Reduced Genus-One Gromov-Witten Invariants

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Abstract

In a previous paper we described a natural closed subset $\overline{\mathfrak{M}}^{0}_{1,k}(X, A; J)$ of the moduli space $\overline{\mathfrak{M}}_{1,k}(X, A; J)$ of stable genus-one J-holomorphic maps into a symplectic manifold X. In this paper we generalize the definition of the main component to moduli spaces of perturbed, in a restricted way, J-holomorphic maps. This generalization implies that $\overline{\mathfrak{M}}^{0}_{1,k}(X, A; J)$, just like $\overline{\mathfrak{M}}_{1,k}(X, A; J)$, carries a virtual fundamental class and can be used to define symplectic invariants. These truly genus-one invariants constitute part of the standard genus-one Gromov-Witten invariants, which arise from the entire moduli space $\overline{\mathfrak{M}}_{1,k}(X, A; J)$. The new invariants are more geometric and can be used to compute the genus-one GW-invariants of complete intersections, as shown in a separate paper.

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1 Introduction

1.1 Background and Motivation

Let (X, ω, J) be a compact almost Kahler manifold. In other words, (X, ω) is a symplectic manifold and J is an almost complex structure on X tamed by ω, i.e.

$$\omega(v, Jv) > 0 \qquad \forall v \in TX - X.$$

If g, k are nonnegative integers and $A \in H_2(X; \mathbb{Z})$, we denote by $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ the moduli space of (equivalence classes of) stable J-holomorphic maps from genus-g Riemann surfaces with k marked points in the homology class A. Let $\mathfrak{M}_{g,k}^0(X, A; J)$ be the subspace of $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ consisting of the stable maps $[\mathcal{C}, u]$ such that the domain $\mathcal{C}$ is a smooth Riemann surface. The compact moduli space $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ was constructed in order to "compactify" $\mathfrak{M}_{g,k}^0(X, A; J)$ and to define invariants of (X, ω) enumerating J-holomorphic curves of genus g in X. If $g = 0$, $(X, \omega; A)$ is positive in a certain sense, and J is generic, then $\mathfrak{M}_{g,k}^0(X, A; J)$ is a dense open subset of $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ and the corresponding Gromov-Witten invariants do indeed count genus-zero J-holomorphic curves in X; see [McSa, Chapter 7] and [RT, Sections 1,9], for example. However, if $g \geq 1$, it is usually the case that $\mathfrak{M}_{g,k}^0(X, A; J)$ is not dense in $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ and the genus-g GW-counts include J-holomorphic curves of lower genera.

If $g = 1$ and $(X, \omega; A)$ is positive, the above deficiencies are due exclusively to the presence of large subspaces of stable maps $[\mathcal{C}, u]$ in $\overline{\mathfrak{M}}_{1,k}(X, A; J)$ such that u is constant on the principal components of $\mathcal{C}$, i.e. the irreducible components that carry the genus of $\mathcal{C}$. More precisely, if m is a positive integer, let $\mathfrak{M}_{1,k}^m(X, A; J)$ be the subset of $\overline{\mathfrak{M}}_{1,k}(X, A; J)$ consisting of the stable maps $[\mathcal{C}, u]$ such that $\mathcal{C}$ is a smooth genus-one curve $\mathcal{C}_P$ with m rational components attached directly to $\mathcal{C}_P$, $u|_{\mathcal{C}_P}$ is constant, and the restriction of u to each rational component is non-constant. Figure 1 shows the domain of an element of $\mathfrak{M}_{1,k}^3(X, A; J)$, from the points of view of symplectic topology and of algebraic geometry. In the first diagram, each shaded disc represents a sphere; the homology class next to each rational component $\mathcal{C}_i$ indicates the degree of $u|_{\mathcal{C}_i}$. In the second diagram, the components of $\mathcal{C}$ are represented by curves, and the pair of indices next to each component $\mathcal{C}_i$ shows the genus of $\mathcal{C}_i$ and the degree of $u|_{\mathcal{C}_i}$. We denote by $\overline{\mathfrak{M}}_{1,k}^m(X, A; J)$ the closure of $\mathfrak{M}_{1,k}^m(X, A; J)$ in $\overline{\mathfrak{M}}_{1,k}(X, A; J)$. The image $u(\mathcal{C})$ of an element of $\overline{\mathfrak{M}}_{1,k}^m(X, A; J)$ is a genus-zero, instead of genus-one, J-holomorphic curve in X. We note that if J is sufficiently regular, then

$$\dim \mathfrak{M}_{1,k}^0(X, A; J) = 2\big(\langle c_1(TX), A \rangle + k\big) \equiv \dim_{1,k}(X, A) \qquad \text{and}$$
$$\dim \mathfrak{M}_{1,k}^m(X, A; J) = \dim_{1,k}(X, A) + 2(n-m),$$

where $2n$ is the real dimension of X. Thus, the complement of $\mathfrak{M}_{1,k}^0(X, A; J)$ in $\overline{\mathfrak{M}}_{1,k}(X, A; J)$ contains subspaces of dimension at least as large as the dimension of $\mathfrak{M}_{1,k}^0(X, A; J)$, as long as $n \geq 1$, i.e. X is not a finite collection of points.

In [Z4, Definition 1.1], we describe a subset $\overline{\mathfrak{M}}_{1,k}^0(X, A; J)$ of $\overline{\mathfrak{M}}_{1,k}(X, A; J)$, for an arbitrary compact almost Kahler manifold (X, ω, J), obtained from $\overline{\mathfrak{M}}_{1,k}(X, A; J)$ by discarding most elements of the spaces $\overline{\mathfrak{M}}_{1,k}^m(X, A; J)$ with $m \leq n$. In particular, $\overline{\mathfrak{M}}_{1,k}^0(X, A; J)$ contains $\mathfrak{M}_{1,k}^0(X, A; J)$. By [Z4, Theorem 1.2], $\overline{\mathfrak{M}}_{1,k}^0(X, A; J)$ is a closed subset of $\overline{\mathfrak{M}}_{1,k}(X, A; J)$ and thus is compact. If $(X, \omega; A)$ is positive in the same sense as in the genus-zero case and J is generic, $\mathfrak{M}_{1,k}^0(X, A; J)$ is a dense open
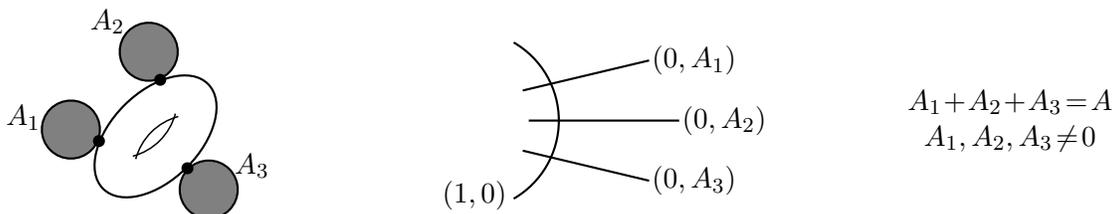
Figure 1: The domain of an element of $\mathfrak{M}_{1,k}^3(X, A; J)$

subset of $\overline{\mathfrak{M}}_{1,k}^0(X, A; J)$. In addition, $\overline{\mathfrak{M}}_{1,k}^0(X, A; J)$ carries a rational fundamental class, which can be used to define a symplectic invariant of (X, ω) counting genus-one J-holomorphic curves in X, without any genus-zero contribution in contrast to the standard Gromov-Witten invariants; see [Z4, Subsection 1.3]. Unlike the genus-zero case, $\overline{\mathfrak{M}}_{1,k}^0(X, A; J)$ has the topological structure of a singular, instead of smooth, orbivariety.

A J-holomorphic map into X is a smooth map u from a Riemann surface (Σ, j) that satisfies the Cauchy-Riemann equation corresponding to (J, j):

$$\bar{\partial}_{J,j} u \equiv \frac{1}{2}\big(du + J \circ du \circ j\big) = 0.$$

The Riemann surface (Σ, j) may have simple nodes. In this paper we generalize the results of [Z4] to smooth maps u, from genus-one Riemann surfaces, that satisfy a family of perturbed Cauchy-Riemann equations:

$$\bar{\partial}_{J,j} u + \nu(u) = 0.$$

The perturbation term $\nu(u)$ is a section of the vector bundle

$$\Lambda_{J,j}^{0,1} T^*\Sigma \otimes u^*TX \equiv \big\{\eta \in \mathrm{Hom}_{\mathbb{R}}(T\Sigma, u^*TX) \colon J \circ \eta = -\eta \circ j\big\} \longrightarrow \Sigma.$$

We will study the moduli space $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu)$ of (J, ν)-holomorphic maps, i.e. of solutions to the perturbed Cauchy-Riemann equations, for a continuous family $\nu = \nu(u)$ chosen from a proper linear subspace of the space of all such families; see Definition 1.2. A key condition on ν will be that if the degree of u restricted to the principal components of Σ is zero, then the restriction of $\nu(u)$ to the principal components and all nearby degree-zero bubble components is also zero. Such a family $\nu = \nu(u)$ will be called effectively supported.

We will show that if ν is sufficiently small and effectively supported, then the moduli space $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu)$ contains a natural closed subspace $\overline{\mathfrak{M}}_{1,k}^0(X, A; J, \nu)$ containing $\mathfrak{M}_{1,k}^0(X, A; J, \nu)$, i.e. the subspace of maps with smooth domains; see Definition 1.3 and Theorem 1.4. For a generic choice of ν, the "boundary" of $\overline{\mathfrak{M}}_{1,k}^0(X, A; J, \nu)$ is of real codimension two and thus $\overline{\mathfrak{M}}_{1,k}^0(X, A; J, \nu)$ determines a rational homology class. This virtual fundamental class (VFC) for $\overline{\mathfrak{M}}_{1,k}^0(X, A; J)$ does not change under small changes in ν and is an invariant of (X, ω). It can be used to define new GW-style invariants, which we denote by $\mathrm{GW}_{1,k}^0$. These invariants differ from the standard GW-invariants by a combination of the genus-zero GW-invariants of X; see Subsection 1.2 below for some special cases.

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We note that effectively supported families $\nu = \nu(u)$ are in no sense generic in the space of all families. If fact, for a generic ν, $\mathfrak{M}_{1,k}^0(X, A; J, \nu)$ is dense in $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu)$, and the latter space determines the standard GW-invariants of (X, ω). In particular, the statements of the previous paragraph do not hold for a generic family ν of perturbations.

An algebraic approach to reduced genus-one GW-invariants is suggested by Vakil and the author at the end of [VaZ2]. It still remains to verify that the resulting algebraic invariants agree with the symplectic ones defined in this paper (whenever the target space is a smooth algebraic variety), but this should be deducible from the desingularizations for certain natural sheaves constructed by Vakil and the author in [VaZ1, Section 5].

Since the symplectic invariants arising from the moduli space $\overline{\mathfrak{M}}_{1,k}^0(X, A; J, \nu)$, with ν effectively supported, are closely related to the standard GW-invariants, they do not in principle carry any new information. In practice, they behave better geometrically. In particular, Li and the author show in [LZ] that there is a simple relation between *reduced* genus-one GW-invariants of a projective complete intersection and twisted *reduced* genus-one GW-invariants of the ambient space. This relation mimics the corresponding well-known relation in genus zero (see [LZ, (1.2)], for example), but no relation in positive genera had been even conjectured until [LZ]. Combining [LZ, Theorem 1.1] for the reduced genus-one invariants constructed in this paper with the desingularization of Vakil and the author in [VaZ1] and Theorem 1.1 below, the author finally confirms the 1993 mirror symmetry of Bershadsky-Cecotti-Ooguri-Vafa [BCOV] for the genus-one GW-invariants of a quintic threefold[1] in [Z5].

In Subsection 1.3, we describe a geometric reinterpretation of the VFC constructions of Fukaya-Ono [FuOn] and Li-Tian [LT] which is well suited for defining a VFC for $\overline{\mathfrak{M}}_{1,k}^0(X, A; J)$. We state the main results of this paper in Subsection 1.4. In Subsections 2.1 and 2.2, we generalize the setup of [Z4] for J-holomorphic maps to (J, ν)-holomorphic maps. In Subsection 2.3, we state three propositions that together are equivalent to Theorem 1.4. They are proved in Subsections 2.4 and 2.5 by extending some of the analytic arguments of [Z4, Sections 3,4] to the present situation. The difference between the standard and reduced GW-invariants is analyzed in Section 3; see also the next subsection.

1.2 Standard vs. Reduced Gromov-Witten Invariants

From the construction of VFC for $\overline{\mathfrak{M}}_{1,k}^0(X, A; J)$ in Subsection 1.4, it is immediate that the difference between the standard and reduced genus-one GW-invariants of X must be a combination of the genus-zero GW-invariants of X. The exact form of this combination can be determined in each specific case from Proposition 3.1. In this subsection, we give an explicit expression for the difference between the standard and reduced genus-one GW-invariants in the two simplest cases.

For each $l = 1, \ldots, k$, let

$$\mathrm{ev}_l \colon \overline{\mathfrak{M}}_{g,k}(X, A; J) \longrightarrow X, \qquad \big[\Sigma, y_1, \ldots, y_k; u\big] \longrightarrow u(y_l),$$

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[1] This is the genus-one analogue of the 1991 genus-zero mirror symmetry prediction of Candelas-de la Ossa-Green-Parkes [CDGP], which was proved in several different ways in the mid to late 90s.

the evaluation map at the lth marked point. We will call a cohomology class ψ on $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ geometric if ψ is a product of the classes $\mathrm{ev}_l^* \mu_l$ for $\mu_l \in H^*(X; \mathbb{Z})$. We denote by $\bar{\mathbb{Z}}^+$ the set of nonnegative integers.

Theorem 1.1 *Suppose (X, ω) is a compact symplectic manifold, $A \in H_2(X; \mathbb{Z})^*$, $k \in \bar{\mathbb{Z}}^+$. If J is an ω-compatible almost complex structure on X and ψ is a geometric cohomology class on $\overline{\mathfrak{M}}_{1,k}(X, A; J)$, then*

$$
\mathrm{GW}_{1,k}^X(A; \psi) - \mathrm{GW}_{1,k}^{0;X}(A; \psi) = \begin{cases} 0, & \text{if } \dim_{\mathbb{R}} X = 4; \\ \frac{2 - \langle c_1(TX), A \rangle}{24} \mathrm{GW}_{0,k}^X(A; \psi), & \text{if } \dim_{\mathbb{R}} X = 6. \end{cases}
$$

Theorem 1.1 is proved in Section 3 by studying the obstruction theory along each stratum of the moduli space $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu)$, after capping it with ψ. This proof generalizes to higher-dimensional manifolds X and more general cohomology classes; an explicit formula is obtained by the author in [Z6]. In fact, for geometric cohomology classes and higher-dimensional manifolds X, the difference is given by an expression similar to the correction term in [Z3, Theorem 1.1]; this can be seen a priori from Proposition 3.1 and [Z3, Subsection 3.2]. A special case of Theorem 1.1 is [Ge, Theorem A]; its proof has not yet appeared.

Theorem 1.1 has a natural, but rather speculative, generalization to higher-genus invariants. Suppose that the main component

$$
\overline{\mathfrak{M}}_{g,k}^0(X, A; J) \subset \overline{\mathfrak{M}}_{g,k}(X, A; J)
$$

is well-defined, as is the main component

$$
\overline{\mathfrak{M}}_{g,k}^0(X, A; J, \nu) \subset \overline{\mathfrak{M}}_{g,k}(X, A; J, \nu)
$$

for a sufficiently large subspace of perturbations ν of the $\bar{\partial}_J$-operator so that $\overline{\mathfrak{M}}_{g,k}(X, A; J, \nu)$ has a regular structure for a generic ν in this subspace; see Subsection 1.4 for the $g = 1$ case. If so, $\overline{\mathfrak{M}}_{g,k}^0(X, A; J)$ carries a virtual fundamental class and determines reduced genus-g GW-invariants $\mathrm{GW}_{g,k}^{0;X}(A; \psi)$. Theorem 1.1 and its proof should then generalize to higher-genus invariants. If $\dim_{\mathbb{R}} X = 6$, the expected relationship is

$$
\mathrm{GW}_{g,k}^X(A; \psi) - \mathrm{GW}_{g,k}^{0;X}(A; \psi) = \sum_{g'=0}^{g-1} C_g^{g'}(\langle c_1(TX), A \rangle) \mathrm{GW}_{g',k}^{0;X}(A; \psi),
$$

where $\mathrm{GW}_{0,k}^0 \equiv \mathrm{GW}_{0,k}$. The coefficients $C_g^{g'}(\langle c_1(TX), A \rangle)$ are given by Hodge integrals, i.e. integrals of natural cohomology classes on the moduli spaces $\overline{\mathcal{M}}_{*,*}$ of curves. They are of the form expected from the usual obstruction bundle approach. For example,

$$
C_2^1((5 - a)d) = -\frac{d(a - 5)}{24},
$$

$$
C_2^0((5 - a)d) = \frac{1}{2} \left(\frac{2 + d(a - 5)}{24} \right)^2 + \left\langle c(\mathbb{E}^* \otimes TX) c(L_{2,1} \otimes T\mathbb{P}^1)^{-1}, \left[\overline{\mathcal{M}}_{2,1} \right] \times [\mathbb{P}^1] \right\rangle,
$$

where $L_{2,1} \longrightarrow \overline{\mathcal{M}}_{2,1}$ is the universal tangent line bundle, $\mathbb{E} \longrightarrow \overline{\mathcal{M}}_{2,1}$ is the rank-two Hodge bundle, and $\mathbb{P}^1$ is viewed as a smooth degree-d curve in Y. The coefficients $C_g^{g'}(\langle c_1(TX), A \rangle)$ can be expressed in terms of the numbers $C_{g'}(g - g', X, A)$ of [Pa] and vice versa.

1.3 Configuration Spaces

In this subsection we recall certain configuration spaces that are standard in the theory of Gromov-Witten invariants. We then define what we mean by effectively supported perturbations of the $\bar\partial_J$-operator that are central to this paper.

Fix $p > 2$. Suppose X is a compact manifold, $A \in H_2(X; \mathbb{Z})$, and $g, k \in \bar{\mathbb{Z}}^+$. We denote by $\mathfrak{X}_{g,k}(X, A)$ the space of equivalence classes of stable L_1^p-maps $u \colon \Sigma \longrightarrow X$ from genus-g Riemann surfaces with k marked points, which may have simple nodes, to X of degree A, i.e.

$$u_*[\Sigma] = A \in H_2(X; \mathbb{Z}).$$

Let $\mathfrak{X}_{g,k}^0(X, A)$ be the subset of $\mathfrak{X}_{g,k}(X, A)$ consisting of the stable maps with smooth domains. The spaces $\mathfrak{X}_{g,k}(X, A)$ are topologized using L_1^p-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes; see [LT, Section 3]. The spaces $\mathfrak{X}_{g,k}(X, A)$ can be stratified by the smooth infinite-dimensional orbifolds $\mathfrak{X}_{\mathcal{T}}(X)$ of stable maps from domains of the same geometric type and with the same degree distribution between the components of the domain; see Subsections 2.1 and 2.2. The closure of the main stratum, $\mathfrak{X}_{g,k}^0(X, A)$, is $\mathfrak{X}_{g,k}(X, A)$.

If J is an almost complex structure on X, let

$$\Gamma_{g,k}^{0,1}(X, A; J) \longrightarrow \mathfrak{X}_{g,k}(X, A)$$

be the bundle of (TX, J)-valued $(0, 1)$ L^p-forms. In other words, the fiber of $\Gamma_{g,k}^{0,1}(X, A; J)$ over a point $[b] = [\Sigma, j; u]$ in $\mathfrak{X}_{g,k}(X, A)$ is the space

$$\Gamma_{g,k}^{0,1}(X, A; J)\big|_{[b]} = \Gamma^{0,1}(b; J)/\mathrm{Aut}(b), \quad \text{where} \quad \Gamma^{0,1}(b; J) = L^p\big(\Sigma; \Lambda_{J,j}^{0,1} T^*\Sigma \otimes u^* TX\big).$$

Here j is the complex structure on Σ, the domain of the smooth map u. The bundle $\Lambda_{J,j}^{0,1} T^*\Sigma \otimes u^* TX$ over Σ consists of (J, j)-antilinear homomorphisms:

$$\Lambda_{J,j}^{0,1} T^*\Sigma \otimes u^* TX = \big\{\eta \in \mathrm{Hom}(T\Sigma, u^* TX) \colon J \circ \eta = -\eta \circ j\big\}.$$

The total space of the bundle $\Gamma_{g,k}^{0,1}(X, A; J) \longrightarrow \mathfrak{X}_{g,k}(X, A)$ is topologized using L^p-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes. The restriction of $\Gamma_{g,k}^{0,1}(X, A; J)$ to each stratum $\mathfrak{X}_{\mathcal{T}}(X)$ is a smooth vector orbibundle of infinite rank. Let

$$\mathfrak{S}_{g,k}^{0,1}(X, A; J) = \Gamma\big(\mathfrak{X}_{g,k}(X, A), \Gamma_{g,k}^{0,1}(X, A; J)\big)$$

denote the space of all continuous multisections[2] ν of $\Gamma_{g,k}^{0,1}(X, A; J)$ such that the restriction of ν to each stratum $\mathfrak{X}_{\mathcal{T}}(X)$ is smooth.

We define a continuous section of the bundle $\Gamma_{g,k}^{0,1}(X, A; J) \longrightarrow \mathfrak{X}_{g,k}(X, A)$ by

$$\bar\partial_J\big([\Sigma, j; u]\big) = \bar\partial_{J,j} u = \frac{1}{2}\big(du + J \circ du \circ j\big).$$

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[2] Our term multisection corresponds to locally liftable multisection described by [FuOn, Definition 3.5].

By definition, the zero set of this section is the moduli space $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ of equivalence classes of stable J-holomorphic degree-A maps from genus-g curves with k marked points into X. The restriction of $\bar{\partial}_J$ to each stratum of $\mathfrak{X}_{g,k}(X, A)$ is smooth. The section $\bar{\partial}_J$ of $\Gamma_{g,k}^{0,1}(X, A; J)$ is Fredholm, i.e. the linearization of its restriction to every stratum $\mathfrak{X}_{\mathcal{T}}(X)$ has finite-dimensional kernel and cokernel at every point of $\bar{\partial}_J^{-1}(0) \cap \mathfrak{X}_{\mathcal{T}}(X)$. The index of the linearization of $\bar{\partial}_J$ at an element of $\mathfrak{M}_{g,k}^0(X, A; J)$ is the expected dimension $\dim_{g,k}(X, A)$ of the moduli space $\overline{\mathfrak{M}}_{g,k}(X, A; J)$.

If ν is a sufficiently small element of $\Gamma_{g,k}^{0,1}(X, A; J)$, the space

$$\overline{\mathfrak{M}}_{g,k}(X, A; J, \nu) \equiv \left\{\bar{\partial}_J + \nu\right\}^{-1}(0) \subset \mathfrak{X}_{g,k}(X, A)$$

is compact, since $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ is. For a small generic choice of ν, $\overline{\mathfrak{M}}_{g,k}(X, A; J, \nu)$ admits a stratification by orbifolds of even dimensions; see the first remark below. The main stratum,

$$\mathfrak{M}_{g,k}^0(X, A; J, \nu) = \overline{\mathfrak{M}}_{g,k}(X, A; J, \nu) \cap \mathfrak{X}_{g,k}^0(X, A),$$

is a smooth orbifold of dimension $\dim_{g,k}(X, A)$. Since $\mathfrak{X}_{g,k}(X, A)$ is locally a Banach space, there exist arbitrary small neighborhoods U of

$$\overline{\mathfrak{M}}_{g,k}(X, A; J, \nu) - \mathfrak{M}_{g,k}^0(X, A; J, \nu)$$

in $\mathfrak{X}_{g,k}(X, A)$ such that

$$H_l\big(U; \mathbb{Q}\big) = \{0\} \qquad \forall\, l \geq \dim_{g,k}(X, A) - 1.$$

Since $\overline{\mathfrak{M}}_{g,k}(X, A; J, \nu) - U$ is compact, via the pseudocycle construction of [McSa, Chapter 7] and [RT, Section 1], $\mathfrak{M}_{g,k}^0(X, A; J, \nu)$ determines a homology class

$$\big[\overline{\mathfrak{M}}_{g,k}(X, A; J, \nu)\big] \in H_{\dim_{g,k}(X,A)}(W, U; \mathbb{Q})$$
$$\approx H_{\dim_{g,k}(X,A)}(W; \mathbb{Q}),$$

for any small neighborhood W of $\overline{\mathfrak{M}}_{g,k}(X, A; J, \nu)$ in $\mathfrak{X}_{g,k}(X, A)$. The isomorphism between the two homology groups is induced by inclusion. Since ν can be chosen to be arbitrarily small, this procedure defines a rational homology class in an arbitrary small neighborhood of $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ in $\mathfrak{X}_{g,k}(X, A)$. This topological reinterpretation of the VFC constructions done in [FuOn] and [LT] turns out to be very suitable for constructing a VFC for the moduli space $\overline{\mathfrak{M}}_{1,k}^0(X, A; J)$.

Remark 1: The strata of $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu)$ locally are unions of finitely many smooth suborbifolds of a smooth orbifold. The branches of the strata correspond to the branches of ν. We will call such objects orbifolds, nevertheless, as these generalized orbifolds are just as suitable for the topological purposes of [FuOn], [LT], and this paper; see in [FuOn, Sections 3,4] for details.

Remark 2: The above construction defines a homology class

$$\Omega_W \in H_{\dim_{g,k}(X,A)}(W; \mathbb{Q})$$

for every neighborhood W of $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ in $\mathfrak{X}_{g,k}(X, A)$. Furthermore, if

$$\iota_{W',W} \colon W \longrightarrow W'$$

is the inclusion map of a neighborhood W into a larger neighborhood W', then

$$\iota_{W',W*}\,\Omega_W = \Omega_{W'}.$$

Thus, the above construction defines VFC for $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ as an element of the inverse limit of the homology groups $H_*(W;\mathbb{Q})$ under inclusion, taken over all neighborhoods of $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ in $\mathfrak{X}_{g,k}(X,A)$. If (X,J) is algebraic, $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ is a deformation retract of a neighborhood W, and one can then define VFC for $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ as a homology class in such a neighborhood W. However, these formalities are not essential for defining GW-invariants as intersection numbers of $\overline{\mathfrak{M}}_{g,k}(X,A;J,\nu)$ with certain natural classes on $\mathfrak{X}_{g,k}(X,A)$.

For a small *generic* perturbation ν of $\bar{\partial}_J$, the closure of $\mathfrak{M}^0_{g,k}(X,A;J,\nu)$ is the entire moduli space $\overline{\mathfrak{M}}_{g,k}(X,A;J,\nu)$. In particular, the results of [Z4], that are summarized in Subsection 1.1, cannot possibly generalize to $\mathfrak{M}^0_{g,k}(X,A;J,\nu)$, even with $g=1$, for a generic ν. Instead, for $g=1$, we consider *non-generic* perturbations ν of $\bar{\partial}_J$, which we now describe.

An element $[\Sigma;u]$ of $\mathfrak{X}_{1,k}(X,A)$ is an equivalence class of pairs consisting of a prestable genus-one Riemann surface Σ and a smooth map $u\colon \Sigma \longrightarrow X$. The prestable surface Σ is a union of the principal component(s) Σ_P, which is either a smooth torus or a circle of spheres, and trees of rational bubble components, which together will be denoted by Σ_B. Let

$$\mathfrak{X}^{\{0\}}_{1,k}(X,A) = \left\{[\Sigma;u]\in\mathfrak{X}_{1,k}(X,A)\colon u_*[\Sigma_P]\neq 0 \in H_2(X;\mathbb{Z})\right\}.$$

Suppose

$$[\Sigma;u] \in \mathfrak{X}_{1,k}(X,A) - \mathfrak{X}^{\{0\}}_{1,k}(X,A), \tag{1.1}$$

i.e. the degree of $u|_{\Sigma_P}$ is zero. Let $\chi^0(\Sigma;u)$ be the set of components Σ_i of Σ such that for every bubble component Σ_h that lies between Σ_i and Σ_P, including Σ_i itself, the degree of $u|_{\Sigma_h}$ is zero. The set $\chi^0(\Sigma;u)$ includes the principal component(s) of Σ. We give an example of the set $\chi^0(\Sigma;u)$ in Figure 2. In this figure, as in Figure 1, we show the domain Σ of the stable map $(\Sigma;u)$ and shade the components of the domain on which the degree of the map u is not zero. Let

$$\Sigma^0_u = \bigcup_{i\in\chi^0(\Sigma;u)} \Sigma_i.$$

Definition 1.2 *Suppose (X,ω) is a compact symplectic manifold, $\underline{J}\equiv(J_t)_{t\in[0,1]}$ is a continuous family of ω-tamed almost structures on X, $A\in H_2(X;\mathbb{Z})$, and $k\in\bar{\mathbb{Z}}^+$. A continuous family of multisections $\underline{\nu}\equiv(\nu_t)_{t\in[0,1]}$, with $\nu_t\in\mathfrak{G}^{0,1}_{1,k}(X,A;J_t)$ for all $t\in[0,1]$, is* **effectively supported** *if for every element*

$$b\equiv[\Sigma,u]\in\mathfrak{X}_{1,k}(X,A)-\mathfrak{X}^{\{0\}}_{1,k}(X,A)$$

there exists a neighborhood $\mathcal{W}_b$ of Σ^0_u in a semi-universal family of deformations for b such that

$$\nu_t(\Sigma';u')\big|_{\Sigma'\cap\mathcal{W}_b} = 0 \qquad \forall\ [\Sigma';u']\in\mathfrak{X}_{1,k}(X,A),\ t\in[0,1].$$

We use the C^1-topology on the space of all almost complex structures on X, in this definition and throughout the rest of the paper. The bundles $\Gamma^{0,1}_{1,k}(X, A; J_t)$ are contained in the bundle $\Gamma^1_{1,k}(X, A)$ over $\mathfrak{X}_{1,k}(X, A)$ with the fibers

$$\Gamma^1_{1,k}(X, A)\big|_{[b]} = \Gamma^1(b)/\mathrm{Aut}(b), \quad \text{where} \quad \Gamma^1(b) = L^p\big(\Sigma; T^*\Sigma \otimes_{\mathbb{R}} u^*TX\big),$$

and with the topology constructed as for $\Gamma^{0,1}_{g,k}(X, A; J)$. Finally, let $b = [\Sigma; u]$ be an element of $\mathfrak{X}_{1,k}(X, A)$. A semi-universal universal family of deformations for b is a fibration

$$\sigma_b \colon \tilde{\mathcal{U}}_b \longrightarrow \Delta_b$$

such that $\Delta_b/\mathrm{Aut}(b)$ is a neighborhood of b in $\mathfrak{X}_{1,k}(X, A)$ and the fiber of σ_b over a point $[\Sigma'; u']$ is Σ'.

If $\underline{\nu}$ is effectively supported and $[\Sigma; u]$ is as in (1.1), then the restriction of ν_t to a neighborhood W of Σ^0_u in Σ is zero for all t. Furthermore, if $\{[\Sigma_k; u_k]\}$ is a sequence converging to $[\Sigma; u]$ in $\mathfrak{X}_{1,k}(X, A)$, then for all k sufficiently large and a choice of representatives $(\Sigma_k; u_k)$ there is an open subset W_k of Σ_k such that $\nu_t(\Sigma_k; u_k)|_{W_k} = 0$ and the open sets W_k converge to W; see the beginning of Section 3 in [LT] for a detailed setting.

For example, if $[\Sigma; u]$ is as indicated in Figure 2 and $\underline{\nu}$ is effectively supported, then $\nu_t(\Sigma; u)$ vanishes on a neighborhood of $\Sigma_P \cup \Sigma_{h_3}$ in Σ. On the other hand, even if Σ_{h_2} had not been shaded, i.e. the degree of $u|_{\Sigma_{h_2}}$ were zero, there still would have been no condition on $\nu_t(\Sigma; u)|_{\Sigma_{h_2}}$ because the degree of $u|_{\Sigma_{h_1}}$ is not zero.

If $\underline{J} \equiv (J_t)_{t \in [0,1]}$ is a continuous family of ω-tamed almost structures on X, we denote the space of effectively supported families $\underline{\nu}$ as in Definition 1.2 by $\mathfrak{G}^{\mathrm{es}}_{1,k}(X, A; \underline{J})$. Similarly, if J is an almost complex structure on X, we denote by $\mathfrak{G}^{\mathrm{es}}_{1,k}(X, A; J)$ the subspace of elements ν of $\mathfrak{G}^{0,1}_{1,k}(X, A; J)$ such that the family $\nu_t = \nu$ is effectively supported.

Remark: Since ν is a multisection of $\Gamma^{0,1}_{g,k}(X, A; J)$, which is a union of orbi-vector spaces
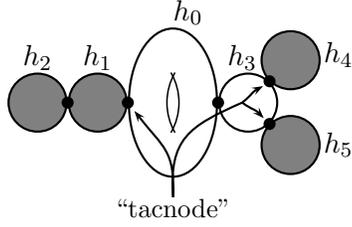
$$\Gamma^{0,1}_{g,k}(X, A; J)\big|_{[b]} = \Gamma^{0,1}(b; J)/\mathrm{Aut}(b),$$

ν is a family of equivalence classes of elements of $\Gamma^{0,1}_{g,k}(X, A; J)$ and can be locally represented by a family of elements of $\Gamma^{0,1}(\cdot; J)$. In order to simplify notation, we will use the same symbol for both, as the exact meaning will be determined by the context.

1.4 Main Results

In this subsection we state the main results of this paper. We begin by describing the subspace $\overline{\mathfrak{M}}^0_{1,k}(X, A; J, \nu)$ of $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu)$. We then state the main compactness result, i.e. Theorem 1.4. One of its consequences is that for a small generic choice of ν the moduli space $\overline{\mathfrak{M}}^0_{1,k}(X, A; J, \nu)$ determines a virtual fundamental class for $\overline{\mathfrak{M}}^0_{1,k}(X, A; J)$, which is independent of J; see Theorem 1.5 and Corollary 1.6.

Suppose $[\Sigma; u]$ is an element of $\mathfrak{X}_{1,k}(X, A)$. Every bubble component $\Sigma_i \subset \Sigma_B$ is a sphere and has a distinguished singular point, which will be called the *attaching node of Σ_i*. This is the node of

$$\chi^0(\Sigma; u) = \{h_0, h_3\}$$

$$\chi(\Sigma; u) = \{h_1, h_4, h_5\}$$

Figure 2: An illustration of Definition 1.3

Σ_i that lies either on Σ_P or on a bubble Σ_h that lies between Σ_i and Σ_P. For example, if Σ is as shown in Figure 2, the attaching node of Σ_{h_3} is the node Σ_{h_3} shares with the torus. If $[\Sigma; u]$ is as in (1.1), we denote by $\chi(\Sigma; u)$ the set of bubble components Σ_i such that the attaching node of Σ_i lies on Σ_u^0 and the degree of $u|_{\Sigma_i}$ is not zero, i.e. Σ_i is not an element of $\chi^0(\Sigma; u)$; see Figure 2. These components are called *first-level $(\Sigma; u)$-effective* in [Z4, Subsection 1.2].

Suppose $\nu \in \mathfrak{G}_{1,k}^{\mathrm{es}}(X, A; J)$ and $[\Sigma; u]$ is an element of $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu)$ as in (1.1). Since $\Sigma_i \subset \Sigma_B$ is a sphere, we can represent every element of $\mathfrak{X}_{1,k}(X, A)$ by a pair $(\Sigma; u)$ such that the attaching node of every bubble component $\Sigma_i \subset \Sigma_B$ is the south pole, or the point $\infty = (0, 0, -1)$, of $S^2 \subset \mathbb{R}^3$. Let $e_\infty = (1, 0, 0)$ be a nonzero tangent vector to S^2 at the south pole. If $i \in \chi(\Sigma; u)$, we put

$$\mathcal{D}_i(\Sigma; u) = d\{u|_{\Sigma_i}\}\big|_\infty e_\infty \in T_{u|_{\Sigma_i}(\infty)}X.$$

Since $u|_{\Sigma_i}$ is J-holomorphic on a neighborhood of ∞ in Σ_i, the linear subspace $\mathbb{C} \cdot \mathcal{D}_i(\Sigma; u)$ is determined by $[\Sigma; u]$, just as in the $\nu = 0$ case, which is considered [Z4, Subsection 1.2]. We also note that $u|_{\Sigma_u^0}$ is a degree-zero holomorphic map and thus constant. Thus, u maps the attaching nodes of all elements of $\chi(\Sigma; u)$ to the same point in X, just as in the $\nu = 0$ case of [Z4, Subsection 1.2].

Definition 1.3 *Suppose (X, ω, J) is a compact almost Kahler manifold, $A \in H_2(X; \mathbb{Z})^*$, and $k \in \bar{\mathbb{Z}}^+$. If $\nu \in \mathfrak{G}_{1,k}^{\mathrm{es}}(X, A; J)$ is an effectively supported perturbation of the $\bar{\partial}_J$-operator, the* **main component** *of the space $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu)$ is the subset $\overline{\mathfrak{M}}_{1,k}^0(X, A; J, \nu)$ consisting of the elements $[\Sigma; u]$ of $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu)$ such that*
   (a) the degree of $u|_{\Sigma_P}$ is not zero, or
   (b) the degree of $u|_{\Sigma_P}$ is zero and $\dim_\mathbb{C} Span_{(\mathbb{C}, J)}\{\mathcal{D}_i(\Sigma; u) : i \in \chi(\Sigma; u)\} < |\chi(\Sigma; u)|$.

This definition generalizes [Z4, Definition 1.1]. As in [Z4], we let

$$H_2(X; \mathbb{Z})^* = H_2(X; \mathbb{Z}) - \{0\}.$$

If $[\Sigma; u]$ is as in (1.1), $[\Sigma; u]$ belongs to $\overline{\mathfrak{M}}_{1,k}^0(X, A; J, \nu)$ if and only if the branches of $u(\Sigma)$ corresponding to the attaching nodes of the first-level effective bubbles of $[\Sigma; u]$ form a *generalized tacnode*. In the case of Figure 2, this means that the complex dimension of the span of the images of du at the attaching nodes of the bubbles h_1, h_4, and h_5 is at most two.

Theorem 1.4 *Suppose (X, ω) is a compact symplectic manifold, $\underline{J} \equiv (J_t)_{t \in [0,1]}$ is a continuous family of ω-tamed almost complex structures on X, $A \in H_2(X; \mathbb{Z})^*$, and $k \in \bar{\mathbb{Z}}^+$. If*

$$\underline{\nu} \equiv (\nu_t)_{t \in [0,1]} \in \mathfrak{G}_{1,k}^{\mathrm{es}}(X, A; \underline{J})$$

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is a family of sufficiently small perturbations of the $\bar{\partial}_{J_t}$-operators on $\mathfrak{X}_{1,k}(X,A)$, then

$$\overline{\mathfrak{M}}^0_{1,k}(X,A;\underline{J},\underline{\nu}) \equiv \bigcup_{t\in[0,1]} \overline{\mathfrak{M}}^0_{1,k}(X,A;J_t,\nu_t)$$

is compact.

The requirement that ν_t be sufficiently small means that it lies in a neighborhood of the zero section with respect to a C^1-type of topology, with appropriate interpretations of the rate of change in the normal directions to the boundary strata of $\mathfrak{X}_{1,k}(X,A)$. This topology will be made apparent in the proof.

Theorem 1.4 follows immediately from Propositions 2.5-2.7; see also the beginning of Subsection 2.3. These propositions generalize [Z4, Propositions 5.1-5.3].

Theorem 1.5 *Suppose (X,ω,J) is a compact almost Kahler manifold, $A \in H_2(X;\mathbb{Z})^*$, $k \in \bar{\mathbb{Z}}^+$, and W is a neighborhood of $\overline{\mathfrak{M}}^0_{1,k}(X,A;J)$ in $\mathfrak{X}_{1,k}(X,A)$. If $\nu \in \mathfrak{G}^{\text{es}}_{1,k}(X,A;J)$ is a sufficiently small generic perturbation of the $\bar{\partial}_J$-operator on $\mathfrak{X}_{1,k}(X,A)$, then $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$ determines a rational homology class in W. Furthermore, if $\underline{J}=(J_t)_{t\in[0,1]}$ is a family of ω-tamed almost complex structures on X, such that $J_0 = J$ and J_t is sufficiently close to J for all t, and ν_0 and ν_1 are sufficiently small generic perturbations of $\bar{\partial}_{J_0}$ and $\bar{\partial}_{J_1}$ on $\mathfrak{X}_{1,k}(X,A)$, then there exists a homotopy*

$$\underline{\nu}=(\nu_t)_{t\in[0,1]} \in \mathfrak{G}^{\text{es}}_{1,k}(X,A;\underline{J})$$

between ν_0 and ν_1 such that $\overline{\mathfrak{M}}^0_{1,k}(X,A;\underline{J},\underline{\nu})$ determines a chain in W and

$$\partial\overline{\mathfrak{M}}^0_{1,k}(X,A;\underline{J},\underline{\nu}) = \overline{\mathfrak{M}}^0_{1,k}(X,A;J_1,\nu_1) - \overline{\mathfrak{M}}^0_{1,k}(X,A;J_0,\nu_0).$$

Corollary 1.6 *If (X,ω,J) is a compact almost Kahler manifold, $A \in H_2(X;\mathbb{Z})^*$, and $k \in \bar{\mathbb{Z}}^+$, the moduli space $\overline{\mathfrak{M}}^0_{1,k}(X,A;J)$ carries a well-defined virtual fundamental class of the expected dimension. This class is an invariant of (X,ω).*

It is straightforward to see that for a generic $\nu\in\mathfrak{G}^{\text{es}}_{1,k}(X,A;J)$ the space $\overline{\mathfrak{M}}_{1,k}(X,A;J,\nu)$ is stratified by smooth orbifolds of even dimensions. The strata of

$$\mathfrak{M}^{\{0\}}_{1,k}(X,A;J,\nu) \equiv \overline{\mathfrak{M}}_{1,k}(X,A;J,\nu)\cap\mathfrak{X}^{\{0\}}_{1,k}(X,A)$$

have the expected dimension, based on the index of a certain elliptic operator. In particular, the dimension of the main stratum of $\mathfrak{M}^{\{0\}}_{1,k}(X,A;J,\nu)$ is $\dim_{1,k}(X,A)$, while the dimensions of all other strata of $\mathfrak{M}^{\{0\}}_{1,k}(X,A;J,\nu)$ are smaller than $\dim_{1,k}(X,A)$.

On the other hand, suppose $\mathcal{U}_{\mathcal{T},\nu}(X;J)$ is a stratum of the complement of $\mathfrak{M}^{\{0\}}_{1,k}(X,A;J,\nu)$ in $\overline{\mathfrak{M}}_{1,k}(X,A;J,\nu)$; see Subsection 2.2 for more details. The sets $\chi^0(\Sigma;u)$ and $\chi(\Sigma;u)$ are independent of the choice of $[\Sigma;u]$ in $\mathcal{U}_{\mathcal{T},\nu}(X;J)$. We denote them by $\chi^0(\mathcal{T})$ and $\chi(\mathcal{T})$, respectively. By Definition 1.3, for every $[\Sigma,u]\in\mathcal{U}_{\mathcal{T},\nu}(X;J)$ and $i\in\chi^0(\mathcal{T})$, $u|_{\Sigma_i}$ is constant. Thus,

$$\mathcal{U}_{\mathcal{T},\nu}(X;J) \subset \mathcal{M}_{\mathcal{T}} \times \mathfrak{X}_{\bar{\mathcal{T}}}(X),$$

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where $\mathcal{M}_{\mathcal{T}}$ is a product of $|\chi^0(\mathcal{T})|$ moduli spaces of smooth genus-zero and genus-one curves and $\mathfrak{X}_{\bar{\mathcal{T}}}(X)$ is a certain collection of $|\chi(\mathcal{T})|$-tuples of stable smooth genus-zero bubble maps. For example, if the elements of $\mathcal{U}_{\mathcal{T},\nu}(X;J)$ are described by Figure 2,

$$\mathcal{M}_{\mathcal{T}} = \mathcal{M}_{1,2} \times \mathcal{M}_{0,3}.$$

In this case, $\mathfrak{X}_{\bar{\mathcal{T}}}(X)$ consists of triples of stable genus-zero bubble maps each with a special marked point, corresponding to the attaching nodes of the elements of $\chi(\mathcal{T})$, such that the values of three maps at the special marked points are the same. If $\nu \in \mathfrak{G}_{1,k}^{\mathrm{es}}(X,A;J)$ is generic, we have a fiber bundle

$$\pi_0 : \mathcal{U}_{\mathcal{T},\nu}(X;J) \longrightarrow \mathcal{M}_{\mathcal{T}},$$

with fibers of the expected dimension. An index computation then shows that

$$\dim \mathcal{U}_{\mathcal{T},\nu}(X;J) \le \dim_{1,k}(X,A) + 2\big(n - |\chi(\mathcal{T})|\big), \tag{1.2}$$

where $2n$ is the dimension of X as before.

We denote by $E_{\mathcal{T}} \longrightarrow \mathcal{U}_{\mathcal{T},\nu}(X;J)$ the direct sum of the $|\chi(\mathcal{T})|$ universal tangent line bundles for the special marked points of the elements of each $\chi(\mathcal{T})$-tuple in $\mathfrak{X}_{\bar{\mathcal{T}}}(X)$ and by

$$\mathrm{ev}_P : \mathcal{U}_{\mathcal{T},\nu}(X;J) \longrightarrow X$$

the map sending an element $[\Sigma;u]$ of $\mathcal{U}_{\mathcal{T},\nu}(X;J)$ to the value of u on Σ_P. Let $\gamma_{E_{\mathcal{T}}} \longrightarrow \mathbb{P}E_{\mathcal{T}}$ be the tautological line bundle. By Definition 1.3,

$$\mathcal{U}_{\mathcal{T},\nu}(X;J) \cap \overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu) = \pi_{\mathcal{T}}(\mathcal{Z}_{\mathcal{T}}),$$

where

$$\pi_{\mathcal{T}} : \mathbb{P}E_{\mathcal{T}} \longrightarrow \mathcal{U}_{\mathcal{T},\nu}(X;J)$$

is the bundle projection map and $\mathcal{Z}_{\mathcal{T}}$ is the zero set of the section of the vector bundle

$$\gamma_{E_{\mathcal{T}}}^* \otimes \mathrm{ev}_P^* TX \longrightarrow \mathbb{P}E_{\mathcal{T}}$$

induced by the differentials $\mathcal{D}_i$, with $i \in \chi(\mathcal{T})$, defined above. It is straightforward to see that this section is transverse to the zero set if $\nu \in \mathfrak{G}_{1,k}^{\mathrm{es}}(X,A;J)$ is generic. Thus,

$$\begin{aligned}
\dim \mathcal{U}_{\mathcal{T},\nu}(X;J) \cap \overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu) &\le \dim \mathcal{Z}_{\mathcal{T}} \\
&= \dim \mathcal{U}_{\mathcal{T},\nu}(X;J) + 2\big(\mathrm{rk}_{\mathbb{C}} E_{\mathcal{T}} - 1\big) - 2\,\mathrm{rk}_{\mathbb{C}}\big(\gamma_{E_{\mathcal{T}}}^* \otimes \mathrm{ev}_P^* TX\big) \\
&\le \dim_{1,k}(X,A) - 2,
\end{aligned}$$

by (1.2).

By the above, for a generic $\nu \in \mathfrak{G}_{1,k}^{\mathrm{es}}(X,A;J)$, $\overline{\mathfrak{M}}_{1,k}^0(X,A;J,\nu)$ is stratified by smooth orbifolds of even dimensions, such that the main stratum is of dimension $\dim_{1,k}(X,A)$, while all other strata have smaller dimensions. Thus, the first claim of Theorem 1.5 follows from Theorem 1.4 by the same topological construction as in Subsection 1.3. The second claim of Theorem 1.5 is obtained by a similar argument.

By the first claim of Theorem 1.5, we can define a homology class for $\overline{\mathfrak{M}}^0_{1,k}(X,A;J)$, which is induced by $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$, for any J. By the last statement of Theorem 1.5, this class is independent of the choice ν and does not change under small changes in J. Since the space of ω-tamed almost complex structures on X is path-connected, it follows that the virtual fundamental class of $\overline{\mathfrak{M}}^0_{1,k}(X,A;J)$ is an invariant of (X,ω).

Remark: It is simplest to view the last statement above as the independence of all numbers $\mathrm{GW}^{0;X}_{1,k}(\mu)$ obtained by evaluating natural cohomology classes on $\overline{\mathfrak{M}}^0_{1,k}(X,A;J,\nu)$.

2 Proof of Theorem 1.4

2.1 Notation: Genus-Zero Maps

We now describe our notation for bubble maps from genus-zero Riemann surfaces and for the spaces of such bubble maps that form the standard stratifications of moduli spaces of stable maps. We also state analogues of Definition 1.2 for genus-zero maps with one and two special marked points.

In general, moduli spaces of stable maps can stratified by the dual graph. However, in the present situation, it is more convenient to make use of *linearly ordered sets*:

Definition 2.1 *(1) A finite nonempty partially ordered set I is a **linearly ordered set** if for all $i_1, i_2, h \in I$ such that $i_1, i_2 < h$, either $i_1 \le i_2$ or $i_2 \le i_1$.*
*(2) A linearly ordered set I is a **rooted tree** if I has a unique minimal element, i.e. there exists $\hat{0} \in I$ such that $\hat{0} \le i$ for all $i \in I$.*

If I is a linearly ordered set, let $\hat{I}$ be the subset of the non-minimal elements of I. For every $h \in \hat{I}$, denote by $\iota_h \in I$ the largest element of I which is smaller than h, i.e. $\iota_h = \max\{i \in I : i < h\}$.

We identify $\mathbb{C}$ with $S^2 - \{\infty\}$ via the stereographic projection mapping the origin in $\mathbb{C}$ to the north pole, or the point $(0,0,1)$, in S^2. If M is a finite set, a **genus-zero X-valued bubble map with M-marked points** is a tuple

$$b = \big(M, I; x, (j,y), u\big),$$

where I is a rooted tree, and

$$x \colon \hat{I} \longrightarrow \mathbb{C} = S^2 - \{\infty\}, \quad j \colon M \longrightarrow I, \quad y \colon M \longrightarrow \mathbb{C}, \quad \text{and} \quad u \colon I \longrightarrow C^\infty(S^2; X) \qquad (2.1)$$

are maps such that $u_h(\infty) = u_{\iota_h}(x_h)$ for all $h \in \hat{I}$. We associate such a tuple with Riemann surface

$$\Sigma_b = \Big(\bigsqcup_{i \in I} \Sigma_{b,i}\Big)\Big/\sim, \quad \text{where} \quad \Sigma_{b,i} = \{i\} \times S^2 \quad \text{and} \quad (h,\infty) \sim (\iota_h, x_h) \;\; \forall h \in \hat{I}, \qquad (2.2)$$

with marked points

$$y_l(b) \equiv (j_l, y_l) \in \Sigma_{b,j_l} \qquad \text{and} \qquad y_0(b) \equiv (\hat{0}, \infty) \in \Sigma_{b,\hat{0}},$$

and continuous map $u_b \colon \Sigma_b \longrightarrow X$, given by $u_b|_{\Sigma_{b,i}} = u_i$ for all $i \in I$. The general structure of bubble maps is described by tuples $\mathcal{T} = (M, I; j, \underline{A})$, where

$$A_i = u_{i*}[S^2] \in H_2(X; \mathbb{Z}) \qquad \forall i \in I.$$

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We call such tuples **bubble types**. Let $\mathfrak{X}_{\mathcal{T}}(X)$ denote the subset of $\mathfrak{X}_{0,\{\hat{0}\}\sqcup M}(X,A)$ consisting of stable maps $[\mathcal{C};u]$ such that

$$[\mathcal{C};u] = \big[(\Sigma_b,(\hat{0},\infty),(j_l,y_l)_{l\in M});u_b\big],$$

for some bubble map b of type $\mathcal{T}$ as above, where $\hat{0}$ is the minimal element of I; see [Z2, Section 2] for details. For $l \in \{0\}\sqcup M$, let

$$\mathrm{ev}_l\colon \mathfrak{X}_{\mathcal{T}}(X) \longrightarrow X$$

be the evaluation map corresponding to the marked point y_l.

With notation as above, suppose

$$[b] \equiv \big[M,I;x,(j,y),u\big] \in \mathfrak{X}_{0,\{0\}\sqcup M}(X,A).$$

Let $\chi^0(b)$ be the set of components $\Sigma_{b,i}$ of Σ_b such that for every component $\Sigma_{b,h}$ that lies between Σ_i and $\Sigma_{b,\hat{0}}$, including $\Sigma_{b,i}$ and $\Sigma_{b,\hat{0}}$, the degree of $u|_{\Sigma_{b,h}}$ is zero. For example, if b is as indicated by Figure 4 on page 22, the set $\chi^0(b)$ consists of the two components that are not shaded. The set $\chi^0(b)$ is empty if and only if the degree of the restriction of u_b to the component containing the special marked point is not zero. Let

$$\Sigma_b^0 = \big\{(\hat{0},\infty)\big\} \cup \bigcup_{i\in\chi^0(b)} \Sigma_{b,i}.$$

We denote by $\chi(b)$ the set of components $\Sigma_{b,i}$ such that the attaching node of $\Sigma_{b,i}$ lies on Σ_b^0 and the degree of $u_b|_{\Sigma_{b,i}}$ is not zero, i.e. $\Sigma_{b,i}$ is not an element of $\chi^0(b)$. If the degree of $u_b|_{\Sigma_{b,\hat{0}}}$ is not zero, $\chi(b) = \{\hat{0}\}$. If $A \neq 0$ and the degree of $u_b|_{\Sigma_{b,\hat{0}}}$ is zero, the set $\chi(b)$ is not empty, but does not contain $\hat{0}$.

Definition 2.2 *Suppose (X,ω) is a compact symplectic manifold, $\underline{J} \equiv (J_t)_{t\in[0,1]}$ is a continuous family of ω-tamed almost structures on X, $A \in H_2(X;\mathbb{Z})^*$, and M is a finite set. A continuous family of multisections $\underline{\nu} \equiv (\nu_t)_{t\in[0,1]}$, with $\nu_t \in \mathfrak{G}_{0,\{0\}\sqcup M}^{0,1}(X,A;J_t)$ for all $t \in [0,1]$, is **effectively supported** if for every element b of $\mathfrak{X}_{0,\{0\}\sqcup M}(X,A)$ there exists a neighborhood $\mathcal{W}_b$ of Σ_b^0 in a semi-universal family of deformations for b such that*

$$\nu_t(b')\big|_{\Sigma_{b'}\cap\mathcal{W}_b} = 0 \qquad \forall \ [b'] \in \mathfrak{X}_{0,\{0\}\sqcup M}(X,A), \ t\in[0,1].$$

Definition 2.3 *Suppose (X,ω), $\underline{J} \equiv (J_t)_{t\in[0,1]}$, A, and M are as in Definition 2.2. A continuous family of multisections $\underline{\nu} \equiv (\nu_t)_{t\in[0,1]}$, with $\nu_t \in \mathfrak{G}_{0,\{0,1\}\sqcup M}^{0,1}(X,A;J_t)$ for all $t\in[0,1]$, is **semi-effectively supported** if for every element b of $\mathfrak{X}_{0,\{0,1\}\sqcup M}(X,A)$ such that the marked point $y_1(b)$ lies on Σ_b^0 there exists a neighborhood $\mathcal{W}_b$ of Σ_b^0 in a semi-universal family of deformations for b such that*

$$\nu_t(b')\big|_{\Sigma_{b'}\cap\mathcal{W}_b} = 0 \qquad \forall \ [b'] \in \mathfrak{X}_{0,\{0,1\}\sqcup M}(X,A), \ t\in[0,1].$$

We denote the spaces of effectively and semi-effectively supported families $\underline{\nu}$ as in Definitions 2.2 and 2.3 by

$$\mathfrak{G}^{\mathrm{es}}_{0,\{0\}\sqcup M}(X,A;\underline{J}) \qquad \text{and} \qquad \mathfrak{G}^{\mathrm{ses}}_{0,\{0,1\}\sqcup M}(X,A;\underline{J}),$$

respectively. Similarly to the genus-one case, if J is an almost complex structure on X, we denote by

$$\mathfrak{G}^{\mathrm{es}}_{0,\{0\}\sqcup M}(X,A;J) \qquad \text{and} \qquad \mathfrak{G}^{\mathrm{ses}}_{0,\{0,1\}\sqcup M}(X,A;J)$$

the subspaces of elements ν of $\mathfrak{G}^{0,1}_{0,\{0\}\sqcup M}(X,A;J)$ and $\mathfrak{G}^{0,1}_{0,\{0,1\}\sqcup M}(X,A;J)$ such that the family $\nu_t = \nu$ is effectively supported or semi-effectively supported, respectively.

If $[b] = [\Sigma_b; u_b]$ is an element of $\mathfrak{X}_{0,\{0\}\sqcup M}(X,A)$ is as above and $i \in \chi(b)$, we put

$$\mathcal{D}_i b = d\big\{ u_b |_{\Sigma_{b,i}} \big\}\big|_\infty e_\infty \in T_{u_b|_{\Sigma_{b,i}}(\infty)} X.$$

If $\nu \in \mathfrak{G}^{\mathrm{es}}_{0,\{0\}\sqcup M}(X,A;J)$ and b is an element of

$$\overline{\mathfrak{M}}_{0,\{0\}\sqcup M}(X,A;J,\nu) \equiv \big\{ \bar\partial_J + \nu \big\}^{-1}(0),$$

then $u_b|_{\Sigma_{b,i}}$ is J-holomorphic on a neighborhood of ∞ in $\Sigma_{b,i}$ and $\mathbb{C}\cdot_J \mathcal{D}_i b$ is determined by b, just as in Subsection 1.4. This is also the case if $\nu \in \mathfrak{G}^{\mathrm{ses}}_{0,\{0,1\}\sqcup M}(X,A;J)$ and $[b]$ is an element of $\overline{\mathfrak{M}}_{0,\{0,1\}\sqcup M}(X,A;J,\nu)$ such that $y_1(b) \in \Sigma_b^0$. In both of these cases, $u_b|_{\Sigma_b^0}$ is a degree-zero holomorphic map and thus constant. Thus, u_b maps the attaching nodes of all elements of $\chi(b)$ to the same point in X, as in the genus-one case of Subsection 1.4.

2.2 Notation: Genus-One Maps

We next set up analogous notation for maps from genus-one Riemann surfaces. In this case, we also need to specify the structure of the principal component. Thus, we index the strata of $\mathfrak{X}_{1,M}(X,A)$ by *enhanced linearly ordered sets*:

Definition 2.4 *An* enhanced linearly ordered set *is a pair* $(I,\aleph)$, *where* I *is a linearly ordered set,* $\aleph$ *is a subset of* $I_0 \times I_0$, *and* I_0 *is the subset of minimal elements of* I, *such that if* $|I_0| > 1$,

$$\aleph = \big\{ (i_1,i_2),(i_2,i_3),\ldots,(i_{n-1},i_n),(i_n,i_1) \big\}$$

for some bijection $i\colon \{1,\ldots,n\} \longrightarrow I_0$.

An enhanced linearly ordered set can be represented by an oriented connected graph. In Figure 3, the dots denote the elements of I. The arrows outside the loop, if there are any, specify the partial ordering of the linearly ordered set I. In fact, every directed edge outside of the loop connects a non-minimal element h of I with ι_h. Inside of the loop, there is a directed edge from i_1 to i_2 if and only if $(i_1,i_2) \in \aleph$.

The subset $\aleph$ of $I_0 \times I_0$ will be used to describe the structure of the principal curve of the domain of stable maps in a stratum of $\mathfrak{X}_{1,M}(X,A)$. If $\aleph = \emptyset$, and thus $|I_0| = 1$, the corresponding principal
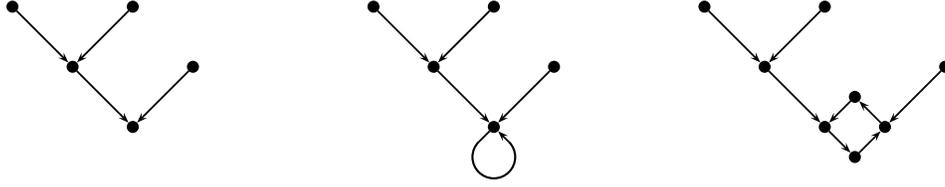
Figure 3: Some enhanced linearly ordered sets

curve Σ_P is a smooth torus, with some complex structure. If $\aleph \neq \emptyset$, the principal components form a circle of spheres:

$$\Sigma_P = \left(\bigsqcup_{i \in I_0} \{i\} \times S^2 \right) \Big/ \sim, \qquad \text{where} \qquad (i_1, \infty) \sim (i_2, 0) \ \text{ if } \ (i_1, i_2) \in \aleph.$$

A genus-one X-valued bubble map with M-marked points is a tuple

$$b = \big(M, I, \aleph; S, x, (j, y), u \big),$$

where S is a smooth Riemann surface of genus one if $\aleph = \emptyset$ and the circle of spheres Σ_P otherwise. The objects x, j, y, u, and (Σ_b, u_b) are as in (2.1) and (2.2), except the sphere $\Sigma_{b,\hat{0}}$ is replaced by the genus-one curve $\Sigma_{b;P} \equiv S$. Furthermore, if $\aleph = \emptyset$, and thus $I_0 = \{\hat{0}\}$ is a single-element set, $u_{\hat{0}} \in C^\infty(S; X)$ and $y_l \in S$ if $j_l = \hat{0}$. In the genus-one case, the general structure of bubble maps is encoded by the tuples of the form $\mathcal{T} = (M, I, \aleph; j, \underline{A})$. Similarly to the genus-zero case, we denote by $\mathfrak{X}_\mathcal{T}(X)$ the subset of $\mathfrak{X}_{1,M}(X, A)$ consisting of stable maps $[\mathcal{C}; u]$ such that

$$[\mathcal{C}; u] = \big[(\Sigma_b, (j_l, y_l)_{l \in M}); u_b \big],$$

for some bubble map b of type $\mathcal{T}$ as above. If ν is an element of $\mathfrak{G}^{es}_{1,M}(X, A)$, we put

$$\mathcal{U}_{\mathcal{T}, \nu}(X; J) = \big\{ [b] \in \mathfrak{X}_\mathcal{T}(X) \colon \{\bar{\partial}_J + \nu\}(b) = 0 \big\}.$$

All vector orbi-bundles we encounter will be assumed to be normed. Some will come with natural norms; for others, we choose a norm, sometimes implicitly, once and for all. If $\mathfrak{F} \longrightarrow \mathfrak{X}$ is a normed vector bundle and $\delta \in \mathbb{R}^+$, let

$$\mathfrak{F}_\delta = \big\{ v \in \mathfrak{F} \colon |v| < \delta \big\}.$$

If Ω is any subset of $\mathfrak{F}$, we take $\Omega_\delta = \Omega \cap \mathfrak{F}_\delta$.

2.3 Outline of the Proof of Theorem 1.4

Suppose (X, ω) is a compact symplectic manifold, $\underline{J} \equiv (J_t)_{t \in [0,1]}$ is a continuous family of ω-tamed almost complex structures on X, $A \in H_2(X; \mathbb{Z})^*$, M is a finite set, and

$$\underline{\nu} \equiv (\nu_t)_{t \in [0,1]} \in \mathfrak{G}^{es}_{1,M}(X, A; \underline{J})$$

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is a family of sufficiently small perturbations of the $\bar\partial_{J_t}$-operators on $\mathfrak{X}_{1,M}(X,A)$. Let t_r and b_r be sequences of elements in $[0,1]$ and in $\overline{\mathfrak{M}}^0_{1,M}(X,A;J_{t_r},\nu_{t_r})$ such that

$$\lim_{r\longrightarrow\infty} t_r = 0 \quad\text{and}\quad \lim_{r\longrightarrow\infty} b_r = b \in \overline{\mathfrak{M}}_{1,M}(X,A;J_0,\nu_0).$$

We need to show that $b\in\overline{\mathfrak{M}}^0_{1,M}(X,A;J_0,\nu_0)$. By Definition 1.3, it is sufficient to assume that b is an element of $\mathcal{U}_{\mathcal{T},\nu_0}(X;J_0)$ for a bubble type

$$\mathcal{T} = \big(M,I,\aleph;j,\underline{A}\big)$$

such that $A_i\!=\!0$ for all minimal elements $i\!\in\!I$.

We can also assume that for some bubble type

$$\mathcal{T}' = \big(M,I',\aleph';j',\underline{A}'\big)$$

$b_r\in\mathcal{U}_{\mathcal{T}',\nu_{t_r}}(X;J_{t_r})$ for all r. If $A'_i\!=\!0$ for all minimal elements $i\!\in\!I'$, the desired conclusion follows Proposition 2.5 below, as it implies that the second condition in Definition 1.3 is closed with respect to the stable map topology. If $A'_i\!\neq\!0$ for some minimal element $i\!\in\!I'$ and $\aleph'\!\neq\!\emptyset$, i.e. the principal component of Σ_{b_r} is a circle of spheres, Proposition 2.6 implies that b satisfies the second condition in Definition 1.3. Finally, if $\aleph'\!=\!\emptyset$ and $A'_i\!\neq\!0$ for the unique minimal element i of I', the desired conclusion follows from Proposition 2.7. We note that the three propositions are applied with b and b_r that are components of the ones above.

Let $[n] = \big\{1,\ldots,n\big\}$.

Proposition 2.5 *Suppose (X,ω) is a compact symplectic manifold, $\underline{J}\equiv(J_t)_{t\in[0,1]}$ is a continuous family of ω-tamed almost complex structures on X, $A\in H_2(X;\mathbb{Z})^*$, M is a finite set, and*

$$\underline{\nu}\equiv(\nu_t)_{t\in[0,1]} \in \mathfrak{G}^{\mathrm{es}}_{0,\{0\}\sqcup M}(X,A;\underline{J})$$

is a family of sufficiently small perturbations of the $\bar\partial_{J_t}$-operators on $\mathfrak{X}_{0,\{0\}\sqcup M}(X,A)$. If t_r and $[b_r]$ are sequences of elements in $[0,1]$ and in $\mathfrak{M}^0_{0,\{0\}\sqcup M}(X,A;J_{t_r},\nu_{t_r})$ such that

$$\lim_{r\longrightarrow\infty} t_r = 0 \quad and \quad \lim_{r\longrightarrow\infty} [b_r] = [b] \in \overline{\mathfrak{M}}_{0,\{0\}\sqcup M}(X,A;J_0,\nu_0),$$

then either
(a) $\dim_{\mathbb{C}} Span_{(\mathbb{C},J_0)}\{\mathcal{D}_i b: i\in\chi(b)\} < |\chi(b)|$, *or*
(b) $\bigcap_{r=1}^{\infty} \overline{\bigcup_{r'>r} \mathbb{C}\cdot_{J_{r'}}\mathcal{D}_{\hat 0}b_{r'}} \subset Span_{(\mathbb{C},J_0)}\{\mathcal{D}_i b: i\in\chi(b)\}$.

Proposition 2.6 *Suppose (X,ω) and $\underline{J}$ are as in Proposition 2.5, $n\in\mathbb{Z}^+$, $A_1,\ldots,A_n\in H_2(X;\mathbb{Z})^*$, $M_1,\ldots,M_n$ are finite sets, and for each $k\in[n]$*

$$\underline{\nu}_k\equiv(\nu_{k,t})_{t\in[0,1]} \in \mathfrak{G}^{\mathrm{ses}}_{0,\{0,1\}\sqcup M_k}(X,A_k;\underline{J})$$

is a family of sufficiently small perturbations of the $\bar\partial_{J_t}$-operators on $\mathfrak{X}_{0,\{0,1\}\sqcup M_k}(X,A)$. Let t_r and $[b_{k,r}]$ be sequences of elements in $[0,1]$ and in $\mathfrak{M}^0_{0,\{0,1\}\sqcup M_k}(X,A_k;J_{t_r},\nu_{k,t_r})$ for $k\in[n]$ such that

$$\mathrm{ev}_1(b_{k,r}) = \mathrm{ev}_0(b_{k+1,r}) \quad\forall\,k\in[n-1], \qquad \mathrm{ev}_1(b_{n,r}) = \mathrm{ev}_0(b_{1,r}),$$

$$\lim_{r\longrightarrow\infty} t_r = 0, \quad and \quad \lim_{r\longrightarrow\infty} [b_{k,r}] = [b_k] \in \overline{\mathfrak{M}}_{0,\{0,1\}\sqcup M_k}(X,A_k;J_0,\nu_{k,0}) \quad\forall\,k\in[n].$$

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If $y_1(b_k) \in \Sigma_{b_k}^0$ for all $k \in [n]$, then

$$\dim_{\mathbb{C}} Span_{(\mathbb{C}, J_0)} \{\mathcal{D}_i b_k : i \in \chi(b_k), \ k \in [n]\} < \sum_{k=1}^{k=n} |\chi(b_k)|.$$

Proposition 2.7 *Suppose (X, ω), $\underline{J}$, A, and M are as in Proposition 2.5 and*

$$\underline{\nu} \equiv (\nu_t)_{t \in [0,1]} \in \mathfrak{G}_{1,M}^{es}(X, A; \underline{J})$$

is a family of sufficiently small perturbations of the $\bar{\partial}_{J_t}$-operators on $\mathfrak{X}_{1,M}(X, A)$. Let t_r and $[b_r]$ be sequences of elements in $[0, 1]$ and in $\mathfrak{M}_{1,M}^0(X, A; J_{t_r}, \nu_{t_r})$ such that

$$\lim_{r \to \infty} t_r = 0 \quad and \quad \lim_{r \to \infty} [b_r] = [b] \in \overline{\mathfrak{M}}_{1,M}(X, A; J_0, \nu_0).$$

If $b = (\Sigma; u)$ is such that the degree of $u|_{\Sigma_P}$ is zero, then

$$\dim_{\mathbb{C}} Span_{(\mathbb{C}, J_0)} \{\mathcal{D}_i b : i \in \chi(b)\} < |\chi(b)|.$$

Propositions 2.5, 2.6, and 2.7 follow immediately from the estimates (2.12), (2.17), and (2.35) below. These estimates are obtained by combining the approach of [Z4, Sections 3,4] with some aspects of the local setting of [LT, Section 3]. A key step is [Z4, Lemma 3.5] that gives power series expansions for the behavior of derivatives of J-holomorphic genus-zero maps under gluing. They lead to estimates on obstructions to smoothing genus-one J-holomorphic maps from singular domains in [Z4, Lemma 4.4]. While the maps we encounter are not J-holomorphic on the entire domain, they are J-holomorphic around the part of the domain which is essential for the estimates of [Z4, Lemmas 3.5,4.4], i.e. Σ_b^0 in the notation of Subsections 1.3 and 2.1 above. The argument in the next two subsections is in fact an extension of [Z4, Section 5], but is far more detailed (as promised in [Z4]).

2.4   Proofs of Propositions 2.5 and 2.6

Let (X, ω), $\underline{J}$, A, M, $\underline{\nu}$,

$$b = (M, I; x, (j, y), u), \qquad and \qquad u_i \equiv u_b|_{\Sigma_{b,i}}$$

be as in the statement of Proposition 2.5. For each $i \in I$, we put

$$\Gamma(b; i) = \{\xi \in L_1^p(\Sigma_{b,i}; u_i^* TX) : \xi(\infty) = 0\} \qquad and \qquad \Gamma^{0,1}(b; i) = L^p(\Sigma_{b,i}; \Lambda_{J_0,j}^{0,1} T^* \Sigma_{b,i} \otimes u_i^* TX),$$

where j is the complex structure on Σ_b. We denote by

$$D_{J_0, \nu_0; b, i} : \Gamma(b; i) \longrightarrow \Gamma^{0,1}(b; i)$$

the linear operator induced by the linearization $D_{J_0, \nu_0; b}$ of the section $\bar{\partial}_{J_0} + \nu_0$ at b.

We put
$$I^+ = \{i \in I : A_i \neq 0\}. \tag{2.3}$$
For each $i \in I^+$, choose a finite-dimensional linear subspace
$$\tilde{\Gamma}^{0,1}_-(b; i) \subset \Gamma\big(\Sigma_{b,i} \times X; \Lambda^{0,1}_{J_0,j} \pi_1^* T^* \Sigma_{b,i} \otimes \pi_2^* TX\big)$$
such that
$$\Gamma^{0,1}(b; i) = \operatorname{Im} D_{J_0, \nu_0; b, i} \oplus \big\{ \{\mathrm{id} \times u_i\}^* \eta \colon \eta \in \tilde{\Gamma}^{0,1}_-(b; i) \big\}$$
and every element of $\tilde{\Gamma}^{0,1}_-(b; i)$ vanishes on a neighborhood of $\infty \in \Sigma_{b,i}$ and the nodes $x_h(b) \in \Sigma_{b,i}$ with $\iota_h = i$. If $i \in I - I^+$, we denote by $\tilde{\Gamma}^{0,1}_-(b; i)$ the zero vector space. Let $\mathcal{T}$ be the bubble type of the map b. We put
$$\tilde{\mathcal{U}} = \big\{ b' \equiv (M, I; x', (j, y'), u') \colon \ [b'] \in \mathfrak{X}_{\mathcal{T}}(X),$$
$$\pi_i \{\bar{\partial}_{J_0, j} + \nu_0\} u_{b'} \in \{\mathrm{id} \times u_i'\}^* \tilde{\Gamma}^{0,1}_-(b; i) \ \forall i \in I \big\},$$
where
$$\pi_i \colon \Gamma^{0,1}(b'; J_0) \longrightarrow \Gamma^{0,1}(b'; i)$$
is the natural projection map. By the Implicit Function Theorem, $\tilde{\mathcal{U}}$ is a smooth manifold near b. Let
$$\mathrm{ev}_0 \colon \tilde{\mathcal{U}} \longrightarrow X, \qquad b' \longrightarrow u_{b'}(\hat{0}, \infty),$$
be the evaluation map for the special marked point 0; see also Subsection 2.1. Let
$$\widetilde{\mathcal{F}} \equiv \tilde{\mathcal{U}} \times \mathbb{C}^{\hat{I}}$$
be the bundle of smoothing parameters. We denote by $\widetilde{\mathcal{F}}^\emptyset$ the subset of $\widetilde{\mathcal{F}}$ consisting of the elements with all components nonzero. For each $\upsilon = (b', v)$, where $b' \in \tilde{\mathcal{U}}$ and $v = (v_i)_{i \in \hat{I}}$, and $i \in \chi(b)$, we put
$$\rho_i(\upsilon) = \prod_{\hat{0} < h \leq i} v_h \in \mathbb{C} \qquad \text{and} \qquad x_i(\upsilon) = \sum_{\hat{0} < i' \leq i} \Big(x_{i'}(b') \prod_{\hat{0} < h < i'} v_h \Big) \in \mathbb{C},$$
where $x_i(b')$ is the point of Σ_{b', ι_i} to which the bubble $\Sigma_{b',i}$ is attached; see (2.2) and Figure 4 on page 22.

For each sufficiently small element $\upsilon = (b', v)$ of $\widetilde{\mathcal{F}}^\emptyset$, let
$$q_\upsilon \colon \Sigma_\upsilon \longrightarrow \Sigma_{b'}$$
be the basic gluing map constructed in [Z2, Subsection 2.2]. In this case, Σ_υ is the projective line $\mathbb{P}^1$ with $|M|+1$ marked points. The map q_υ collapses $|\hat{I}|$ circles on Σ_υ. It induces a metric g_υ on Σ_υ such that $(\Sigma_\upsilon, g_\upsilon)$ is obtained from $\Sigma_{b'}$ by replacing the $|\hat{I}|$ nodes of $\Sigma_{b'}$ by thin necks. Let
$$u_\upsilon = u_{b'} \circ q_\upsilon.$$
We fix a J_0-compatible metric g on X and denote the corresponding J_0-compatible connection by ∇. The map q_υ induces norms $\| \cdot \|_{\upsilon, p, 1}$ and $\| \cdot \|_{\upsilon, p}$ on the spaces
$$\Gamma(\Sigma_\upsilon; u_\upsilon^* TX) \qquad \text{and} \qquad \Gamma(\Sigma_\upsilon; \Lambda^{0,1}_{J_0, j} T^* \Sigma_\upsilon \otimes u_\upsilon^* TX),$$

respectively; see [Z2, Subsection 3.3]. We denote the corresponding completions by $\Gamma(\upsilon)$ and $\Gamma^{0,1}(\upsilon)$. The norms $\|\cdot\|_{\upsilon,p,1}$ and $\|\cdot\|_{\upsilon,p}$ are equivalent to the ones used in Section 3 of [LT].

Let t_r and b_r be as in Proposition 2.5. Since the sequence $[b_r]$ converges to $[b]$, for all r sufficiently large there exist

$$b'_r \in \tilde{\mathcal{U}}, \qquad \upsilon_r = (b'_r, v_r) \in \widetilde{\mathcal{F}}^{\emptyset}, \qquad \text{and} \qquad \xi_r \in \Gamma(\upsilon_r)$$

such that

$$\lim_{r \to \infty} b'_r = b, \qquad \lim_{r \to \infty} |v_r| = 0, \qquad \xi_r(\infty) = 0 \quad \forall r, \qquad \lim_{r \to \infty} \|\xi_r\|_{\upsilon_r,p,1} = 0, \qquad (2.4)$$

$$\text{and} \qquad b_r \equiv \left(\Sigma_{b_r}; u_{b_r}\right) = \left(\Sigma_{\upsilon_r}; \exp_{u_{\upsilon_r}} \xi_r\right).$$

The last equality holds for a representative b_r for $[b_r]$.

Remark: The existence of b'_r, υ_r, and ξ_r as above can be shown by an argument similar to the surjectivity argument in [Z2, Section 4], with significant simplifications. In fact, the only facts about the bubble maps b'_r we use below are that they are constant on the degree-zero components and holomorphic on fixed neighborhoods of the attaching nodes of the first-level effective bubbles. Such bubble maps b'_r, along with v_r and ξ_r, can be constructed directly from the maps b_r; see the beginning of Subsection 4.4 in [Z2].

If $\delta \in \mathbb{R}^+$, $b' \in \tilde{\mathcal{U}}$, and $\upsilon = (b', v) \in \widetilde{\mathcal{F}}^{\emptyset}$ is sufficiently small, we put

$$\Sigma_{b'}^0(\delta) = \Sigma_{b'}^0 \cup \bigcup_{i \in \chi(b)} A_{b',i}(\delta), \quad \text{where} \quad A_{b',i}(\delta) = \left\{(i,z) \colon |z| \geq \delta^{-1/2}/2\right\} \subset \Sigma_{b',i} \approx S^2, \qquad (2.5)$$

$$\text{and} \qquad \Sigma_{\upsilon}^0(\delta) = q_{\upsilon}^{-1}\left(\Sigma_{b'}^0(\delta)\right). \qquad (2.6)$$

Choose $\delta \in \mathbb{R}^+$ such that for all $i \in \chi(b)$ all elements of $\tilde{\Gamma}_-^{0,1}(b;i)$ vanish on $A_{b,i}(2\delta)$ and for all r sufficiently large

$$\nu_t(b'_r)\big|_{\Sigma_{b'_r}^0(2\delta)} = 0 \quad \text{and} \quad \nu_t(b_r)\big|_{\Sigma_{\upsilon_r}^0(2\delta)} = 0 \qquad \forall \, t \in [0,1].$$

Such a positive number δ exists by our assumptions on the spaces $\tilde{\Gamma}_-^{0,1}(b;i)$ and the family of perturbations $\underline{\nu}$; see Definition 2.2.

For every element $b' = (\Sigma_{b'}; u_{b'})$ of $\tilde{\mathcal{U}}$ and every sufficiently small element $\upsilon = (b', v)$ of $\widetilde{\mathcal{F}}^{\emptyset}$, we denote by

$$\text{Hol}_{J_0}\left(\Sigma_{b'}^0(\delta); T_{\text{ev}_0(b')}X\right) \qquad \text{and} \qquad \text{Hol}_{J_0}\left(\Sigma_{\upsilon}^0(\delta); T_{\text{ev}_0(b')}X\right)$$

the spaces of holomorphic maps from $\Sigma_{b'}^0(\delta)$ and $\Sigma_{\upsilon}^0(\delta)$ into the complex vector space $(T_{\text{ev}_0(b')}X; J_0)$. Let exp be the ∇-exponential map. For every $b' \in \tilde{\mathcal{U}}$ as above, $u_{b'}|_{\Sigma_{b'}^0}$ is constant and $u_{b'}|_{\Sigma_{b'}^0(2\delta)}$ is J_0-holomorphic. Thus, if δ is sufficiently small, there exist continuous families of maps

$$\Phi_{b'} \in L_1^p\left(\Sigma_{b'}^0(\delta); \text{End}(T_{\text{ev}_0(b')}X)\right) \qquad \text{and} \qquad \vartheta_{b'} \in \text{Hol}_{J_0}\left(\Sigma_b^0(\delta); T_{\text{ev}_0(b')}X\right)$$

with $b' \in \tilde{\mathcal{U}}$ such that for all b' sufficiently close to b'

$$\Phi_{b'}|_{\Sigma^0_{b'}} = \mathrm{Id}, \qquad \left\|\Phi_{b'} - \mathrm{Id}\right\|_{b',p,1} \leq \frac{1}{2}, \qquad \text{and}$$

$$\exp_{\mathrm{ev}_0(b')}\big(\Phi_{b'}(z)\vartheta_{b'}(z)\big) = u_{b'}(z) \qquad \forall z \in \Sigma^0_{b'}(\delta).$$

This statement follows immediately from the proof of Theorem 2.2 in [FlHS]. Similarly, for every

$$b_r \equiv \big(\Sigma_{b_r}; u_{b_r}\big) = \big(\Sigma_{v_r}; \exp_{u_{v_r}}\xi_r\big)$$

with r sufficiently large, $u_{b_r}|_{\Sigma^0_{v_r}(2\delta)}$ is J_{t_r}-holomorphic. Since $\|\xi_r\|_{v_r,p,1}$ tends to zero as r approaches ∞, if δ is sufficiently small and r is sufficiently large, there exist

$$\Phi_{b_r} \in L^p_1\big(\Sigma^0_{v_r}(\delta); \mathrm{End}(T_{\mathrm{ev}_0(b'_r)}X)\big), \qquad \vartheta_{b_r} \in \mathrm{Hol}_{J_0}\big(\Sigma^0_{v_r}(\delta); T_{\mathrm{ev}_0(b'_r)}X\big)$$

such that

$$\left\|\Phi_{b_r} - \Phi_{b'_r} \circ q_{v_r}\right\|_{v_r,p,1} \leq C\big(\|J_0 - J_{t_r}\|_{C^1} + |v_r|^{1/p} + \|\xi_r\|_{v_r,p,1}\big) \qquad \text{and}$$

$$\exp_{\mathrm{ev}_0(b'_r)}\big(\Phi_{b_r}(z)\vartheta_{b_r}(z)\big) = u_{b_r}(z) \qquad \forall z \in \Sigma^0_{v_r}(\delta).$$

In the inequality above, both norms $\|\cdot\|_{v_r,p,1}$ are the norms induced from the pregluing construction as in Subsection 3.3 of [Z2]. With these norms, the existence of Φ_{b_r} and ϑ_{b_r} follows easily from the proof of Theorem 2.2 in [FlHS]; see the paragraph following Lemma 3.3 in [Z4].

If $i \in \chi(b)$ and $b' \in \tilde{\mathcal{U}}$, let w_i be the standard holomorphic coordinate centered at the point ∞ in $\Sigma_{b',i} = S^2$. If $m \in \mathbb{Z}^+$, we put

$$\mathcal{D}^{(m)}_i \vartheta_{b'} = \frac{1}{m!}\frac{d^m}{dw^m_i}\vartheta_{b',i}(w_i)\Big|_{w_i=0} \in T_{\mathrm{ev}_0(b')}X, \qquad \text{where} \qquad \vartheta_{b',i} = \vartheta_{b'}|_{\Sigma_{b',i}}.$$

Similarly, for all r sufficiently large, we put

$$\mathcal{D}^{(m)}_{\hat{0}} \vartheta_{b_r} = \frac{1}{m!}\frac{d^m}{dw^m}\vartheta_{b_r,i}(w)\Big|_{w=0} \in T_{\mathrm{ev}_0(b'_r)}X,$$

where w is the standard holomorphic coordinate centered at the point ∞ in $\Sigma_{v_r} \approx S^2$. The key step in the proof of Propositions 2.5 and 2.6 is the power series expansion

$$\mathcal{D}^{(m)}_{\hat{0}}\vartheta_{b_r} = \sum_{k=1}^{k=m}\binom{m-1}{k-1}\sum_{i\in\chi(b)}x_i^{m-k}(v_r)\rho_i^k(v_r)\big\{\mathcal{D}^{(k)}_i\vartheta_{b'_r} + \varepsilon^{(k)}_{i,r}\big\} \in (T_{\mathrm{ev}_0(b'_r)}X, J_0), \qquad (2.7)$$

for some $\varepsilon^{(k)}_{i,r} \in T_{\mathrm{ev}_0(b'_r)}X$ such that

$$\big|\varepsilon^{(k)}_{i,r}\big| \leq C\delta^{-k/2}\big(\|J_{t_r} - J_0\|_{C^1} + |v_r|^{1/p} + \|\xi_r\|_{v_r,p,1}\big). \qquad (2.8)$$

The expansion (2.7) is obtained by exactly the same integration-by-parts argument as the expansion in (2a) of Lemma 3.4 in [Z4]. We point out that $\varepsilon^{(k)}_{i,r}$ is independent of m. The $m=1$ case of the estimates (2.7) and (2.8) is illustrated in Figure 4.

$$\chi(b) = \{h_1, h_4, h_5\}$$
$$\rho(v) = (v_{h_1}, v_{h_3}v_{h_4}, v_{h_3}v_{h_5})$$
$$x_{h_5}(v) = x_{h_3}(b'_r) + v_{h_3}x_{h_5}(b'_r)$$

$$\mathcal{D}^{(1)}_{\hat{0}}\vartheta_{b_r} \cong v_{h_1}\big(\mathcal{D}^{(1)}_{h_1}\vartheta_{b'_r}\big) + v_{h_3}v_{h_4}\big(\mathcal{D}^{(1)}_{h_4}\vartheta_{b'_r}\big) + v_{h_3}v_{h_5}\big(\mathcal{D}^{(1)}_{h_5}\vartheta_{b'_r}\big)$$

Figure 4: An example of the estimates (2.7) and (2.8)

We now complete the proof of Proposition 2.5. By the $m=1$ case of (2.7) and (2.8),

$$\Big|\mathcal{D}^{(1)}_{\hat{0}}\vartheta_{b_r} - \sum_{i\in\chi(b)}\rho_i(v_r)\big(\mathcal{D}^{(1)}_i\vartheta_{b'_r}\big)\Big| \le C\big(\|J_{t_r}-J_0\|_{C^1}+|v_r|^{1/p}+\|\xi_r\|_{v_r,p,1}\big)\sum_{i\in\chi(b)}\big|\rho_i(v_r)\big|. \qquad (2.9)$$

On the other hand, since $\Phi_{b'_r}(i,\infty)=\mathrm{Id}$ and $\vartheta_{b'_r}(i,\infty)=0$ for all $i\in\chi(b)$,

$$\mathcal{D}_i b'_r = \mathcal{D}^{(1)}_i\vartheta_{b'_r} \qquad \forall\, i\in\chi(b). \qquad (2.10)$$

Furthermore, since

$$\big|\Phi_{b_r}(\hat{0},\infty)-\mathrm{Id}\big| \le C\big\|\Phi_{b_r}-\Phi_{b'_r}\circ q_{v_r}\big\|_{v_r,p,1} \le C'\big(\|J_0-J_{t_r}\|_{C^1}+|v_r|^{1/p}+\|\xi_r\|_{v_r,p,1}\big)$$

and $\vartheta_{b_r}(\hat{0},\infty)=0$,

$$\big|\mathcal{D}_{\hat{0}}b_r - \mathcal{D}^{(1)}_{\hat{0}}\vartheta_{b_r}\big| \le C'\big(\|J_0-J_{t_r}\|_{C^1}+|v_r|^{1/p}+\|\xi_r\|_{v_r,p,1}\big)\big|\mathcal{D}^{(1)}_{\hat{0}}\vartheta_{b_r}\big|. \qquad (2.11)$$

By (2.9)-(2.11),

$$\Big|\mathcal{D}_{\hat{0}}b_r - \sum_{i\in\chi(b)}\rho_i(v_r)\big(\mathcal{D}_i b'_r\big)\Big| \le C\big(\|J_0-J_{t_r}\|_{C^1}+|v_r|^{1/p}+\|\xi_r\|_{v_r,p,1}\big)\sum_{i\in\chi(b)}\big|\rho_i(v_r)\big| \qquad (2.12)$$

for all r sufficiently large. Since

$$\lim_{r\longrightarrow\infty}\big(\|J_0-J_{t_r}\|_{C^1}+|v_r|^{1/p}+\|\xi_r\|_{v_r,p,1}\big) = 0$$

and $\mathcal{D}_i b'_r \longrightarrow \mathcal{D}_i b$ for all $i\in\chi(b)$, (2.12) implies that b must satisfy one of the two conditions in the statement of Proposition 2.5.

We next complete the proof of Proposition 2.6. By the assumption on the bubble maps b_k made in Proposition 2.6 and by Definition 2.3, $\mathrm{ev}_0(b_k)=\mathrm{ev}_1(b_k)$ for all k. Thus,

$$\mathrm{ev}_1(b_k) = \mathrm{ev}_0(b_k) = \mathrm{ev}_1(b_l) \qquad \forall\, k,l \in [n].$$

Let q denote the point $\mathrm{ev}_0(b_1)$. We identify a small neighborhood of q in X with a small neighborhood of q in T_qX via the exponential map $\exp$ and the tangent space to X at a point close to q with T_qX via the parallel transport with respect to the J_0-linear connection ∇.

For each pair (k, r), with r sufficiently large, let $(b'_{k,r}, v_{k,r}, \xi_{k,r})$ be an analogue of (b'_r, v_r, ξ_r) for $b_{k,r}$. We put

$$\zeta_{k,r} = \mathrm{ev}_0(b'_{k,r}) \in T_q X \qquad \text{and}$$

$$\tilde{\zeta}_{k,r} = \mathrm{ev}_1(b_{k,r}) - \mathrm{ev}_0(b_{k,r}) = \mathrm{ev}_1(b_{k,r}) - \mathrm{ev}_0(b'_{k,r}) \in T_q X.$$

By the assumption on the maps $b_{k,r}$ made in the statement of Proposition 2.6,

$$\left|\zeta_{k,r} + \tilde{\zeta}_{k,r} - \zeta_{k+1,r}\right| \le C\left|\zeta_{k,r}\right| \cdot \left|\tilde{\zeta}_{k,r}\right| \qquad \forall\, k \in [n-1],$$
$$\left|\zeta_{n,r} + \tilde{\zeta}_{n,r} - \zeta_{1,r}\right| \le C\left|\zeta_{n,r}\right| \cdot \left|\tilde{\zeta}_{n,r}\right|;$$

$$\Longrightarrow \qquad \left|\tilde{\zeta}_{1,r} + \ldots + \tilde{\zeta}_{n,r}\right| \le \epsilon_r \sum_{k=1}^{k=n} \left|\tilde{\zeta}_{k,r}\right|, \tag{2.13}$$

for a sequence ϵ_r converging to 0.

On the other hand, the marked point $y_1(b_{k,r}) = y_1(v_{k,r})$ of the bubble map $b_{k,r}$ lies in $\Sigma^0_{v_{k,r}}(\delta)$. Thus,

$$\tilde{\zeta}_{k,r} = \vartheta_{b_{k,r}}\big(y_1(v_{k,r})\big) = \sum_{m=1}^{\infty} y_1(v_{k,r})^{-m} \big(\mathcal{D}^{(m)}_{\hat{0}} \vartheta_{b_{k,r}}\big), \tag{2.14}$$

where $y_1(v_{k,r}) \in S^2 - \{\infty\}$ is viewed as a complex number. Combining (2.14) with (2.7) and (2.8) and then taking the lowest-order terms, we obtain an expression of the form

$$\left|\tilde{\zeta}_{k,r} - \sum_{i \in \chi(b_k)} (y_{1;i}(b'_{k,r}) - x_{i;1}(b'_{k,r}))^{-1} \rho_{i;1}(v_{k,r}) \big(\mathcal{D}^{(1)}_i \vartheta_{b'_{k,r}}\big)\right|$$
$$\le C\big(\|J_0 - J_{t_r}\|_{C^1} + |v_{k,r}|^{1/p} + \|\xi_{k,r}\|_{v_{k,r},p,1}\big) \sum_{i \in \chi(b_k)} \left|\rho_{i;1}(v_{k,r})\right|; \tag{2.15}$$

see the proof of Corollary 3.7 in [Z4] for a derivation and the notation involved. For the present purposes, the only fact we need to know about (2.15) is that

$$0 < \left|\rho_{i;1}(v_{k,r})\right| \le \left|v_{k,r}\right| \qquad \forall\, i \in \chi(b_k),\ k \in [n]. \tag{2.16}$$

In particular, $\rho_{i;1}(v_{k,r})$ is a sequence of nonzero complex numbers that approaches zero as r tends to infinity. By (2.13) and (2.15),

$$\left|\sum_{k=1}^{k=n} \sum_{i \in \chi(b_k)} (y_{1;i}(b'_{k,r}) - x_{i;1}(b'_{k,r}))^{-1} \rho_{i;1}(v_{k,r}) \big(\mathcal{D}^{(1)}_i \vartheta_{b'_{k,r}}\big)\right| \le \tilde{\epsilon}_r \sum_{k=1}^{k=n} \sum_{i \in \chi(b_k)} \left|\rho_{i;1}(v_{k,r})\right|, \tag{2.17}$$

for a sequence $\{\tilde{\epsilon}_r\}$ converging to 0. Since

$$\mathcal{D}^{(1)}_i \vartheta_{b'_{k,r}} = \mathcal{D}_i b'_{k,r} \longrightarrow \mathcal{D}_i b_k \qquad \text{as} \quad r \longrightarrow \infty,$$

(2.16) and (2.17) imply the conclusion of Proposition 2.6.

2.5 Proof of Proposition 2.7

We prove Proposition 2.7 by combining elements of the previous subsection with a version of the two-stage gluing construction of [Z4, Section 4]. At the first stage, we smooth out all nodes of an element $[\Sigma'; u']$ close to b that lie away from the principal component(s) Σ'_P of Σ'. This stage will be unobstructed. The objective of the second stage of the gluing construction is to smooth out the remaining nodes of Σ'. We obtain Proposition 2.7 by estimating the obstruction to achieving this objective.

Let (X, ω), $\underline{J}$, A, M, $\underline{\nu}$,

$$b = (M, I, \aleph; S, x, (j, y), u), \quad \text{and} \quad u_i \equiv u_b|_{\Sigma_{b,i}}$$

be as in the statement of Proposition 2.7. For each $i \in I$, we define

$$D_{J_0, \nu_0; b, i} \colon \Gamma(b; i) \longrightarrow \Gamma^{0,1}(b; i)$$

as at the beginning of the previous subsection. With $I^+ \subset I$ as in (2.3), choose

$$\tilde{\Gamma}_-^{0,1}(b; i) \subset \Gamma\big(\Sigma_{b,i} \times X; \Lambda_{J_0,j}^{0,1} \pi_1^* T^* \Sigma_{b,i} \otimes \pi_2^* TX\big)$$

as in Subsection 2.4. Let $\mathcal{T}$ be the bubble type of the map b. We put

$$\tilde{\mathcal{U}} = \big\{b' \equiv (M, I, \aleph; S', x', (j, y'), u') \colon [b'] \in \mathfrak{X}_{\mathcal{T}}(X),$$
$$\pi_i\{\bar{\partial}_{J_0,j} + \nu_0\}u_{b'} \in \{\mathrm{id} \times u_i'\}^* \tilde{\Gamma}_-^{0,1}(b; i) \ \forall i \in I\big\},$$

where

$$\pi_i \colon \Gamma^{0,1}(b'; J_0) \longrightarrow \Gamma^{0,1}(b'; i)$$

is the natural projection map. By the Implicit Function Theorem, $\tilde{\mathcal{U}}$ is a smooth manifold near b. If $b' \in \tilde{\mathcal{U}}$, $u_{b'}|_{\Sigma_{b';P}}$ is a degree-zero J_0-holomorphic map and thus is constant. Let

$$\mathrm{ev}_P \colon \tilde{\mathcal{U}} \longrightarrow X, \qquad b' \longrightarrow u_{b'}(\Sigma_{b';P}),$$

be the map sending each element b' of $\tilde{\mathcal{U}}$ to the image of the main component of its domain.

For each $b' \in \tilde{\mathcal{U}}$, let

$$\Gamma_-(b') = \big\{\xi \in \Gamma(b') \colon \pi_i D_{J_0, \nu_0; b'} \xi \in \{\mathrm{id} \times u_{b',i}\}^* \tilde{\Gamma}_-^{0,1}(b; i) \ \forall i \in I\big\},$$

where $u_{b',i} = u_{b'}|_{\Sigma_{b',i}}$. We denote by

$$\tilde{\Gamma}_-^{0,1}(b'; X) \subset \Gamma\big(\Sigma_{b'} \times X; \Lambda_{J_0,j}^{0,1} \pi_1^* T^* \Sigma_{b'} \otimes \pi_2^* TX\big)$$

the subspace obtained by extending all elements of each of the spaces

$$\tilde{\Gamma}_-^{0,1}(b'; i) \equiv \tilde{\Gamma}_-^{0,1}(b; i)$$

with $i \in \hat{I}$ by zero outside of the component $\Sigma_{b',i}$ of $\Sigma_{b'}$.

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We put
$$I_1 = \{h \in \hat{I} \colon \iota_h \in I_0\},$$
where I_0 is the subset of minimal elements of I. Let
$$\widetilde{\mathcal{F}} \longrightarrow \tilde{\mathcal{U}}$$
be the bundle of gluing parameters. In this case, $\widetilde{\mathcal{F}}$ has three distinguished components:
$$\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_{\aleph} \oplus \widetilde{\mathcal{F}}_0 \oplus \widetilde{\mathcal{F}}_1, \qquad \text{where}$$
$$\widetilde{\mathcal{F}}_{\aleph} = \tilde{\mathcal{U}} \times \mathbb{C}^{\aleph}, \qquad \widetilde{\mathcal{F}}_1 = \tilde{\mathcal{U}} \times \mathbb{C}^{\hat{I}-I_1}, \quad \text{and} \quad \widetilde{\mathcal{F}}_0\big|_{b'} = \bigoplus_{h \in I_1} T_{x_h(b')}\Sigma_{b';P} \quad \forall\, b' \in \tilde{\mathcal{U}}.$$

The total space of $\widetilde{\mathcal{F}}_0$ has a natural topology; see [Z4, Subsection 2.2]. We denote by $\widetilde{\mathcal{F}}^{\emptyset}$ the subset of $\widetilde{\mathcal{F}}$ consisting of the elements with all components nonzero. If $i \in \hat{I}$, let $h(i) \in I_1$ be the unique element such that $h(i) \leq i$. For each $\upsilon = (b', v)$, where $b' \in \tilde{\mathcal{U}}$ and $v = (v_i)_{i \in \aleph \sqcup \hat{I}}$, and $i \in \chi(b)$, we put

$$\upsilon_0 = \big(b', (v_i)_{i \in \aleph \sqcup I_1}\big), \qquad \upsilon_1 = \big(b', (v_i)_{i \in \hat{I}-I_1}\big),$$
$$\tilde{\rho}_i(\upsilon) = \prod_{h(i)<h\leq i} v_h \in \mathbb{C}, \qquad \text{and} \qquad \rho_i(\upsilon) = \tilde{\rho}_i(\upsilon) \cdot v_{h(i)} \in T_{x_{h(i)}(b')}\Sigma_{b';P}.$$

The component υ_1 of υ consists of the smoothings of the nodes of Σ_b that lie away from the principal component.

For each sufficiently small element $\upsilon = (b', v)$ of $\widetilde{\mathcal{F}}^{\emptyset}$, let
$$q_{\upsilon_1} \colon \Sigma_{\upsilon_1} \longrightarrow \Sigma_{b'}$$
be the basic gluing map constructed in [Z2, Subsection 2.2]. In this case, the principal component $\Sigma_{\upsilon_1;P}$ of Σ_{υ_1} is the same as principal component $\Sigma_{b';P}$ of $\Sigma_{b'}$, and Σ_{υ_1} has $|I_1|$ bubble components $\Sigma_{\upsilon_1,h}$, with $h \in I_1$, attached directly to $\Sigma_{\upsilon_1;P}$. The map q_{υ_1} collapses $|\hat{I}-I_1|$ circles on the bubble components of Σ_{υ_1}. It induces a metric g_{υ_1} on Σ_{υ_1} such that $(\Sigma_{\upsilon_1}, g_{\upsilon_1})$ is obtained from $\Sigma_{b'}$ by replacing $|\hat{I}-I_1|$ nodes by thin necks. Let
$$u_{\upsilon_1} = u_{b'} \circ q_{\upsilon_1}.$$

The map q_{υ_1} induces norms $\|\cdot\|_{\upsilon_1,p,1}$ and $\|\cdot\|_{\upsilon_1,p}$ on the spaces

$$\big\{\xi \in \Gamma(\Sigma_{\upsilon_1}; u_{\upsilon_1}^* TX)\colon \xi|_{\Sigma_{\upsilon_1;P}} = 0\big\} \qquad \text{and} \qquad \big\{\eta \in \Gamma(\Sigma_{\upsilon_1}; \Lambda^{0,1}_{J_0,j} T^*\Sigma_{\upsilon_1} \otimes u_{\upsilon_1}^* TX)\colon \eta|_{\Sigma_{\upsilon_1;P}} = 0\big\}$$

respectively; see [Z2, Subsection 3.3]. We denote the corresponding completions by $\Gamma_B(\upsilon_1)$ and $\Gamma_B^{0,1}(\upsilon_1)$.

Remark: The weights for the norms $\|\cdot\|_{\upsilon_1,p,1}$ and $\|\cdot\|_{\upsilon_1,p}$ are constructed as [Z2, Subsection 3.3], but on each of the $|I_1|$ bubbles separately. The restrictions of these norms to each of the bubbles are equivalent to the norms used in [LT, Section 3].

Fix $\epsilon_b \in \mathbb{R}^+$ such that for every $h \in I_1$ the disk of radius of $8\epsilon_b$ in $\Sigma_{b;P}$ around the node $x_h(b)$ contains no other special, i.e. singular or marked, point of Σ_b. For each $(b', v) \in \tilde{\mathcal{F}}^\emptyset$ with $b' \in \tilde{\mathcal{U}}$ sufficiently close to b and v sufficiently small, let

$$q_{v_0;2} \colon \Sigma_v \longrightarrow \Sigma_{v_1} \qquad \text{and} \qquad \tilde{q}_{v_0;2} \colon \Sigma_v \longrightarrow \Sigma_{v_1}$$

be the basic gluing map of [Z2, Subsection 2.2] corresponding to the gluing parameter v_0 and the modified basic gluing map defined in the middle of Subsection 4.2 in [Z4] with the collapsing radius ϵ_b. In this case, Σ_v is a smooth genus-one curve. For each $h \in I_1$, the maps $q_{v_0;2}$ and $\tilde{q}_{v_0;2}$ collapse the circles of radii $|v_h|^{1/2}$ and ϵ_b, respectively, around the point $x_h(b') \in \Sigma_{v_1;P}$. Once again, the map

$$q_v \equiv q_{v_0;2} \circ q_{v_1} \colon \Sigma_v \longrightarrow \Sigma_{b'}$$

induces a metric g_v on Σ_v such that (Σ_v, g_v) is obtained from $\Sigma_{b'}$ by replacing all nodes by thin necks.

For each $t \in [0, 1]$, let g_t be the J_t-compatible symmetric two-tensor on X given by

$$g_t(\cdot, \cdot) = \frac{1}{2}\big(g(J_0\cdot, J_t\cdot) + g(J_t\cdot, J_0\cdot)\big).$$

If t is sufficiently close to 0, g_t is positive-definite, i.e. it is a J_t-compatible metric on X. We denote the J_t-compatible connection induced by the Levi-Civita connection of the metric g_t by ∇^t and the corresponding exponential map by $\exp^t$.

If W is a small neighborhood of b in $\tilde{\mathcal{U}}$ and $\delta \in \mathbb{R}^+$ is sufficiently small, let

$$\mathfrak{X}_1(W, \delta) = \big\{ (\Sigma_{v_1}; u_{v_1,\xi}) \equiv \big(\Sigma_{v_1}; \exp_{u_{v_1}}\xi\big) \colon v = (b', v) \in \tilde{\mathcal{F}}^\emptyset_\delta|_W; \ \xi \in \Gamma_B(v_1), \ \|\xi\|_{v_1,p,1} \leq \delta \big\}.$$

For each element $(\Sigma_{v_1}; u_{v_1,\xi})$ of $\mathfrak{X}_1(W, \delta)$, we put

$$u_{v,\xi} = u_{v_1,\xi} \circ \tilde{q}_{v_0;2}. \tag{2.18}$$

The map $q_{v_0;2}$ induces norms $\|\cdot\|_{v,p,1}$ and $\|\cdot\|_{v,p}$ on the spaces

$$\Gamma(\Sigma_v; u_{v,\xi}^* TX) \qquad \text{and} \qquad \Gamma(\Sigma_v; \Lambda^{0,1}_{J_t,j} T^*\Sigma_v \otimes u_{v,\xi}^* TX),$$

respectively. For t sufficiently small, we use the metric g_t on X to define a norm on the latter space. Let $\Gamma(v; \xi)$ and $\Gamma^{0,1}_t(v; \xi)$ be the corresponding completions. If $v = (b', v)$, we put

$$\Gamma_-(v; \xi) = \big\{ \big(\Pi_\xi(\zeta \circ q_{v_1})\big) \circ \tilde{q}_{v_0;2} \colon \zeta \in \Gamma_-(b') \big\} \subset \Gamma(v; \xi),$$

where Π_ξ is the ∇-parallel transport along the ∇-geodesics $\tau \longrightarrow \exp_{u_{v_1}}\tau\xi$. Let

$$\Gamma_+(v; \xi) \subset \Gamma(v; \xi)$$

be the L^2-orthogonal complement of $\Gamma_-(v; \xi)$ defined with respect to the metrics g_v on Σ_v and g on X. For every $t \in [0, 1]$, we denote by

$$\pi^{0,1}_t \colon \Gamma(\Sigma_v; T^*\Sigma_v \otimes_{\mathbb{R}} u_{v,\xi}^* TX) \longrightarrow \Gamma(\Sigma_v; \Lambda^{0,1}_{J_t,j} T^*\Sigma_v \otimes u_{v,\xi}^* TX)$$

the natural projection map.

For each $b'\in\tilde{\mathcal{U}}$, each sufficiently small element $\upsilon=(b',v)$ of $\widetilde{\mathcal{F}}^\emptyset$, and $\delta\in\mathbb{R}^+$, we define

$$A_{b',i}(\delta), \Sigma^0_{b'}(\delta) \subset \Sigma_{b'}, \qquad \Sigma^0_{v_1}(\delta) \subset \Sigma_{v_1}, \quad\text{and}\quad \Sigma^0_v(\delta) \subset \Sigma_v$$

by (2.5) and (2.6). Choose a neighborhood W of b in $\tilde{\mathcal{U}}$, $\delta_b\in(0,\epsilon_b)$, and $\delta\in(0,\delta_b^2/2)$ such that
   (i) the maps q_v and $\tilde{q}_{v_0;2}$ are defined for all $\upsilon\in\widetilde{\mathcal{F}}^\emptyset_\delta|_W$;
   (ii) for all $i\in\chi(b)$ all elements of $\tilde{\Gamma}^{0,1}_-(b;i)$ vanish on $A_{b,i}(\delta_b)$;
   (iii) $\nu_t(b')\big|_{\Sigma^0_{b'}(2\delta_b)}=0$ for every $t\in[0,1]$ and $b'\in W$;
   (iv) $\nu_t(\tilde{b})\big|_{\Sigma^0_v(2\delta_b)}=0$ for every $t\in[0,1]$ and every $\tilde{b}=(\Sigma_v;\exp_{u_{v_1,\xi}}\zeta)$ such that
   $\upsilon\in\widetilde{\mathcal{F}}^\emptyset_\delta|_W$, $(\Sigma_{v_1};u_{v_1,\xi})\in\mathfrak{X}_1(W,\delta)$, $\zeta\in\Gamma(v;\xi)$, and $\|\zeta\|_{v,p,1}\le\delta$.
Such a positive number δ exists by our assumptions on the spaces $\tilde{\Gamma}^{0,1}_-(b;i)$ and the family of perturbations $\underline{\nu}$; see Definition 1.2.

Suppose $\upsilon\in\widetilde{\mathcal{F}}^\emptyset_\delta|_W$ and $(\Sigma_{v_1};u_{v_1,\xi})\in\mathfrak{X}_1(W,\delta)$. By the construction of the map $\tilde{q}_{v_0;2}$ in Subsection 4.2 of [Z4] and the assumptions that $\delta\le\delta_b^2/2$ and $\delta_b\le\epsilon_b$,

$$\tilde{q}_{v_0;2}\colon \Sigma_v - \Sigma^0_v(\delta_b/2) \longrightarrow \Sigma_{v_1} - \Sigma^0_{v_1}(\delta_b/2)$$

is a biholomorphism. Thus, by the assumption (iv) above with $\zeta=0$, for every $t\in[0,1]$, we can define

$$\nu_{t;v_1,\xi} \in \Gamma^{0,1}\big(\Sigma_{v_1};\Lambda^{0,1}_{J,j}T^*\Sigma_{v_1}\otimes u^*_{v_1,\xi}TX\big) \quad\text{by}\quad \tilde{q}^*_{v_0;2}\nu_{t;v_1,\xi} = \nu_t(\Sigma_v;u_{v,\xi}), \quad \nu_{t;v_1,\xi}\big|_{\Sigma^0_{v_1}(\delta_b)} = 0.$$

If $\upsilon=(b',v)$, we put

$$\Gamma_{t,\delta}(\upsilon) = \big\{\xi\in\Gamma_B(v_1)\colon \|\xi\|_{v_1,p,1}\le\delta;\ \{\bar{\partial}_J+\nu_{t;v_1,\xi}\}u_{v_1,\xi} \in \pi_t\{q_{v_1}\times u_{v_1,\xi}\}^*\tilde{\Gamma}^{0,1}_-(b';X)\big\}.$$

Let t_r and b_r be as in Proposition 2.7. Since the sequence $[b_r]$ converges to $[b]$, for all r sufficiently large there exist

$$b'_r \in W, \quad \upsilon_r = (b'_r,v_r) \in \widetilde{\mathcal{F}}^\emptyset_\delta, \qquad \xi_r \in \Gamma_{t,\delta}(\upsilon_r), \quad\text{and}\quad \zeta_r \in \Gamma_+(\upsilon_r;\xi_r)$$

such that

$$\lim_{r\longrightarrow\infty} b'_r = b, \quad \lim_{r\longrightarrow\infty} |v_r| = 0, \quad \lim_{r\longrightarrow\infty} \|\xi_r\|_{v_r;1,p,1} = 0, \quad \lim_{r\longrightarrow\infty} \|\zeta_r\|_{v_r,p,1} = 0, \tag{2.19}$$

$$\text{and} \qquad b_r\equiv\big(\Sigma_{b_r};u_{b_r}\big) = \big(\Sigma_{v_r};\exp^t_{u_{v_r,\xi_r}}\zeta_r\big). \tag{2.20}$$

The last equality holds for a representative b_r for $[b_r]$. The existence of $(\upsilon_r,\xi_r,\zeta_r)$ for

$$b_r \in \mathfrak{M}^0_{1,M}(X,A;J_{t_r},\nu_{t_r}) \tag{2.21}$$

as above will imply that b satisfies the second property in Definition 1.3.

Remark: Similarly to the genus-zero case, the existence of (v_r, ξ_r, ζ_r) can be shown by a variation on the surjectivity argument of [Z2, Section 4]; see also the paragraph following Lemma 4.4 in [Z4]. This is also the case if the map $q_{v_0;2}$ is used instead of $\tilde{q}_{v_0;2}$ in (2.18) . However, using the map $\tilde{q}_{v_0;2}$ in (2.18) makes the maps $u_{v,\xi}$, with $\xi \in \Gamma_{t,\delta}(v)$, closer to being (J_t, ν_t)-holomorphic. Since $u_{b'}|_{\Sigma_{b';P}}$ is constant, the choice of $\tilde{q}_{v_0;2}$ for constructing approximately (J_t, ν_t)-holomorphic maps is analogous to that of Section 3 in [LT].

For each $v \in \widetilde{\mathcal{F}}_\delta^\emptyset|_W$ and $\xi \in \Gamma_{t,\delta}(v)$, let

$$D_{v;\xi}^t \colon \Gamma(v;\xi) \longrightarrow \Gamma_t^{0,1}(v;\xi)$$

be the linearization of section $\bar{\partial}_{J_t} + \nu_t$ at $(\Sigma_v; u_{v,\xi})$ defined via the connection ∇^t. We denote by

$$\Gamma_{t;+}^{0,1}(v;\xi) \subset \Gamma_t^{0,1}(v;\xi)$$

the image $\Gamma_+(v;\xi)$ under $D_{v;\xi}^t$. By the same argument as in [Z2, Subsection 5.4], there exists $C \in \mathbb{R}^+$ such that

$$C^{-1}\|\zeta\|_{v,p,1} \le \left\|D_{v;\xi}^t \zeta\right\|_{v,p} \le C\|\zeta\|_{v,p,1} \quad \forall\, t \in [0,\delta],\ v \in \widetilde{\mathcal{F}}_\delta^\emptyset|_W,\ \xi \in \Gamma_{t,\delta}(v),\ \zeta \in \Gamma_+(v;\xi), \qquad (2.22)$$

provided W and δ are sufficiently small.

Put

$$\tilde{\Gamma}_{t;-}^{0,1}(v;\xi) = \left\{\pi_t\{q_v \times u_{v,\xi}\}^*\eta \colon \eta \in \tilde{\Gamma}_-^{0,1}(b';X)\right\} \subset \Gamma_t^{0,1}(v;\xi).$$

If δ is sufficiently small, by the same argument as in [Z2, Subsection 3.5] and our assumptions on the spaces $\tilde{\Gamma}_-(b;i)$,

$$\Gamma_t^{0,1}(v;\xi) = \Gamma_{t;+}^{0,1}(v;\xi) \oplus \tilde{\Gamma}_{t;-}^{0,1}(v;\xi) \oplus \Gamma_{t;-}^{0,1}(v;\xi), \qquad (2.23)$$

for some subspace $\Gamma_{t;-}^{0,1}(v;\xi)$ of $\Gamma_t^{0,1}(v;\xi)$ isomorphic to the cokernel of the composition:

$$\pi \circ D_{J_0,\nu_0;b'} \colon \Gamma(b') \longrightarrow \Gamma^{0,1}(b';J_0) \longrightarrow \Gamma^{0,1}(b';J_0)/\{\mathrm{id} \times u_{b'}\}^*\tilde{\Gamma}_-^{0,1}(b';X).$$

This cokernel is naturally isomorphic to

$$\Gamma_-^{0,1}(b';J_0) \equiv \mathcal{H}_{b'}^{0,1} \otimes_{J_0} T_{\mathrm{ev}_P(b')}X \subset \Gamma^{0,1}(b';J_0),$$

where $\mathcal{H}_{b'}^{0,1}$ is the one-dimensional complex vector space of $(0,1)$-harmonic forms on the principal component $\Sigma_{b';P}$ of $\Sigma_{b'}$. If $\Sigma_{b';P}$ is a circle of spheres, the elements of $\mathcal{H}_{b'}^{0,1}$ have simple poles at the nodes of $\Sigma_{b';P}$ with the residues adding up to zero at each node. Recall that (Σ_v, g_v) is obtained from Σ_b by replacing the nodes of $\Sigma_{b'}$ with thin necks. The map

$$q_v \colon \Sigma_v \longrightarrow \Sigma_{b'}$$

collapses each neck at its thinnest position to the corresponding node. For each element η of $\Gamma_-^{0,1}(b';J_t)$, we can construct an element $R_{v,\xi}\eta$ of $\Gamma_t^{0,1}(v;\xi)$ by parallel transporting η along the restriction of $u_{v,\xi}$ to $q_v^{-1}(A_{b,h}(\delta_b^2))$ for each $h \in I_1$ and cutting it off with a smooth function that drops from 1 to 0 over the annulus $q_v^{-1}(A_{b,h}(\delta_b^2) - A_{b,h}(\delta_b^2/4))$; see the middle of Subsection 4.2 in [Z4] for details. We denote by $\Gamma_{t;-}^{0,1}(v;\xi)$ the image of $\Gamma_-^{0,1}(b';J_t)$ under R_v.

Remark: In the construction of the map $\tilde{q}_{v_0;2}$ in [Z4, Subsection 4.2], δ_K corresponds to ϵ_b^2 above. In the construction of $R_{v,\xi}\eta$ in [Z4, Subsection 4.2], δ_K corresponds to $\delta_b^2/4$ above.

Let $\langle\!\langle\cdot,\cdot\rangle\!\rangle_t$ denote the Hermitian inner-product on $\Gamma_t^{0,1}(v;\xi)$ induced by the J_t-compatible metric g_t on X. For each $\eta\in\Gamma_-^{0,1}(b';J_t)$, let

$$\|\eta\| = \sum_{h\in I_1}\left\langle\eta_{x_h(b')},\eta_{x_h(b')}\right\rangle_t,$$

where $\langle\cdot,\cdot\rangle_t$ is the hermitian inner-product on $\mathcal{H}_{b'}^{0,1}\otimes_{J_t}T_{\mathrm{ev}_P(b')}X$ defined via the metric g on X and the original metric $g_{b'}$ on $\Sigma_{b'}$. From the construction of R_v in [Z4, Subsection 4.2] and Holder's inequality, it is immediate that

$$\left|\langle\!\langle\eta',R_{v,\xi}\eta\rangle\!\rangle_t\right| \le C\|\eta\|\|\eta'\|_{v,p} \qquad \forall\,\eta\in\Gamma_-^{0,1}(b';J_t),\ \eta'\in\Gamma_t^{0,1}(v;\xi); \tag{2.24}$$

see [Z4, (4.11)]. Another essential property of the above construction is that

$$\left|\langle\!\langle D_{v;\xi}^t\zeta,R_{v,\xi}\eta\rangle\!\rangle_t\right| \le C|v|^{1/2}\|\eta\|\|\zeta\|_{v,p,1} \quad \forall\,t\in[0,\delta],\ v\in\widetilde{\mathcal{F}}_\delta^\emptyset|_W,\ \xi\in\Gamma_{t,\delta}(v),\ \zeta\in\Gamma(v;\xi); \tag{2.25}$$

see part (7) of Lemma 4.4 in [Z4].

Due to the assumption (2.20), the condition (2.21) is equivalent to

$$\left\{\bar{\partial}_{J_{t_r}}+\nu_{t_r}\right\}u_{v_r,\xi_r} + D_{v_r;\xi_r}^{t_r}\zeta_r + N_{v_r;\xi_r}^{t_r}\zeta_r = 0. \tag{2.26}$$

The quadratic term $N_{v;\xi}^t$ satisfies

$$\left\|N_{v;\xi}^t\zeta\right\|_{v,p} \le C\|\zeta\|_{v,p,1}^2 \quad \forall\,t\in[0,\delta],\ v\in\widetilde{\mathcal{F}}_\delta^\emptyset|_W,\ \xi\in\Gamma_{t,\delta}(v),\ \zeta\in\Gamma(v;\xi)\text{ s.t. }\|\zeta\|_{v,p,1}\le\delta. \tag{2.27}$$

We will obtain Proposition 2.7 by estimating the inner-product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{t_r}$ of each term in (2.26) with each element of $\Gamma_{t_r;-}^{0,1}(v_r;\xi_r)$.

First, for every $v\in\widetilde{\mathcal{F}}_\delta^\emptyset|_W$ and $\xi\in\Gamma_{t,\delta}(v)$, let

$$\tilde{\pi}_{v;\xi}^{t;+}: \Gamma_t^{0,1}(v;\xi) \longrightarrow \Gamma_{t;+}^{0,1}(v;\xi) \oplus \Gamma_{t;-}^{0,1}(v;\xi)$$

be the projection map corresponding to the decomposition (2.23). Then,

$$\begin{aligned}
\tilde{\pi}_{v;\xi}^{t;+}\left\{\bar{\partial}_{J_t}+\nu_t\right\}u_{v,\xi} &= \tilde{\pi}_{v;\xi}^{t;+}\left\{\bar{\partial}_{J_t}+\tilde{q}_{v_0;2}^*\nu_{t;v_1,\xi}\right\}\left(u_{v_1,\xi}\circ\tilde{q}_{v_0;2}\right) \\
&= \tilde{\pi}_{v;\xi}^{t;+}\left(\left(\{\bar{\partial}_{J_t}+\nu_{t;v_1,\xi}\}u_{v_1,\xi}\right)\circ\partial\tilde{q}_{v_0;2}\right) + \tilde{\pi}_{v;\xi}^{t;+}\bar{\partial}_{J_t}(u_{v,\xi}\circ\tilde{q}_{v_0;2}) \\
&= \tilde{\pi}_{v;\xi}^{t;+}\bar{\partial}_{J_t}(u_{v,\xi}\circ\tilde{q}_{v_0;2}).
\end{aligned} \tag{2.28}$$

The reason for the second equality is that the map $\tilde{q}_{v_0;2}$ is holomorphic over the support of $\nu_{t;v_1,\xi}$. This last equality follows from the definition of $\Gamma_{t,\delta}(v)$ above. By our assumptions on b' and ξ,

$$u_{v_1,\xi}\big|_{q_{v_1}^{-1}(\Sigma_{b';P})} = \mathrm{const} \qquad \Longrightarrow \qquad du_{v_1,\xi}\circ\bar{\partial}\tilde{q}_{v_0;2}\big|_{\tilde{q}_{v_0;2}^{-1}q_{v_1}^{-1}(\Sigma_{b';P})} = 0.$$

By the construction of the map $\tilde{q}_{v_0;2}$, the restriction of $\tilde{q}_{v_0;2}$ to the complement of $\tilde{q}_{v_0;2}^{-1}q_{v_1}^{-1}(\Sigma_{b';P})$ in Σ_v is holomorphic outside of the annuli

$$\mathcal{A}_{v,h} \equiv \tilde{q}_{v_0;2}^{-1}q_{v_1}^{-1}\big(A_{b',h}(2|v_h|^2/\epsilon_b^2)\big),$$

with $h \in I_1$. The map q_v maps such an annulus isomorphically onto the annulus of radii $\epsilon_b/2$ and ϵ_b around the point $x_{h(b')} \in \Sigma_{b';P}$. The key advantage of using the map $\tilde{q}_{v_0;2}$ instead of $q_{v_0;2}$ in (2.18) is that

$$\big\| d\tilde{q}_{v_0;2} \big\|_{C^0(\mathcal{A}_{v,h})} \leq C|v_h| \qquad \forall\, h \in I_1, \tag{2.29}$$

where C^0-norm of $d\tilde{q}_{v_0;2}$ is computed with respect to the metrics g_{v_1} and g_v on Σ_{v_1} and Σ_v, respectively.

Since $u_{v_1,\xi}|_{\tilde{q}_{v_0;2}(\mathcal{A}_{v,h})}$ is J_t-holomorphic, (2.7) and (2.8) give

$$\big\| du_{v_1,\xi}|_{\tilde{q}_{v_0;2}(\mathcal{A}_{v,h})} \big\|_{v_1,p} \leq C|v_h|^{2/p} \sum_{i \in \chi(\mathcal{T}), h(i)=h} \big|\tilde{\rho}_i(v)\big| \qquad \forall\, h \in I_1; \tag{2.30}$$

see part (2b) of Corollary 3.8 in [Z4]. From (2.28)-(2.30), we conclude that

$$\big\| \tilde{\pi}_{v;\xi}^{t;+} \{\bar{\partial}_{J_t} + \nu_t\} u_{v,\xi} \big\|_{v,p} \leq C \sum_{i \in \chi(\mathcal{T})} |\rho_i(v)| \qquad \forall\, t \in [0,\delta],\ v \in \widetilde{\mathcal{F}}_\delta^\emptyset|_W,\ \xi \in \Gamma_{t,\delta}(v), \tag{2.31}$$

if W and δ are sufficiently small. This estimate is the analogue of the first estimate in part (3) of Lemma 4.4 in [Z4]. Separately, by the construction of the homomorphism $R_{v,\xi}$,

$$\operatorname{supp} R_{v,\xi}\eta \cap \operatorname{supp} \tilde{\eta} = \emptyset \qquad \forall\, R_{v,\xi}\eta \in \Gamma_{t;-}^{0,1}(v;\xi),\ \tilde{\eta} \in \tilde{\Gamma}_{t;-}^{0,1}(v;\xi).$$

Thus, by the J_t-holomorphicity of $u_{v_1,\xi}|_{\tilde{q}_{v_0;2}(\mathcal{A}_{v,h})}$, (2.7), (2.8), (2.28), and integration by parts,

$$\begin{aligned}
\Big| \big\langle\!\big\langle \{\bar{\partial}_{J_t} + \nu_t\} u_{v,\xi}, R_{v,\xi}\eta \big\rangle\!\big\rangle_t &+ 2\pi\mathrm{i} \sum_{i \in \chi(\mathcal{T})} \big\langle \mathcal{D}_i b', \eta_{x_{h(i)}(b')}(\rho_i(v)) \big\rangle \Big| \\
&\leq C\big(\|J_t - J_0\|_{C^1} + |v|^{1/p} + |v|^{(p-2)/p} + \|\xi\|_{v,p,1} \big) \|\eta\| \sum_{i \in \chi(\mathcal{T})} |\rho_i(v)|;
\end{aligned} \tag{2.32}$$

see the proof of part (6) of Lemma 4.4 at the end of Subsection 4.2 of [Z4]. Here $\langle\cdot,\cdot\rangle$ is the Hermitian inner-product on $(T_{\mathrm{ev}_P(b')}X, J_0)$ defined via the metric g on X.

We now finish the proof of Proposition 2.7. By (2.22), (2.26), (2.27), (2.31), and the definition of $\Gamma_+^{0,1}(v;\xi)$,

$$\|\zeta_r\|_{v_r,p,1} \leq C \sum_{i \in \chi(\mathcal{T})} |\rho_i(v_r)|, \tag{2.33}$$

for all r sufficiently large. Combining this estimate with (2.24), (2.25), and (2.27), we obtain that for all $\eta \in \Gamma_-^{0,1}(b'_r; J_0)$,

$$\begin{aligned}
\big| \big\langle\!\big\langle D_{v_r;\xi_r}^{t_r} \zeta_r, R_{v_r,\xi_r}\eta \big\rangle\!\big\rangle_{t_r} \big| &\leq C|v_r|^{1/2} \|\eta\| \sum_{i \in \chi(\mathcal{T})} |\rho_i(v_r)|; \\
\big| \big\langle\!\big\langle N_{v_r;\xi_r}^{t_r} \zeta_r, R_{v_r,\xi_r}\eta \big\rangle\!\big\rangle_{t_r} \big| &\leq C|v_r| \|\eta\| \sum_{i \in \chi(\mathcal{T})} |\rho_i(v_r)|.
\end{aligned} \tag{2.34}$$

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Finally, by (2.19), (2.26), (2.32), and (2.34), for a sequence ϵ_r converging to 0

$$\left| \sum_{i\in\chi(\mathcal{T})} \langle \mathcal{D}_i b'_r, \eta_{x_{h(i)}(b'_r)}(\rho_i(v_r)) \rangle \right| \leq \epsilon_r \|\eta\| \sum_{i\in\chi(\mathcal{T})} |\rho_i(v_r)| \qquad \forall \, \eta \in \mathcal{H}^{0,1}_{b'_r} \otimes_{J_0} T_{\mathrm{ev}_P(b')} X. \qquad (2.35)$$

Since $v_r \in \widetilde{\mathcal{F}}^\emptyset$ for all r, $\rho_i(v_r) \neq 0$ for all $i \in \chi(b)$. Thus, (2.35) implies the conclusion of Proposition 2.7, since $\mathcal{D}_i b'_r \longrightarrow \mathcal{D}_i b$ as $r \longrightarrow \infty$.

3 Proof of Theorem 1.1

3.1 Summary

Suppose (X, ω), A, k, and J are as in the statement of Theorem 1.1 and

$$\psi \in H^{\dim_{1,k}(X,A)}\big(\overline{\mathfrak{M}}_{1,k}(X, A; J); \mathbb{Q}\big)$$

is a geometric cohomology class. By definition,

$$\psi = \prod_{l=1}^{l=k} \mathrm{ev}_l^* \mu_l \qquad \text{for some} \qquad \mu_l \in H^{2n-d_l}(X; \mathbb{Z}), \quad d_l < 2n.$$

For each $l \in [k]$, choose a pseudocycle representative

$$f_l \colon \bar{Y}_l \longrightarrow X$$

for $\mathrm{PD}_X \mu_l$. In particular, $\bar{Y}_l$ is a disjoint union of smooth manifolds. The dimension of one of them, $Y_{l;mn}$, is d_l; the dimensions of all others are at most $d_l - 2$. The map f_l is continuous, and its restriction to each smooth manifold is smooth; see Chapter 7 in [McSa] or Section 1 in [RT]. Let

$$\mathrm{ev} = \prod_{l=1}^{l=k} \mathrm{ev}_l \colon \mathfrak{X}_{1,k}(X, A) \longrightarrow \prod_{l=1}^{l=k} X,$$

$$f = \prod_{l=1}^{l=k} f_l \colon \bar{Y} \equiv \prod_{l=1}^{l=k} \bar{Y}_l \longrightarrow \prod_{l=1}^{l=k} X, \qquad \text{and} \qquad Y_{mn} = \prod_{l=1}^{l=k} Y_{l;mn}.$$

With $(f_l)_{l\in[k]}$ as above, for any $\nu \in \mathfrak{G}^{0,1}_{1,k}(X, A; J)$ and a bubble type $\mathcal{T}$ as in Subsection 2.2, let

$$\mathfrak{X}_{1,k}(X, A; f) = \big\{ (b, z) \in \mathfrak{X}_{1,k}(X, A) \times \bar{Y} : \mathrm{ev}(b) = f(z) \big\},$$
$$\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu; f) = \big(\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu) \times \bar{Y}\big) \cap \mathfrak{X}_{1,k}(X, A; f), \qquad \text{and}$$
$$\mathcal{U}_{\mathcal{T},\nu}(X; J; f) = \big(\mathcal{U}_{\mathcal{T},\nu}(X; J) \times \bar{Y}\big) \cap \mathfrak{X}_{1,k}(X, A; f).$$

If ν is sufficiently small, the space $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu; f)$ is compact. Let

$$\Delta_X^k = \prod_{l=1}^{l=k} \Delta_X \subset \prod_{l=1}^{l=k} X \times X$$

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be the k-fold product of the diagonals. If ν and f_l are chosen generically, then

$$\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu; f) \subset \mathfrak{M}^0_{1,k}(X, A; J, \nu) \times Y_{\mathrm{mn}} \tag{3.1}$$

and the map

$$\mathrm{ev} \times f : \mathfrak{M}^0_{1,k}(X, A; J, \nu) \times Y_{\mathrm{mn}} \longrightarrow (X^2)^k$$

is transverse to Δ^k_X. Thus, $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu; f)$ is a compact zero-dimensional orbifold. By definition,

$$\mathrm{GW}^X_{1,k}(A; \psi) = {}^{\pm}\big|\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu; f)\big|, \tag{3.2}$$

if ν is sufficiently small and ν and f_l are generic.

Similarly to the previous paragraph, if $\nu_{\mathrm{es}} \in \mathfrak{G}^{\mathrm{es}}_{1,k}(X, A; J)$, let

$$\overline{\mathfrak{M}}^0_{1,k}(X, A; J, \nu_{\mathrm{es}}; f) = \big(\overline{\mathfrak{M}}^0_{1,k}(X, A; J, \nu_{\mathrm{es}}) \times \bar{Y}\big) \cap \overline{\mathfrak{M}}_{1,k}(X, A; J, \nu_{\mathrm{es}}; f).$$

For generic $\nu_{\mathrm{es}} \in \mathfrak{G}^{\mathrm{es}}_{1,k}(X, A; J)$ and f_l,

$$\overline{\mathfrak{M}}^0_{1,k}(X, A; J, \nu_{\mathrm{es}}; f) \subset \mathfrak{M}^0_{1,k}(X, A; J, \nu_{\mathrm{es}}) \times Y_{\mathrm{mn}} \tag{3.3}$$

and the map

$$\mathrm{ev} \times f : \mathfrak{M}^0_{1,k}(X, A; J, \nu) \times Y_{\mathrm{mn}} \longrightarrow (X^2)^k$$

is transverse to Δ^k_X, by the second half of Subsection 1.4. Thus, $\overline{\mathfrak{M}}^0_{1,k}(X, A; J, \nu_{\mathrm{es}}; f)$ is a compact zero-dimensional orbifold. By definition,

$$\mathrm{GW}^{0;X}_{1,k}(A; \psi) = {}^{\pm}\big|\overline{\mathfrak{M}}^0_{1,k}(X, A; J, \nu_{\mathrm{es}}; f)\big|, \tag{3.4}$$

if ν_{es} is sufficiently small and $\nu_{\mathrm{es}} \in \mathfrak{G}^{\mathrm{es}}_{1,k}(X, A; J)$ and f_l are generic.

If $\nu_{\mathrm{es}} \in \mathfrak{G}^{\mathrm{es}}_{1,k}(X, A; J)$ and f_l are generic, (3.1) and (3.2) do not generally hold for $\nu = \nu_{\mathrm{es}}$ because the restriction of the bundle section $\bar{\partial} + \nu_{\mathrm{es}}$ is not transverse to the zero set along some strata

$$\mathfrak{X}_{\mathcal{T}}(X, A) \subset \mathfrak{X}_{1,k}(X, A) - \mathfrak{X}^{\{0\}}_{1,k}(X, A).$$

Instead, we will apply (3.1) and (3.2) with ν replaced by $\nu_{\mathrm{es}} + \nu$ for a generic $\nu \in \mathfrak{G}^{0,1}_{1,k}(X, A; J)$ which is sufficiently small relatively to ν_{es}. In such a case, the compact zero-dimensional orbifold

$$\overline{\mathfrak{M}}_{1,k}\big(X, A; J, \nu + \nu_{\mathrm{es}}; f\big) \tag{3.5}$$

will lie in a small neighborhood of

$$\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu_{\mathrm{es}}; f) \subset \mathfrak{X}_{1,k}(X, A; f).$$

We will express the cardinality of this orbifold in terms of data intrinsic to $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu_{\mathrm{es}}; f)$. From the transversality of the relevant maps, it is straightforward to see that there is a unique element of (3.5) close to each of the elements of $\overline{\mathfrak{M}}^0_{1,k}(X, A; J, \nu_{\mathrm{es}}, f)$. Thus,

$$\mathrm{GW}^X_{1,k}(A; \psi) - \mathrm{GW}^{0;X}_{1,k}(A; \psi)$$

is the number of elements of (3.5) that lie close to the closed subset

$$\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu_{\mathrm{es}}; f) - \overline{\mathfrak{M}}^0_{1,k}(X, A; J, \nu_{\mathrm{es}}) \times \bar{Y} \tag{3.6}$$

of $\overline{\mathfrak{M}}_{1,k}(X, A; J, \nu_{\mathrm{es}}) \times \bar{Y}$. The contribution of (3.6) to (3.4) can in fact be split into contributions of the subspaces $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X; J; f)$ of (3.6). By studying local obstructions similarly to [Z1], each of these contributions will be shown to be equal to the number of zeros of an affine bundle map between two finite-rank vector bundles over $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X; J; f)$; see Proposition 3.1 below. Such numbers can be determined using the procedure described in [Z1, Subsections 3.2,3.3].

If $\dim_{\mathbb{R}} X = 4$, only two strata of (3.6) are nonempty for a good choice of ν_{es}. They are isomorphic to

$$\mathcal{M}_{1,1} \times \mathfrak{M}^0_{0,\{0\}\cup[k]}(X, A; J, \nu_B; f) \qquad \text{and} \qquad \big(\overline{\mathcal{M}}_{1,1} - \mathcal{M}_{1,1}\big) \times \mathfrak{M}^0_{0,\{0\}\cup[k]}(X, A; J, \nu_B; f)$$

for some $\nu_B \in \mathfrak{G}^{0,1}_{0,\{0\}\cup[k]}(X, A; J)$, where

$$\mathcal{M}_{1,1} \subset \overline{\mathcal{M}}_{1,1}$$

is the complement of the equivalence class of the singular elliptic curve. The dimension of $\mathfrak{M}^0_{0,\{0\}\cup[k]}(X, A; J, \nu_B; f)$ is zero, even though no constraint has been imposed on the zeroth marked point. In other words, $\mathfrak{M}^0_{0,\{0\}\cup[k]}(X, A; J, \nu_B; f)$ is "virtually empty". It is thus not too surprising that neither of these strata contributes to $\mathrm{GW}^X_{1,k}(A; \psi)$.

Remark: If J is a genus-one A-regular almost complex structure in the sense of [Z4, Definition 1.3], we can take $\nu_{\mathrm{es}} = 0$. If $\dim_{\mathbb{R}} X = 4$, we then find that the space (3.6) is empty, since

$$\mathfrak{M}^0_{0,\{0\}\cup[k]}(X, A; J; f) \equiv \mathfrak{M}^0_{0,\{0\}\cup[k]}(X, A; J, 0; f)$$

cannot be zero-dimensional. Thus, if (X, ω) admits a genus-one A-regular almost complex structure, the first case of Theorem 1.1 is immediate from dimension-counting, once it is known that $\mathrm{GW}^{0;X}_{1,k}(A; \psi)$ is well defined.

If $\dim_{\mathbb{R}} X = 6$, for a good choice of ν_{es} only a few strata of (3.6) are nonempty. All, but two of them, are either virtually empty or $\bar{\partial}_J$-hollow, in the sense of [Z1, Subsection 3.1]. In either of these cases, $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X; J; f)$ does not contribute to (3.2). The two remaining strata are isomorphic to

$$\mathcal{M}_{1,1} \times \mathfrak{M}^0_{0,\{0\}\cup[k]}(X, A; J, \nu_B; f) \qquad \text{and} \qquad \mathcal{M}_{1,2} \times \mathfrak{M}^0_{0,k}(X, A; J, \nu_B; f)$$

for some $\nu_B \in \mathfrak{G}^{0,1}_{0,\{0\}\cup[k]}(X, A; J)$ or $\nu_B \in \mathfrak{G}^{0,1}_{0,k}(X, A; J)$, respectively. In the second case, one of the elements of $[k]$ corresponds to the attaching node of the only bubble component of each element in $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X; J; f)$; we will denote it by 0. Let

$$L_1 \longrightarrow \overline{\mathcal{M}}_{1,1}, \overline{\mathcal{M}}_{1,2} \qquad \text{and} \qquad L_0 \longrightarrow \overline{\mathfrak{M}}_{0,\{0\}\sqcup k}(X, A; J, \nu_B; f), \mathfrak{M}^0_{0,k}(X, A; J, \nu_B; f)$$

be the universal tangent line bundles for the marked points labeled by 1 and 0, respectively. Let

$$\mathcal{D}_0 : L_0 \longrightarrow \mathrm{ev}^*_0 TX$$

be the bundle homomorphism over $\overline{\mathfrak{M}}_{0,\{0\}\sqcup k}(X,A;J,\nu_B;f)$ or $\mathfrak{M}_{0,k}^0(X,A;J,\nu_B;f)$ given by

$$\mathcal{D}_0\big([b,v]\big) = du_b\big|_{y_0(b)}v.$$

We denote by

$$\pi_P, \pi_B \colon \overline{\mathcal{M}}_{1,1}\times\overline{\mathfrak{M}}_{0,\{0\}\cup[k]}(X,A;J,\nu_B;f) \longrightarrow \overline{\mathcal{M}}_{1,1},\ \overline{\mathfrak{M}}_{0,\{0\}\cup[k]}(X,A;J,\nu_B;f) \qquad \text{and}$$
$$\pi_P, \pi_B \colon \overline{\mathcal{M}}_{1,2}\times\mathfrak{M}_{0,k}^0(X,A;J,\nu_B;f) \longrightarrow \overline{\mathcal{M}}_{1,2},\ \mathfrak{M}_{0,k}^0(X,A;J,\nu_B;f)$$

the two projection maps. In both cases, the linear part of the affine map determining the contribution of the stratum $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f)$ to (3.2) is

$$\mathcal{D}_{\mathcal{T}} \colon \pi_P^*L_1\otimes\pi_B^*L_0 \longrightarrow \pi_P^*\mathbb{E}^*\otimes\pi_B^*\mathrm{ev}_0^*TX,$$
$$\big\{\mathcal{D}_{\mathcal{T}}[b_P,b_B,v_P\otimes v_B]\big\}\big([b_B,\psi]\big) = \psi_{x_1(b_P)}(v_P) \cdot_J \mathcal{D}_0\big([b_B,v_B]\big),$$
$$\text{if} \qquad [b_P,b_B,v_P\otimes v_B]\in\pi_P^*L_1\otimes\pi_B^*L_0,\ [b_B,\psi]\in\pi_P^*\mathbb{E}.$$

The constant term $\bar{\nu}$ of each of the affine maps is generic. In the second case, $\mathfrak{M}_{0,k}^0(X,A;J,\nu_B;f)$ is a finite collection of points. It is then straightforward to see that for a generic ν, the affine bundle map $\mathcal{D}_{\mathcal{T}}+\bar{\nu}$ does not vanish. Thus, the corresponding stratum $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f)$ does not contribute to (3.2). We will show in Subsection 3.4 that the number $N(\mathcal{D}_{\mathcal{T}})$ of zeros of $\mathcal{D}_{\mathcal{T}}+\bar{\nu}$ in the first case is

$$N(\mathcal{D}_{\mathcal{T}}) = \frac{2-\langle c_1(TX),A\rangle}{24}\,\mathrm{GW}_{0,k}^X(A;\psi),$$

proving the second case of Theorem 1.1.

Remark: If J is a genus-one A-regular almost complex structure and $\dim_{\mathbb{R}}X=6$, the space (3.6) is the union of the spaces

$$\overline{\mathfrak{M}}_{1,k}\big(\kappa,1;J|_{\kappa};\underline{f}|_{\kappa}\big) \tag{3.7}$$

taken over all degree-A genus-zero curves κ in X that intersect $f_l(Y_{l;\mathrm{mn}})$ for every $l\in[k]$. Based on [Pa], one would expect that each of the spaces (3.7) contributes $\frac{(2-\langle c_1(TX),A\rangle}{24}$ to $\mathrm{GW}_{1,k}^X(A;\psi)$. The total number of the curves κ is $\mathrm{GW}_{0,k}^X(A;\psi)$. In particular, the second case of Theorem 1.1, just like the first, is consistent with geometric expectations.

3.2 Analytic Setup

Let $\mathcal{T}=([k],I,\aleph;j,\underline{A})$ be a bubble type such that

$$A_i = 0 \ \ \forall i\in I_0 \qquad \text{and} \qquad \sum_{i\in I}A_i = A. \tag{3.8}$$

For each $i\in I-I_0$, let

$$H_i\mathcal{T} = \big\{h\in\hat{I}\colon \iota_h=i\big\} \qquad \text{and} \qquad M_i\mathcal{T} = \big\{l\in[k]\colon j_l=i\big\}.$$

We denote by

$$\pi_{\mathcal{T};i}\colon \mathfrak{X}_{\mathcal{T}}(X) \longrightarrow \mathfrak{X}_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A_i)$$

the map sending each element $[b]$ of $\mathfrak{X}_{\mathcal{T}}(X)$ to its restriction to $\Sigma_{b,i}$:

$$\big[(S,[k],I,\aleph;x,(j,y),u)\big] \longrightarrow \big[(H_i\mathcal{T}\sqcup M_i\mathcal{T},\{i\};,(\iota,x)|_{H_i\mathcal{T}}\sqcup(j,y)|_{M_i\mathcal{T}},u_i)\big].$$

Let $\mathfrak{G}^{\mathrm{gd}}_{1,k}(X,A;J)$ be the subspace of elements ν in $\mathfrak{G}^{\mathrm{es}}_{1,k}(X,A;J)$ such that for every bubble type $\mathcal{T}$ as above,

$$\nu|_{\mathfrak{X}_{\mathcal{T}}(X)} = \sum_{i\in I-I_0} \pi^*_{\mathcal{T};i}\nu_{\mathcal{T};i} \qquad \text{for some} \qquad \nu_{\mathcal{T};i} \in \mathfrak{G}^{0,1}_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A;J)$$

and for every $[b]\in\mathfrak{M}^0_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A_i;J,\nu_{\mathcal{T};i})$ the linearization

$$D_{J,\nu_{\mathcal{T};i};b}\colon \big\{\xi\in\Gamma(\Sigma_b;u_b^*TX)\colon\xi(y_0(b))=0\big\} \longrightarrow \Gamma\big(\Sigma_b;\Lambda^{0,1}_{J,j}T^*\Sigma_b\otimes u_b^*TX\big) \tag{3.9}$$

of the section $\bar\partial_J+\nu_{\mathcal{T};i}$ at b is surjective.

For a generic element $\nu_{\mathcal{T};i}\in\mathfrak{G}^{\mathrm{es}}_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A_i;J)$, the operator (3.9) is surjective for every

$$[b]\in\overline{\mathfrak{M}}_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A_i;J,\nu_{\mathcal{T};i}).$$

This implies that the closure of $\mathfrak{G}^{\mathrm{gd}}_{1,k}(X,A;J)$ in $\mathfrak{G}^{0,1}_{1,k}(X,A;J)$ contains the zero section, since we can construct an element ν of $\mathfrak{G}^{\mathrm{gd}}_{1,k}(X,A;J)$ inductively starting from the highest-codimension strata of $\mathfrak{X}_{1,k}(X,A)$. If $\mathcal{T}$ is a bubble type as above and ν has been defined on

$$\overline{\mathfrak{X}}_{\mathcal{T}}(X)-\mathfrak{X}_{\mathcal{T}}(X)\subset\mathfrak{X}_{1,k}(X,A)$$

subject to the above restriction and regularity conditions, then ν induces a multisection $\nu_{\mathcal{T};i}$ of

$$\Gamma^{0,1}_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A_i;J) \longrightarrow \mathfrak{X}_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A_i)-\mathfrak{X}^0_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A_i).$$

It extends continuously to an effectively supported multisection over all of $\mathfrak{X}_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A_i)$. By perturbing this extension outside of

$$\mathfrak{X}_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A_i)-\mathfrak{X}^0_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A_i),$$

we obtain an element $\nu_{\mathcal{T};i}$ of $\mathfrak{G}^{\mathrm{es}}_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A_i;J)$ such that the operator (3.9) is surjective for every

$$[b]\in\overline{\mathfrak{M}}_{0,\{0\}\sqcup H_i\mathcal{T}\sqcup M_i\mathcal{T}}(X,A_i;J,\nu_{\mathcal{T};i}).$$

We fix small generic elements

$$\nu_{\mathrm{es}}\in\mathfrak{G}^{\mathrm{gd}}_{1,k}(X,A;J) \qquad \text{and} \qquad \nu\in\mathfrak{G}^{0,1}_{1,k}(X,A;J)$$

such that for all $t\in\mathbb{R}^+$ sufficiently small the section

$$\big\{\bar\partial_J+\nu_{\mathrm{es}}+t\nu\big\}\big|_{\mathfrak{X}_{\mathcal{T}}(X)}$$

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is transverse to the zero set in $\Gamma_{1,k}^{0,1}(X,A;J)|_{\mathfrak{X}_{\mathcal{T}}(X)}$ for every stratum $\mathfrak{X}_{\mathcal{T}}(X)$ of $\mathfrak{X}_{1,k}(X,A)$. Let $\{Y_\lambda\}_{\lambda\in\mathcal{A}}$ be the strata of $\bar{Y}$ induced by the partitions of each $\bar{Y}_l$ into smooth manifolds. By our assumptions,

$$\dim_\mathbb{R} Y_{\mathrm{mn}} = 2nk - \dim_{1,k}(X,A) \qquad \text{and} \qquad \dim_\mathbb{R} Y_\lambda \le \dim_\mathbb{R} Y_{\mathrm{mn}} - 2 \quad \forall\lambda\in\mathcal{A}-\{0\}.$$

For each bubble type $\mathcal{T}$ as above and $\lambda\in\mathcal{A}$, let

$$\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda) = \big(\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J)\times Y_\lambda\big) \cap \overline{\mathfrak{M}}_{1,k}(X,A;J,\nu_{\mathrm{es}};f).$$

We will call a bubble type $\mathcal{T}$ as above simple if

$$\aleph = \emptyset \qquad \text{and} \qquad \hat{I} = \chi(\mathcal{T}).$$

In other words, $\mathcal{T}$ is simple if and only if for every element $[b]\in\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X,A;J)$ the domain Σ_b consists of a smooth principal component $\Sigma_{b;P}$, on which the map u_b is necessarily constant, and $|\hat{I}|$ bubble components, all of which are attached directly to $\Sigma_{b;P}$ and on which the map u_b is not constant.

Suppose $\overline{\mathcal{M}}$ is a compact topological space which is a disjoint union of smooth orbifolds, one of which, $\mathcal{M}$, is a dense open subset of $\overline{\mathcal{M}}$, and the dimensions of all others do not exceed $\dim\mathcal{M}-2$. Let

$$E, \mathcal{O} \longrightarrow \overline{\mathcal{M}}$$

be vector bundles such that the restrictions of E and $\mathcal{O}$ to every stratum of $\overline{\mathcal{M}}$ is smooth and

$$\mathrm{rk}\,\mathcal{O} - \mathrm{rk}\,E = \frac{1}{2}\dim_\mathbb{R}\mathcal{M}.$$

If

$$\alpha \in \Gamma\big(\overline{\mathcal{M}};\mathrm{Hom}(E,\mathcal{O})\big)$$

is a regular section in the sense of [Z1, Definition 3.9], then the cardinality of the zero set of the affine bundle map

$$\psi_{\alpha,\bar{\nu}}\equiv\alpha+\bar{\nu}\colon E \longrightarrow \mathcal{O}$$

is finite and independent of a generic choice of $\bar{\nu}\in\Gamma(\overline{\mathcal{M}};\mathcal{O})$, by [Z1, Lemma 3.14]. We denote it by $N(\alpha)$. A key step in our proof of Theorem 1.1 is the following proposition.

Proposition 3.1 *Suppose* (X,ω,J), A, k, ψ, f_l, ν_{es}, ν, *and* $\{Y_\lambda\}_{\lambda\in\mathcal{A}}$ *are as above. If* $\mathcal{T}$ *is a bubble as above and* $\lambda\in\mathcal{A}$, *there exist*

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)}(\bar{\partial}_J) \in \mathbb{Q}, \qquad \epsilon_\nu \in \mathbb{R}^+,$$

and a compact subset K_ν *of* $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)$ *with the following property. For every compact subset* K *of* $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)$ *and open subset* U *of* $\mathfrak{X}_{1,k}(X,A;f)$, *there exist an open neighborhood* $U_\nu(K)$ *of* K *in* $\mathfrak{X}_{1,k}(X,A;f)$ *and* $\epsilon_\nu(U)\in(0,\epsilon_\nu)$, *respectively, such that*

$$\pm\big|\overline{\mathfrak{M}}_{1,k}\big(X,A;J,\nu_{\mathrm{es}}+t\nu;f\big)\cap U\big| = \mathcal{C}_{\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)}(\bar{\partial}_J) \quad \text{if} \quad t\in(0,\epsilon_\nu(U)),\ K_\nu\subset K\subset U\subset U_\nu(K).$$

Furthermore, if $\mathcal{T}$ *is simple and* $\lambda=\mathrm{mn}$,

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)}(\bar{\partial}_J) = N(\mathcal{D}_\mathcal{T})$$

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for some regular vector bundle homomorphism $\mathcal{D}_T$ over $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f)$. Otherwise,

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)}(\bar{\partial}_J) = 0.$$

This proposition is proved in the next subsection. In the previous subsection we described the homomorphism $\mathcal{D}_{\mathcal{T}}$ for a simple bubble type $\mathcal{T}$ such that $|\chi(\mathcal{T})| = 1$. Below we describe this homomorphism for an arbitrary bubble $\mathcal{T}$ satisfying (3.8).

For each $i \in \hat{I}$, let

$$\tilde{H}_i\mathcal{T} = \big\{h \in \hat{I} \colon \iota_h \geq i\big\}, \qquad \tilde{M}_i\mathcal{T} = \big\{l \in [k] \colon j_l \geq i\big\}, \quad \text{and} \quad \tilde{A}_i = \sum_{h \geq i} A_h.$$

We denote by

$$\tilde{\pi}_{\mathcal{T};i} \colon \mathfrak{X}_{\mathcal{T}}(X) \longrightarrow \mathfrak{X}_{0,\{0\}\sqcup\tilde{M}_i\mathcal{T}}(X,\tilde{A}_i)$$

the map sending each element $[b]$ of $\mathfrak{X}_{\mathcal{T}}(X)$ to its restriction to the tree of bubble components beginning with $\Sigma_{b,i}$:

$$\big[(S,[k],I,\aleph;x,(j,y),u)\big] \longrightarrow \big[(\tilde{M}_i\mathcal{T},\{i\}\cup\tilde{H}_i\mathcal{T};(\iota,x)|_{\tilde{H}_i\mathcal{T}},(j,y)|_{\tilde{M}_i\mathcal{T}},u|_{\{i\}\cup\tilde{H}_i\mathcal{T}})\big].$$

By our assumptions on ν_{es},

$$\tilde{\pi}_{\mathcal{T};i} \colon \mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J) \longrightarrow \overline{\mathfrak{M}}_{0,\{0\}\sqcup\tilde{M}_i\mathcal{T}}\big(X,\tilde{A}_i;J,\tilde{\nu}_{B;i}\big)$$

for some $\tilde{\nu}_{B;i} \in \mathfrak{G}^{\mathrm{es}}_{0,\{0\}\sqcup\tilde{M}_i\mathcal{T}}(X,\tilde{A}_i;J)$. Let

$$\mathfrak{F}\mathcal{T} = \bigg(\bigoplus_{i\in\chi(\mathcal{T})} \pi_P^* L_{h(i)} \otimes \tilde{\pi}_{\mathcal{T};i}^* L_0\bigg)\Big/\mathrm{Aut}(\mathcal{T}) \longrightarrow \mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J), \mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f).$$

If M_0 is a finite set and $h \in M_0$, let

$$s_h \in \Gamma\big(\overline{\mathcal{M}}_{1,M_0};\mathrm{Hom}(L_h,\mathbb{E}^*)\big)$$

be the the section given by

$$\big\{s_h(b;v)\big\}(\psi) = \psi_{x_h(b)}v \in \mathbb{C} \qquad \text{if} \quad v \in L_h|_b, \ \psi \in \mathbb{E}_b,$$

where $x_h(b) \in \Sigma_b$ is the hth marked point. We define the homomorphism

$$\mathcal{D}_{\mathcal{T}} \colon \mathfrak{F}\mathcal{T} \longrightarrow \big(\pi_P^*\mathbb{E}^* \otimes \mathrm{ev}_P^* TX\big)\big/\mathrm{Aut}(\mathcal{T})$$

over $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J)$ or $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f)$ by

$$\mathcal{D}_{\mathcal{T}} = \sum_{i\in\chi(\mathcal{T})} \pi_P^* s_{h(i)} \otimes \tilde{\pi}_{\mathcal{T};i}^* \mathcal{D}_0.$$

3.3 Proof of Proposition 3.1

We continue with the notation of Subsections 2.4 and 3.2. By our assumptions on ν_{es}, the operators $D_{J,\nu_{\mathrm{es}};b,i}$ are surjective for all $[b] \in \mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J)$ and $i \in \hat{I}$. Thus, we can take

$$\tilde{\Gamma}_{-}^{0,1}(b;i) = \{0\} \qquad \forall\, [b] \in \mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J),\ i \in I.$$

The corresponding space $\tilde{\mathcal{U}}$ of Subsection 2.4 is a smooth manifold of (J,ν_{es})-holomorphic maps. In Subsections 2.5 of [Z2] and 4.2 of [Z4], we describe a space $\mathcal{U}_{\mathcal{T}}^{(0)}(X;J)$ of balanced J-holomorphic maps, not of equivalence classes of such maps. If ν_{es} is sufficiently small, the same definitions can be used to describe a submanifold $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}^{(0)}(X;J)$ of $\tilde{\mathcal{U}}$. In particular,

$$\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J) = \mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}^{(0)}(X;J) / \mathrm{Aut}(\mathcal{T}) \propto (S^1)^{\hat{I}}$$

for a natural action of $\mathrm{Aut}(\mathcal{T})$ on $(S^1)^{\hat{I}}$ and of $\mathrm{Aut}(\mathcal{T}) \propto (S^1)^{\hat{I}}$ on $\tilde{\mathcal{U}}$. Let

$$\tilde{\mathcal{F}}\mathcal{T} = \tilde{\mathcal{F}}\big|_{\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}^{(0)}(X;J)},$$

where $\tilde{\mathcal{F}} \longrightarrow \tilde{\mathcal{U}}$ is the vector bundle defined in Subsection 2.5. The above group action on $\tilde{\mathcal{U}}$ lifts to an action on $\tilde{\mathcal{F}}$ so that

$$\mathcal{F}\mathcal{T} \equiv \tilde{\mathcal{F}}\mathcal{T} / \mathrm{Aut}(\mathcal{T}) \propto (S^1)^{\hat{I}}$$

is the bundle of gluing parameters for $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J)$.

We will apply the construction of Subsection 2.5, with some refinements, to the entire space $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}^{(0)}(X;J)$, instead of a small open subset of $\tilde{\mathcal{U}}$. We will view $\mathbb{R}$-valued functions on $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J)$ as functions on $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}^{(0)}(X;J)$ via the quotient projection map

$$\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}^{(0)}(X;J) \longrightarrow \mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J).$$

Fix small

$$\delta, \epsilon \in C^\infty(\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J); \mathbb{R}^+)$$

such that the basic gluing map

$$q_v \colon \Sigma_v \longrightarrow \Sigma_b$$

of [Z2, Subsection 2.2] and the modified gluing map

$$\tilde{q}_{v_0;2} \colon \Sigma_v \longrightarrow \Sigma_{v_1}$$

of [Z4, Subsection 4.2] with the collapsing radius $\epsilon(b)$ are defined for all $v \equiv (b,v) \in \tilde{\mathcal{F}}\mathcal{T}_\delta$. For all

$$\xi \in \Gamma(v_1) \qquad \text{s.t.} \qquad \|\xi\|_{v_1,p,1} \leq \delta(b),$$

let $u_{v_1,\xi},\ u_{v,\xi},\ \Gamma_+(v;\xi)$,

$$D_{J,\nu_{\mathrm{es}};v;\xi} \colon \Gamma(v;\xi) \longrightarrow \Gamma^{0,1}(v;\xi), \qquad R_{v,\xi} \colon \Gamma_{-}^{0,1}(b;J) \longrightarrow \Gamma^{0,1}(v;\xi),$$

and $\nu_{\mathrm{es};v_1,\xi}$ be as in Subsection 2.5, with J_t, ν_t, and ∇^t replaced by J, ν_{es}, and a J-compatible connection ∇, respectively. The estimates (2.24), (2.25), (2.27), and (2.29)-(2.32) continue to hold if $C \in \mathbb{R}^+$, $D_{v;\xi}^t$, J_t, and ν_t are replaced by

$$C \in C^\infty(\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J);\mathbb{R}^+),$$

$D_{J,\nu_{\mathrm{es}};v;\xi}$, J, and ν_{es}, respectively. In (2.31) and (2.32), $\tilde{\pi}_{v;\xi}^{t;+}=\mathrm{id}$ and $J_0=J$.

With notation as in Subsection 2.5, for each $v \in \tilde{\mathcal{F}}\mathcal{T}_\delta^\emptyset$ let

$$\Gamma_{B;-}(v_1) = \left\{ \xi \circ q_{v_1} : \xi \in \ker D_{J,\nu_{\mathrm{es}};b}, \ \xi|_{\Sigma_{b;P}}=0 \right\}.$$

We denote by $\Gamma_{B;+}(v_1)$ the L^2-orthogonal complement of $\Gamma_{B;-}(v_1)$ in $\Gamma_B(v_1)$. Let

$$D_{J,\nu_{\mathrm{es}};v_1}^B : \Gamma_B(v_1) \longrightarrow \Gamma_B^{0,1}(v_1)$$

be the linearization of section $\bar{\partial}_J+\nu_{\mathrm{es}}$ at $(\Sigma_{v_1};u_{v_1})$ defined via the connection ∇. Similarly to (2.22),

$$C(b)^{-1}\|\xi\|_{v_1,p,1} \leq \left\|D_{J,\nu_{\mathrm{es}};v_1}^B\xi\right\|_{v_1,p} \leq C(b)\|\xi\|_{v_1,p,1} \quad \forall \ v=(b,v)\in\tilde{\mathcal{F}}\mathcal{T}_\delta^\emptyset, \quad \xi\in\Gamma_{B;+}(v_1), \qquad (3.10)$$

for some $C \in C^\infty(\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J);\mathbb{R}^+)$, provided that $\delta \in C^\infty(\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J);\mathbb{R}^+)$ is sufficiently small. In particular, the operator

$$D_{J,\nu_{\mathrm{es}};v_1}^B : \Gamma_{B;+}(v_1) \longrightarrow \Gamma_B^{0,1}(v_1)$$

is an isomorphism. Its norm and the norm of its inverse are dependent only on $[b] \in \mathcal{U}_{\mathcal{T};\nu_{\mathrm{es}}}(X;J)$. Thus, by the Contraction Principle, for each $v\in\tilde{\mathcal{F}}\mathcal{T}_\delta^\emptyset$, the equation

$$\left\{\bar{\partial}_J+\nu_{\mathrm{es};v_1,\xi}\right\}u_{v_1,\xi} = 0, \qquad \xi \in \Gamma_{B;+}(v_1),$$

has a unique small solution $\xi_{\nu_{\mathrm{es}}}(v_1)$.

Remark: Since ν_{es} is a multisection, the uniqueness statement above, as well as similar statements below, should be interpreted in terms of local branches of ν_{es} as defined in [FuOn, Section 3].

Lemma 3.2 *If $\mathcal{T}$ is a bubble type as in (3.8) and $\nu_{\mathrm{es}}\in\mathfrak{S}_{1,k}^{\mathrm{gd}}(X,A;J)$ is a sufficiently small generic perturbation, there exist*

$$\delta \in C^\infty\big(\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J);\mathbb{R}^+\big)$$

and an open neighborhood $U_{\mathcal{T}}$ of $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J)$ in $\mathfrak{X}_{1,k}(X,A)$ such that the map

$$\left\{(v,\zeta) : v=(b,v)\in\tilde{\mathcal{F}}\mathcal{T}_\delta^\emptyset; \zeta\in\Gamma_+(v;\xi_{\nu_{\mathrm{es}}}(v_1)), \ \|\zeta\|_{v,p,1}<\delta(b)\right\}/\mathrm{Aut}(\mathcal{T})\propto(S^1)^{|\hat{I}|}$$

$$\longrightarrow \mathfrak{X}_{1,k}^0(X,A)\cap U_{\mathcal{T}},$$

$$\big[(v,\zeta)\big] \longrightarrow \big[(\Sigma_v, \exp_{u_{v,\xi_{\nu_{\mathrm{es}}}(v_1)}}\zeta)\big],$$

is a diffeomorphism.

It is immediate from the construction that the map

$$(v,\zeta) \longrightarrow \big[(\Sigma_v, \exp_{u_{v,\xi_{\nu_{\mathrm{es}}}(v_1)}}\zeta)\big]$$

is $\mathrm{Aut}(\mathcal{T})\propto(S^1)^{|\hat{I}|}$-invariant and smooth. The injectivity and surjectivity of the induced map on the quotient are proved by arguments similar to Subsections 4.2 and 4.3-4.5 in [Z2], respectively; see also the paragraph following Lemma 4.4 in [Z4].

For each $v=(b,v)\in\tilde{\mathcal{F}}\mathcal{T}_\delta^\emptyset$, we define the homomorphism

$$\pi_{v;-}^{0,1}:\Gamma^{0,1}\big(v;\xi_{\nu_{\mathrm{es}}}(v_1)\big)\longrightarrow\Gamma_-^{0,1}(b;J)\approx\mathbb{E}_{\pi_P(b)}^*\otimes T_{\mathrm{ev}_P(b)}X$$

as follows. If $\{\eta_r\}_{r\in[n]}$ is an orthonormal basis for $\Gamma_-^{0,1}(b;J)$, we put

$$\pi_{v;-}^{0,1}\eta'=\sum_{r=1}^{r=n}\big\langle\!\big\langle\eta',R_{v,\xi_{\nu_{\mathrm{es}}}(v_1)}\eta_r\big\rangle\!\big\rangle\eta_r\qquad\forall\,\eta'\in\Gamma^{0,1}\big(v;\xi_{\nu_{\mathrm{es}}}(v_1)\big).$$

This map is well defined.

For each (v,ζ) as above, let

$$\Pi_\zeta:L^p(\Sigma_v;\Lambda_{J,j}^{0,1}T^*\Sigma_v\otimes u_{v,\xi_{\nu_{\mathrm{es}}}(v_1)}^*TX)\longrightarrow L^p(\Sigma_v;\Lambda_{J,j}^{0,1}T^*\Sigma_v\otimes\{\exp_{u_{v,\xi_{\nu_{\mathrm{es}}}(v_1)}}\zeta\}^*TX)$$

be the isomorphism induced by the ∇-parallel transport along the ∇-geodesics

$$\tau\longrightarrow\exp_{u_{v,\xi_{\nu_{\mathrm{es}}}(v_1)}}\tau\zeta,\qquad\tau\in[0,1].$$

Similarly to (2.26),

$$\begin{aligned}\Phi_t(v,\zeta)&\equiv\Pi_\zeta^{-1}\{\bar{\partial}_J+\nu_{\mathrm{es}}+t\nu\}\exp_{u_{v,\xi_{\nu_{\mathrm{es}}}(v_1)}}\zeta\\&=\{\bar{\partial}_J+\nu_{\mathrm{es}}\}u_v+D_{J,\nu_{\mathrm{es}};v;\xi_{\nu_{\mathrm{es}}}(v_1)}\zeta+t\,\nu\big(\Sigma_v,u_{v,\xi_{\nu_{\mathrm{es}}}(v_1)}\big)+N_{v;t}\xi,\end{aligned}\tag{3.11}$$

where the quadratic term $N_{v;t}$ satisfies

$$N_{v;t}0=0,\qquad\big\|N_{v;t}\zeta-N_{v;t}\zeta'\big\|_{v,p}\leq C(b)\big(t+\|\zeta\|_{v,p,1}+\|\zeta'\|_{v,p,1}\big)\big\|\zeta-\zeta'\big\|_{v,p,1}\tag{3.12}$$
$$\forall\quad t\in[0,\delta(b)],\;v=(b,v)\in\tilde{\mathcal{F}}\mathcal{T}_\delta^\emptyset,\;\zeta,\zeta'\in\Gamma\big(v;\xi_{\nu_{\mathrm{es}}}(v_1)\big)\;\text{s.t.}\;\|\zeta\|_{v,p,1},\|\zeta'\|_{v,p,1}\leq\delta(b).$$

Thus, by Lemma 3.2, the analogues of (2.22) and (2.31) mentioned above, for every precompact open subset K of $\mathcal{U}_{\mathcal{T},\nu}(X;A)$ there exist $\delta_K,C_K\in\mathbb{R}^+$ and a neighborhood of U_K of K in $\mathfrak{X}_{1,k}(X,A)$ such that for all $t\in[0,\delta_K]$

$$\mathfrak{M}_{1,k}^0(X,A;J,\nu_{\mathrm{es}}+t\nu)\cap U_K\approx\big\{[(v,\zeta)]\in\Omega_K(t):\Phi_t(v,\zeta)=0\big\},\tag{3.13}$$
$$\text{where}\quad\Omega_K(t)=\big\{[(v,\zeta)]:v=(b,v)\in\tilde{\mathcal{F}}\mathcal{T}_{\delta_K}^\emptyset|_K;\;\zeta\in\Gamma_+(v;\xi_{\nu_{\mathrm{es}}}(v_1)),\,\|\zeta\|_{v,p,1}\leq C_K\big(|\rho(v)|+t\big)\big\}.$$

For each $b\in\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}^{(0)}(X;J)$, let

$$\bar{\nu}_b\in\Gamma_-^{0,1}(b;J)$$

be the L^2-projection of $\nu(b)$. We note that the map

$$\mathfrak{G}_{1,k}^{0,1}(X,A;J)\longrightarrow\Gamma\big(\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J);\pi_P^*\mathbb{E}^*\otimes\mathrm{ev}_P^*TX\big),\qquad\nu\longrightarrow\bar{\nu},$$

is surjective for every bubble type $\mathcal{T}$. By (3.11), (3.12), and the analogues of (2.24), (2.25), and (2.32),

$$\big\| \pi^{0,1}_{\upsilon;-}\Phi_t(\upsilon,\zeta) - (\mathcal{D}_{\mathcal{T}}\rho(\upsilon)+t\bar\nu_b)\big\| \leq \varepsilon(\upsilon)\big(|\rho(\upsilon)|+t\big) \tag{3.14}$$
$$\forall \quad t\in[0,\delta_K],\ t\in[0,\delta_K],\ \upsilon=(b,v)\in\tilde{\mathcal{F}}\mathcal{T}^{\emptyset}_{\delta_K}|_K,\ [(\upsilon,\zeta)]\in\Omega_K(t),$$

for some function

$$\varepsilon\colon \tilde{\mathcal{F}}\mathcal{T}^{\emptyset}\longrightarrow\mathbb{R}^+ \qquad \text{s.t.} \qquad \lim_{|\upsilon|\longrightarrow 0}\varepsilon(\upsilon)=0.$$

We denote by

$$\pi_{\mathbb{P}\mathfrak{F}\mathcal{T}}\colon \mathbb{P}\mathfrak{F}\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J) \qquad \text{and} \qquad \gamma_{\mathbb{P}\mathfrak{F}\mathcal{T}}\longrightarrow\mathbb{P}\mathfrak{F}\mathcal{T}$$

the bundle projection map and the tautological line bundle. If $\nu_{\mathrm{es}}\in\mathfrak{S}^{\mathrm{gd}}_{1,k}(X,A;J)$ is generic, the section

$$\tilde{\mathcal{D}}_{\mathcal{T}}\in\Gamma\big(\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J);\mathrm{Hom}(\gamma_{\mathbb{P}\mathfrak{F}\mathcal{T}};\pi^*_{\mathbb{P}\mathfrak{F}\mathcal{T}}(\pi^*_P\mathbb{E}^*\otimes\mathrm{ev}^*_P TX))\big)$$

induced by $\mathcal{D}_{\mathcal{T}}$ is transverse to the zero set. For a generic choice of the pseudocycles f_l, this is also the case for the restriction of $\tilde{\mathcal{D}}_{\mathcal{T}}$ to $\mathbb{P}\mathfrak{F}\mathcal{T}\big|_{\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)}$ for every $\lambda\in\mathcal{A}$. On the other hand,

$$\dim\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda) = \dim\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J) + \big(\dim Y_\lambda - nk\big)$$
$$\leq \big(\dim_{1,k}(X,A) + 2(n-|\hat{I}|-|\aleph|)\big) - \dim_{1,k}(X,A) \tag{3.15}$$
$$= 2(n-|\chi(\mathcal{T})|) - 2\big(|\hat{I}-\chi(\mathcal{T})|+|\aleph|\big).$$

The middle inequality is an equality if and only if $\lambda=\mathrm{mn}$. Thus, the section $\tilde{\mathcal{D}}_{\mathcal{T}}$ does not vanish over $\mathbb{P}\mathfrak{F}\mathcal{T}\big|_{\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)}$. This is equivalent to saying that the bundle homomorphism

$$\mathcal{D}_{\mathcal{T}}\colon \mathfrak{F}\mathcal{T}\longrightarrow\pi^*_P\mathbb{E}^*\otimes\mathrm{ev}^*_P TX$$

is nondegenerate over $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)$, i.e. is injective on every fiber over $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)$.

Suppose $\mathcal{T}$ is not a simple bubble type or $\lambda\neq\mathrm{mn}$. By (3.15),

$$\frac{1}{2}\dim\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda) + \mathrm{rk}\,\mathfrak{F}\mathcal{T} < \mathrm{rk}\,\pi^*_P\mathbb{E}^*\otimes\mathrm{ev}^*_P TX.$$

Thus, for a generic $\nu\in\mathfrak{S}^{\mathrm{gd}}_{1,k}(X,A;J)$, the affine bundle map

$$\mathfrak{F}\mathcal{T}\longrightarrow\pi^*_P\mathbb{E}^*\otimes\mathrm{ev}^*_P TX, \qquad \upsilon=[b,v]\longrightarrow\mathcal{D}_{\mathcal{T}}\upsilon+\bar\nu_b,$$

has no zeros over $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)$. Since $\mathcal{D}_{\mathcal{T}}$ is nondegenerate over $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)$, (3.14) and the proof of Lemma 3.2 in [Z1] then imply that for every compact subset K of $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_\lambda)$ there exist $\delta_K\in\mathbb{R}^+$ and a neighborhood U'_K of K in $\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J)\times\bar{Y}$ such that

$$\big\{[(\upsilon,\zeta),z]\in\Omega_K(t)|_{U'_K}\colon \pi^{0,1}_{\upsilon;-}\Phi_t(\upsilon,\zeta)=0\big\}=\emptyset \qquad \forall\, t\in(0,\delta_K).$$

Thus, there exists a neighborhood U_K of K in $\mathfrak{X}_{1,k}(X,A)\times\bar{Y}$ such that

$$\big(\mathfrak{M}^0_{1,k}(X,A;J,\nu_{\mathrm{es}}+t\nu)\times\bar{Y}\big)\cap U_K=\emptyset.$$

The proof of Proposition 3.1 in the case $\mathcal{T}$ is not a simple bubble type or $\lambda \neq \mathrm{mn}$ is now complete.

For remainder of this subsection we assume that $\mathcal{T}$ is a simple bubble type and $\lambda = \mathrm{mn}$. If $\upsilon = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}_\delta^\emptyset$, we denote by

$$\Gamma_-^{0,1}(\upsilon) \subset \Gamma^{0,1}\big(\upsilon; \xi_{\nu_{\mathrm{es}}}(\upsilon_1)\big)$$

the image of $\Gamma_-^{0,1}(b; J)$ under R_υ and by $\Gamma_+^{0,1}(\upsilon)$ the L^2-orthogonal complement of $\Gamma_-^{0,1}(\upsilon)$ in $\Gamma^{0,1}\big(\upsilon; \xi_{\nu_{\mathrm{es}}}(\upsilon_1)\big)$. Let

$$\pi_{\upsilon;+}^{0,1} : \Gamma^{0,1}\big(\upsilon; \xi_{\nu_{\mathrm{es}}}(\upsilon_1)\big) \longrightarrow \Gamma_+^{0,1}(\upsilon)$$

the L^2-projection map. Since $\aleph = \emptyset$,

$$\|R_{\upsilon, \xi_{\nu_{\mathrm{es}}}(\upsilon_1)} \eta\|_{\upsilon, p} \leq C(b) \|R_{\upsilon, \xi_{\nu_{\mathrm{es}}}(\upsilon_1)} \eta\|_{\upsilon, 2} \qquad \forall \ \upsilon = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}_\delta^\emptyset, \ \eta \in \Gamma_-^{0,1}(b; J),$$

for some $C \in C^\infty(\mathcal{U}_{\mathcal{T}, \nu_{\mathrm{es}}}(X; J); \mathbb{R}^+)$. It follows that

$$\big\|\pi_+^{0,1} \eta'\big\|_{\upsilon, p} \leq C(b) \|\eta'\|_{\upsilon, p} \qquad \forall \ \upsilon = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}_\delta^\emptyset, \ \eta' \in \Gamma^{0,1}\big(\upsilon; \xi_{\nu_{\mathrm{es}}}(\upsilon_1)\big).$$

Thus, by the analogues of (2.22) and (2.25),

$$C(b)^{-1} \|\zeta\|_{\upsilon, p, 1} \leq \big\|\pi_+^{0,1} D_{J, \nu_{\mathrm{es}}; \upsilon; \xi_{\nu_{\mathrm{es}}}(\upsilon_1)} \zeta\big\|_{\upsilon, p} \leq C(b) \|\zeta\|_{\upsilon, p, 1} \tag{3.16}$$
$$\forall \quad \upsilon = (b, v) \in \tilde{\mathcal{F}}\mathcal{T}_\delta^\emptyset, \ \zeta \in \Gamma_+\big(\upsilon; \xi_{\nu_{\mathrm{es}}}(\upsilon_1)\big).$$

In particular, the operator

$$\pi_+^{0,1} D_{J, \nu_{\mathrm{es}}; \upsilon; \xi_{\nu_{\mathrm{es}}}(\upsilon_1)} : \Gamma_+\big(\upsilon; \xi_{\nu_{\mathrm{es}}}(\upsilon_1)\big) \longrightarrow \Gamma_+^{0,1}(\upsilon)$$

is an isomorphism. Its norm and the norm of its inverse are dependent only on $[b] \in \mathcal{U}_{\mathcal{T}, \nu_{\mathrm{es}}}(X; J)$. By (3.11), (3.12), (3.16), the analogue of (2.31), and the Contraction Principle, for every compact subset K of $\mathcal{U}_{\mathcal{T}, \nu_{\mathrm{es}}}(X; J)$ there exists $\delta_K \in \mathbb{R}^+$ such that for all

$$\upsilon \in \tilde{\mathcal{F}}\mathcal{T}_{\delta_K}^\emptyset|_K \qquad \text{and} \qquad t \in [0, \delta_K]$$

the equation

$$\pi_+^{0,1} \Phi_t(\upsilon, \zeta) = 0, \qquad \zeta \in \Gamma_+\big(\upsilon; \xi_{\nu_{\mathrm{es}}}(\upsilon_1)\big),$$

has a unique small solution $\zeta_t(\upsilon)$. Furthermore, $\zeta_t(\upsilon) \in \Omega_K(t)$.

By the above, for every compact subset K of $\mathcal{U}_{\mathcal{T}, \nu_{\mathrm{es}}}(X; J; f_{\mathrm{mn}})$ there exist $\delta_K \in \mathbb{R}^+$ and small neighborhoods U_K' and U_K of K in $\mathcal{U}_{\mathcal{T}, \nu_{\mathrm{es}}}(X; J) \times Y_{\mathrm{mn}}$ and $\mathfrak{X}_{1,k}(X, A) \times \bar{Y}$, respectively, such that

$$\mathfrak{M}_{1,k}^0(X, A; J, \nu_{\mathrm{es}} + t\nu; f_{\mathrm{mn}}) \cap U_K \approx \big\{([\upsilon], z) \in \mathcal{F}\mathcal{T}_{\delta_K}^\emptyset|_K \times Y_{\mathrm{mn}} : \pi_{\upsilon;-}^{0,1} \Phi_t(\upsilon, \zeta_t(\upsilon)) = 0;$$
$$\big\{\mathrm{ev} \times f\big\}\big((\Sigma_\upsilon, \exp_{u_{\upsilon, \xi_{\nu_{\mathrm{es}}}(\upsilon_1)}} \zeta_t(\upsilon)), z\big) \in \Delta_X^k\big\}$$

for all $t \in [0, \delta_K]$. On the other hand, the bundle homomorphism $\mathcal{D}_{\mathcal{T}}$ is regular over $\bar{\mathcal{U}}_{\mathcal{T}, \nu_{\mathrm{es}}}(X; J; f)$ by the $m = 1$ case of (2.7) and (2.8), i.e. $\mathcal{D}_{\mathcal{T}}$ can be approximated by a polynomial on the normal bundle near every stratum of $\bar{\mathcal{U}}_{\mathcal{T}, \nu_{\mathrm{es}}}(X; J; f)$; see [Z1, Definition 3.9]. Since ν is generic, $\mathcal{D}_{\mathcal{T}}$ is nondegenerate over $\mathcal{U}_{\mathcal{T}, \nu_{\mathrm{es}}}(X; J; f_{\mathrm{mn}})$, and $\mathrm{ev} \times f$ is transverse to Δ_X^k in $(X^2)^k$, (3.13) and the proof of Lemma 3.5 in [Z1] then imply that there exists a compact subset K_ν of $\mathcal{U}_{\mathcal{T}, \nu_{\mathrm{es}}}(X; J; f_{\mathrm{mn}})$ with

the following property. For every compact subset K of $\mathcal{U}_{\mathcal{T},\nu_{es}}(X;J;f_{mn})$ that contains K_ν there exist $\delta_K \in \mathbb{R}^+$ and a neighborhood U_K of K in $\mathfrak{X}_{1,k}(X,A) \times \bar{Y}$ such that

$$\pm\left|\mathfrak{M}_{1,k}^0(X,A;J,\nu_{es}+t\nu;f_{mn})\cap U_K\right| = N(\mathcal{D}_{\mathcal{T}}) \qquad \forall\, t \in [0,\delta_K].$$

We note that K_ν can taken to be any compact subset of $\mathcal{U}_{\mathcal{T},\nu_{es}}(X;J;f_{mn})$ such that all of the finitely many zeros of the affine map

$$\mathcal{D}_{\mathcal{T}}+\bar{\nu}\colon \mathfrak{F}\mathcal{T} \longrightarrow \pi_P^*\mathbb{E}^*\otimes\mathrm{ev}_P^*TX$$

over $\mathcal{U}_{\mathcal{T},\nu_{es}}(X;J;f_{mn})$ lie in $\mathfrak{F}\mathcal{T}|_{K_\nu}$.

3.4 Counting Zeros of Affine Bundle Maps

In this subsection we conclude the proof of Theorem 1.1 by computing the numbers $N(\mathcal{D}_{\mathcal{T}})$ when the dimension of X is 4 or 6. Using the method of [Z1, Section 3], $N(\mathcal{D}_{\mathcal{T}})$ can be computed for arbitrary-dimensional symplectic manifolds X and more general cohomology classes ψ; this is the main subject of [Z6]. In order to avoid introducing quite a bit of additional notation in this paper we restrict the computation to the special cases of Theorem 1.1.

If $\mathcal{T}$ is a bubble type as in (3.8), let

$$M_P\mathcal{T} = \left\{l \in [k]\colon j_l \notin \hat{I}\right\} \quad \text{and} \quad \mathrm{Aut}^*(\mathcal{T}) = \mathrm{Aut}(\mathcal{T})/\{g \in \mathrm{Aut}(\mathcal{T})\colon g\cdot h=h \ \forall\, h \in I_1\}.$$

For each $i \in I_1$, let

$$\tilde{\mathcal{T}}_i = \left(\tilde{M}_i\mathcal{T}, \{i\}\cup\tilde{H}_i\mathcal{T}; j|_{\tilde{M}_i\mathcal{T}}, \underline{A}|_{\{i\}\cup\tilde{H}_i\mathcal{T}}\right).$$

If $\nu_{es} \in \mathfrak{G}_{1,k}^{gd}(X,A;J)$ and $\aleph=\emptyset$, then

$$\mathcal{U}_{\mathcal{T},\nu_{es}}(X;J) \approx \left(\mathcal{M}_{1,M_P\mathcal{T}\sqcup I_1}\times\mathcal{U}_{\bar{\mathcal{T}},\nu_{es}}(X;J)\right)\big/\mathrm{Aut}^*(\mathcal{T}), \tag{3.17}$$

where $\mathcal{M}_{1,M_P\mathcal{T}\sqcup I_1}$ is the subspace of the moduli space $\overline{\mathcal{M}}_{1,M_P\mathcal{T}\sqcup I_1}$ consisting of smooth curves and

$$\mathcal{U}_{\bar{\mathcal{T}},\nu_{es}}(X;J) = \left\{(b_i)_{i\in I_1} \in \prod_{i\in I_1}\mathcal{U}_{\mathcal{T}_i,\nu_{B;i}}\colon \mathrm{ev}_0(b_{i_1})=\mathrm{ev}_0(b_{i_2}) \ \forall\, i_1,i_2\in I_1\right\}$$
$$\subset \prod_{i\in I_1}\overline{\mathfrak{M}}_{0,\{0\}\cup\tilde{M}_i\mathcal{T}}(X,\tilde{A}_i;J,\nu_{B;i}), \tag{3.18}$$

for some $\nu_{B;i} \in \mathfrak{G}_{0,\{0\}\cup\tilde{M}_i\mathcal{T}}^{es}(X,\tilde{A}_i;J)$. This decomposition is illustrated in Figure 5. In this figure, we represent an entire stratum $\mathcal{U}_{\mathcal{T},\nu_{es}}(X;J)$ of bubble maps by the domain of the stable maps in $\mathcal{U}_{\mathcal{T},\nu_{es}}(X;J)$. The right-hand side of Figure 5 represents the subset of the cartesian product of the three spaces of bubble maps, corresponding to the three drawings, on which the appropriate evaluation maps agree, as indicated by the solid line and defined in (3.18).

We next define a similar splitting for the space $\mathcal{U}_{\mathcal{T},\nu_{es}}(X;J;f_{mn})$. We can assume that the maps $\{f_{l;mn}\}_{l\in M_P\mathcal{T}}$ intersect transversally. Let

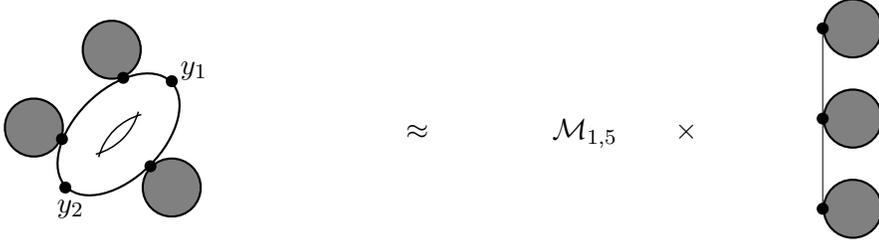$$f_0^{\mathcal{T}}\colon \bar{Y}_0^{\mathcal{T}} \longrightarrow \bigcap_{l\in M_P\mathcal{T}}f_l(\bar{Y}_l) \subset X$$

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Figure 5: An example of the decomposition (3.17)

be a pseudocycle representative for

$$\bigcap_{l\in M_P\mathcal{T}}\mathrm{PD}_X\psi_l \ \in H_*(X;\mathbb{Z})$$

such that

$$f_{0;\mathrm{mn}}^{\mathcal{T}}\colon Y_{0;\mathrm{mn}}^{\mathcal{T}} \longrightarrow \bigcap_{l\in M_P\mathcal{T}}f_l(\bar{Y}_{l;\mathrm{mn}})$$

is one-to-one. We put

$$\mathrm{ev}^{\mathcal{T}}=\mathrm{ev}_0\times\prod_{l\in[k]-M_P\mathcal{T}}\mathrm{ev}_l\colon\mathcal{U}_{\bar{\mathcal{T}},\nu_{\mathrm{es}}}(X;J)\longrightarrow X\times\prod_{l\in[k]-M_P\mathcal{T}}X\ ,$$

$$f^{\mathcal{T}}=f_0^{\mathcal{T}}\times\prod_{l\in[k]-M_P\mathcal{T}}f_l\colon\bar{Y}^{\mathcal{T}}\equiv\bar{Y}_0^{\mathcal{T}}\times\prod_{l\in[k]-M_P\mathcal{T}}\bar{Y}_l\longrightarrow X\times\prod_{l\in[k]-M_P\mathcal{T}}X\ ,\quad\text{and}\quad Y_{\mathrm{mn}}^{\mathcal{T}}=Y_{0;\mathrm{mn}}^{\mathcal{T}}\times\prod_{l\in[k]-M_P\mathcal{T}}Y_{l;\mathrm{mn}}.$$

Similarly to Subsection 3.1, let

$$\bar{\mathcal{U}}_{\bar{\mathcal{T}},\nu}(X;J;f)=\big\{(b,z)\in\bar{\mathcal{U}}_{\bar{\mathcal{T}},\nu}(X;J)\times\bar{Y}^{\mathcal{T}}\colon\mathrm{ev}^{\mathcal{T}}(b)=f^{\mathcal{T}}(z)\big\},\qquad\text{and}$$

$$\mathcal{U}_{\bar{\mathcal{T}},\nu}(X;J;f_{\mathrm{mn}})=\big(\mathcal{U}_{\bar{\mathcal{T}},\nu}(X;J)\times Y_{\mathrm{mn}}^{\mathcal{T}}\big)\cap\bar{\mathcal{U}}_{\bar{\mathcal{T}},\nu}(X;J;f).$$

We have

$$\mathcal{U}_{\mathcal{T},\nu_{\mathrm{es}}}(X;J;f_{\mathrm{mn}})\approx\big(\mathcal{M}_{1,M_P\mathcal{T}\sqcup I_1}\times\mathcal{U}_{\bar{\mathcal{T}},\nu_{\mathrm{es}}}(X;J;f_{\mathrm{mn}})\big)\big/\mathrm{Aut}^*(\mathcal{T}).\tag{3.19}$$

If $\mathcal{T}$ is a simple bubble type, we define the homomorphism

$$\widetilde{\mathcal{D}}_{\mathcal{T}}\colon\widetilde{\mathfrak{F}}\mathcal{T}\longrightarrow\pi_P^*\mathbb{E}^*\otimes\mathrm{ev}_0^*TX$$

over $\overline{\mathcal{M}}_{1,M_P\mathcal{T}\sqcup I_1}\times\bar{\mathcal{U}}_{\bar{\mathcal{T}},\nu_{\mathrm{es}}}(X;J;f)$ similarly to the homomorphism $\mathcal{D}_{\mathcal{T}}$. By (3.19),

$$N(\mathcal{D}_{\mathcal{T}})=N(\widetilde{\mathcal{D}}_{\mathcal{T}})\big/\big|\mathrm{Aut}^*(\mathcal{T})\big|.\tag{3.20}$$

By Proposition 3.1 and (3.20), the difference between the standard and reduced genus-one GW-invariants of Theorem 1.1 is determined by the numbers $N(\widetilde{\mathcal{D}}_{\mathcal{T}})$, where $\mathcal{T}$ is a simple bubble type as in (3.8). We will compute these numbers in the two special cases of Theorem 1.1.

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First, we note that if $\mathcal{T}$ is a bubble type as in (3.8), not necessarily simple, and $\aleph = \emptyset$,

$$\dim \bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu_{\mathrm{es}}}(X; J; f) = \dim \bar{\mathcal{U}}_{\mathcal{T}, \nu_{\mathrm{es}}}(X; J; f) - \dim \overline{\mathcal{M}}_{1, M_P \mathcal{T} \sqcup I_1}$$
$$= 2(n - |\chi(\mathcal{T})| - |I_1|) - 2(|\hat{I} - \chi(\mathcal{T})| + |M_P \mathcal{T}|), \tag{3.21}$$

by (3.15). In particular, if $n = 2$, then

$$\bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu_{\mathrm{es}}}(X; J; f) \neq \emptyset \quad \Longrightarrow \quad |\chi(\mathcal{T})| = 1, \ \ \chi(\mathcal{T}) = I_1 = \hat{I}, \ \ M_P \mathcal{T} = \emptyset.$$

Furthermore, if $\bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu_{\mathrm{es}}}(X; J; f)$ is nonempty, it is a finite collection of points. In this case, $\widetilde{\mathcal{D}}_{\mathcal{T}}$ is the bundle homomorphism

$$\pi_P^*\big(s_1 \oplus 0\big) \colon \pi_P^* L_1 \longrightarrow \pi_P^*\big(\mathbb{E}^* \oplus \mathbb{E}^*\big)$$

over $\overline{\mathcal{M}}_{1,1} \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu_{\mathrm{es}}}(X; J; f)$. Thus,

$$N(\widetilde{\mathcal{D}}_{\mathcal{T}}) = -\frac{1}{24} \,{}^\pm\big|\bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu_{\mathrm{es}}}(X; J; f)\big|.$$

Since $\,{}^\pm\big|\bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu_{\mathrm{es}}}(X; J; f)\big| = \,{}^\pm\big|\bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu'}(X; J; f)\big|$ for any sufficiently small ν' such that the restriction of the bundle section $\bar{\partial} + \nu'$ to every stratum of $\mathfrak{X}_{0, \{0\} \sqcup [k]}(X, A)$ is transverse to the zero set, we can take $\nu' = \pi^* \nu_0$, where

$$\pi \colon \mathfrak{X}_{0, \{0\} \sqcup [k]}(X, A) \longrightarrow \mathfrak{X}_{0, [k]}(X, A)$$

is the forgetful map and ν_0 is a small generic deformation of $\bar{\partial}$ on $\mathfrak{X}_{0, [k]}(X, A)$. Since

$$\overline{\mathfrak{M}}_{0, k}(X, A; J, \nu_0; f) = \emptyset$$

for a generic ν_0 for dimensional reasons, it follows $\bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu'}(X; J; f) = \emptyset$ as well and thus $N(\widetilde{\mathcal{D}}_{\mathcal{T}}) = 0$, as claimed by $n = 2$ case of Theorem 1.1.

Suppose $n = 3$. By (3.21),

$$\bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu_{\mathrm{es}}}(X; J; f) \neq \emptyset \quad \Longrightarrow \quad |\chi(\mathcal{T})| = 1, \ \ \big(|I_1| - |\chi(\mathcal{T})|\big) + \big|\hat{I} - \chi(\mathcal{T})\big| + |M_P \mathcal{T}| \in \{0, 1\}.$$

If $\mathcal{T}$ is a simple bubble type and $\bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu_{\mathrm{es}}}(X; J; f)$ is nonempty, it follows that

$$|\chi(\mathcal{T})| = 1 \quad \text{and} \quad |M_P \mathcal{T}| \in \{0, 1\}.$$

If $|M_P \mathcal{T}| = 1$, $\bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu_{\mathrm{es}}}(X; J; f)$ is again a finite collection of points. In this case, $\widetilde{\mathcal{D}}_{\mathcal{T}}$ is the bundle homomorphism

$$\pi_P^*\big(s_1 \oplus 0 \oplus 0\big) \colon \pi_P^* L_1 \longrightarrow \pi_P^*\big(\mathbb{E}^* \oplus \mathbb{E}^* \oplus \mathbb{E}^*\big)$$

over $\overline{\mathcal{M}}_{1,2} \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu_{\mathrm{es}}}(X; J; f)$. Thus, as in the $n = 2$ case above, $N(\widetilde{\mathcal{D}}_{\mathcal{T}}) = 0$.[3]

Finally, suppose $n = 3$, $\mathcal{T}$ is a simple bubble type, and $|M_P \mathcal{T}| = 0$. If i is the unique element of $\chi(\mathcal{T})$, $\widetilde{\mathcal{D}}_{\mathcal{T}}$ is the bundle homomorphism

$$\pi_P^* s_1 \otimes \pi_i^* \mathcal{D}_0 \colon \pi_P^* L_1 \otimes \pi_i^* L_0 \longrightarrow \pi_P^* \mathbb{E}^* \otimes \pi_i^* \mathrm{ev}_0^* TX$$

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[3] As in the $n = 2$ case, $\,{}^\pm\big|\bar{\mathcal{U}}_{\bar{\mathcal{T}}, \nu_{\mathrm{es}}}(X; J; f)\big| = 0$. Furthermore, since $c_1(\mathbb{E}^*)^2 = 0$ on $\overline{\mathcal{M}}_{1,2}$, we can choose ν so that $\bar{\nu}$ does not vanish and thus $N(\widetilde{\mathcal{D}}_{\mathcal{T}}) = 0$ by definition.

over

$$\overline{\mathcal{M}}_{1,1} \times \bar{\mathcal{U}}_{\bar{\mathcal{T}},\nu_{\mathrm{es}}}(X;J;f) = \overline{\mathcal{M}}_{1,1} \times \overline{\mathfrak{M}}_{0,\{0\}\cup[k]}(X,A;J,\nu_{B;i};f).$$

It can be assumed that $\nu_{B,i} = \pi^*\nu_B$ for a generic element $\nu_B \in \mathfrak{G}^{\mathrm{es}}_{0,\{0\}}(X,A;J)$, where

$$\pi\colon \mathfrak{X}_{0,\{0\}\sqcup[k]}(X,A) \longrightarrow \mathfrak{X}_{0,\{0\}}(X,A)$$

is the forgetful map. Then,

$$\pi\colon \overline{\mathfrak{M}}_{0,\{0\}\cup[k]}(X,A;J,\nu_{B;i};f) \longrightarrow \overline{\mathfrak{M}}_{0,\{0\}}(X,A;J,\nu_B) \tag{3.22}$$

and there are identifications $L_0 = \pi^*L_0$ and $\mathcal{D}_0 = \pi^*\mathcal{D}_0$. Thus,

$$N(\widetilde{\mathcal{D}}_{\mathcal{T}}) = N\big(\pi_P^*s_1 \otimes \pi_i^*\pi^*\mathcal{D}_0\big). \tag{3.23}$$

On the other hand, if ν_B is generic, the image of the projection π in (3.22) is contained in $\mathfrak{M}^{\{0\}}_{0,\{0\}}(X,A;J,\nu_B)$, while the restriction of $\mathcal{D}_0$ to every stratum of $\mathfrak{M}^{\{0\}}_{0,\{0\}}(X,A;J,\nu_B)$ is transverse to the zero set. Thus, $\pi^*\mathcal{D}_0$ does not vanish over $\overline{\mathfrak{M}}_{0,\{0\}\cup[k]}(X,A;J,\nu_{B;i};f)$ for dimensional reasons if ν_B is generic. Since s_1 does not vanish over $\overline{\mathcal{M}}_{1,1}$, the bundle homomorphism

$$\pi_P^*s_1 \otimes \pi_i^*\pi^*\mathcal{D}_0\colon \pi_P^*L_1 \otimes \pi_i^*\pi^*L_0 \longrightarrow \pi_P^*\mathbb{E}^* \otimes \pi_i^*\pi^*\mathrm{ev}_0^*TX$$

does not vanish over $\overline{\mathcal{M}}_{1,1} \times \overline{\mathfrak{M}}_{0,\{0\}\cup[k]}(X,A;J,\nu_{B;i};f)$. Thus, by Lemma 3.14 in [Z1],

$$\begin{aligned}
N(\widetilde{\mathcal{D}}_{\mathcal{T}}) &= \big\langle c(\pi_P^*\mathbb{E}^* \otimes \pi_i^*\mathrm{ev}_0^*TX)c(\pi_P^*L_1 \otimes \pi_i^*\pi^*L_0)^{-1}, \overline{\mathcal{M}}_{1,1} \times \overline{\mathfrak{M}}_{0,\{0\}\cup[k]}(X,A;J,\nu_{B;i};f)\big\rangle \\
&= -\frac{1}{24}\big\langle c_1(TX) + \pi^*c_1(L_0^*), \overline{\mathfrak{M}}_{0,\{0\}\cup[k]}(X,A;J,\nu_{B;i};f)\big\rangle.
\end{aligned} \tag{3.24}$$

By the divisor and dilaton relations for GW-invariants,

$$\begin{aligned}
\big\langle c_1(TX), \overline{\mathfrak{M}}_{0,\{0\}\cup[k]}(X,A;J,\nu_{B;i};f)\big\rangle &= \langle c_1(TX),A\rangle \cdot \mathrm{GW}^X_{0,k}(A;\psi), \\
\big\langle \pi^*c_1(L_0^*), \overline{\mathfrak{M}}_{0,\{0\}\cup[k]}(X,A;J,\nu_{B;i};f)\big\rangle &= -2 \cdot \mathrm{GW}^X_{0,k}(A;\psi);
\end{aligned} \tag{3.25}$$

see Section 26.3 in [MirSym], for example. Combining (3.23) with (3.24) and (3.25), we conclude that

$$N(\widetilde{\mathcal{D}}_{\mathcal{T}}) = \frac{2 - \langle c_1(TX),A\rangle}{24}\mathrm{GW}^X_{0,k}(A;\psi).$$

The proof of Theorem 1.1 is now complete.

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