcal{T})$, and $i\in\chi_h(\mathcal{T})$,

$$\begin{gathered}\big|\varepsilon_{\upsilon,h;i}(\xi)\big|\le C(b)|\upsilon|^{1/p}\|\xi\|_{\upsilon_{\langle s+1\rangle},p,1}\qquad\textit{and}\\ \mathfrak{D}^{(1)}_{s;h}\big\{\tilde{R}'_{\upsilon,s}\xi\big\}=\sum_{h'\in\chi'_h(\mathcal{T})}v_{h'}\mathfrak{D}^{(1)}_{s+1;h'}\xi+\sum_{i\in\chi_h(\mathcal{T})}\rho_{h;i}(\upsilon)\varepsilon_{\upsilon,h;i}(\xi).\end{gathered}\tag{4.20}$$

Furthermore, the maps $\upsilon\longrightarrow\tilde{R}'_{\upsilon,s}$ and $\upsilon\longrightarrow\varepsilon_{\upsilon,h;i}$ are $\mathrm{Aut}(\mathcal{T})\propto(S^1)^I$-invariant and smooth over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset$. They extend continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$. These extensions satisfy

$$\tilde{R}'_{b,s}=\mathrm{id}\quad\forall\,b\in\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^n;J)\qquad\textit{and}\qquad\varepsilon_{\upsilon,h;i}=0\quad\forall\,h\in\mathcal{I}^0_{s-1}(\upsilon),\,i\in\chi_h(\mathcal{T}).\tag{4.21}$$

Isomorphisms $\tilde{R}'_{\upsilon,s}$ satisfying (4.18) and (4.19) have already been constructed. The estimate (4.20) is obtained by applying the integration-by-parts argument in the proof of Theorem 2.8 in [Z2] to the holomorphic functions

$$\{\tilde{\Pi}^{\upsilon,s}_{\infty,\cdot}\}^{-1}\tilde{R}'_{\upsilon,s}\xi\colon\Sigma^{h;0}_{\upsilon_{\langle s\rangle}}\big(\delta(b)\big)\longrightarrow\mathfrak{L}_{\mathrm{ev}_P(b)}\qquad\text{and}\qquad\{\tilde{\Pi}^{\upsilon,s+1}_{\infty,\cdot}\}^{-1}\xi\colon\Sigma^{h';0}_{\upsilon_{\langle s+1\rangle}}\big(\delta(b)\big)\longrightarrow\mathfrak{L}_{\mathrm{ev}_P(b)}.$$

The homomorphism $\varepsilon_{\upsilon,h;i}$ is given by

$$\varepsilon_{\upsilon,h;i}(\xi)=\frac{1}{2\pi\mathfrak{i}}\oint_{\partial^-A^-_{\upsilon_{\langle s\rangle},i}(\delta(b))}\big\{\{\Pi^{\upsilon,s}_{\infty,\cdot}\}^{-1}\varepsilon_{\upsilon,s;h}(\xi)\big\}(w_i)\frac{dw_i}{w_i^2},\tag{4.22}$$

where w_i is the coordinate on a neighborhood of the circle $\partial^-A^-_{\upsilon_{\langle s\rangle},i}(\delta(b))$ induced from the standard holomorphic coordinate centered at ∞ in $\Sigma_{b,i}=S^2$; see the proof of Lemma 3.5 in [Z5] for details. By the continuity of the maps

$$\upsilon\longrightarrow\zeta_{\upsilon,s},\ \nabla^{\upsilon,s}$$

over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$ and the same argument as in Subsection 4.1 of [Z3], the homomorphisms $\varepsilon_{\upsilon,s;h}$ extend continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$. Thus, by (4.22), the homomorphisms $\varepsilon_{\upsilon,h;i}$ also extend continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$. By (4.5), (4.7), and (4.17),

$$\varepsilon_{\upsilon,s;h}=0 \qquad \forall\, h\in\mathcal{I}^0_{s-1}(\upsilon).$$

This observation, along with (4.22), implies the second claim in (4.21). The first claim in (4.21) follows from (4.4), (4.6), and the construction of $\tilde{R}'_{\upsilon,s}$.

Suppose $\upsilon=(b,v)\in\tilde{\mathcal{F}}^1\mathcal{T}^\emptyset_\delta$, $s\in[\langle\mathcal{T}\rangle]$, and we have constructed an isomorphism

$$\tilde{R}_{\upsilon,s+1}\colon\Gamma_-(b;\mathfrak{L})\longrightarrow\tilde{\Gamma}_-\big(\upsilon_{\langle s+1\rangle};\mathfrak{L}\big)$$

that satisfies (4.11)-(4.13). We note that for every $s\in[\langle\mathcal{T}\rangle]$ and $h\in\mathcal{I}^*_{s-1}\cap\chi^-(\mathcal{T})$,

$$\rho_{s;h}(\upsilon)=\big(v_{h'}\rho_{s+1;h'}(\upsilon)\big)_{\iota_{h'}=h} \qquad \forall\ \upsilon\equiv\big(b,(v_i)_{i\in\aleph\cup\hat{I}}\big)\in\tilde{\mathcal{F}}\mathcal{T}.$$

Thus, by (4.11) and (4.12) with s replaced by $s{+}1$, (4.19), and (4.20), there exists a homomorphism

$$\tilde{\varepsilon}_{\upsilon,s;h}\colon\tilde{\mathfrak{F}}_h\mathcal{T}\longrightarrow\mathrm{Hom}\big(\Gamma_-(b;\mathfrak{L}),\mathfrak{L}_{\mathrm{ev}_P(b)}\big)$$

such that

$$\big|\tilde{\varepsilon}_{\upsilon,s;h}\big|\le C(b)|\upsilon|^{1/p},\quad \tilde{\varepsilon}_{\upsilon,s;h}(w_h;\xi)=0\quad \forall\, w_h\in\tilde{\mathfrak{F}}_h\mathcal{T},\ \xi\in\Gamma_{h';-}(b;\mathfrak{L}),\ h'\in\mathcal{I}^*_{s-1}-\{h\}, \tag{4.23}$$

$$\mathfrak{D}^{(1)}_{s;h}\big\{\tilde{R}'_{\upsilon,s}\tilde{R}_{\upsilon,s+1}\xi\big\}=\alpha_{s;h}\big(\rho_{s;h}(\upsilon);\xi\big)+\tilde{\varepsilon}_{\upsilon,s;h}\big(\rho_{s;h}(\upsilon);\xi\big)\qquad\forall\,\xi\in\Gamma_-(b;\mathfrak{L}). \tag{4.24}$$

We note that for $h\in\mathcal{I}^*_{s-1}-\chi^-(\mathcal{T})$, the existence of such $\tilde{\varepsilon}_{\upsilon,s;h}$ is immediate from (4.12) with s replaced by $s{+}1$, (4.18), and (4.19). Let $[\rho_{s;h}(\upsilon)]$ denote the image of $\rho_{s;h}(\upsilon)$ under the projection map $\tilde{\mathfrak{F}}_h\mathcal{T}-\{0\}\longrightarrow\mathbb{P}\tilde{\mathfrak{F}}_h\mathcal{T}$. Since

$$\alpha_{s;h}\big(w_h;\cdot\big)\colon\Gamma^\perp_{h;-}(b;\mathfrak{L};[\rho_{s;h}(\upsilon)])\longrightarrow\mathfrak{L}_{\mathrm{ev}_P(b)}$$

is an isomorphism for each $h\in\mathcal{I}^*_{s-1}$, by the first bound in (4.23), (4.24), and (4.14) there exists a unique homomorphism

$$\mu_{\upsilon,s;h}\colon\Gamma_-(b;\mathfrak{L})\longrightarrow\Gamma^\perp_{h;-}(b;\mathfrak{L};[\rho_{s;h}(\upsilon)]),$$

such that

$$\mathfrak{D}^{(1)}_{s;h}\big\{\tilde{R}'_{\upsilon,s}\tilde{R}_{\upsilon,s+1}\big(\xi+\mu_{\upsilon,s;h}(\xi)\big)\big\}=\alpha_{s;h}\big(\rho_{s;h}(\upsilon);\xi\big). \tag{4.25}$$

Furthermore, by (4.14) and (4.23),

$$\big|\mu_{\upsilon,s;h}\big|\le C(b)|\upsilon|^{1/p},\qquad \mu_{\upsilon,s;h}(\xi)=0\quad\forall\,\xi\in\Gamma_{h';-}(b;\mathfrak{L}),\ h'\in\mathcal{I}^*_{s-1}-\{h\}. \tag{4.26}$$

We define

$$\tilde{R}_{\upsilon,s}\colon\Gamma_-(b;\mathfrak{L})\longrightarrow\tilde{\Gamma}_-\big(\upsilon_{\langle s\rangle};\mathfrak{L}\big)\qquad\text{by}\qquad \tilde{R}_{\upsilon,s}(\xi)=\tilde{R}'_{\upsilon,s}\tilde{R}_{\upsilon,s+1}\Big(\xi+\sum_{h\in\mathcal{I}^*_{s-1}}\mu_{\upsilon,s;h}(\xi)\Big).$$

By (4.11) with s replaced by $s{+}1$, (4.19), and the second statement in (4.26), $\tilde{R}_{\upsilon,s}$ satisfies (4.11). Since

$$\mathfrak{D}^{(1)}_{s;h}\xi=0\qquad\forall\,\xi\in\tilde{\Gamma}_{h';-}\big(\upsilon_{\langle s\rangle};\mathfrak{L}\big),\ h'\in\mathcal{I}^*_{s-1}-\{h\},$$

$\tilde{R}_{\upsilon,s}$ satisfies (4.12) by (4.25), along with (4.11) with s replaced by $s+1$, (4.19), and the second statement in (4.26).

It remains to show that for every $h\in\mathcal{I}^*_{s-1}$ the family of homomorphisms

$$\mu_{\upsilon,s;h}\colon \Gamma_-(b;\mathfrak{L}) \longrightarrow \Gamma^{\perp}_{h;-}(b;\mathfrak{L};[\rho_{s;h}(\upsilon)]) \subset \Gamma_{h;-}(b;\mathfrak{L}), \qquad \upsilon\in\tilde{\mathcal{F}}^1\mathcal{T}^{\emptyset}_{\delta},$$

extends continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$. Each homomorphism $\tilde{\varepsilon}_{\upsilon,s;h}$ of the previous paragraph extends continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$, as this is case for the homomorphisms $\varepsilon_{\upsilon,h;i}$ by Lemma 4.2. Furthermore,

$$\tilde{\varepsilon}_{b,s;h}=0 \quad \forall\, b\in\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^n;J),\ h\in\mathcal{I}^*_{s-1}, \quad \text{and} \quad \varepsilon_{\upsilon,s;h}=0 \quad \forall\,\upsilon\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta,\ h\in\mathcal{I}^0_{s-1}(\upsilon). \tag{4.27}$$

The first claim above follows from (4.13) with s replaced by $s+1$ and first statement in (4.21). The second claim in (4.27) follows from the second statement in (4.21). If

$$\upsilon\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta \qquad \text{and} \qquad h\in\mathcal{I}^*_{s-1}-\mathcal{I}^0_{s-1}(\upsilon),$$

we define $\mu_{\upsilon,s;h}$ as in the previous paragraph. This extension is continuous at υ since $\tilde{\varepsilon}_{\upsilon,s;h}$ is. If $h\in\mathcal{I}^0_{s-1}(\upsilon)$, we take $\mu_{\upsilon,s;h}=0$. This extension is continuous by the continuity of $\tilde{\varepsilon}_{\upsilon,s;h}$ *and* the second statement in (4.27). Finally, $\tilde{R}_{\upsilon,s}$ satisfies (4.13) by the first statement in (4.27), along with (4.13) with s replaced by $s+1$ and the first statement in (4.21).

Remark: The key point in the previous paragraph is the second statement in (4.27), because the lines $\Gamma^{\perp}_{h;-}(b;\mathfrak{L};[\rho_{s;h}(\upsilon)])$ may *not* extend continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$.

Corollary 4.3 *If n, d, k, a, and $\mathfrak{L}$ are as in Proposition 3.3, there exists $\delta_n(d)\in\mathbb{R}^+$ such that for every almost complex structure J on $\mathbb{P}^n$, such that $\|J-J_0\|_{C^1}\le\delta_n(d)$, and a bubble type $\mathcal{T}$ as above, there exist $\delta,C\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$ such that the requirement of Lemma 4.1 is satisfied. In addition, for every $\upsilon=(b,v)\in\tilde{\mathcal{F}}^1\mathcal{T}^{\emptyset}_{\delta}$ there exists an isomorphism*

$$\tilde{R}_{\upsilon,1}\colon \Gamma_-(b;\mathfrak{L}) \longrightarrow \tilde{\Gamma}_-(\upsilon_{\langle 1\rangle};\mathfrak{L})$$

such that for every $\xi\in\Gamma_-(b;\mathfrak{L})$, $h\in\mathcal{I}^*_0$, and $\epsilon\in(0,2\delta(b))$,

$$\big\|\nabla^{\upsilon,1}\tilde{R}_{\upsilon,1}\xi\big\|_{C^0(A^-_{\upsilon_{\langle 1\rangle},h}(\delta(b))),g_{\upsilon_{\langle 1\rangle}}} \le C(b)\big|\rho_{1;h}(\upsilon)\big|\cdot\|\xi\|_{b,p,1}, \qquad \textit{and} \tag{4.28}$$

$$\oint_{\partial^-A^-_{\upsilon_{\langle 1\rangle},h}(\epsilon)}\big\{\{\tilde{\Pi}^{\upsilon,1}_{\infty,\cdot}\}^{-1}\tilde{R}_{\upsilon,1}\xi\big\}(w_h)\frac{dw_h}{w_h^2}=2\pi\mathfrak{i}\sum_{i\in\chi_h(\mathcal{T})}\rho_{h;i}(\upsilon)\mathfrak{D}_{\mathcal{T},i}\xi, \tag{4.29}$$

where w_h is the standard holomorphic on the neighborhood of ∞ in $\Sigma_{\upsilon_{\langle 1\rangle},h}=S^2$. Finally, the map $\upsilon\longrightarrow\tilde{R}_{\upsilon,1}$ is $\mathrm{Aut}(\mathcal{T})\propto(S^1)^I$-invariant and smooth on $\tilde{\mathcal{F}}^1\mathcal{T}^{\emptyset}_{\delta}$. It extends continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$. This extension satisfies

$$\tilde{R}_{b,1}=\mathrm{id} \quad \forall\, b\in\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^n;J). \tag{4.30}$$

The homomorphism $\tilde{R}_{\upsilon,1}$ constructed above satisfies the extension requirements of the corollary. Since $\{\tilde{\Pi}^{\upsilon,1}_{\infty,\cdot}\}^{-1}\tilde{R}_{\upsilon,1}\xi$ is holomorphic on $A^-_{\upsilon_{\langle 1\rangle},h}(\epsilon)$, (4.29) is equivalent to the $s\!=\!1$ case of (4.12).

It remains to verify (4.28). Let

$$u_{\upsilon_{\langle 1\rangle}}=u_b\circ q_{\upsilon_{\langle 1\rangle}}.$$

For each $h\in\mathcal{I}_0$ and $z\in\Sigma^{h;0}_{\upsilon_{\langle 1\rangle}}(\delta(b))$, we denote by $\Pi^{\upsilon_{\langle 1\rangle}}_{\infty,z}$ the parallel transport in the line bundle $u^*_{\upsilon_{\langle 1\rangle}}\mathfrak{L}$ along a path from ∞ to z in $\Sigma^{h;0}_{\upsilon_{\langle s\rangle}}(\delta(b))$ with respect to the connection $q^*_{\upsilon_{\langle 1\rangle}}\nabla^{\upsilon,\langle\mathcal{T}\rangle+1}$. By the construction of the homomorphism $\tilde{R}_{\upsilon,1}$ above,

$$\{\tilde{\Pi}^{\upsilon,1}_{\infty,\cdot}\}^{-1}\tilde{R}_{\upsilon,1}\xi\big|_{\Sigma^{h;0}_{\upsilon_{\langle 1\rangle}}(\delta(b))}=\{\Pi^{\upsilon_{\langle 1\rangle}}_{\infty,\cdot}\}^{-1}(\xi\circ q_{\upsilon_{\langle 1\rangle}})\big|_{\Sigma^{h;0}_{\upsilon_{\langle 1\rangle}}(\delta(b))}+\varepsilon_{\upsilon}(\xi)\qquad\forall\,\xi\in\Gamma_-(b;\mathfrak{L}),$$

for some homomorphism

$$\varepsilon_{\upsilon}\colon\Gamma_-(b;\mathfrak{L})\longrightarrow C^{\infty}\big(\Sigma^{h;0}_{\upsilon_{\langle 1\rangle}}(\delta(b));\mathfrak{L}_{\mathrm{ev}_P(b)}\big)\qquad\text{s.t.}$$
$$\big\|\varepsilon_{\upsilon}(\xi)\big\|_{C^0(\Sigma^{h;0}_{\upsilon_{\langle s\rangle}}(\delta(b)))}\le C(b)|\upsilon|^{1/p}\|\xi\|_{b,p,1}\qquad\forall\,\xi\in\Gamma_-(b;\mathfrak{L}).$$

Thus, by the same integration-by-parts argument as in the proof of Theorem 2.2 in [Z2], there exist homomorphisms

$$\varepsilon^{(l)}_{\upsilon,h;i}\colon\Gamma_-(b;\mathfrak{L})\longrightarrow\mathfrak{L}_{\mathrm{ev}_P(b)},\qquad\forall\;h\!\in\!\mathcal{I}_0^*,\;i\!\in\!\chi_h(\mathcal{T}),\;l\!\in\!\mathbb{Z}^+,$$

such that for all $h\!\in\!\mathcal{I}_0^*$, $i\!\in\!\chi_h(\mathcal{T})$, $l,m\!\in\!\mathbb{Z}^+$, and $\xi\!\in\!\Gamma_-(b;\mathfrak{L})$

$$\big|\varepsilon^{(l)}_{\upsilon,h;i}(\xi)\big|\le C(b)\delta(b)^{-l/2}\|\xi\|_{b,p,1},$$
$$\mathfrak{D}^{(m)}_{s;h}\{\tilde{R}_{\upsilon,1}\xi\}=\sum_{l=1}^{l=m}\binom{m-1}{l-1}\sum_{i\in\chi_h(\mathcal{T})}x_i^{m-l}(\upsilon)\rho^l_{h;i}(\upsilon)\big(\mathfrak{D}^{(l)}_{\langle\mathcal{T}\rangle+1;i}\xi+\varepsilon^{(l)}_{\upsilon,h;i}(\xi)\big).\tag{4.31}$$

The number $x_i(\upsilon)\in\mathbb{C}$ is given explicitly in the paragraph preceding Lemma 3.4 in [Z5]. It is close to

$$x_{h'}(b)\in\mathbb{C}=\Sigma_{b,h}-\{\infty\},\qquad\text{where}\qquad h'\!\le\! i,\;\iota_{h'}\!=\!h.$$

The estimate (4.28) is obtained by summing up the derivatives of $\tilde{R}_{\upsilon,1}\xi|_{\Sigma^h_{\upsilon_{\langle 1\rangle}}}$ at ∞ with the appropriate coefficients, using (4.31); see the proof of Lemma 4.2 in [Z4] for a similar argument.

4.3 Smoothing Bundle Sections, II

In this subsection, we take the inductive construction of the previous subsection one step further to define a homomorphism $\tilde{R}_{\upsilon}\equiv\tilde{R}_{\upsilon,0}$. However, in this case we will encounter an obstruction bundle. The homomorphism $\tilde{R}_{\upsilon}$ will not extend continuously over $\tilde{\mathcal{F}}^1\mathcal{T}$, but its restriction to a cone contained in $\tilde{\mathcal{V}}^d_{1,k}$ will.

We first recall certain facts concerning the modified gluing map

$$\tilde{q}_{\upsilon_0;\langle\mathcal{T}\rangle+1}\colon\Sigma_{\upsilon}\longrightarrow\Sigma_{\upsilon_{\langle 1\rangle}}$$

corresponding to the parameter $\delta(b)^{1/2}$, as constructed in Subsection 4.2 of [Z5]. Suppose

$$\upsilon \equiv \big(b, v_{\aleph}, (v_h)_{h\in\hat{I}}\big) \in \tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset.$$

The map $\tilde{q}_{\upsilon_0;\langle\mathcal{T}\rangle+1}$ is biholomorphic outside $|\aleph|$ thin necks $A_{\upsilon,h}$, with $h\in\aleph$, of $(\Sigma_\upsilon, g_\upsilon)$ and the $|I_1|$ annuli

$$\tilde{\mathcal{A}}_{b,h} \equiv \tilde{\mathcal{A}}_{b,h}^- \cup \tilde{\mathcal{A}}_{b,h}^+,$$

with $h\in I_1$, where

$$\tilde{\mathcal{A}}_{b,h}^\pm \equiv \tilde{\mathcal{A}}_{b,h}^\pm\big(\delta(b)\big) \subset \Sigma_{b;P}\approx\Sigma_\upsilon$$

are annuli independent of υ. In addition,

$$\begin{gathered} \tilde{u}_{\upsilon,1}|_{\tilde{q}_{\upsilon_0;\langle\mathcal{T}\rangle+1}(A_{\upsilon,h})} = \text{const} \quad \forall\, h\in\aleph, \\ \tilde{u}_{\upsilon,1}|_{\tilde{q}_{\upsilon_0;\langle\mathcal{T}\rangle+1}(\tilde{\mathcal{A}}_{b,h})} = \text{const} \quad \forall\, h\in I_1-\mathcal{I}_0, \qquad \tilde{u}_{\upsilon,1}|_{\tilde{q}_{\upsilon_0;\langle\mathcal{T}\rangle+1}(\tilde{\mathcal{A}}_{b,h}^+)} = \text{const} \quad \forall\, h\in\mathcal{I}_0; \end{gathered} \tag{4.32}$$

$$\tilde{q}_{\upsilon_0;\langle\mathcal{T}\rangle+1}(\tilde{\mathcal{A}}_{b,h}^-) \subset A^-_{\upsilon_{\langle1\rangle},h}\big(|v_h|^2/\delta(b)\big) \quad \text{and} \quad \big\|d\tilde{q}_{\upsilon_0;\langle\mathcal{T}\rangle+1}\big\|_{C^0(\tilde{\mathcal{A}}_{b,h}^-)} \le C(b)|v_h| \quad \forall\, h\in\mathcal{I}_0, \tag{4.33}$$

if the C^0-norm of $d\tilde{q}_{\upsilon_0;\langle\mathcal{T}\rangle+1}$ is computed with respect to the metrics g_υ on Σ_υ and $g_{\upsilon_{\langle1\rangle}}$ on $\Sigma_{\upsilon_{\langle1\rangle}}$. Furthermore,

$$\big\|d\tilde{q}_{\upsilon_0;\langle\mathcal{T}\rangle+1}\big\|_{C^0} \le C(b). \tag{4.34}$$

We now proceed similarly to the previous subsection. If $\upsilon\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset$, we denote the completions of the spaces

$$\Gamma\big(\Sigma_\upsilon; u_{\upsilon,0}^*\mathfrak{L}\big) \qquad \text{and} \qquad \Gamma\big(\Sigma_\upsilon; \Lambda^{0,1}_{\mathfrak{i},j}T^*\Sigma_\upsilon\otimes u_{\upsilon,0}^*\mathfrak{L}\big)$$

with respect to the Sobolev norms $\|\cdot\|_{\upsilon,p,1}$ and $\|\cdot\|_{\upsilon,p}$ by $\Gamma(\upsilon;\mathfrak{L})$ and $\Gamma^{0,1}(\upsilon;\mathfrak{L})$. Let

$$\Gamma_-(\upsilon;\mathfrak{L}) = \big\{R'_{\upsilon,0}\xi : \xi\in\tilde{\Gamma}_-(\upsilon_{\langle1\rangle};\mathfrak{L})\big\}, \qquad \text{where} \quad R'_{\upsilon,0}\xi = \xi\circ\tilde{q}_{\upsilon_0;\langle\mathcal{T}\rangle+1}.$$

Let $R_{\upsilon,0}=R'_{\upsilon,0}\tilde{R}_{\upsilon,1}$. By (4.28), (4.32), and (4.33),

$$\begin{aligned} \big\|\bar{\partial}_{\nabla,b_0(\upsilon)}R'_{\upsilon,0}\xi\big\|_{\upsilon,p} &\le C(b)|\rho(\upsilon)|\|\xi\|_{\upsilon_{\langle1\rangle},p,1} \\ &\le C'(b)|\rho(\upsilon)|\|R'_{\upsilon,0}\xi\|_{\upsilon,p,1} \qquad \forall\,\xi\in\Gamma_-(\upsilon_{\langle1\rangle};\mathfrak{L}). \end{aligned} \tag{4.35}$$

Let $\Gamma_+(\upsilon;\mathfrak{L})$ denote the L^2-orthogonal complement of $\Gamma_-(\upsilon;\mathfrak{L})$ in $\Gamma(\upsilon;\mathfrak{L})$. Similarly to (4.15),

$$C(b)^{-1}\|\xi\|_{\upsilon,p,1} \le \big\|\bar{\partial}_{\nabla,b_0(\upsilon)}\xi\big\|_{\upsilon,p} \le C(b)\|\xi\|_{\upsilon,p,1} \qquad \forall\;\xi\in\Gamma_+(\upsilon;\mathfrak{L}) \tag{4.36}$$

for some $C\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$, provided $\delta\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$ is sufficiently small. Let $\Gamma_+^{0,1}(\upsilon;\mathfrak{L})$ be the image of $\Gamma_+(\upsilon;\mathfrak{L})$ under $\bar{\partial}_{\nabla,b_0(\upsilon)}$.

In contrast to the previous subsection, the operator $\bar{\partial}_{\nabla,b_0(\upsilon)}$ is not surjective. We next describe a complement of $\Gamma_+^{0,1}(\upsilon;\mathfrak{L})$ in $\Gamma^{0,1}(\upsilon;\mathfrak{L})$. Since the operator $\bar{\partial}^B_{\nabla,b}$ is surjective, the cokernel of $\bar{\partial}_{\nabla,b}$ can be identified with the vector space

$$\Gamma_-^{0,1}(b;\mathfrak{L}) \equiv \mathcal{H}_{b;P}\otimes\mathfrak{L}_{\text{ev}_P(b)} \approx \mathbb{E}^*_{\pi_P(b)}\otimes\mathfrak{L}_{\text{ev}_P(b)},$$

where $\mathcal{H}_{b;P}$ is the space of harmonic antilinear differentials on the main component $\Sigma_{b;P}$ of Σ_b. If $\aleph\neq\emptyset$, i.e. $\Sigma_{b;P}$ is a circle of spheres, the elements of $\mathcal{H}_{b;P}$ have simple poles at the nodes of $\Sigma_{b;P}$ with the residues adding up to zero at each node. Since the Riemann surfaces Σ_υ, with $\upsilon\in\mathcal{F}^1\mathcal{T}_\delta$, are deformations of Σ_b, with $b\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$, there exists a family of isomorphisms

$$R_{\upsilon;P}^{0,1}\colon \mathcal{H}_{b;P}\longrightarrow\mathcal{H}_{\upsilon;P}\equiv\mathcal{H}_{b_0(\upsilon);P},\qquad \upsilon=(b,v)\in\mathcal{F}^1\mathcal{T}_\delta,$$

such that the family of induced homomorphisms

$$\mathcal{H}_{b;P}\longrightarrow\Gamma^{0,1}(\upsilon;\mathbb{C})^*,\qquad \{R_{\upsilon;P}^{0,1}\eta\}(\eta')=\langle\!\langle R_{\upsilon;P}^{0,1}\eta,\eta'\rangle\!\rangle_2\quad\forall\,\eta\in\mathcal{H}_{b;P},\ \eta'\in\Gamma^{0,1}(\upsilon;\mathbb{C}),$$

is $\mathrm{Aut}(\mathcal{T})\propto(S^1)^I$-invariant and smooth on $\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset$, continuous on $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$, and

$$R_{\upsilon;P}^{0,1}|_b=\mathrm{id}\qquad\forall\,b\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J).\tag{4.37}$$

With notation as in (4.8), we define $\tilde{\beta}_b\in C^\infty(\Sigma_b;\mathbb{R})$ by

$$\tilde{\beta}_b(z)=\begin{cases}1, & \text{if } z\in\Sigma_{b,i},\ i\in\chi^0(\mathcal{T});\\ 1-\beta_{\delta(b)}(r(z)), & \text{if } z\in\Sigma_{b,i},\ i\in\chi(\mathcal{T});\\ 0, & \text{otherwise.}\end{cases}$$

In other words, $\tilde{\beta}_b=1$ on $\Sigma_b^0(\delta(b)/2)$ and vanishes outside of $\Sigma_b^0(2\delta(b))\subset\Sigma_b$. Let $\tilde{\beta}_\upsilon=\tilde{\beta}_b\circ q_\upsilon$. If $z\in\Sigma_\upsilon^0(2\delta(b))$, we denote by $\Pi_z^{\upsilon,0}$ the parallel transport in the line bundle $u_{\upsilon,0}^*\mathfrak{L}$ along a path from $x\in\tilde{q}_{\upsilon_0;\langle\mathcal{T}\rangle+1}^{-1}(\Sigma_{\upsilon_{\langle 1\rangle};P})$ to z in $\Sigma_\upsilon^0(2\delta(b))$ with respect to the connection $\tilde{q}_{\upsilon_0;\langle\mathcal{T}\rangle+1}^*\nabla^{\upsilon,1}$. For each

$$\upsilon=(b,v)\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset\qquad\text{and}\qquad\eta\in\Gamma_-^{0,1}(b;\mathfrak{L}),\tag{4.38}$$

let $R_\upsilon^{0,1}\eta\in\Gamma^{0,1}(\upsilon;\mathfrak{L})$ be given by

$$\{R_\upsilon^{0,1}\eta\}_z w=\tilde{\beta}_\upsilon(z)\,\Pi_z^{\upsilon,0}\eta_z(w)\in\mathfrak{L}_{u_{\upsilon,0}(z)}\qquad z\in\Sigma_\upsilon,\ w\in T_z\Sigma_\upsilon.$$

The image of $\Gamma_-^{0,1}(b;\mathfrak{L})$ in $\Gamma^{0,1}(\upsilon;\mathfrak{L})$ is a complement of $\Gamma_+^{0,1}(\upsilon;\mathfrak{L})$ in $\Gamma^{0,1}(\upsilon;\mathfrak{L})$, as can be seen from Lemma 4.4 below.

If $\eta\in\Gamma_-^{0,1}(b;\mathfrak{L})$, we put

$$\|\eta\|=\sum_{h\in\mathcal{I}_0}|\eta|_{x_h(b)},$$

where $|\eta|_{x_h(b)}$ is the norm of $\eta|_{x_h(b)}$ with respect to the metric $g_{\pi_P(b)}$ on $\Sigma_{b;P}$. If υ and η are as in (4.38) and $\|\eta\|=1$, we define by

$$\pi_{\upsilon;-}^{0,1}\colon\Gamma^{0,1}(\upsilon;\mathfrak{L})\longrightarrow\Gamma_-^{0,1}(b;\mathfrak{L})\qquad\text{by}\qquad\pi_{\upsilon;-}^{0,1}(\eta')=\langle\!\langle\eta',R_\upsilon^{0,1}\eta\rangle\!\rangle_2\eta\quad\forall\,\eta'\in\Gamma^{0,1}(\upsilon;\mathfrak{L}).$$

Since the space $\Gamma_-^{0,1}(b;\mathfrak{L})$ is one-dimensional, $\pi_{\upsilon;-}^{0,1}$ is independent of the choice of η. We note that since $p>2$, by Holder's inequality

$$\big\|\pi_{\upsilon;-}^{0,1}\eta'\big\|\le C(b)\|\eta'\|_{\upsilon,p}\qquad\forall\,\eta'\in\Gamma^{0,1}(\upsilon;\mathfrak{L}).\tag{4.39}$$

Lemma 4.4 *If n, d, k, a, and $\mathfrak{L}$ are as in Proposition 3.3, there exists $\delta_n(d)\in\mathbb{R}^+$ such that for every almost complex structure J on $\mathbb{P}^n$, such that $\|J-J_0\|_{C^1}\le\delta_n(d)$, and a bubble type $\mathcal{T}$ as above, there exist $\delta, C\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$ such that the requirements of Corollary 4.3 are satisfied. Furthermore, with notation as above, for all $\upsilon=(b,v)\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset$,*

$$\pi^{0,1}_{\upsilon;-}\bar{\partial}_{\nabla,b_0(\upsilon)}R_{\upsilon,0}\xi = -2\pi\mathfrak{i}\,\mathfrak{D}_{\mathcal{T}}\big(\xi\otimes\rho(\upsilon)\big) \qquad \forall\,\xi\in\Gamma_-(b;\mathfrak{L}); \tag{4.40}$$

$$\big\|\pi^{0,1}_{\upsilon;-}\bar{\partial}_{\nabla,b_0(\upsilon)}\xi\big\| \le C(b)\big|\rho(\upsilon)\big|\|\xi\|_{\upsilon,p,1} \qquad \forall\,\xi\in\Gamma(\upsilon;\mathfrak{L}). \tag{4.41}$$

Finally, the map $\upsilon\longrightarrow\pi^{0,1}_{\upsilon;-}$ is $\mathrm{Aut}(\mathcal{T})\propto(S^1)^I$-invariant and smooth on $\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset$. It extends continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$.

The identity (4.40) requires the restriction on the homomorphisms $R^{0,1}_{\upsilon;P}$ and identification of gluing parameters described in Subsection 4.2 of [Z5]. It follows from (4.29) by the same integration-by-parts argument as used in the proof of Proposition 4.4 in [Z2]. The estimate (4.41) is obtained by computing $\bar{\partial}^*_{\nabla,b_0(\upsilon)}R^{0,1}_{\upsilon}\eta$; see the proof of Lemma 2.2 in [Z2].

With notation as in the two previous subsections, let

$$\Pi_{\zeta_{\upsilon,0}}\colon u^*_{\upsilon,0}\mathfrak{L}\longrightarrow\tilde{u}^*_{\upsilon,0}\mathfrak{L}$$

be the ∇-parallel transport along the geodesics $\tau\longrightarrow\exp_{u_{\upsilon,0}(z)}\zeta_{\upsilon,0}(z)$, with $\tau\in[0,1]$. We put

$$L_{\upsilon,0}=\Pi^{-1}_{\zeta_{\upsilon,0}}\circ\bar{\partial}_{\nabla,\tilde{b}_0(\upsilon)}\circ\Pi_{\zeta_{\upsilon,0}}-\bar{\partial}_{\nabla,b_0(\upsilon)}\colon\Gamma\big(\upsilon;\mathfrak{L}\big)\longrightarrow\Gamma^{0,1}\big(\upsilon;\mathfrak{L}\big);$$
$$\tilde{\Gamma}'_-(\upsilon;\mathfrak{L})=\big\{\Pi^{-1}_{\zeta_{\upsilon,0}}\xi\colon\xi\in\tilde{\Gamma}_-(\upsilon;\mathfrak{L})\big\}\subset\Gamma\big(\upsilon;\mathfrak{L}\big).$$

We denote by

$$\pi_{\upsilon;-}\colon\Gamma(\upsilon;\mathfrak{L})\longrightarrow\Gamma_-(\upsilon;\mathfrak{L}) \qquad\text{and}\qquad \tilde{\pi}_{\upsilon;-}\colon\Gamma(\upsilon;\mathfrak{L})\longrightarrow\tilde{\Gamma}'_-(\upsilon;\mathfrak{L})$$

the L^2-projection maps. Let $\Gamma'_-(\upsilon;\mathfrak{L})$ be the image of $\tilde{\Gamma}'_-(\upsilon;\mathfrak{L})$ under $\pi_{\upsilon;-}$. By the analogue of (3.2) for $\zeta_{\upsilon,0}$ and (4.3),

$$\big\|L_{\upsilon,0}\xi\big\|_{\upsilon,p}\le C(b)\big|\rho(\upsilon)\big|^2\|\xi\|_{\upsilon,p,1} \qquad \forall\,\xi\in\Gamma(\upsilon;\mathfrak{L}). \tag{4.42}$$

By (4.35), (4.36), and (4.42),

$$\big\|\xi-\pi_{\upsilon;-}\xi\big\|_{\upsilon,p,1}\le C(b)\big|\rho(\upsilon)\big|\|\xi\|_{\upsilon,p,1} \qquad \forall\ \xi\in\tilde{\Gamma}'_-(\upsilon;\mathfrak{L}). \tag{4.43}$$

By (4.39)-(4.43),

$$\big|\mathfrak{D}_{\mathcal{T}}(\xi\otimes\rho(\upsilon))\big|\le C(b)\big|\rho(\upsilon)\big|^2\|R_{\upsilon,0}\xi\|_{\upsilon,p,1} \qquad \forall\ R_{\upsilon,0}\xi\in\Gamma'_-(\upsilon;\mathfrak{L}). \tag{4.44}$$

For each $b\in\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^n;J)$ and $[w]\in\mathbb{P}\tilde{\mathfrak{F}}\mathcal{T}|_b$, let

$$\Gamma_-\big(b;\mathfrak{L};[w]\big)=\big\{\xi\in\Gamma_-(b;\mathfrak{L});\ \mathfrak{D}_{\mathcal{T}}(\xi\otimes w)=0\big\}.$$

Similarly to the previous subsection, the map $\mathfrak{D}_{\mathcal{T}}$ is surjective. Thus, the L^2-orthogonal complement $\Gamma_-^{\perp}(b;\mathfrak{L};[w])$ of $\Gamma_-(b;\mathfrak{L};[w])$ in $\Gamma_-(b;\mathfrak{L})$ is one-dimensional. Furthermore, there exists $C\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$ such that

$$C(b)^{-1}|w|\cdot\|\xi\|_{b,p,1}\le\big|\mathfrak{D}_{\mathcal{T}}(\xi\otimes w)\big|\le C(b)|w|\cdot\|\xi\|_{b,p,1}\qquad\forall\ \xi\in\Gamma_-^{\perp}\big(b;\mathfrak{L};[w]\big).\tag{4.45}$$

If $\upsilon\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta^{\emptyset}$, let

$$\Gamma_-\big(\upsilon;\mathfrak{L};[w]\big)=\big\{R_{\upsilon,0}\xi:\xi\in\Gamma_-(b;\mathfrak{L};[w])\big\}\subset\Gamma_-(\upsilon;\mathfrak{L}).$$

We denote by $\Gamma_-^{\perp}(\upsilon;\mathfrak{L};[w])$ the L^2-orthogonal complement of $\Gamma_-(\upsilon;\mathfrak{L};[w])$ in $\Gamma_-(\upsilon;\mathfrak{L})$. Since $R_{\upsilon,0}$ is close to an isometry on $\Gamma_-(b;\mathfrak{L})$ with respect to the L^2 and L^p_1-norms,

$$\big|\mathfrak{D}_{\mathcal{T}}(\xi\otimes w\big|\ge C(b)^{-1}|w|\|R_{\upsilon,0}\xi\|_{\upsilon,p,1}\qquad\forall\ R_\upsilon\xi\in\Gamma_-^{\perp}\big(\upsilon;\mathfrak{L};[\rho(\upsilon)]\big),\tag{4.46}$$

by (4.45). We note that

$$\dim\Gamma_-\big(\upsilon;\mathfrak{L};[\rho(\upsilon)]\big)=\dim\tilde{\Gamma}'_-(\upsilon;\mathfrak{L})=\dim\Gamma'_-(\upsilon;\mathfrak{L}).$$

Thus, by (4.43), (4.44), and (4.46) applied with $w=\rho(\upsilon)$, the map

$$\tilde{\pi}_{\upsilon;-}:\Gamma_-\big(\upsilon;\mathfrak{L};[\rho(\upsilon)]\big)\longrightarrow\tilde{\Gamma}'_-(\upsilon;\mathfrak{L})$$

is an isomorphism. Furthermore,

$$\big\|\xi-\tilde{\pi}_{\upsilon;-}\xi\big\|_{\upsilon,p,1}\le C(b)\big|\rho(\upsilon)\big|\|\xi\|_{\upsilon,p,1}\qquad\forall\ \xi\in\Gamma_-\big(\upsilon;\mathfrak{L};[\rho(\upsilon)]\big).\tag{4.47}$$

If $b\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J)$, let

$$\tilde{\Gamma}_-(b;\mathfrak{L})=\big\{\xi\in\Gamma_-(b;\mathfrak{L}):\mathfrak{D}_{\mathcal{T}}(\xi\otimes w)=0\ \forall\,w\in\tilde{\mathfrak{F}}^1\mathcal{T}|_b\big\}.$$

Corollary 4.5 *If n, d, k, a, and $\mathfrak{L}$ are as in Proposition 3.3, there exists $\delta_n(d)\in\mathbb{R}^+$ such that for every almost complex structure J on $\mathbb{P}^n$, such that $\|J-J_0\|_{C^1}\le\delta_n(d)$, and a bubble type $\mathcal{T}$ as above, there exist $\delta,C\in C(\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J);\mathbb{R}^+)$ with the following property. For every $\upsilon=(b,v)\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta$ there exists a homomorphism*

$$\tilde{R}_\upsilon:\Gamma_-(b;\mathfrak{L})\longrightarrow\tilde{\Gamma}_-(\upsilon;\mathfrak{L})$$

such that the map $\upsilon\longrightarrow\tilde{R}_\upsilon$ is $\mathrm{Aut}(\mathcal{T})\propto(S^1)^I$-invariant and smooth on $\tilde{\mathcal{F}}^1\mathcal{T}_\delta^{\emptyset}$. Furthermore, the map $\upsilon\longrightarrow\tilde{R}_\upsilon|_{\tilde{\Gamma}_-(b;\mathfrak{L})}$ is continuous on $\tilde{\mathcal{F}}^1\mathcal{T}_\delta^{\emptyset}$ and

$$\tilde{R}_b=\mathrm{id}\quad\forall\,b\in\mathcal{U}_{\mathcal{T}}^{(0)}(\mathbb{P}^n;J).\tag{4.48}$$

If $\upsilon\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta^{\emptyset}$, the homomorphism $\tilde{R}_\upsilon$ is defined by

$$\tilde{R}_\upsilon\xi=\Pi_{\zeta_{\upsilon,0}}\tilde{\pi}_{\upsilon;-}R_{\upsilon,0}\xi\qquad\forall\,\xi\in\Gamma_-(b;\mathfrak{L}).$$

Since the maps

$$\upsilon \longrightarrow b_0(\upsilon),\ \zeta_{\upsilon,0},\ R_{\upsilon,0},\ \Gamma_-\big(\upsilon;\mathfrak{L};[\rho(\upsilon)]\big)$$

are continuous over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta-\rho^{-1}(0)$, this family of homomorphisms extends continuously over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta-\rho^{-1}(0)$, as can be seen by an argument similar to Subsection 3.9 and 4.1 in [Z3]. This extension is formally described in the same way as the homomorphisms $\tilde{R}_\upsilon$ for $\upsilon\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset$. On the other hand, if $\rho(\upsilon)=0$, we put

$$\tilde{R}_\upsilon\xi = \Pi_{\zeta_{\upsilon,0}}R_{\upsilon,0}\xi = R_{\upsilon,0}\xi \qquad \forall\,\xi\in\tilde{\Gamma}_-(b;\mathfrak{L}).$$

The second equality above holds by (4.4). By (4.30), the requirement (4.48) is satisfied.

It remains to check that the extension described above is continuous at every

$$\upsilon^*\equiv(b^*,v^*)\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta\cap\rho^{-1}(0).$$

We note that by (4.47),

$$\tilde{R}_\upsilon\xi = \Pi_{\zeta_{\upsilon,0}}\big(R_{\upsilon,0}\xi+\varepsilon_{\upsilon,0}(\xi)\big) \qquad \forall\,\xi\in\Gamma_-(b;\mathfrak{L}),\ \upsilon\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset, \tag{4.49}$$

for some homomorphism

$$\varepsilon_{\upsilon,0}\colon\Gamma_-(b;\mathfrak{L})\longrightarrow\Gamma(\upsilon;\mathfrak{L})$$

such that

$$\big\|\varepsilon_{\upsilon,0}(\xi)\big\|_{\upsilon,p,1}\le C(b)\big|\rho(\upsilon)\big|\|\xi\|_{b,p,1} \qquad \forall\,\xi\in\Gamma_-\big(b;\mathfrak{L};[\rho(\upsilon)]\big). \tag{4.50}$$

Suppose $\upsilon_r\equiv(b_r,v_r)\in\tilde{\mathcal{F}}^1\mathcal{T}_\delta^\emptyset$ and $\xi_r\in\tilde{\Gamma}(b_r;\mathfrak{L})$ are sequence such that

$$\lim_{r\longrightarrow\infty}\upsilon_r=b^* \qquad\text{and}\qquad \lim_{r\longrightarrow\infty}\xi_r=\xi^*\in\Gamma_-(b^*;\mathfrak{L}).$$

Since $\tilde{\Gamma}(b_r;\mathfrak{L})\subset\Gamma_-(b_r;\mathfrak{L};[\rho(\upsilon_r)])$ and the maps

$$\upsilon\longrightarrow b_0(\upsilon),\ \zeta_{\upsilon,0},\ R_{\upsilon,0}$$

are continuous over $\tilde{\mathcal{F}}^1\mathcal{T}_\delta$,

$$\lim_{r\longrightarrow\infty}\tilde{R}_{\upsilon_r}\xi_r = \lim_{r\longrightarrow\infty}\Pi_{\zeta_{\upsilon_r,0}}\big(R_{\upsilon_r,0}\xi+\varepsilon_{\upsilon_r,0}(\xi)\big)=\tilde{R}_\upsilon\xi^*,$$

by (4.49) and (4.50), as needed.

Corollary 4.5 concludes the proof of Lemma 3.4. It remains to finish the proof of Proposition 3.3. By Corollary 4.5, $\tilde{R}_\upsilon$ induces an injective homomorphism

$$R_{[\upsilon]}\colon\mathcal{V}^{d;m}_{1,k;\mathcal{T}}\big|_b\longrightarrow\mathcal{V}^d_{1,k}\big|_{\phi_\mathcal{T}([\upsilon])}$$

for $b\in\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$ and $[\upsilon]=[b,v]\in\mathcal{F}^1\mathcal{T}_\delta$. If U is an open subset of $\mathcal{U}_\mathcal{T}(\mathbb{P}^n;J)$ and $\mathcal{W}\longrightarrow U$ is a smooth subbundle of $\mathcal{V}^d_{1,k}|_U$ such that

$$\mathcal{W}_b\subset\mathcal{V}^{d;m}_{1,k;\mathcal{T}}\big|_b \qquad \forall\ b\in U\cap\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J),\ m\in\big(\max(|\chi(\mathcal{T})|-n,1),|\chi(\mathcal{T})|\big),$$

then the map $[v]\longrightarrow R_{[v]}$ induces a continuous injective bundle homomorphism

$$\tilde{\phi}_{\mathcal{W}}\colon \pi^*_{\mathcal{F}^1\mathcal{T}_\delta|_U}\mathcal{W}\longrightarrow \mathcal{V}^d_{1,k}$$

that restricts to the identity over U and is smooth over $\mathcal{F}^1\mathcal{T}^{\emptyset}_{\delta}$.

Finally, for each $m\in(\max(|\chi(\mathcal{T})|-n,1),|\chi(\mathcal{T})|)$, let $U^m_{\mathcal{T}}\subset U_{\mathcal{T}}$ be a small neighborhood of $\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$ in $\mathfrak{X}_{1,k}(\mathbb{P}^n,d)$ and let

$$\mathcal{W}^{d;m}_{1,k;\mathcal{T}}\longrightarrow \mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)\cap U^m_{\mathcal{T}}$$

be a subbundle of $\mathcal{V}^d_{1,k}$ such that

$$\mathcal{W}^{d;m}_{1,k;\mathcal{T}}\big|_{\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)}=\mathcal{V}^{d;m}_{1,k;\mathcal{T}}\big|_{\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)} \qquad\text{and} \tag{4.51}$$

$$\mathcal{W}^{d;m}_{1,k;\mathcal{T}}\big|_b\subset \mathcal{V}^{d;m'}_{1,k;\mathcal{T}}\big|_b \qquad \forall\, b\in \mathcal{U}^{m'}_{\mathcal{T};1}(\mathbb{P}^n;J)\cap U^m_{\mathcal{T}},\ m'\in\big(\max(|\chi(\mathcal{T})|-n,1),|\chi(\mathcal{T})|\big). \tag{4.52}$$

By the next paragraph, such an extension of $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}\big|_{\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)}$ to $\mathcal{U}_{\mathcal{T}}(\mathbb{P}^n;J)\cap U^m_{\mathcal{T}}$ exists if $U^m_{\mathcal{T}}$ is sufficiently small. By the previous paragraph, the bundle homomorphism

$$\tilde{\phi}^m_{\mathcal{T}}\equiv\tilde{\phi}_{\mathcal{W}^{d;m}_{1,k;\mathcal{T}}}\colon \pi^*_{\mathcal{F}^1\mathcal{T}_\delta|_U}\mathcal{W}^{d;m}_{1,k;\mathcal{T}}\longrightarrow \mathcal{V}^d_{1,k}$$

is continuous and injective, restricts to the identity over $\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)\cap U_{\mathcal{T}}$, and is smooth over $\mathcal{F}^1\mathcal{T}^{\emptyset}_{\delta}|_U$. We define the bundle

$$\mathcal{V}^{d,m}_{1,k;\mathcal{T}}\longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d;J)\cap U^m_{\mathcal{T}}$$

to be the image of $\tilde{\phi}^m_{\mathcal{T}}$. This bundle has the claimed rank by the last statement of Lemma 3.4. The last condition of Proposition 3.3 is satisfied by the definition of the bundles $\mathcal{V}^{d;m}_{1,k;\mathcal{T}}|_{\mathcal{U}^m_{\mathcal{T};1}(\mathbb{P}^n;J)}$ following Proposition 3.3. The proof of Proposition 3.3 is now complete.

We now prove the extension claim used in the previous paragraph. By definition,

$$\tilde{\mathfrak{F}}^1\mathcal{T}=\big\{w\in\tilde{\mathfrak{F}}\mathcal{T}\colon \mathcal{D}_{\mathcal{T}}w=0\big\}.$$

Since $\mathcal{D}_{\mathcal{T}}$ is a continuous bundle section, if $\tilde{U}$ is a sufficiently small neighborhood of $\tilde{\mathcal{U}}^m_{\mathcal{T};1}(\mathbb{P}^n;J)$ in $\mathcal{U}^{(0)}_{\mathcal{T}}(\mathbb{P}^n;J)$, there exists a vector bundle $\tilde{\mathfrak{F}}^1\mathcal{T}^m\longrightarrow\tilde{U}$ such that

$$\tilde{\mathfrak{F}}^1\mathcal{T}^m\big|_{\tilde{\mathcal{U}}^m_{\mathcal{T};1}(\mathbb{P}^n;J)}=\tilde{\mathfrak{F}}^1\mathcal{T}\big|_{\tilde{\mathcal{U}}^m_{\mathcal{T};1}(\mathbb{P}^n;J)} \qquad\text{and}\qquad \tilde{\mathfrak{F}}^1\mathcal{T}|_{\tilde{U}}\subset\tilde{\mathfrak{F}}^1\mathcal{T}^m\subset\tilde{\mathfrak{F}}\mathcal{T}. \tag{4.53}$$

The neighborhood $\tilde{U}$ and the bundle $\tilde{\mathfrak{F}}^1\mathcal{T}^m$ can be chosen so that they are preserved by the actions of $\mathrm{Aut}(\mathcal{T})\propto(S^1)^I$. We then define the vector bundle $\mathcal{W}^{d;m}_{1,k;\mathcal{T}}\longrightarrow U$ by

$$U=\big\{[b]\in\mathcal{U}_{\mathcal{T};1}(\mathbb{P}^n;J)\colon b\in\tilde{U}\big\} \qquad\text{and}$$
$$\mathcal{W}^{d;m}_{1,k;\mathcal{T}}=\big\{[\xi]\in\mathcal{V}^d_{1,k}|_b\colon b\in U;\ \mathfrak{D}_{\mathcal{T}}(\xi\otimes w)=0\ \forall\, w\in\tilde{\mathfrak{F}}^1\mathcal{T}^m|_b\big\}.$$

By the same argument as at the end of Subsection 3.3, $\mathcal{W}^{d;m}_{1,k;\mathcal{T}}\longrightarrow U$ is a vector bundle of rank $da+1-m$. By the middle statement of Lemma 3.4 and (4.53), this vector bundle satisfies the requirements (4.51) and (4.52), as needed.

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