Mirror Symmetry for
Closed, Open, and Unoriented Gromov-Witten Invariants

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Abstract

In the first part of this paper, we obtain mirror formulas for twisted genus 0 two-point Gromov-Witten (GW) invariants of projective spaces and for the genus 0 two-point GW-invariants of Fano and Calabi-Yau complete intersections. This extends previous results for projective hypersurfaces, following the same approach, but we also completely describe the structure coefficients in both cases and obtain relations between these coefficients that are vital to the applications to mirror symmetry in the rest of this paper. In the second and third parts of this paper, we confirm Walcher's mirror symmetry conjectures for the annulus and Klein bottle GW-invariants of Calabi-Yau complete intersection threefolds; these applications are the main results of this paper. In a separate paper, the genus 0 two-point formulas are used to obtain mirror formulas for the genus 1 GW-invariants of all Calabi-Yau complete intersections.

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1 Introduction

Gromov-Witten invariants of projective varieties are counts of curves that are conjectured (and known in some cases) to possess a rich structure. In [Gi96], [Gi99], and [LLY], the original mirror prediction of [CdGP] for the genus 0 GW-invariants of a quintic threefold is verified and shown to be a special case of mirror formulas satisfied by the genus 0 one-point GW-invariants of complete intersections. In [BeK], [Z10], [C], and [GhT], these results are used to obtain mirror formulas for two-point genus 0 GW-invariants of projective hypersurfaces. We begin this paper by extending the approach of [Z10] to Fano and Calabi-Yau projective complete intersections, give a complete description of the (equivariant) structure coefficients in both cases, and obtain relations between them. The mirror formulas and relations of Sections 3 and 4 are used in Sections 6 and 7 to confirm the mirror symmetry predictions of Walcher [W] concerning the annulus and Klein bottle invariants of Calabi-Yau complete intersection threefolds in the presence of an anti-holomorphic involution; these are the main results of this paper. Unlike proofs of mirror symmetry in other settings (such as in [Po], [PSoW], and [Z09b]), our arguments do not rely on a priori knowledge that the final answers are independent of the toric weights (i.e. are purely non-equivariant).

Throughout this paper, $\mathbf{a}=(a_1,a_2,\ldots,a_l)$ denotes a tuple of positive integers and $X_{\mathbf{a}}\subset\mathbb{P}^{n-1}$, a smooth complete intersection of multi-degree $\mathbf{a}$; for example, $X_\emptyset=\mathbb{P}^{n-1}$. Let

$$\langle\mathbf{a}\rangle\equiv\prod_{k=1}^{l}a_k,\qquad \mathbf{a}!\equiv\prod_{k=1}^{l}a_k!,\qquad \mathbf{a}^{\mathbf{a}}\equiv\prod_{k=1}^{l}a_k^{a_k},\qquad |\mathbf{a}|\equiv\sum_{k=1}^{l}a_k,\qquad \nu_{\mathbf{a}}\equiv n-|\mathbf{a}|.$$

We consider only the Fano cases, $\nu_{\mathbf{a}}>0$, and the Calabi-Yau cases, $\nu_{\mathbf{a}}=0$. All cohomology groups in this paper are with rational coefficients unless specified otherwise. Let $\mathrm{H}\in H^2(\mathbb{P}^{n-1})$ denote the hyperplane class.

Mirror formulas relate GW-invariants of $X_{\mathbf{a}}$ to the hypergeometric series

$$F(w,q)\equiv\sum_{d=0}^{\infty}q^d w^{\nu_{\mathbf{a}}d}\frac{\prod_{k=1}^{l}\prod_{r=1}^{a_kd}(a_kw+r)}{\prod_{r=1}^{d}(w+r)^n}\,. \tag{1.1}$$

This is a power series in q with constant term 1 whose coefficients are rational functions in w which are regular at $w=0$. As in [ZaZ], we denote the subgroup of all such power series by $\mathcal{P}$ and define

$$\begin{aligned} &\mathbf{D}\colon \mathbb{Q}(w)\big[\big[q\big]\big] \longrightarrow \mathbb{Q}(w)\big[\big[q\big]\big], && \mathbf{M}:\mathcal{P}\longrightarrow\mathcal{P} \qquad \text{by}\\ &\mathbf{D}H(w,q)\equiv\left\{1+\frac{q}{w}\frac{\mathrm{d}}{\mathrm{d}q}\right\}H(w,q), && \mathbf{M}H(w,q)\equiv\mathbf{D}\left(\frac{H(w,q)}{H(0,q)}\right). \end{aligned} \tag{1.2}$$

If $\nu_{\mathbf{a}}=0$ and $p\in\mathbb{Z}^{\geq 0}$, let

$$I_p(q)\equiv\mathbf{M}^pF(0,q). \tag{1.3}$$

For example,

$$I_0(q)=\sum_{d=0}^{\infty}q^d\frac{(a_1d)!(a_2d)!\dots(a_ld)!}{(d!)^n} \quad \text{if} \quad \nu_{\mathbf{a}}=0.$$

If $\nu_{\mathbf{a}}=0$, let

$$J(q)\equiv\frac{1}{I_0(q)}\left\{\sum_{d=1}^{\infty}q^d\frac{\prod_{k=1}^{l}(a_kd)!}{(d!)^n}\left(\sum_{k=1}^{l}\sum_{r=d+1}^{a_kd}\frac{a_k}{r}\right)\right\} \quad \text{and} \quad Q\equiv q\,\mathrm{e}^{J(q)}. \tag{1.4}$$

Thus, the map $q\longrightarrow Q$ is a change of variables; it will be called the mirror map.

1.1 Mirror formulas for closed GW-invariants

In light of previous work on genus 0 two-point GW-invariants of projective complete intersections, the precise statements of Theorems 5 and 6 concerning these invariants are primarily stepping stones to the results on open and unoriented GW-invariants of Theorems 3 and 4. Nevertheless, in this section, we illustrate Theorems 5 and 6 with explicit examples and describe relations with other work on formulas for genus 0 two-point GW-invariants.

Given a smooth subvariety $X\subset\mathbb{P}^{n-1}$, we will write $\overline{\mathfrak{M}}_{0,m}(X,d)$ for the moduli space of stable degree d maps into X from genus 0 curves with m marked points and

$$\mathrm{ev}_i:\overline{\mathfrak{M}}_{0,m}(X,d)\longrightarrow X$$

for the evaluation map at the i-th marked point; see [MirSym, Chapter 24]. For each $i=1,2,\dots,m$, let $\psi_i\in H^2(\overline{\mathfrak{M}}_{0,m}(X,d))$ be the first Chern class of the universal cotangent line bundle for the i-th marked point. Gromov-Witten invariants are obtained by integration of classes against the virtual fundamental class of $\overline{\mathfrak{M}}_{0,m}(X,d)$:

$$\begin{aligned} \big\langle\tau_{p_1}(\mathrm{H}^{b_1}),\dots,\tau_{p_m}(\mathrm{H}^{b_m})\big\rangle_d^X &\equiv \big\langle\psi^{p_1}\mathrm{H}^{b_1},\dots,\psi^{p_m}\mathrm{H}^{b_m}\big\rangle_d^X\\ &\equiv\int_{[\overline{\mathfrak{M}}_{0,m}(X,d)]^{vir}}\big(\psi_1^{p_1}\mathrm{ev}_1^*\mathrm{H}^{b_1}\big)\dots\big(\psi_m^{p_m}\mathrm{ev}_m^*\mathrm{H}^{b_m}\big), \end{aligned} \tag{1.5}$$

where $H\in H^2(\mathbb{P}^{n-1})$ is the hyperplane class.

The two-point mirror formulas take the simplest shape in the two extremal cases: $\nu_{\mathbf{a}}=0,n$.

Theorem 1. *The degree $d\ge 1$ genus 0 two-point descendant invariants of $\mathbb{P}^{n-1}$ with $n\ge 2$ are given by the following identity in* $(\mathbb{Q}[\mathrm{H}_1,\mathrm{H}_2]/\{\mathrm{H}_1^n,\mathrm{H}_2^n\})\,[[\hbar_1^{-1},\hbar_2^{-1}]]$*:*

$$\sum_{p_1,p_2\ge0}\left\langle\frac{\mathrm{H}^{n-1-p_1}}{\hbar_1-\psi},\frac{\mathrm{H}^{n-1-p_2}}{\hbar_2-\psi}\right\rangle_d^{\mathbb{P}^{n-1}}\mathrm{H}_1^{p_1}\mathrm{H}_2^{p_2} = \frac{1}{\hbar_1+\hbar_2}\sum_{\substack{p_1+p_2=n-1\\ p_1,p_2\ge0}}\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge0}}\frac{(\mathrm{H}_1+d_1\hbar_1)^{p_1}\,(\mathrm{H}_2+d_2\hbar_2)^{p_2}}{\prod\limits_{r=1}^{d_1}(\mathrm{H}_1+r\hbar_1)^n\,\prod\limits_{r=1}^{d_2}(\mathrm{H}_2+r\hbar_2)^n}\,.$$

Theorem 2. *The genus 0 two-point descendant invariants of a Calabi-Yau complete intersection $X_{\mathbf{a}}$ in $\mathbb{P}^{n-1}$ are given by the following identity in* $\big(\mathbb{Q}[\mathrm{H}_1,\mathrm{H}_2]/\{\mathrm{H}_1^{n-l},\mathrm{H}_2^{n-l}\}\big)\,[[\hbar_1^{-1},\hbar_2^{-1},q]]$*:*

$$\sum_{p_1,p_2\ge0}\sum_{d=1}^{\infty}Q^d\left\langle\frac{\mathrm{H}^{n-1-l-p_1}}{\hbar_1-\psi},\frac{\mathrm{H}^{n-1-l-p_2}}{\hbar_2-\psi}\right\rangle_d^{X_{\mathbf{a}}}\mathrm{H}_1^{p_1}\mathrm{H}_2^{p_2} = \frac{\langle\mathbf{a}\rangle}{\hbar_1+\hbar_2}\sum_{\substack{p_1+p_2=n-1-l\\ p_1,p_2\ge0}}\left(-1+\mathrm{e}^{-J(q)\left(\frac{\mathrm{H}_1}{\hbar_1}+\frac{\mathrm{H}_2}{\hbar_2}\right)}\frac{\mathbf{M}^{p_1}F\left(\frac{\mathrm{H}_1}{\hbar_1},q\right)}{I_{p_1}(q)}\frac{\mathbf{M}^{p_2}F\left(\frac{\mathrm{H}_2}{\hbar_2},q\right)}{I_{p_2}(q)}\right)\mathrm{H}_1^{p_1}\mathrm{H}_2^{p_2}\,,$$

with Q and q related by the mirror map (1.4).

Remark 1. In both theorems, the sums on the right-hand side of the identities are power series in $\hbar_1^{-1}$ and $\hbar_2^{-1}$ (to see this in Theorem 1, divide both the numerator and denominator of each (p_1,p_2,d_1,d_2)-summand by $\hbar_1^{nd_1}\hbar_2^{nd_2}$). The two theorems state in particular that these sums are divisible by $\hbar_1+\hbar_2$. This can be seen directly in the case of Theorem 1 as follows. Divisibility by $\hbar_1+\hbar_2$ of a series in $\hbar_1^{-1}$ and $\hbar_2^{-1}$ is equivalent to the vanishing of the series evaluated at $(\hbar_1,\hbar_2)=(\hbar,-\hbar)$. The sum on the right-hand side of Theorem 1 evaluated at $(\hbar_1,\hbar_2)=(\hbar,-\hbar)$ and multiplied by $H_1-H_2+d\hbar$ is

$$\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge0}}\frac{(\mathrm{H}_1+d_1\hbar)^n-(\mathrm{H}_2-d_2\hbar)^n}{\prod\limits_{r=1}^{d_1}(\mathrm{H}_1+r\hbar)^n\prod\limits_{r=1}^{d_2}(\mathrm{H}_2-r\hbar)^n} = \frac{1}{\prod\limits_{r=1}^{d-1}(\mathrm{H}_1+r\hbar)^n}-\frac{1}{\prod\limits_{r=1}^{d-1}(\mathrm{H}_2-r\hbar)^n} + \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}}\left(\frac{1}{\prod\limits_{r=1}^{d_1-1}(\mathrm{H}_1+r\hbar)^n\prod\limits_{r=1}^{d_2}(\mathrm{H}_2-r\hbar)^n}-\frac{1}{\prod\limits_{r=1}^{d_1}(\mathrm{H}_1+r\hbar)^n\prod\limits_{r=1}^{d_2-1}(\mathrm{H}_2-r\hbar)^n}\right)=0;$$

the first equality above uses $\mathrm{H}_1^n,\mathrm{H}_2^n=0$.

Theorem 2 leads to a simple identity for the primary GW-invariants of a Calabi-Yau complete intersection $X_{\mathbf{a}}$. Applying $Q\frac{\mathrm{d}}{\mathrm{d}Q}=\frac{q}{I_1(q)}\frac{\mathrm{d}}{\mathrm{d}q}$ to both sides of the identity in Theorem 2 and considering the coefficient of $\mathrm{H}_1^{n-1-l-b}\mathrm{H}_2^{b+1}$, we obtain

$$\frac{1}{\hbar_1\hbar_2}\sum_{d=1}^{\infty}d\langle\mathrm{H}^b,\mathrm{H}^{n-2-l-b}\rangle_d^{X_{\mathbf{a}}}Q^d=\frac{\langle\mathbf{a}\rangle}{\hbar_1+\hbar_2}\left(\frac{I_{n-1-l-b}(q)}{\hbar_1\,I_1(q)}+\frac{I_{b+1}(q)}{\hbar_2\,I_1(q)}-\frac{1}{\hbar_1}-\frac{1}{\hbar_2}\right),$$

since $1+q\frac{\mathrm{d}J(q)}{\mathrm{d}q}=I_1(q)$. Multiplying both sides by $\hbar_1+\hbar_2$ and setting $\hbar_2=-\hbar_1$, we obtain

$$I_{b+1}(q) = I_{n-1-l-b}(q) \quad \forall\ b=0,1,\ldots,n-l-2. \tag{1.6}$$

The last two equations give

$$\langle\mathbf{a}\rangle + \sum_{d=1}^{\infty} d\big\langle \mathrm{H}^{b_1},\mathrm{H}^{b_2}\big\rangle_d^{X_{\mathbf{a}}} Q^d = \langle\mathbf{a}\rangle\frac{I_{b_1+1}(q)}{I_1(q)} \qquad \text{if}\ \ b_1+b_2=n-l-2. \tag{1.7}$$

Taking $(b_1,b_2)=(1,n-l-3)$ in (1.7) and applying the divisor equation [MirSym, Section 26.3], we obtain the following identity for one-point primary GW-invariants of a Calabi-Yau complete intersection $X_{\mathbf{a}}$:

$$\langle\mathbf{a}\rangle + \sum_{d=1}^{\infty} d^2\big\langle \mathrm{H}^{n-l-3}\big\rangle_d^{X_{\mathbf{a}}} Q^d = \langle\mathbf{a}\rangle\frac{I_2(q)}{I_1(q)}. \tag{1.8}$$

Tables 1-5 show low-degree two-point genus 0 BPS numbers, defined from the GW-invariants by equation (2) in [KP], for all multi-degree $\mathbf{a}$ complete intersections $X_{\mathbf{a}}$ in $\mathbb{P}^{n-1}$ with $n\le 10$. As predicted by Conjecture 0 in [KP], these numbers are integers; using a computer program, we have confirmed this conjecture for all degree $d\le 100$ two-point BPS counts in all Calabi-Yau complete intersections $X_{\mathbf{a}}$ in $\mathbb{P}^{n-1}$ with $n\le 10$. The degree 1 and 2 BPS numbers in these cases match the usual Schubert calculus computations on $G(2,n)$ and $G(3,n)$, respectively; see [Ka]. The degree 3 numbers for the hypersurfaces agree with [ES]; it should be possible to verify our degree 3 numbers for the other complete intersections in the tables by the approach of [ES] as well.

The genus 0 GW-invariants of the form (1.5) with $m=1$ are often assembled into a generating function, known as the small J-function or Givental's J-function. Explicit closed formulas for the small J-function are obtained in [Gi96], [Gi99], and [LLY] and used in many computations throughout GW-theory. In light of [LP, Theorem 1], the small J-function determines all genus 0 GW-invariants of projective complete intersections of the form (1.5). However, [LP, Theorem 1] is yet to be directly used to express a generating function for the GW-invariants of the form (1.5), even with $m=2$, in terms of the small J-function. The relation [BeK, Formula 1.1] determines a generating function for the GW-invariants of the form (1.5) with $m=2$ in terms of the small J-function and indirectly encodes the consequences of [LP, Theorem 1] relevant to the $m=2$ case. Unfortunately, [BeK, Formula 1.1] is not a completely explicit relationship. A different approach, more in the spirit of [Gi96] and [Gi99], is used in [Z10] to express a generating function for the GW-invariants of the form (1.5) with $m=2$ as a linear combination of derivatives in terms of the small J-function, with details sufficient for the computation of genus 1 invariants in [Z09b]. The same approach is used in Section 4 of this paper to obtain a more precise description of the structure coefficients in Theorems 5 and 6, which is vital to the computations of open and oriented invariants in Sections 6 and 7. In [C], the approach of [Z10] is incorporated into the Mirror Principle of [LLY].

In the theory of Frobenius structures, all genus 0 GW-invariants with descendants at only one marked point, i.e. as in (1.5) with arbitrary m, but with pull-backs of arbitrary elements of $H^*(X)$, not just powers of the hyperplane class, and with $p_i=0$ for all $i<m$, are assembled into a generating function, called the big J-function; see [Gi98, Section 1]. According to Dubrovin's Reconstruction Formula, the big J-function determines a generating function for all genus 0 GW-invariants of a symplectic manifold X, i.e. as in (1.5), but with pull-backs of arbitrary elements of $H^*(X)$; see [Du, (6.46),(6.48)], [Gi98, Section 1], [Gi04, Theorem 1]. However, there is no simple closed formula for the big J-function, even when restricted to the pull-back of $H^*(\mathbb{P}^{n-1})$

d	1	2	3	4
X_7	1707797	510787745643	222548537108926490	113635631482486991647224
X_{26}	616896	41762262528	4088395365564096	468639130901813987328
X_{35}	344925	10528769475	465037227025650	24049433312314947000
X_{44}	284672	6749724672	231518782306304	9297639201854554112
X_{225}	257600	4672315200	121622886740800	3703337959222528000
X_{234}	169344	1695326976	24368988329856	409711274829020160
X_{333}	134865	959370561	9805843550034	117225412143917130
X_{2224}	126976	755572736	6403783700480	63420292743217152
X_{2233}	101088	427633344	2578114145376	18160214808655872

Table 1: Low-degree genus 0 BPS numbers $(\mathrm{H}^2, \mathrm{H}^2)$ for some Calabi-Yau 5-folds

d	1	2	3	4
X_8	37502976	224340704157696	2000750410187341381632	21122119007324663457380794368
X_{27}	12302724	14461287750168	25229820971457458076	52062878981745707203195872
X_{36}	5983632	2687545163520	1790676521197504848	1410987322122907728701952
X_{45}	4207200	1199825510400	507532701727557600	253883290498940295168000
X_{226}	4568832	1218545282304	480017733854171904	223463727594724776026112
X_{235}	2556900	308135971800	54819457086152700	11523817961861217228000
X_{244}	2113536	197815492608	27330245107728384	4461495054506601185280
X_{334}	1682208	112043367936	11011993317434016	1278661763157122064384

Table 2: Low-degree genus 0 BPS numbers $(\mathrm{H}^2, \mathrm{H}^3)$ for some Calabi-Yau 6-folds

d	2	3	4
X_9	93777295128674544	17873898563070361396216980	4116769336772585598746250465113376
X_{28}	4927955151077376	162926148665902467481600	6500105641339003383917401800704
X_{37}	705385191838824	7728929806910065428150	102149074253694894133257041184
X_{46}	232110378925056	1366213248304683678720	9698512727764286393809084416
X_{55}	161520243390000	777366857564506697500	4511987527454184551984500000

Table 3: Low-degree genus 0 BPS numbers $(\mathrm{H}^2, \mathrm{H}^4)$ for some Calabi-Yau 7-folds

d	2	3	4
X_9	156037426159482684	33815935806268253433549768	8638744084627099110538662706812804
X_{28}	7991674345455616	299081290134892802629632	13191988997947686388859151876096
X_{37}	1140060797165178	14119492055187150903348	206104052757048604579337400666
X_{46}	374346228782592	2489348580867704950272	19510528916120073780261924864
X_{55}	260419900772500	1415758838048143140000	9071479905327228206518687500

Table 4: Low-degree genus 0 BPS numbers $(\mathrm{H}^3, \mathrm{H}^3)$ for some Calabi-Yau 7-folds

d	2	3	4
$(\mathrm{H}^2, \mathrm{H}^5)$	40342298386119224000	174824389112955477418055016000	942582519217090098297647146585590400000
$(\mathrm{H}^3, \mathrm{H}^4)$	100290980400305376000	546627811934015785499223984000	3538531932815556807325167617597092800000

Table 5: Low-degree genus 0 BPS numbers for $X_{10} \subset \mathbb{P}^9$

in $H^*(X)$. Unfortunately, the big J-function is sometimes called the one-point J-function; this has lead to some confusion in GW-theory as to whether [BeK, Formula 1.1] and Theorem 6 in this paper are somehow contained in Dubrovin's Reconstruction Formula. In light of the divisor and string relations [MirSym, Section 26.3], the small J-function is essentially the restriction of the big J-function to $H^0(X)\oplus H^2(\mathbb{P}^{n-1})|_X$; the analogous statement holds for the small and big generating functions with ψ-classes at two marked points. While (3.10) can be viewed as the restriction to $H^0(X)\oplus H^2(\mathbb{P}^{n-1})|_X$ of a similar relationship for big generating functions, this is not the case with (3.11) as Dubrovin's Reconstruction Formula mixes invariants with different numbers of marked points. In fact, [GhT] is essentially dedicated to showing that Dubrovin's Reconstruction Formula restricts to the collection of GW-invariants of the form (1.5), with $m=2$ and with pull-backs of powers of the hyperplane class, and leads to some version of (3.11); the argument in [GhT] uses a version of the differential operators that appear in our Theorem 6. In summary, [BeK, Formula 1.1] and Theorem 6 in this paper are *not* contained in Dubrovin's Reconstruction Formula.

1.2 Mirror formulas for open and unoriented GW-invariants

Given a symplectic manifold (X,ω) endowed with an anti-symplectic involution $\Omega\colon X\longrightarrow X$, it is natural to fix an ω-compatible almost complex structure J anti-commuting with $\mathrm{d}\Omega$ and consider J-holomorphic maps $\tilde{f}:\tilde{C}\longrightarrow X$ from (possibly nodal) Riemann surfaces $\tilde{C}$ endowed with anti-holomorphic involutions τ so that

$$\tilde{f}\circ\tau=\Omega\circ\tilde{f}\colon\tilde{C}\longrightarrow X.$$

Such a triple $(\tilde{C},\tau,\tilde{f})$ will be called an Ω-invariant map to X. The notion of isomorphic stable maps in GW-theory naturally extends to that of isomorphic stable Ω-invariant maps, by requiring compatibility with the anti-holomorphic involutions. Such triples have three discrete parameters: the degree β of $\tilde{f}$, the genus g of $\tilde{C}$, and the number h of fixed components of τ, which roughly correspond to the boundary components of $\tilde{C}/\tau$. Each moduli space $\overline{\mathfrak{M}}_{g,h,0}(X,\Omega,\beta)$ of stable Ω-invariant maps with fixed distinct parameters is expected to carry a virtual fundamental class, giving rise to open and unoriented GW-invariants.[1]

If $\tilde{C}$ is a smooth Riemann surface of genus 0 (i.e. $\tilde{C}=\mathbb{P}^1$), the only possible quotients $\tilde{C}/\tau$ are $\mathbb{R}P^2$ $(h=0)$ and the disk $(h=1)$. The former invariants should vanish according to [W, Section 3.3]. Disk invariants of symplectic 4- and 6-folds endowed with anti-symplectic involutions are defined in [So]. Such manifolds include smooth Calabi-Yau complete intersection threefolds $X_{\mathbf{a}}\subset\mathbb{P}^{n-1}$ preserved by the involution

$$\Omega:\mathbb{P}^{n-1}\longrightarrow\mathbb{P}^{n-1},\qquad \Omega([z_1,z_2,\ldots,z_n])\equiv\begin{cases}[\bar{z}_2,\bar{z}_1,\ldots,\bar{z}_n,\bar{z}_{n-1}], & \text{if } 2|n;\\ [\bar{z}_2,\bar{z}_1,\ldots,\bar{z}_{n-1},\bar{z}_{n-2},\bar{z}_n], & \text{if } 2\not|n.\end{cases} \tag{1.9}$$

The disk invariants are computed in [PSoW] in the case $\mathbf{a}=(5)$ and for other CY CI threefolds $X_{\mathbf{a}}$ in [Sh]. In these cases, the disk invariants are related to Euler classes of certain vector bundles over moduli spaces of stable Ω-invariant maps to $\mathbb{P}^{n-1}$. The Localization Theorem [ABo] then reduces these invariants to sums over graphs.

Positive-genus analogues of the disk invariants of [So] have not yet been defined mathematically. However, physical considerations of [W] lead to explicit localization data for such invariants

[1] The last subscript in $\overline{\mathfrak{M}}_{g,h,0}(X,\Omega,\beta)$ indicates no marked points.

of $(X_{\mathbf{a}}, \Omega)$, whenever $X_{\mathbf{a}}$ is a Calabi-Yau complete intersection threefold. As in the disk case, the localization data describes the invariants as sums of rational functions in several variables over graphs; the resulting sums are in particular predicted to be weight-independent, i.e. not dependent on the variables involved. If $\tilde{C}$ is a smooth Riemann surface of genus 1 (i.e. $\tilde{C}$ is a two-torus), the only possible quotients $\tilde{C}/\tau$ are the Klein bottle ($h=0$), the Mobius band ($h=1$), and the annulus ($h=2$). According to [W, Section 3.3], the Mobius band invariants vanish. In the other two genus 1 subcases, the invariants are predicted to be described by explicit mirror formulas.

After reviewing the equivariant setting of [W] in Section 3.2, we recall the graph-sum description of the annulus invariants in Section 6.1 and of the Klein bottle invariants in Section 7.1. We then confirm the mirror symmetry predictions of [W] for these invariants, *without* assuming weight independence; see Theorems 3 and 4 below. This confirmation implies that the annulus and Klein bottle invariants of Calabi-Yau complete intersections threefolds $X_{\mathbf{a}}$ are well-defined (independent of the torus weights). Since the power series $I_0(q)$, $I_1(q)$, $I_2(q)$, and $J(q)$ defined by (1.3) and (1.4) are the same for the tuples $(a_1,\ldots,a_l)$ and $(a_1,\ldots,a_l,1)$, these invariants for $X_{(a_1,\ldots,a_l)}$ and $X_{(a_1,\ldots,a_l,1)}$ are the same, as expected. Both types of invariants vanish for odd-degree maps; thus, both theorems concern only even-degree invariants.[2]

Theorem 3. *The degree $2d$ annulus invariants A_{2d} as described in Definition 6.1 are weight-independent and satisfy*

$$Q\frac{\mathrm{d}}{\mathrm{d}Q}\Big[\sum_{d=1}^{\infty} Q^d A_{2d}\Big] = -\frac{1}{2\langle\mathbf{a}\rangle}\frac{I_1(q)}{I_2(q)}\left[\Big\{Q\frac{\mathrm{d}}{\mathrm{d}Q}\Big\}^2 Z_{disk}(Q)\right]^2, \tag{1.10}$$

where q and Q are related by the mirror map (1.4) and $Z_{disk}(Q)$ is the disk potential given by (1.11).

This confirms the prediction of [W, (5.21),(5.22)] with $f_t^{(0,2)}=0$. In [W], $\tilde{n}_d^{(0,2)}=A_d$, the variables $(z, q=\mathrm{e}^t)$ are our variables (q,Q),

$$-\mathrm{i}\Delta(q) = Z_{disk}(Q) \equiv \sum_{d\in\mathbb{Z}^+\text{ odd}} Q^{\frac{d}{2}} N_d^{disk} \tag{1.11}$$

is the disk potential (describing the disk invariants N_d^{disk}), and the power series C is the genus 0 generating function whose third derivative with respect to t is given by (1.8). It is shown in [Sh], as well as in [PSoW] in the $\mathbf{a}=(5)$ case, that

$$Z_{disk}(Q) = \frac{2}{I_0(q)}\sum_{d\in\mathbb{Z}^+\text{ odd}} q^{\frac{d}{2}}\frac{\prod_{r=1}^{l}(a_r d)!!}{(d!!)^n} \tag{1.12}$$

whenever all components of $\mathbf{a}$ are odd; otherwise, $Z_{disk}(Q)=0$. This formula is obtained from equation (6.10), which is the disk analogue of the graph-sum definition (6.9) of the annulus invariants. The argument in [PSoW] deducing (1.12) from (6.10) is rather delicate, limited to the case $\mathbf{a}=(5)$, and relies on the weight independence of the right-hand side in (6.11), which is an a priori fact in the disk case. The $(b,p)=(0,0)$ case of Lemma 6.2 (which is used in the proof of Theorem 3) gives a simple direct argument for this step in [PSoW]; this approach works for all

[2] On the other hand, the disk invariants in even degrees are expected to vanish according to [W]; see also [PSoW, Section 1.5].

tuples $\mathbf{a}$ and does not presume weight independence. A similar argument is used in [Sh], based on an independently discovered variation of Lemma 6.2 which is applicable in an overlapping set of cases.

Theorem 4. *The degree 2d one-point Klein bottle invariants $\tilde{K}_{2d}$ as described in Definition 7.1 are weight-independent and satisfy*

$$\sum_{d=1}^{\infty} Q^d \tilde{K}_{2d} = -Q\frac{\mathrm{d}}{\mathrm{d}Q} \ln\left((1-\mathbf{a}^{\mathbf{a}}q)^{1/4} I_1(q)\right), \tag{1.13}$$

where q and Q are related by the mirror map (1.4).

By the divisor relation, this corresponds to the prediction of [W, (5.26),(5.27)]. In [W], the variables (z,q) are our variables (q,Q),

$$\tilde{n}_d^{(1,0)_k} = \tilde{K}_d/d, \qquad \text{and} \quad diss = 1-\mathbf{a}^{\mathbf{a}}q.$$

1.3 Outline of the paper

Theorems 1 and 2 are immediate consequences of Theorem 5 in Section 2. Theorem 5 follows immediately from Theorem 6; the latter is an equivariant version of the former and extends [Z10, Theorem 1.1] from line bundles to split bundles of arbitrary rank. Theorem 5 is preceded by an explicit recursive formula which facilitates the computation of genus 0 descendant invariants of the Fano projective complete intersections as well. An example of such a computation is given for the primary GW-invariants of $X_3 \subset \mathbb{P}^4$. The equivariant setting is introduced in Section 3, where we state Theorem 6.

It is well-known that equivariant one-point GW-invariants of $X_{\mathbf{a}}$ are expressed in terms of an equivariant version of the hypergeometric series F; see (4.3). It is shown in [Z10] that closed formulas for two-point genus 0 GW-invariants of hypersurfaces are explicit transforms of the one-point formulas; by Theorem 6 in this paper, this is the case for all projective complete intersections. The first part of Section 4 extends the proof of the analogous result in [Z10] to the proof of Theorem 6. It consists of showing that both sides satisfy certain good properties which guarantee uniqueness. The proof that the two-point GW-invariants satisfy these properties is nearly identical to the analogous statement of [Z10] which uses the Atiyah-Bott Localization Theorem [ABo]; details are given in Section 4.3. The main difference with [Z10] occurs in constructing the equivariant hypergeometric series with required properties; see (3.6) and (3.7). Unlike [Z10], we pay close attention to the Fano case as well, giving explicit recursions for the structure coefficients of Theorem 6 and thus for the non-equivariant structure coefficients of Theorem 5. In Section 5, we obtain relations between the equivariant coefficients appearing in Theorem 6; these relations are used in the proofs of Theorems 3 and 4 in the rest of the paper.

In Section 6, we use the explicit equivariant recursions of Theorem 6 in the case $X_{\mathbf{a}}$ is a Calabi-Yau complete intersection threefold to study a recent prediction of Walcher [W] concerning annulus GW-invariants. In particular, we use the Residue Theorem on S^2 to show that the localization formulas given in [W] are indeed weight-independent and sum up to the simple expression obtained in [W] based on physical considerations; see Theorem 3. Along the way, we streamline one of the steps used in [PSoW] to obtain a mirror formula for the disk invariants of the quintic threefold $X_{(5)}$.

In Section 7, we show that the natural one-point analogues of the localization formulas in [W] for Klein bottle invariants are indeed weight-independent and yield the closed formula predicted in [W]; see Theorem 4. Unlike the annulus case, this case has a truly genus one flavor; the proof of Theorem 4 thus has little (if any) similarity to the proof of Theorem 3. We prove Theorem 4 by breaking the graphs at the special vertex of each loop and using a special property of the generating functions for one-point genus 0 GW-invariants, as in [Z09b] and [Po]. In contrast to [Z09b] and [Po], we do not presuppose that the sums are weight-independent and thus carry out the final step in the computation in a completely different way, without using the Residue Theorem on S^2 (it is still used in earlier steps).

Sections 6 and 7 can thus be seen as the analogues for the annulus and Klein bottle invariants of the localization computations confirming mirror symmetry predictions for the closed genus 0 invariants ([Gi96], [Gi99], [LLY]), the closed genus 1 invariants ([Z09b], [Po]), and the disk invariants ([PSoW]). However, the localization setup which serves as the starting point for these computations still requires a full mathematical justification for the annulus and Klein bottle invariants and cannot be presumed to be weight-independent (the last property is used in [Z09b], [Po], and [PSoW]).

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2 Main Theorem for Closed GW-Invariants

Gromov-Witten invariants of a complete intersection $X_{\mathbf{a}}\subset\mathbb{P}^{n-1}$ can be computed from twisted GW-invariants of $\mathbb{P}^{n-1}$. Let $\pi\colon\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d)$ be the universal curve and $\mathrm{ev}:\mathfrak{U}\longrightarrow\mathbb{P}^{n-1}$ the natural evaluation map; see [MirSym, Section 24.3]. Denote by

$$\mathcal{V}_{\mathbf{a}}\longrightarrow\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d)$$

the vector bundle corresponding to the locally free sheaf

$$\bigoplus_{k=1}^{l}\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^{n-1}}(a_k)\longrightarrow\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d).$$

The Euler class $e(\mathcal{V}_{\mathbf{a}})$ relates genus 0 GW-invariants of $X_{\mathbf{a}}$ to genus 0 GW-invariants of $\mathbb{P}^{n-1}$ by

$$\int_{[\overline{\mathfrak{M}}_{0,m}(X_{\mathbf{a}},d)]^{vir}}\eta=\int_{[\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d)]}\eta\,e(\mathcal{V}_{\mathbf{a}})\qquad\forall\,\eta\in H^*\big(\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d)\big);\tag{2.1}$$

see [BDPP, Section 2.1.2].

For each $i=1,\ldots,m$, there is a well-defined bundle map

$$\widetilde{\mathrm{ev}}_i\colon\mathcal{V}_{\mathbf{a}}\longrightarrow\mathrm{ev}_i^*\bigoplus_{k=1}^{l}\mathcal{O}_{\mathbb{P}^{n-1}}(a_k),\qquad\widetilde{\mathrm{ev}}_i\big([\mathcal{C},f;\xi]\big)=\big[\xi(x_i(\mathcal{C}))\big],$$

where $x_i(\mathcal{C})$ is the i-th marked point of $\mathcal{C}$. Since it is surjective, its kernel is again a vector bundle. Let

$$\mathcal{V}'_{\mathbf{a}}\equiv\ker\widetilde{\mathrm{ev}}_1\longrightarrow\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d)\qquad\text{and}\qquad\mathcal{V}''_{\mathbf{a}}\equiv\ker\widetilde{\mathrm{ev}}_2\longrightarrow\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d),$$

whenever $m\ge1$ and $m\ge2$, respectively.[3] With $\mathrm{ev}_1,\mathrm{ev}_2:\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)\longrightarrow\mathbb{P}^{n-1}$ denoting the evaluation maps at the two marked points, define

$$Z_p(\hbar,Q)\equiv \mathrm{H}^{l+p}+\sum_{d=1}^{\infty}Q^d\mathrm{ev}_{1*}\bigg[\frac{e(\mathcal{V}''_{\mathbf{a}})\mathrm{ev}_2^*\mathrm{H}^{l+p}}{\hbar-\psi_1}\bigg]\in\big(H^*(\mathbb{P}^{n-1})\big)[\hbar^{-1}]\big[\big[Q\big]\big] \tag{2.2}$$

for $p\in\mathbb{Z}$ with $p\ge -l$, and set

$$Z^*(\hbar_1,\hbar_2,Q)\equiv\sum_{d=1}^{\infty}Q^d\,(\mathrm{ev}_1\times\mathrm{ev}_2)_*\bigg[\frac{e(\mathcal{V}_{\mathbf{a}})}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}\bigg]\in\big(H^*(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1})\big)[\hbar_1^{-1},\hbar_2^{-1}]\big[\big[Q\big]\big]. \tag{2.3}$$

These power series determine all numbers (2.1) with $\eta=(\psi_1^{p_1}\mathrm{ev}_1^*\mathrm{H}^{b_1})(\psi_2^{p_2}\mathrm{ev}_2^*\mathrm{H}^{b_2})$. The motivation behind the choice of indexing in (2.2) is analogous to the $l=1$ case in [Z10, Section 1.1].

Define

$$F_{-l}(w,q)\equiv\sum_{d=0}^{\infty}q^dw^{d\nu_{\mathbf{a}}}\frac{\prod\limits_{k=1}^{l}\prod\limits_{r=0}^{a_kd-1}(a_kw+r)}{\prod\limits_{r=1}^{d}(w+r)^n}\in\mathcal{P}; \tag{2.4}$$

$$F_{-l+p}\equiv\mathbf{D}^pF_{-l}=\mathbf{M}^pF_{-l}\qquad\forall\ p=1,2,\ldots,l.$$

In particular, $F_0=F$. For $\nu_{\mathbf{a}}>0$, we also define $\mathrm{c}_{p,s}^{(d)},\widetilde{\mathrm{c}}_{p,s}^{(d)}\in\mathbb{Q}$ with $p,d,s\ge0$ by

$$\sum_{d=0}^{\infty}\sum_{s=0}^{\infty}\mathrm{c}_{p,s}^{(d)}w^sq^d=\sum_{d=0}^{\infty}q^d\frac{(w+d)^p\prod\limits_{k=1}^{l}\prod\limits_{r=1}^{a_kd}(a_kw+r)}{\prod\limits_{r=1}^{d}(w+r)^n}=w^p\mathbf{D}^pF\big(w,q/w^{\nu_{\mathbf{a}}}\big), \tag{2.5}$$

$$\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge0}}\sum_{r=0}^{p-\nu_{\mathbf{a}}d_1}\widetilde{\mathrm{c}}_{p,r}^{(d_1)}\mathrm{c}_{r,s}^{(d_2)}=\delta_{d,0}\delta_{p,s}\qquad\forall\,d,s\in\mathbb{Z}^{\ge0},\ s\le p-\nu_{\mathbf{a}}d. \tag{2.6}$$

Since $\mathrm{c}_{p,s}^{(0)}=\delta_{p,s}$, (2.6) expresses $\widetilde{\mathrm{c}}_{p,s}^{(d)}$ in terms of the numbers $\widetilde{\mathrm{c}}_{p,r}^{(d_1)}$ with $d_1<d$; the numbers $\widetilde{\mathrm{c}}_{p,s}^{(d)}$ with $s>p-\nu_{\mathbf{a}}d$ will not be needed. For example,

$$\widetilde{\mathrm{c}}_{p,s}^{(0)}=\delta_{p,s}\,,\qquad \sum_{s=0}^{p-\nu_{\mathbf{a}}}\widetilde{\mathrm{c}}_{p,s}^{(1)}w^s+\langle\mathbf{a}\rangle\frac{\prod\limits_{k=1}^{l}\prod\limits_{r=1}^{a_k-1}(a_kw+r)}{(w+1)^{n-l-p}}\in w^{p-\nu_{\mathbf{a}}+1}\mathbb{Q}[[w]]. \tag{2.7}$$

For $p\ge1$, set

$$F_p(w,q)\equiv\begin{cases}\mathbf{M}^pF(w,q), & \text{if } \nu_{\mathbf{a}}=0;\\ \sum\limits_{d=0}^{\infty}\sum\limits_{s=0}^{p-\nu_{\mathbf{a}}d}\frac{\widetilde{\mathrm{c}}_{p,s}^{(d)}\,q^d}{w^{p-\nu_{\mathbf{a}}d-s}}\mathbf{D}^sF(w,q), & \text{if } \nu_{\mathbf{a}}>0.\end{cases}$$

Thus, $F_p=\mathbf{D}^pF$ unless $p\ge\nu_{\mathbf{a}}$.

[3] In [BDPP, Part 4], $\mathcal{V}_{\mathbf{a}}=W_{m,d}$, $\varepsilon_{m,d}=e(W_{m,d})$, and $\varepsilon'_{2,d}=e(W'_{2,d})$. In [Gi96, Section 9], $e(\mathcal{V}_{\mathbf{a}})=E_d$ and $e(\mathcal{V}'_{\mathbf{a}})=E'_d$.

Theorem 5. *For every l-tuple of positive integers $\mathbf{a}=(a_1,a_2,\ldots,a_l)$,*

$$Z^*(\hbar_1,\hbar_2,Q)=\frac{\langle\mathbf{a}\rangle}{\hbar_1+\hbar_2}\sum_{\substack{p_1+p_2=n-1-l\\ p_1,p_2\ge0}}\big(-\pi_1^*\mathrm{H}^{l+p_1}\pi_2^*\mathrm{H}^{l+p_2}+\pi_1^*Z_{p_1}(\hbar_1,Q)\pi_2^*Z_{p_2}(\hbar_2,Q)\big),$$

where $\pi_1,\pi_2\colon\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}\longrightarrow\mathbb{P}^{n-1}$ are the two projection maps. For every $p\ge -l$,

$$Z_p(\hbar,Q)=\mathrm{H}^{l+p}\begin{cases}\mathrm{e}^{-J(q)\frac{\mathrm{H}}{\hbar}}\frac{F_p\left(\frac{\mathrm{H}}{\hbar},q\right)}{I_p(q)}, & \text{if } \nu_{\mathbf{a}}=0,\\ \mathrm{e}^{-\mathbf{a}!q\frac{\mathrm{H}}{\hbar}}F_p\left(\frac{\mathrm{H}}{\hbar},q\right), & \text{if } \nu_{\mathbf{a}}=1,\\ F_p\left(\frac{\mathrm{H}}{\hbar},q\right), & \text{if } \nu_{\mathbf{a}}\ge2,\end{cases}\qquad\text{with}\qquad Q=\begin{cases}q\,\mathrm{e}^{J(q)}, & \text{if } \nu_{\mathbf{a}}=0,\\ \mathrm{H}^{\nu_{\mathbf{a}}}q, & \text{if } \nu_{\mathbf{a}}\ge1,\end{cases}$$

where J is as in (1.4).

Since dropping a component of $\mathbf{a}$ equal to 1 has no effect on the power series F in (1.1), this also has no effect on the right-hand sides of the two formulas in Theorem 5, as expected. If $\mathbf{a}=\emptyset$ and $n\ge2$,

$$\mathbf{D}^pF(w,q)=1+\sum_{d=1}^{\infty}q^d\frac{(w+d)^pw^{nd-p}}{\prod_{r=1}^{d}(w+r)^n},\qquad Z_p(\hbar,Q)=\mathrm{H}^p\,\mathbf{D}^pF\left(\frac{\mathrm{H}}{\hbar},\frac{Q}{\mathrm{H}^n}\right)\qquad\forall\,p\le n-1,$$

giving Theorem 1. Theorem 2 is just the $\nu_{\mathbf{a}}=0$ case of Theorem 5.

Remark 2. If $n-1-l<2\nu_{\mathbf{a}}$, only the coefficients $\widetilde{\mathrm{c}}_{p,s}^{(1)}$ matter for the purposes of Theorem 5; these are given by (2.7). For example, if $n=5$ and $\mathbf{a}=(3)$, then $\widetilde{\mathrm{c}}_{2,0}^{(1)}=\widetilde{\mathrm{c}}_{3,0}^{(1)}=-6$ and $\widetilde{\mathrm{c}}_{3,1}^{(1)}=-21$. Thus,

$$F_p(w,q)=\sum_{d=0}^{\infty}q^dw^{2d-p}(w+d)^p\frac{\prod_{r=1}^{3d}(3w+r)}{\prod_{r=1}^{d}(w+r)^5}\qquad\text{if }\ p=0,1;$$

$$F_2(w,q)=1+3q\sum_{d=0}^{\infty}q^dw^{2d}(w+d)\frac{(7w+7d+5)\prod_{r=1}^{3d}(3w+r)}{(w+d+1)^2\prod_{r=1}^{d}(w+r)^5};$$

$$F_3(w,q)=1+6q\sum_{d=0}^{\infty}q^dw^{2d-1}(w+d)^2\frac{\prod_{r=1}^{3d}(3w+r)}{(w+d+1)\prod_{r=1}^{d}(w+r)^5}.$$

Thus, modulo $(\hbar^{-1})^2$,

$$F_0\left(\frac{\mathrm{H}}{\hbar},\frac{Q}{\mathrm{H}^2}\right)\cong1,\qquad F_1\left(\frac{\mathrm{H}}{\hbar},\frac{Q}{\mathrm{H}^2}\right)\cong1+6\frac{Q}{\mathrm{H}}\hbar^{-1},$$

$$F_2\left(\frac{\mathrm{H}}{\hbar},\frac{Q}{\mathrm{H}^2}\right)\cong1+15\frac{Q}{\mathrm{H}}\hbar^{-1},\qquad F_3\left(\frac{\mathrm{H}}{\hbar},\frac{Q}{\mathrm{H}^2}\right)\cong1+6\frac{Q}{\mathrm{H}}\hbar^{-1}+18\frac{Q^2}{\mathrm{H}^3}\hbar^{-1}.$$

The two identities in Theorem 5 give

$$\begin{aligned}\sum_{d=1}^{\infty} Q^d \sum_{p_1,p_2\ge 0} \left\langle \frac{\mathrm{H}^{3-p_1}}{\hbar_1-\psi}, \frac{\mathrm{H}^{3-p_2}}{\hbar_2-\psi} \right\rangle_d^{X_3} \mathrm{H}_1^{p_1}\mathrm{H}_2^{p_2} \\ = \frac{3}{\hbar_1+\hbar_2} \sum_{\substack{p_1+p_2=3\\ p_1,p_2\ge 0}} \left[-1 + F_{p_1}\left(\frac{\mathrm{H}_1}{\hbar_1}, \frac{Q}{\mathrm{H}_1^2}\right) F_{p_2}\left(\frac{\mathrm{H}_2}{\hbar_2}, \frac{Q}{\mathrm{H}_2^2}\right)\right] \mathrm{H}_1^{p_1}\mathrm{H}_2^{p_2}\,,\end{aligned} \tag{2.8}$$

modulo $\mathrm{H}_1^4, \mathrm{H}_2^4$. Thus, considering the coefficient of $\hbar_1^{-1}\hbar_2^{-1}$ in (2.8), we find that

$$\langle \mathrm{H}^3 \rangle_1^{X_3} = \langle \mathrm{H}, \mathrm{H}^3 \rangle_1^{X_3} = 18, \qquad \langle \mathrm{H}^2, \mathrm{H}^2 \rangle_1^{X_3} = 45, \qquad \langle \mathrm{H}^3, \mathrm{H}^3 \rangle_2^{X_3} = 54.$$

This agrees with the usual Schubert calculus computations on $G(2,5)$ and $G(3,5)$.

In general, if $d\in\mathbb{Z}^{\ge 0}$ and $\nu_{\mathbf{a}}d \le p \le n-1-l$, then

$$\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 0}} \tilde{\mathrm{c}}^{(d_1)}_{p-\nu_{\mathbf{a}}d_2,p-\nu_{\mathbf{a}}d} \tilde{\mathrm{c}}^{(d_2)}_{n-1-l-p+\nu_{\mathbf{a}}d_2,n-1-l-p} = \begin{cases} 1, & \text{if } d=0;\\ -\mathbf{a}^{\mathbf{a}}, & \text{if } d=1;\\ 0, & \text{if } d\ge 2.\end{cases} \tag{2.9}$$

This identity is used to simplify four-pointed formulas in [Z11]; we verify it in the appendix.

3 Equivariant Setting

In Section 3.1 below, we review the equivariant setup used in [Z10], closely following [Z10, Section 3.1]. After defining equivariant versions of the generating functions Z_p and Z^* and of the hypergeometric series F and F_{-l}, we state an equivariant version of Theorem 5; see Theorem 6 below. We conclude Section 3.1 with explicit recursive formulas for the equivariant structure coefficients that appear in Theorem 6. In Section 3.2, we describe the restricted equivariant setup used in [W].

The quotient of the classifying space for the n-torus $\mathbb{T}$ is $B\mathbb{T}\equiv(\mathbb{P}^\infty)^n$. Thus, the group cohomology of $\mathbb{T}$ is

$$H_{\mathbb{T}}^* \equiv H^*(B\mathbb{T}) = \mathbb{Q}[\alpha_1,\ldots,\alpha_n],$$

where $\alpha_i \equiv \pi_i^* c_1(\gamma^*)$, $\gamma\longrightarrow\mathbb{P}^\infty$ is the tautological line bundle, and $\pi_i\colon(\mathbb{P}^\infty)^n\longrightarrow\mathbb{P}^\infty$ is the projection to the i-th component. The field of fractions of $H_{\mathbb{T}}^*$ will be denoted by

$$\mathbb{Q}_\alpha \equiv \mathbb{Q}(\alpha_1,\ldots,\alpha_n).$$

We denote the equivariant $\mathbb{Q}$-cohomology of a topological space M with a $\mathbb{T}$-action by $H_{\mathbb{T}}^*(M)$. If the $\mathbb{T}$-action on M lifts to an action on a complex vector bundle $V\longrightarrow M$, let $\mathbf{e}(V)\in H_{\mathbb{T}}^*(M)$ denote the equivariant Euler class of V. A continuous $\mathbb{T}$-equivariant map $f\colon M\longrightarrow M'$ between two compact oriented manifolds induces a pushforward homomorphism

$$f_*\colon H_{\mathbb{T}}^*(M)\longrightarrow H_{\mathbb{T}}^*(M').$$

3.1 Setting for closed GW-invariants

The standard action of $\mathbb{T}$ on $\mathbb{C}^{n-1}$,

$$\big(\mathrm{e}^{\mathrm{i}\theta_1},\ldots,\mathrm{e}^{\mathrm{i}\theta_n}\big)\cdot(z_1,\ldots,z_n)\equiv\big(\mathrm{e}^{\mathrm{i}\theta_1}z_1,\ldots,\mathrm{e}^{\mathrm{i}\theta_n}z_n\big),$$

descends to a $\mathbb{T}$-action on $\mathbb{P}^{n-1}$. The latter has n fixed points,

$$P_1=[1,0,\ldots,0], \qquad P_2=[0,1,0,\ldots,0], \quad \ldots, \quad P_n=[0,\ldots,0,1]. \tag{3.1}$$

The curves preserved by this action are the lines through the fixed points,

$$\ell_{ij}\equiv\big\{[z_1,z_2,\ldots,z_n]\in\mathbb{P}^{n-1}:\ z_k=0\ \ \forall\,k\notin\{i,j\}\big\}.$$

This standard $\mathbb{T}$-action on $\mathbb{P}^{n-1}$ lifts to a natural $\mathbb{T}$-action on the tautological line bundle $\gamma\longrightarrow\mathbb{P}^{n-1}$, since $\gamma\subset\mathbb{P}^{n-1}\times\mathbb{C}^n$ is preserved by the diagonal $\mathbb{T}$-action. With

$$x\equiv\mathbf{e}(\gamma^*)\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$$

denoting the equivariant hyperplane class, the equivariant cohomology of $\mathbb{P}^{n-1}$ is given by

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1})=\mathbb{Q}[x,\alpha_1,\ldots,\alpha_n]\big/(x-\alpha_1)\ldots(x-\alpha_n). \tag{3.2}$$

Let $x_1,x_2\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1})$ be the pull-backs of x by the two projection maps.

The standard action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$ induces $\mathbb{T}$-actions on $\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d)$, $\mathfrak{U}$, $\mathcal{V}_{\mathbf{a}}$, $\mathcal{V}'_{\mathbf{a}}$, and $\mathcal{V}''_{\mathbf{a}}$; see Sections 1 and 2 for the notation. Thus, $\mathcal{V}_{\mathbf{a}}$, $\mathcal{V}'_{\mathbf{a}}$, and $\mathcal{V}''_{\mathbf{a}}$ have well-defined equivariant Euler classes

$$\mathbf{e}(\mathcal{V}_{\mathbf{a}}),\mathbf{e}(\mathcal{V}'_{\mathbf{a}}),\mathbf{e}(\mathcal{V}''_{\mathbf{a}})\in H^*_{\mathbb{T}}\big(\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d)\big).$$

The universal cotangent line bundle for the i-th marked point also has a well-defined equivariant Euler class, which will still be denoted by ψ_i. In analogy with (2.2) and (2.3), we define

$$\mathcal{Z}_p(x,\hbar,Q)\equiv x^{l+p}+\sum_{d=1}^{\infty}Q^d\mathrm{ev}_{1*}\bigg[\frac{\mathbf{e}(\mathcal{V}''_{\mathbf{a}})\mathrm{ev}_2^*x^{l+p}}{\hbar-\psi_1}\bigg]\in\big(H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\big)\big[\big[\hbar^{-1},Q\big]\big], \tag{3.3}$$

$$\mathcal{Z}^*(x_1,x_2,\hbar_1,\hbar_2,Q)\equiv\sum_{d=1}^{\infty}Q^d\,(\mathrm{ev}_1\times\mathrm{ev}_2)_*\bigg[\frac{\mathbf{e}(\mathcal{V}_{\mathbf{a}})}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}\bigg] \tag{3.4}$$
$$\in\big(H^*_{\mathbb{T}}(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1})\big)\big[\big[\hbar_1^{-1},\hbar_2^{-1},Q\big]\big],$$

where $p\in\mathbb{Z}$ with $p\ge -l$ and $\mathrm{ev}_1,\mathrm{ev}_2:\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)\longrightarrow\mathbb{P}^{n-1}$, as before. By (3.3) and (3.2),

$$\sum_{r=0}^{n}(-1)^r\sigma_r\mathcal{Z}_{p-r}(x,\hbar,Q)$$
$$=x^{l+p-n}\sum_{r=0}^{n}(-1)^r\sigma_r x^{n-r}+\sum_{d=1}^{\infty}Q^d\mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\mathcal{V}''_{\mathbf{a}})\mathrm{ev}_2^*\Big(x^{l+p-n}\sum\limits_{r=0}^{n}(-1)^r\sigma_r x^{n-r}\Big)}{\hbar-\psi_1}\right]=0 \tag{3.5}$$

if $n-l\le p\le n-1$, where $\sigma_r\in\mathbb{Q}_\alpha$ is the r-th elementary symmetric polynomial in $\alpha_1,\ldots,\alpha_n$.

The equivariant versions of the hypergeometric series F and F_{-l} that we need are

$$\mathcal{Y}(x,\hbar,q) \equiv \sum_{d=0}^{\infty} q^d \frac{\prod_{k=1}^{l}\prod_{r=1}^{a_k d}(a_k x + r\hbar)}{\prod_{r=1}^{d}\prod_{k=1}^{n}(x-\alpha_k+r\hbar)}\,, \tag{3.6}$$

$$\mathcal{Y}_{-l}(x,\hbar,q) \equiv \sum_{d=0}^{\infty} q^d \frac{\prod_{k=1}^{l}\prod_{r=0}^{a_k d-1}(a_k x + r\hbar)}{\prod_{r=1}^{d}\prod_{k=1}^{n}(x-\alpha_k+r\hbar)}\,. \tag{3.7}$$

With $I_p(q)\equiv 1$ if $\nu_{\mathbf{a}}\neq 0$ (and given by (1.3) if $\nu_{\mathbf{a}}=0$), for $p\geq 0$ define $\mathfrak{D}^p\mathcal{Y}_0(x,\hbar,q)$ inductively by

$$\begin{aligned}\mathfrak{D}^0\mathcal{Y}_0(x,\hbar,q) &\equiv \mathcal{Y}_0(x,\hbar,q) \equiv \frac{x^l}{I_0(q)}\mathcal{Y}(x,\hbar,q),\\ \mathfrak{D}^p\mathcal{Y}_0(x,\hbar,q) &\equiv \frac{1}{I_p(q)}\left\{x+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}\mathfrak{D}^{p-1}\mathcal{Y}_0(x,\hbar,q) \qquad \forall\; p\geq 1.\end{aligned} \tag{3.8}$$

For $p=0,1,\ldots,l-1$, let

$$\begin{aligned}\mathcal{Y}_{-l+p}(x,\hbar,q) &\equiv \mathfrak{D}^{-l+p}\mathcal{Y}_0(x,\hbar,q) \equiv \left\{x+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}^p \mathcal{Y}_{-l}(x,\hbar,q)\\ &= \sum_{d=0}^{\infty} q^d (x+d\hbar)^p \frac{\prod_{k=1}^{l}\prod_{r=0}^{a_k d-1}(a_k x + r\hbar)}{\prod_{r=1}^{d}\prod_{k=1}^{n}(x-\alpha_k+r\hbar)}\,.\end{aligned} \tag{3.9}$$

Thus, $\mathcal{Y}_0(x,\hbar,q) = \frac{1}{I_0(q)}\left\{x+\hbar\, q\frac{\mathrm{d}}{\mathrm{d}q}\right\}\mathfrak{D}^{-1}\mathcal{Y}_0(x,\hbar,q)$.

Theorem 6. *For every l-tuple of positive integers $\mathbf{a}=(a_1,a_2,\ldots,a_l)$,*

$$\mathcal{Z}^*(x_1,x_2,\hbar_1,\hbar_2,Q) = \frac{\langle\mathbf{a}\rangle}{\hbar_1+\hbar_2} \sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2\geq 0}} (-1)^r\sigma_r\big(-x_1^{l+p_1}x_2^{p_2} + \mathcal{Z}_{p_1}(x_1,\hbar_1,Q)\mathcal{Z}_{p_2-l}(x_2,\hbar_2,Q)\big), \tag{3.10}$$

$\sigma_r\in\mathbb{Q}_\alpha$ is the r-th elementary symmetric polynomial in $\alpha_1,\ldots,\alpha_n$. Furthermore, there exist $\widetilde{\mathcal{C}}^{(r)}_{p,s}(q)\in q\cdot\mathbb{Q}[\alpha_1,\ldots,\alpha_n][[q]]$ such that the coefficient of q^d in $\widetilde{\mathcal{C}}^{(r)}_{p,s}(q)$ is a degree $r-\nu_{\mathbf{a}}d$ homogeneous symmetric polynomial in $\alpha_1,\alpha_2,\ldots,\alpha_n$ and

$$\mathcal{Z}_p(x,\hbar,Q) = \begin{cases}\mathcal{Y}_p(x,\hbar,Q), & \text{if } \nu_{\mathbf{a}}\geq 2,\\ \mathrm{e}^{-\frac{Q}{\hbar}\mathbf{a}!}\mathcal{Y}_p(x,\hbar,Q), & \text{if } \nu_{\mathbf{a}}=1,\\ \mathrm{e}^{-J(q)\frac{x}{\hbar}}\mathrm{e}^{-C_1(q)\frac{\sigma_1}{\hbar}}\mathcal{Y}_p(x,\hbar,q), & \text{if } \nu_{\mathbf{a}}=0,\end{cases} \tag{3.11}$$

where

$$\mathcal{Y}_p(x,\hbar,q) \equiv \mathfrak{D}^p\mathcal{Y}_0(x,\hbar,q) + \sum_{r=1}^{p}\sum_{s=0}^{p-r}\widetilde{\mathcal{C}}^{(r)}_{p,s}(q)\hbar^{p-r-s}\mathfrak{D}^s\mathcal{Y}_0(x,\hbar,q), \tag{3.12}$$

with Q and q related by the mirror map (1.4) and $C_1(q)\in q\cdot\mathbb{Q}[[q]]$ given by (4.2).

Theorem 6 implies Theorem 5 as follows. Setting $\alpha_1,\ldots,\alpha_n=0$ in (3.10) and using that $Z_p(\hbar,Q)=0$ if $p\ge n-l$ by (2.2), we obtain the first identity in Theorem 5. By (1.1), (1.2), (3.6), and (3.8),

$$\mathfrak{D}^p\mathcal{Y}_0(x,\hbar,q)\big|_{\alpha=0}=x^{l+p}\cdot\begin{cases}\frac{1}{I_p(q)}\mathbf{M}^pF(x/\hbar,q), & \text{if } \nu_{\mathbf{a}}=0;\\ \mathbf{D}^pF(x/\hbar,q/x^{\nu_{\mathbf{a}}}), & \text{if } \nu_{\mathbf{a}}>0.\end{cases}\tag{3.13}$$

Thus, if $\nu_{\mathbf{a}}=0$, setting $\alpha_1,\ldots,\alpha_n=0$ in Theorem 6 gives the corresponding case of the second identity in Theorem 5.

We now completely describe the power series $\widetilde{\mathcal{C}}_{p,s}^{(r)}$ of Theorem 6; it will be shown in Section 4 below that they indeed satisfy (3.11) and (3.12). For $p,r,s\ge 0$, define $\mathcal{C}_{p,s}^{(r)},\widetilde{\mathcal{C}}_{p,s}^{(r)}\in\mathbb{Q}[\alpha_1,\ldots,\alpha_n][[q]]$ by

$$x^l\hbar^p\sum_{s=0}^{\infty}\sum_{r=0}^{s}\mathcal{C}_{p,s}^{(r)}(q)x^{s-r}\hbar^{-s}=\mathfrak{D}^p\mathcal{Y}_0(x,\hbar,q)\,,\tag{3.14}$$

$$\sum_{\substack{r_1+r_2=r\\ r_1,r_2\ge 0}}\sum_{t=0}^{p-r_1}\widetilde{\mathcal{C}}_{p,t}^{(r_1)}(q)\mathcal{C}_{t,s-r_1}^{(r_2)}(q)=\delta_{r,0}\delta_{p,s}\quad\forall\, r,s\in\mathbb{Z}^{\ge 0},\ r\le s\le p.\tag{3.15}$$

By (3.6), (3.8), and (3.14), the coefficient of q^d in $\mathcal{C}_{p,s}^{(r)}$ is a degree $r-\nu_{\mathbf{a}}d$ homogeneous symmetric polynomial in α. By (3.13) and $\mathfrak{D}^p\mathcal{Y}_0\in x^{l+p}+q\cdot\mathbb{Q}_\alpha(x,\hbar)[[q]]$,

$$\mathcal{C}_{p,p}^{(0)}(q)=1,\qquad \mathcal{C}_{p,s}^{(0)}(q)=0\ \ \forall\, p>s,\qquad \mathcal{C}_{p,s}^{(r)}(q)\in\delta_{r,0}\delta_{p,s}+q\cdot\mathbb{Q}[\alpha_1,\ldots,\alpha_n]\big[\big[q\big]\big].$$

Thus, (3.15) expresses $\widetilde{\mathcal{C}}_{p,s-r}^{(r)}$ with $r\le s\le p$ in terms of the series $\widetilde{\mathcal{C}}_{p,t}^{(r_1)}$ with $r_1<r$ or $r_1=r$ and $t<s-r$; the series $\widetilde{\mathcal{C}}_{p,s}^{(r)}$ with $r+s>p$ are not needed. In particular, $\widetilde{\mathcal{C}}_{p,s}^{(0)}=\delta_{p,s}$ and the coefficient of q^d in $\widetilde{\mathcal{C}}_{p,s}^{(r)}$ is a degree $r-\nu_{\mathbf{a}}d$ homogeneous symmetric polynomial in α. If $\nu_{\mathbf{a}}>0$,

$$\mathcal{C}_{p,s}^{(\nu_{\mathbf{a}}d)}\big|_{\alpha=0}=\mathrm{c}_{p,s-\nu_{\mathbf{a}}d}^{(d)}q^d\qquad\forall\, s\ge\nu_{\mathbf{a}}d$$

by (3.14), (3.13), and (2.5). Thus, setting $\alpha=0$ in (3.15) and comparing with (2.6) with s replaced by $s-\nu_{\mathbf{a}}d$, we conclude that

$$\widetilde{\mathcal{C}}_{p,s}^{(d\nu_{\mathbf{a}})}(q)|_{\alpha=0}=\widetilde{\mathrm{c}}_{p,s}^{(d)}q^d\qquad\text{if}\quad \nu_{\mathbf{a}}\neq 0.\tag{3.16}$$

Setting $\alpha=0$ in (3.11) and (3.12) and using (3.13) and (3.16), we obtain the $\nu_{\mathbf{a}}\neq 0$ case of the second identity in Theorem 5.

In the Calabi-Yau case, $|\mathbf{a}|=n$, it is a consequence of (3.5) and (3.11) that the structure coefficients in (3.12) satisfy

$$\sum_{r=0}^{p}(-1)^r\sigma_r\widetilde{\mathcal{C}}_{n-l-r,n-l-p}^{(p-r)}(q)=(-1)^p\sigma_p\prod_{s=0}^{n-l-p}I_s(q)\qquad\forall\, p=1,2,\ldots,n-l\,.\tag{3.17}$$

This relation, a special case of which is used in the Klein bottle invariants computation in Section 7.3, is proved in Section 5.2. On the other hand, the only property of the structure coefficients $\widetilde{\mathcal{C}}_{p,s}^{(r)}$ needed to compute the closed genus 1 GW-invariants of Calabi-Yau hypersurfaces in [Z09b] is that they lie in the ideal generated by $\sigma_1,\ldots,\sigma_{n-1}$. It is shown in [Po] that the same is the case for all Calabi-Yau complete intersections.

3.2 Setting for open and unoriented GW-invariants

We now recall the equivariant setting used in the graph-sum definition of open and unoriented invariants in [W]. Throughout this section, Ω denotes the anti-holomorphic involution of $\mathbb{P}^{n-1}$ described by (1.9) and its natural lifts to the tautological line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ and to the vector bundle

$$\mathcal{L} \equiv \bigoplus_{r=1}^{l} \mathcal{O}_{\mathbb{P}^{n-1}}(a_k) \longrightarrow \mathbb{P}^{n-1}\,. \tag{3.18}$$

Define

$$\bar{\cdot}: \{1,\ldots,n\} \longrightarrow \{1,\ldots,n\} \qquad \text{by} \qquad \Omega(P_i) = P_{\bar{i}}\,,$$

where $P_1,\ldots,P_n \in \mathbb{P}^{n-1}$ are the $\mathbb{T}^n$-fixed points; see (3.1). Denote by m the integer part of $n/2$ and by $\lambda_1,\ldots,\lambda_m$ the weights of the standard representation of $\mathbb{T}^m$ on $\mathbb{C}^m$. The embedding

$$\iota\colon \mathbb{T}^m \longrightarrow \mathbb{T}^n, \qquad (u_1,u_2,\ldots,u_m) \longrightarrow \begin{cases} (u_1,u_1^{-1},\ldots,u_m,u_m^{-1}), & \text{if } n=2m; \\ (u_1,u_1^{-1},\ldots,u_m,u_m^{-1},1), & \text{if } n=2m+1; \end{cases}$$

induces $\mathbb{T}^m$-actions on $\mathbb{P}^{n-1}$ and moduli spaces of stable Ω-invariant maps to $\mathbb{P}^{n-1}$. Note that

$$(\alpha_1,\ldots,\alpha_n)\big|_{\mathbb{T}^m} = \begin{cases} (\lambda_1,-\lambda_1,\ldots,\lambda_m,-\lambda_m), & \text{if } n=2m; \\ (\lambda_1,-\lambda_1,\ldots,\lambda_m,-\lambda_m,0), & \text{if } n=2m+1. \end{cases} \tag{3.19}$$

The following lemma describes the $\mathbb{T}^m$-fixed and Ω-fixed zero- and one-dimensional subspaces of $\mathbb{P}^{n-1}$. Recall that a subspace $Y \subset \mathbb{P}^{n-1}$ is called Ω-fixed ($\mathbb{T}^m$-fixed) if $\Omega(Y)=Y$ ($u(Y)=Y$ for all $u\in\mathbb{T}^m$). If $n=2m+1$, $1\le i\le m$, and $a,b\in\mathbb{C}^*$, then

$$\mathcal{T}_i(a,b) \equiv \big\{[z_1,z_2,\ldots,z_n] \in \mathbb{P}^{n-1}\colon\ z_k=0\ \ \forall\, k\notin\{2i-1,2i,n\},\ az_{2i-1}z_{2i}+bz_n^2=0\big\}$$

is a smooth conic contained in the plane spanned by P_{2i-1},P_{2i},P_n and passing through P_{2i-1} and P_{2i}.

Lemma 3.1. *(1) The $\mathbb{T}^m$-fixed points in $\mathbb{P}^{n-1}$ are $P_1,P_2,\ldots,P_n$.*

(2) If $n=2m$, the $\mathbb{T}^m$-fixed irreducible curves in $\mathbb{P}^{n-1}$ are the lines ℓ_{ij} with $1\le i\ne j\le n$. If $n=2m+1$, the $\mathbb{T}^m$-fixed irreducible curves in $\mathbb{P}^{n-1}$ are the lines ℓ_{ij} with $1\le i\ne j\le n$ and the conics $\mathcal{T}_i(a,b)$ with $1\le i\le m$ and $a,b\in\mathbb{C}^*$.

(3) If $n=2m$, the Ω-fixed $\mathbb{T}^m$-fixed irreducible curves in $\mathbb{P}^{n-1}$ are the lines $\ell_{2i-1,2i}$ with $1\le i\le m$. If $n=2m+1$, the Ω-fixed $\mathbb{T}^m$-fixed irreducible curves in $\mathbb{P}^{n-1}$ are the lines $\ell_{2i-1,2i}$ with $1\le i\le m$ and the conics $\mathcal{T}_i(a,b)$ with $1\le i\le m$ and $a\bar{b}=\bar{a}b$.

Proof. If a point $p\in\mathbb{P}^{n-1}$ has two distinct nonzero coordinates, z_i and z_j, then the function z_i/z_j is not constant on the orbit of p; this implies the first claim. If a point $p\in\mathbb{P}^{n-1}$ has three distinct nonzero coordinates, $z_{2i-\epsilon_i}$, $z_{2j-\epsilon_j}$, and z_k with $i\ne j\le m$ and $\epsilon_i,\epsilon_j=0,1$, then the image of the orbit of p under the function $(z_k/z_{2i-\epsilon_i}, z_k/z_{2j-\epsilon_j})$ is the complement of the coordinate axes in $\mathbb{C}^2$. Thus, if C is a $\mathbb{T}^m$-fixed irreducible curve which is not one of the lines ℓ_{ij}, C lies in the plane spanned by P_{2i-1}, P_{2i}, and P_n for some $i\le m$ and $n=2m+1$. If $p\in C$ has three nonzero coordinates, its orbit is then one-dimensional and is contained in some conic $\mathcal{T}_i(a,b)$; since C and $\mathcal{T}_i(a,b)$ are irreducible, it follows that $C=\mathcal{T}_i(a,b)$, confirming the second claim. The third claim follows immediately from the second, since Ω sends the coordinate functions z_{2i-1} and z_{2i} to $\bar{z}_{2i}$ and $\bar{z}_{2i-1}$, respectively, and $\mathcal{T}_i(a,b)=\mathcal{T}_i(\bar{a},\bar{b})$ if and only if $a\bar{b}=\bar{a}b$. $\square$

If $\tilde{C}$ is a genus 1 Riemann surface, let $\tilde{C}_0 \subset \tilde{C}$ be the principal, genus-carrying, component or union of components; this is either a smooth torus or a circle of spheres. Any anti-holomorphic involution $\tau : \tilde{C} \longrightarrow \tilde{C}$ restricts to an anti-holomorphic involution on $\tilde{C}_0$; if in addition, $\tilde{C}_0$ is a smooth, then $\tilde{C}_0/\tau$ is topologically either a Klein bottle, a Mobius band, or an annulus.

Corollary 3.2. *Let $\tilde{C}$ be a genus 1 Riemann surface and $\tilde{f}: (\tilde{C}, \tau) \longrightarrow (\mathbb{P}^{n-1}, \Omega)$ an Ω-invariant map representing a $\mathbb{T}^m$-fixed equivalence class. If the principal component $\tilde{C}_0$ of $\tilde{C}$ is a smooth torus, then $\tilde{f}|_{\tilde{C}_0}$ is constant. If $\tilde{f}|_{\tilde{C}_0}$ is constant or the fixed locus of τ is not smooth, then $n=2m+1$ and $\tilde{f}(p)=P_n$ for some node $p\in\tilde{C}_0$.*

Proof. If $\tilde{C}_0$ is a smooth torus and $\tilde{f}|_{\tilde{C}_0}$ is not constant, $\tilde{f}(\tilde{C}_0)$ is a $\mathbb{T}^m$-fixed irreducible curve in $\mathbb{P}^{n-1}$ and the branch locus $B \subset \tilde{f}(\tilde{C}_0)$ of $\tilde{f}|_{\tilde{C}_0}$ is $\mathbb{T}^m$-fixed as well. However, this is impossible, since B contains at least four points (by (2) in Lemma 3.1 and Riemann-Hurwitz), while a zero-dimensional $\mathbb{T}^m$-fixed subset of $\tilde{f}(\tilde{C}_0)$ contains at most two points (also by (2) in Lemma 3.1). This proves the first claim. The second claim is clear, since $\tilde{f}(\tilde{C}_0)$ and $\tilde{f}(p)$ for every $p\in\tilde{C}$ fixed by τ is an Ω-fixed $\mathbb{T}^m$-fixed subset of $\mathbb{P}^{n-1}$. □

4 Proof of Theorem 6

After minor algebraic modifications, the proofs of (3.5) and (3.10) are nearly identical to the proofs of [Z10, (1.6)] and [Z10, (1.17)], which are the $l=1$ cases of (3.5) and (3.10), respectively. The verification of (3.11) is similar to the verification of [Z10, (1.16)], which is the $\mathbf{a}=(n)$ case of (3.11), once suitable functions $\mathcal{Y}_p$ are constructed.

For any ring R, let

$$R[\![\hbar]\!] \equiv R[[\hbar^{-1}]] + R[\hbar]$$

denote the R-algebra of Laurent series in $\hbar^{-1}$ (with finite principal part). If

$$Z(\hbar, Q) = \sum_{d=0}^{\infty} \bigg(\sum_{r=-N_d}^{\infty} Z_d^{(r)} \hbar^{-r} \bigg) Q^d \in R[\![\hbar]\!]\big[[Q]\big]$$

for some $Z_d^{(r)} \in R$, we define

$$Z(\hbar, Q) \cong \sum_{d=0}^{\infty} \bigg(\sum_{r=-N_d}^{p-1} Z_d^{(r)} \hbar^{-r} \bigg) Q^d \quad (\text{mod } \hbar^{-p}),$$

i.e. we drop $\hbar^{-p}$ and higher powers of $\hbar^{-1}$, instead of higher powers of $\hbar$. If in addition $R\supset\mathbb{Q}$, let

$$R_\alpha \equiv \mathbb{Q}_\alpha \otimes_{\mathbb{Q}} R = R[\alpha_1, \ldots, \alpha_n]_{\langle u | u\in\mathbb{Q}[\alpha_1,\ldots,\alpha_n]-0\rangle},$$

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1}; R) \equiv H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) \otimes_{\mathbb{Q}} R = R[\alpha_1, \ldots, \alpha_n, x] \Big/ \prod_{k=1}^{n} (x-\alpha_k);$$

the latter ring is the $\mathbb{T}$-equivariant cohomology of $\mathbb{P}^{n-1}$ with coefficients in R. If R is a field, let $R^* = R-0$ be the set of invertible elements and

$$R(\hbar) \hookrightarrow R[\![\hbar]\!]$$

be the embedding given by taking the Laurent series of rational functions at $\hbar^{-1}=0$.

4.1 Summary

With $\mathrm{ev}_1 : \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, d) \longrightarrow \mathbb{P}^{n-1}$ as before, let

$$\mathcal{Z}(x,\hbar,Q) \equiv 1 + \sum_{d=1}^{\infty} Q^d \mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a}})}{\hbar-\psi_1}\right] \in \big(H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\big)\big[\big[\hbar^{-1}, Q\big]\big], \tag{4.1}$$

$$C_1(q) \equiv \frac{1}{I_0(q)} \sum_{d=1}^{\infty} q^d \frac{\prod_{k=1}^{l}(a_k d)!}{(d!)^n}\left(\sum_{r=1}^{d}\frac{1}{r}\right). \tag{4.2}$$

By [Gi96, Theorems 9.5, 10.7, 11.8] and (3.6),

$$\mathcal{Z}(x,\hbar,Q) = \begin{cases} \mathcal{Y}(x,\hbar,Q), & \text{if } \nu_{\mathbf{a}} \ge 2, \\ \mathrm{e}^{-\frac{Q}{\hbar}\mathbf{a}!}\mathcal{Y}(x,\hbar,Q), & \text{if } \nu_{\mathbf{a}} = 1, \\ \mathrm{e}^{-J(q)\frac{x}{\hbar}}\mathrm{e}^{-C_1(q)\frac{\sigma_1}{\hbar}}\mathcal{Y}(x,\hbar,q)/I_0(q), & \text{if } \nu_{\mathbf{a}} = 0, \end{cases} \tag{4.3}$$

with Q and q related by the mirror map (1.4).

We will follow the five steps given in [Z10] to verify (3.10) and (3.11):

(M1) if $R \supset \mathbb{Q}$ is a field, $Y, Z \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}; R)[\![\hbar]\!]\big[\big[Q\big]\big]$,

$$Y(x=\alpha_i,\hbar,Q) \in R_\alpha(\hbar)\big[\big[Q\big]\big] \subset R_\alpha[\![\hbar]\!]\big[\big[Q\big]\big] \qquad \forall\, i=1,2,\ldots,n,$$

Z is recursive in the sense of Definition 4.1, and Y and Z satisfy the mutual polynomiality condition (MPC) of Definition 4.2, then the transforms of Z of Lemma 4.4 are also recursive and satisfy the MPC with respect to the corresponding transforms of Y;

(M2) if $R \supset \mathbb{Q}$ is a field, $Y, Z \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}; R)[\![\hbar]\!]\big[\big[Q\big]\big]$,

$$Y(x=\alpha_i,\hbar,Q) \in R^*_\alpha + Q \cdot R_\alpha(\hbar)\big[\big[Q\big]\big] \subset R_\alpha[\![\hbar]\!]\big[\big[Q\big]\big] \qquad \forall\, i=1,2,\ldots,n,$$

Z is recursive in the sense of Definition 4.1, and Y and Z satisfy the mutual polynomiality condition (MPC) of Definition 4.2, then Z is determined by its "mod $\hbar^{-1}$ part";

(M3) $\mathcal{Y}_p(x,\hbar,Q)$ and $\mathcal{Z}_p(x,\hbar,Q)$ are $\mathfrak{C}$-recursive in the sense of Definition 4.1 with $\mathfrak{C}$ as in (4.14);

(M4) $\big(\mathcal{Y}(x,\hbar,Q), \mathcal{Y}_p(x,\hbar,Q)\big)$ and $\big(\mathcal{Z}(x,\hbar,Q), \mathcal{Z}_p(x,\hbar,Q)\big)$ satisfy the MPC;

(M5) the two sides of (3.5) and (3.11), viewed as elements of $H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big]$, agree mod $\hbar^{-1}$.

The first two claims above, (M1) and (M2), sum up Lemma 4.4 and Proposition 4.3, respectively. Section 4.3 extends the arguments in [Z10] to show that the GW-generating function $\mathcal{Z}_p$ is $\mathfrak{C}$-recursive and satisfies the MPC with respect to $\mathcal{Z}$; this confirms the claims of (M3) and (M4) concerning $\mathcal{Z}_p$. It is immediate from (3.3) that

$$\mathcal{Z}_p(x,\hbar,Q) \cong x^{l+p} \pmod{\hbar^{-1}} \qquad \forall\; p=-l,-l+1,\ldots \tag{4.4}$$

In Section 4.4, we show that $\mathcal{Y}_{-l}$ is $\mathfrak{C}$-recursive and satisfies the MPC with respect to $\mathcal{Y}$. The admissibility of transforms (i) and (ii) of Lemma 4.4 implies that the power series $\mathcal{Y}_p$ defined by (3.12) is also $\mathfrak{C}$-recursive and satisfies the MPC with respect to $\mathcal{Y}$ for all p, no matter what the coefficients $\widetilde{\mathcal{C}}_{p,s}^{(r)}$ are; this confirms the claims of (M3) and (M4) concerning $\mathcal{Y}_p$. By (3.8) and (3.9),

$$\mathcal{Y}_p(x,\hbar,Q) \cong x^{l+p} \pmod{\hbar^{-1}} \qquad \forall\, p=-l,-l+1,\ldots,0. \tag{4.5}$$

By (3.14),

$$\sum_{r=0}^{p}\sum_{s=0}^{p-r}\widetilde{\mathcal{C}}_{p,s}^{(r)}(q)\hbar^{p-r-s}\mathfrak{D}^s\mathcal{Y}_0(x,\hbar,q) \cong x^{l+p} \pmod{\hbar^{-1}} \qquad \forall\, p\in\mathbb{Z}^{\ge 0} \tag{4.6}$$

if and only if the coefficients $\widetilde{\mathcal{C}}_{p,s}^{(r)}(q)$ are given by (3.15).[4] Since $\widetilde{\mathcal{C}}_{p,s}^{(0)}=\delta_{p,s}$, (3.11) follows from (M1), (M2), (4.3), (4.4), and (4.6).

The proof of (3.10) follows the same principle, which we apply to a multiple of

$$\mathcal{Z}(x_1,x_2,\hbar_1,\hbar_2,Q) \equiv \frac{\langle\mathbf{a}\rangle x_1^l}{\hbar_1+\hbar_2}\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2\ge 0}}(-1)^r\sigma_r x_1^{p_1}x_2^{p_2} + \mathcal{Z}^*(x_1,x_2,\hbar_1,\hbar_2,Q). \tag{4.7}$$

For each $i=1,2,\ldots,n$, let

$$\phi_i \equiv \prod_{k\neq i}(x-\alpha_k) \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1}). \tag{4.8}$$

By the Localization Theorem [ABo], ϕ_i is the equivariant Poincaré dual of the fixed point $P_i\in\mathbb{P}^{n-1}$; see [Z10, Section 3.1]. Since

$$x|_{P_i}=\alpha_i\,, \qquad \mathbf{e}(\mathcal{V}_{\mathbf{a}})=\langle\mathbf{a}\rangle\,\mathrm{ev}_2^*x^l\,\mathbf{e}(\mathcal{V}_{\mathbf{a}}''), \tag{4.9}$$

by the defining property of the cohomology push-forward [Z10, (3.11)] and the string relation [MirSym, Section 26.3]

$$\begin{aligned}\mathcal{Z}(\alpha_i,\alpha_j,\hbar_1,\hbar_2,Q) &= \int_{P_i\times P_j}\mathcal{Z}(x_1,x_2,\hbar_1,\hbar_2,Q) = \int_{\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}}\mathcal{Z}(x_1,x_2,\hbar_1,\hbar_2,Q)\phi_i\times\phi_j\\ &= \frac{\langle\mathbf{a}\rangle\alpha_i^l}{\hbar_1+\hbar_2}\prod_{k\neq i}(\alpha_j-\alpha_k) + \sum_{d=1}^{\infty}Q^d\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}_{\mathbf{a}})\,\mathrm{ev}_1^*\phi_i\,\mathrm{ev}_2^*\phi_j}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}\\ &= \langle\mathbf{a}\rangle\frac{\hbar_1\hbar_2}{\hbar_1+\hbar_2}\sum_{d=0}^{\infty}Q^d\int_{\overline{\mathfrak{M}}_{0,3}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}_{\mathbf{a}}'')\,\mathrm{ev}_1^*\phi_i\,\mathrm{ev}_2^*(x^l\,\phi_j)}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}.\end{aligned} \tag{4.10}$$

Thus, by Lemmas 4.5 and 4.6, $(\hbar_1+\hbar_2)\mathcal{Z}(x_1,x_2,\hbar_1,\hbar_2,Q)$ is $\mathfrak{C}$-recursive and satisfies the MPC with respect to $\mathcal{Z}(x,\hbar,Q)$ for $(x,\hbar)=(x_1,\hbar_1)$ and $x_2=\alpha_j$ fixed. By symmetry, it is also $\mathfrak{C}$-recursive and satisfies the MPC with respect to $\mathcal{Z}(x,\hbar,Q)$ for $(x,\hbar)=(x_2,\hbar_2)$ and $x_1=\alpha_i$ fixed. By (M2), it is thus sufficient to compare

$$(\hbar_1+\hbar_2)\mathcal{Z}(x_1,x_2,\hbar_1,\hbar_2,Q) \quad\text{and}\quad \langle\mathbf{a}\rangle\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2,r\ge 0}}(-1)^r\sigma_r\mathcal{Z}_{p_1}(x_1,\hbar_1,Q)\mathcal{Z}_{p_2-l}(x_2,\hbar_2,Q) \tag{4.11}$$

[4] LHS of (3.15) is the coefficient of $\hbar^p x^{l-r}(x/\hbar)^s$ in the double sum viewed as an element of $\mathbb{Q}_\alpha[[q]][x][[\hbar]]$.

for all $x_1=\alpha_i$ and $x_2=\alpha_j$ with $i,j=1,2,\ldots,n$ modulo $\hbar_1^{-1}$:

$$\big(\hbar_1+\hbar_2\big)\mathcal{Z}(\alpha_i,\alpha_j,\hbar_1,\hbar_2,Q)\cong\langle\mathbf{a}\rangle\alpha_i^l\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2,r\ge0}}(-1)^r\sigma_r\alpha_i^{p_1}\alpha_j^{p_2}+\sum_{d=1}^{\infty}Q^d\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}_{\mathbf{a}})\mathrm{ev}_1^*\phi_i\mathrm{ev}_2^*\phi_j}{\hbar_2-\psi_2};$$

$$\langle\mathbf{a}\rangle\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2,r\ge0}}(-1)^r\sigma_r\mathcal{Z}_{p_1}(\alpha_i,\hbar_1,Q)\mathcal{Z}_{p_2-l}(\alpha_j,\hbar_2,Q)\cong\langle\mathbf{a}\rangle\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2,r\ge0}}(-1)^r\sigma_r\alpha_i^{l+p_1}\mathcal{Z}_{p_2-l}(\alpha_j,\hbar_2,Q).$$

In order to see that the two right-hand side power series are the same, it is sufficient to compare them modulo $\hbar_2^{-1}$:

$$\langle\mathbf{a}\rangle\alpha_i^l\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2,r\ge0}}(-1)^r\sigma_r\alpha_i^{p_1}\alpha_j^{p_2}+\sum_{d=1}^{\infty}Q^d\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}_{\mathbf{a}})\mathrm{ev}_1^*\phi_i\mathrm{ev}_2^*\phi_j}{\hbar_2-\psi_2}\cong\langle\mathbf{a}\rangle\alpha_i^l\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2,r\ge0}}(-1)^r\sigma_r\alpha_i^{p_1}\alpha_j^{p_2};$$

$$\langle\mathbf{a}\rangle\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2,r\ge0}}(-1)^r\sigma_r\alpha_i^{l+p_1}\mathcal{Z}_{p_2-l}(\alpha_j,\hbar_2,Q)\cong\langle\mathbf{a}\rangle\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2,r\ge0}}(-1)^r\sigma_r\alpha_i^{l+p_1}\alpha_j^{p_2}.$$

From this we conclude that the two expressions in (4.11) are the same; this proves (3.10).

4.2 Recursivity, polynomiality, and admissible transforms

We now describe properties of power series, such as $\mathcal{Y}_p$ and $\mathcal{Z}_p$, that impose severe restrictions on them; see Proposition 4.3. This section sums up the results in [Z10, Sections 2.1, 2.2], extending them slightly. Let

$$[n]=\{1,2,\ldots,n\}.$$

If R is a ring, $f\in R[[Q]]$, and $d\in\mathbb{Z}^{\ge0}$, let $[\![f]\!]_{Q;d}\in R$ denote the coefficient of Q^d in f.

Definition 4.1. *Let $R\supset\mathbb{Q}$ be a field and $C\equiv(C_i^j(d))_{d,i,j\in\mathbb{Z}^+}$ any collection of elements of R_α. A power series $Z\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1};R)[\![\hbar]\!]\big[[Q]\big]$ is C-recursive if the following holds: if $d^*\in\mathbb{Z}^{\ge0}$ is such that*

$$\big[\![Z(x=\alpha_i,\hbar,Q)\big]\!]_{Q;d^*-d}\in R_\alpha(\hbar)\qquad\forall\,d\in[d^*],\ i\in[n],$$

and $\big[\![Z(\alpha_i,\hbar,Q)\big]\!]_{Q;d}$ is regular at $\hbar=(\alpha_i-\alpha_j)/d$ for all $d<d^*$ and $i\neq j$, then

$$\big[\![Z(\alpha_i,\hbar,Q)\big]\!]_{Q;d^*}-\sum_{d=1}^{d^*}\sum_{j\neq i}\frac{C_i^j(d)}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\big[\![Z(\alpha_j,z,Q)\big]\!]_{Q;d^*-d}\Big|_{z=\frac{\alpha_j-\alpha_i}{d}}\in R_\alpha\big[\hbar,\hbar^{-1}\big].\tag{4.12}$$

A power series $Z\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1};R)[\![\hbar]\!]\big[[Q]\big]$ is called recursive if it is C-recursive for some collection $C\equiv(C_i^j(d))_{d,i,j\in\mathbb{Z}^+}$ of elements of R_α.

Thus, if $Z\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1};R)[\![\hbar]\!]\big[[Q]\big]$ is recursive, then

$$Z(x=\alpha_i,\hbar,Q)\in R_\alpha(\hbar)\big[[Q]\big]\qquad\forall\,i\in[n].$$

Definition 4.2. *Let $R\supset\mathbb{Q}$ be a field. For any*

$$Y\equiv Y(x,\hbar,Q), Z\equiv Z(x,\hbar,Q)\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1};R)[\![\hbar]\!]\big[\big[Q\big]\big],$$

define $\Phi_{Y,Z}\in R_\alpha[\![\hbar]\!]\big[\big[z,Q\big]\big]$ by

$$\Phi_{Y,Z}(\hbar,z,Q)=\sum_{i=1}^{n}\frac{\mathrm{e}^{\alpha_i z}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}Y\big(\alpha_i,\hbar,Q\mathrm{e}^{\hbar z}\big)Z(\alpha_i,-\hbar,Q).$$

If $Y,Z\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1};R)[\![\hbar]\!]\big[\big[Q\big]\big]$, the pair (Y,Z) satisfies the mutual polynomiality condition (MPC) if

$$\Phi_{Y,Z}\in R_\alpha[\hbar]\big[\big[z,Q\big]\big].$$

If $Y,Z\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1};R)[\![\hbar]\!]\big[\big[Q\big]\big]$ and

$$Y(x=\alpha_i,\hbar,Q), Z(x=\alpha_i,\hbar,Q)\in R_\alpha(\hbar)\big[\big[Q\big]\big]\qquad\forall\, i\in[n], \tag{4.13}$$

then the pair (Y,Z) satisfies the MPC if and only if the pair (Z,Y) does; see [Z10, Lemma 2.2]. Thus, if (4.13) holds, the statement that Y and Z satisfy the MPC is unambiguous.

Proposition 4.3. *Let $R\supset\mathbb{Q}$ be a field. If $Y,Z\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1};R)[\![\hbar]\!]\big[\big[Q\big]\big]$,*

$$Y(x=\alpha_i,\hbar,Q)\in R_\alpha^*+Q\cdot R_\alpha(\hbar)\big[\big[Q\big]\big]\subset R_\alpha[\![\hbar]\!]\big[\big[Q\big]\big]\qquad\forall\, i\in[n],$$

Z is recursive, and Y and Z satisfy the MPC, then $Z\cong 0 \pmod{\hbar^{-1}}$ if and only if $Z=0$.

This is essentially [Z10, Proposition 2.1], where the recursivity for Z is with respect to a specific collection C.[5] However, the proof of [Z10, Proposition 2.1] does not use any properties of the specific collection needed for the purposes of [Z10] and thus applies to any collection of elements of R_α.

Lemma 4.4. *Let $R\supset\mathbb{Q}$ be a field. If $Y,Z\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1};R)[\![\hbar]\!]\big[\big[Q\big]\big]$,*

$$Y(x=\alpha_i,\hbar,Q)\in R_\alpha(\hbar)\big[\big[Q\big]\big]\subset R_\alpha[\![\hbar]\!]\big[\big[Q\big]\big]\qquad\forall\, i\in[n],$$

Z is recursive, and Y and Z satisfy the MPC, then

(i) $\bar{Z}\equiv\Big\{x+\hbar\, Q\frac{\mathrm{d}}{\mathrm{d}Q}\Big\}Z$ *is recursive and satisfies the MPC with respect to Y;*

(ii) if $f\in R_\alpha[\hbar]\big[\big[Q\big]\big]$, then fZ is recursive and satisfies the MPC with respect to Y;

(iii) if $f\in Q\cdot R_\alpha[[Q]]$, then $\mathrm{e}^{f/\hbar}Z$ is recursive and satisfies the MPC with respect to $\mathrm{e}^{f/\hbar}Y$;

(iv) if $g\in Q\cdot R_\alpha[[Q]]$,

$$\bar{Y}(x,\hbar,Q)\equiv\mathrm{e}^{xg(Q)/\hbar}Y\big(x,\hbar,Q\mathrm{e}^{g(Q)}\big)\quad\text{and}\quad \bar{Z}(x,\hbar,Q)\equiv\mathrm{e}^{xg(Q)/\hbar}Z\big(x,\hbar,Q\mathrm{e}^{g(Q)}\big),$$

then $\bar{Z}$ is recursive and satisfies the MPC with respect to $\bar{Y}$.

---

[5] The assumptions on the second line in [Z10, Proposition 2.1] should have been that

$$Y,Z\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[\![\hbar]\!]\big[\big[Q\big]\big]\qquad\text{and}\qquad Y(x=\alpha_i,\hbar,Q), Z(x=\alpha_i,\hbar,Q)\in\mathbb{Q}_\alpha(\hbar)\big[\big[Q\big]\big]\quad\forall\, i\in[n].$$

Since Z is recursive in our case, $Z(\alpha_i,\hbar,Q)\in R_\alpha(\hbar)[[Q]]$.

Parts (i), (iii), and (iv) of this lemma are (i), (iv)[6], and (v), respectively, of [Z10, Lemma 2.3] with $Q=u=\mathrm{e}^t$ and $\mathbb{Q}$ and $\tilde{\mathbb{Q}}_\alpha$ replaced by R_α. The conclusions in these parts continue to hold because their proofs given in [Z10] do not use anything about the collection $C=(C_i^j(d))_{d,i,j}$ of [Z10]. Moreover, replacing $\mathbb{Q}$ and $\tilde{\mathbb{Q}}_\alpha$ by R_α does not affect the proofs either. Part (ii) of Lemma 4.4 generalizes parts (ii) and (iii) of [Z10, Lemma 2.3] as suggested in the remark following [Z10, Lemma 2.3]; it continues to hold for the same reasons as the other parts do.[7]

4.3 Recursivity and MPC for GW power series

This section establishes that the power series $\mathcal{Z}_p(x,\hbar,Q)$ defined in (3.3) is $\mathfrak{C}$-recursive in the sense of Definition 4.1, where

$$\mathfrak{C}_i^j(d)\equiv\frac{\prod\limits_{k=1}^{l}\prod\limits_{r=0}^{a_kd-1}\Big(a_k\alpha_i+r\,\frac{\alpha_j-\alpha_i}{d}\Big)}{d\prod\limits_{\substack{r=1\\(r,k)\neq(d,j)}}^{d}\prod\limits_{k=1}^{n}\Big(\alpha_i-\alpha_k+r\,\frac{\alpha_j-\alpha_i}{d}\Big)}\in\mathbb{Q}_\alpha\,. \tag{4.14}$$

We also show that $\mathcal{Z}_p(x,\hbar,Q)$ satisfies the MPC of Definition 4.2 with respect to the power series $\mathcal{Z}(x,\hbar,Q)$ defined in (4.1). These statements are special cases of Lemmas 4.5 and 4.6 below, since

$$\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}''_{\mathbf{a}})\mathrm{ev}_2^*\eta}{\hbar-\psi_1}\mathrm{ev}_1^*\phi_i=\hbar\int_{\overline{\mathfrak{M}}_{0,3}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}''_{\mathbf{a}})\mathrm{ev}_2^*\eta}{\hbar-\psi_1}\mathrm{ev}_1^*\phi_i\quad\forall\,\eta\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}),\ d\in\mathbb{Z}^+,$$

by the string relation [MirSym, Section 26.3].

Lemma 4.5. *If $m\ge3$, $\mathrm{ev}_j:\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d)\longrightarrow\mathbb{P}^{n-1}$ is the evaluation at the j-th marked point, and $\eta_j\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$ and $\beta_j\in\mathbb{Z}^{\ge0}$ for $j=1,\ldots,m$, then the power series*

$$\mathcal{Z}_{\eta,\beta}(x,\hbar,Q)\equiv\sum_{d=0}^{\infty}Q^d\mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\mathcal{V}''_{\mathbf{a}})}{\hbar-\psi_1}\prod_{j=2}^{m}\big(\psi_j^{\beta_j}\mathrm{ev}_j^*\eta_j\big)\right]\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!][[Q]]$$

is $\mathfrak{C}$-recursive with $\mathfrak{C}$ given by (4.14).

Lemma 4.6. *For all $m\ge3$, $\eta_j\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$, and $\beta_j\in\mathbb{Z}^{\ge0}$, the power series $\hbar^{m-2}\mathcal{Z}_{\eta,\beta}(x,\hbar,Q)$, with $\mathcal{Z}_{\eta,\beta}(x,\hbar,Q)$ as in Lemma 4.5, satisfies the MPC with respect to $\mathcal{Z}(x,\hbar,Q)$.*

Similarly to (4.10),

$$\mathcal{Z}_{\eta,\beta}(\alpha_i,\hbar,Q)=\sum_{d=0}^{\infty}Q^d\int_{\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d)}\left(\frac{\mathbf{e}(\mathcal{V}''_{\mathbf{a}})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\prod_{j=2}^{m}\big(\psi_j^{\beta_j}\mathrm{ev}_j^*\eta_j\big)\right)$$

[6]There is a typo in the statement of [Z10, Lemma 2.3,(iv)]: $\tilde{\mathbb{Q}}_\alpha[u]$ should be $\tilde{\mathbb{Q}}_\alpha[[u]]$.

[7]There are two typos in the proof of [Z10, Lemma 2.3]: the last term in the last equality of (2.19) should be

$$C_i^j(d)u^d\left\{d+\frac{\mathrm{d}}{\mathrm{d}t}\right\}Z((\alpha_j-\alpha_i)/d,\alpha_j,u)$$

and the third argument of Z on both sides of the first equation in the proof of (v) should be $u\mathrm{e}^{g(u)}$.

for all i. Thus, the $l=1$ cases of Lemmas 4.5 and 4.6 are Lemmas 1.1 and 1.2 in [Z10, Section 1.3].[8] The proof of [Z10, Lemma 1.1] consists of applying the Localization Theorem [ABo] to $\mathcal{Z}_{\eta,\beta}(\alpha_i,\hbar,Q)$ with respect to the $\mathbb{T}$-action on $\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d)$ induced by the standard $\mathbb{T}$-action on $\mathbb{P}^{n-1}$.[9] The proof of [Z10, Lemma 1.2] consists of applying the Localization Theorem on a certain subspace of $\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^1\times\mathbb{P}^{n-1},(1,d))$ with respect to the action of the $(n+1)$-torus $\mathbb{T}\times\mathbb{C}^*$ induced by a certain action of this torus on $\mathbb{P}^1\times\mathbb{P}^{n-1}$.[10] The proofs of Lemmas 4.5 and 4.6 are nearly identical: the base spaces and the tori actions remain the same, while the Euler classes of the vector bundles in our case are products of the Euler classes in [Z10]. So only the following modifications need to be made to the proofs in [Z10]:

(1) $\mathcal{V}_0''$ should be replaced by $\mathcal{V}_{\mathbf{a}}''$ everywhere in [Z10, Sections 3.2-3.4];

(2) the first equation in [Z10, (3.21)] should be replaced by

$$\mathbf{e}(\mathcal{V}_{\mathbf{a}}'')=\prod_{k=1}^{l}\prod_{r=0}^{a_kd_0-1}\left(a_k\alpha_i+r\frac{\alpha_j-\alpha_i}{d_0}\right);$$

(3) on the right hand-sides of [Z10, (3.23), (3.24)] and in the last sentence of [Z10, Section 3.2], $C_i^j(d_0)$ should be replaced by $\mathfrak{C}_i^j(d_0)$.

4.4 Properties of the hypergeometric series $\mathcal{Y}_p$

Below we verify the two claims concerning the power series $\mathcal{Y}_p(x,\hbar,q)$ that remain to be proved:

(a) $\mathcal{Y}_{-l}(x,\hbar,q)$ is $\mathfrak{C}$-recursive;

(b) $\Phi_{\mathcal{Y},\mathcal{Y}_{-l}}\in\mathbb{Q}_\alpha[\hbar]\big[\big[z,q\big]\big]$.

The proofs are similar to [Z10, Section 2.3], which treats the $l=1$ Calabi-Yau case. The proof of (a) in [Z10] is applicable in the Fano case as well and so requires little modification; the consideration of the Fano case in (b) requires only slightly more care.

If $f=f(z)$ is a rational function in $\hbar$ and possibly some other variables, for any $z_0\in\mathbb{P}^1\supset\mathbb{C}$ let

$$\mathop{\mathfrak{R}}_{z=z_0}f(z)\equiv\frac{1}{2\pi\mathfrak{i}}\oint f(z)\mathrm{d}z, \tag{4.15}$$

where the integral is taken over a positively oriented loop around $z=z_0$ with no other singular points of $f\mathrm{d}z$, denote the residue of the 1-form $f\mathrm{d}z$. If $z_1,\ldots,z_k\in\mathbb{P}^1$ is any collection of points, let

$$\mathop{\mathfrak{R}}_{z=z_1,\ldots,z_k}f(z)\equiv\sum_{i=1}^{k}\mathop{\mathfrak{R}}_{z=z_i}f(z). \tag{4.16}$$

If f is regular at $z=0$, let $[\![f]\!]_{z;p}$ denote the coefficient of z^p in the power series expansion of f around $z=0$.

[8] There is a typo in [Z10, (1.19)]: x should be α_i.

[9] There are two typos in the proof of this lemma in [Z10]: in the second factor of the second equation in [Z10, (3.20)], Γ_0 should be Γ_c and on the right-hand side of [Z10, (3.23)], $\mathcal{Z}_\Gamma$ should be $\mathcal{Z}_{\Gamma_c}$.

[10] The portion of [Z10, Section 3.3] following the statement of Lemma 3.1 is unnecessary: in light of [Z10, (3.27)], [Z10, (3.29)] immediately implies the statement of [Z10, Lemma 1.2]. There are also five typos in the proof of this lemma in [Z10]: in the first equation of [Z10, (3.32)] and on the left-hand side of [Z10, (3.33)], γ^* should be γ_1^*; in the second equation of [Z10, (3.32)], Γ_0 should be Γ_1 when it is pulled-back by π_1^* and Γ_2 when it is pulled-back by π_2^*; in the second bracket on the right-hand side of [Z10, (3.33)], the numerator of the integrand should be $\mathbf{e}(\mathcal{V}_0'')\eta^\beta\mathrm{ev}_1^*\phi_i$.

Proof of (a). In this argument, we view $\mathcal{Y}_{-l}$ as an element of $\mathbb{Q}_\alpha(x,\hbar)[[q]]$. By (3.7) and (4.14),

$$\frac{\mathfrak{C}_i^j(d)q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\,\mathcal{Y}_{-l}\big(\alpha_j,(\alpha_j-\alpha_i)/d,q\big)=\underset{z=\frac{\alpha_j-\alpha_i}{d}}{\mathfrak{R}}\left\{\frac{1}{\hbar-z}\mathcal{Y}_{-l}(\alpha_i,z,q)\right\}.$$

Thus, by the Residue Theorem on S^2,

$$\begin{split}\sum_{d=1}^{\infty}\sum_{j\neq i}\frac{\mathfrak{C}_i^j(d)q^d}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathcal{Y}_{-l}\big(\alpha_j,(\alpha_j-\alpha_i)/d,q\big)&=-\underset{z=\hbar,0,\infty}{\mathfrak{R}}\left\{\frac{1}{\hbar-z}\mathcal{Y}_{-l}(\alpha_i,z,q)\right\}\\&=\mathcal{Y}_{-l}(\alpha_i,\hbar,q)-\underset{z=0,\infty}{\mathfrak{R}}\left\{\frac{1}{\hbar-z}\mathcal{Y}_{-l}(\alpha_i,z,q)\right\}.\end{split}\tag{4.17}$$

On the other hand,

$$\underset{z=\infty}{\mathfrak{R}}\left\{\frac{1}{\hbar-z}\mathcal{Y}_{-l}(\alpha_i,z,q)\right\}=1,$$

$$\underset{z=0}{\mathfrak{R}}\left\{\frac{1}{\hbar-z}[\![\mathcal{Y}_{-l}(\alpha_i,z,q)]\!]_{q;d}\right\}=\left[\!\!\left[\frac{1}{\hbar-z}\frac{\prod\limits_{k=1}^{l}\prod\limits_{r=0}^{a_kd-1}(a_k\alpha_i+rz)}{d!\prod\limits_{r=1}^{d}\prod\limits_{k\neq i}(\alpha_i-\alpha_k+rz)}\right]\!\!\right]_{z;d-1}\in\mathbb{Q}_\alpha[\hbar^{-1}]\quad\forall\,d\in\mathbb{Z}^{\ge0}.$$

Thus, (4.17) implies that $\mathcal{Y}_{-l}$ is $\mathfrak{C}$-recursive. □

Proof of (b). In this argument, we view $\mathcal{Y}$ and $\mathcal{Y}_{-l}$ as elements of $\mathbb{Q}_\alpha[x][[\hbar^{-1},q]]$; in particular,

$$\frac{\mathrm{e}^{xz}}{\prod\limits_{k=1}^{n}(x-\alpha_k)}\mathcal{Y}\big(x,\hbar,q\mathrm{e}^{\hbar z}\big)\mathcal{Y}_{-l}(-x,\hbar,q)\in\mathbb{Q}_\alpha(x)[[\hbar^{-1},z,q]]$$

viewed as a function of x has residues only at $x=\alpha_i$ with $i\in[n]$ and $x=\infty$. By (3.6) and (3.7),

$$\frac{\mathrm{e}^{\alpha_iz}}{\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}\mathcal{Y}\big(\alpha_i,\hbar,q\mathrm{e}^{\hbar z}\big)\mathcal{Y}_{-l}(\alpha_i,-\hbar,q)=\underset{x=\alpha_i}{\mathfrak{R}}\left\{\frac{\mathrm{e}^{xz}}{\prod\limits_{k=1}^{n}(x-\alpha_k)}\mathcal{Y}\big(x,\hbar,q\mathrm{e}^{\hbar z}\big)\mathcal{Y}_{-l}(x,-\hbar,q)\right\}.$$

Thus, by the Residue Theorem on S^2,

$$\begin{split}\Phi_{\mathcal{Y},\mathcal{Y}_{-l}}(\hbar,z,q)&=-\underset{x=\infty}{\mathfrak{R}}\left\{\frac{\mathrm{e}^{xz}}{\prod\limits_{k=1}^{n}(x-\alpha_k)}\mathcal{Y}\big(x,\hbar,q\mathrm{e}^{\hbar z}\big)\mathcal{Y}_{-l}(x,-\hbar,q)\right\}\\&=\sum_{d_1,d_2=0}^{\infty}\sum_{p=0}^{\infty}\frac{z^{n-1+p+\nu_{\mathbf{a}}(d_1+d_2)}}{(n-1+p+\nu_{\mathbf{a}}(d_1+d_2))!}q^{d_1+d_2}\mathrm{e}^{\hbar d_1z}\\&\quad\times\left[\!\!\left[\frac{1}{\prod\limits_{k=1}^{n}(1-\alpha_kw)}\frac{\prod\limits_{k=1}^{l}\prod\limits_{r=1}^{a_kd_1}(a_k+r\hbar w)}{\prod\limits_{r=1}^{d_1}\prod\limits_{k=1}^{n}(1-(\alpha_k-r\hbar)w)}\cdot\frac{\prod\limits_{k=1}^{l}\prod\limits_{r=0}^{a_kd_2-1}(a_k-r\hbar w)}{\prod\limits_{r=1}^{d_2}\prod\limits_{k=1}^{n}(1-(\alpha_k+r\hbar)w)}\right]\!\!\right]_{w;p}.\end{split}$$

The (d_1,d_2,p)-summand above is $q^{d_1+d_2}$ times an element of $\mathbb{Q}_\alpha[\hbar]\big[\big[z\big]\big]$. □

5 Some Applications of Theorem 6

Sections 5.1 and 5.2 below contain some corollaries of Theorem 6. While the identities we obtain in these sections appear purely technical in nature, they enter in vital ways in the proofs of Theorem 3 and Theorem 4 in the rest of the paper.

5.1 Differentiating (3.10)

In this section, we obtain a description for a derivative of (3.10). Let $\mathrm{D}=q\frac{\mathrm{d}}{\mathrm{d}q}$.

By (3.12), (3.14), and (3.15),

$$\mathcal{Y}_p(x,\hbar,q)=x^{l+p}+x^l\sum_{s=1}^{\infty}\sum_{r=0}^{p+s}B_{p,s}^{(r)}(q)x^{p+s-r}\hbar^{-s} \tag{5.1}$$

for some $B_{p,s}^{(r)}(q)\in q\cdot\mathbb{Q}_\alpha[[q]]$ such that the coefficient of q^d in $B_{p,s}^{(r)}(q)$ is a homogeneous polynomial in $\alpha_1,\ldots,\alpha_n$ of degree $r-\nu_{\mathbf{a}}d$. If $\nu_{\mathbf{a}}=0$,

$$I_{p+1}(q)=1+DB_{p,1}^{(0)}(q) \tag{5.2}$$

by (3.13), (1.3), and

$$x^lF(x/\hbar,q)=\{x+\hbar\mathrm{D}\}\mathfrak{D}^{-1}\mathcal{Y}_0(x,\hbar,q)\big|_{\alpha=0};$$

if $\nu_{\mathbf{a}}\neq 0$, (5.2) is immediate from $B_{p,1}^{(0)}$ being a constant in q. For any $i,j=1,2,\ldots,n$, let

$$S_{ij}^{(2)}(\hbar_1,\hbar_2,q)=\sum_{\substack{p_1+p_2+r=n-l\\ p_1,p_2\geq 0}}(-1)^r\sigma_r\left[\sum_{r'=0}^{p_1}DB_{p_1-1,1}^{(r')}(q)\mathcal{Y}_{p_1-r'}(\alpha_i,\hbar_1,q)\right]\mathcal{Y}_{p_2}(\alpha_j,\hbar_2,q). \tag{5.3}$$

Lemma 5.1. *For all $i=1,2,\ldots,n$ and $p\geq -l$,*

$$\{\alpha_i+\hbar\mathrm{D}\}\mathcal{Y}_p(\alpha_i,\hbar,q)=\mathcal{Y}_{p+1}(\alpha_i,\hbar,q)+\sum_{r=0}^{p+1}\mathrm{D}B_{p,1}^{(r)}(q)\mathcal{Y}_{p+1-r}(\alpha_i,\hbar,q).$$

Proof. Since both sides of this identity with α_i replaced by x are $\mathfrak{C}$-recursive and satisfy the MPC with respect to $\mathcal{Y}(x,\hbar,q)$, it is sufficient to verify this identity modulo $\hbar^{-1}$. The latter is immediate from (5.1). □

Lemma 5.2. *For all $i,j=1,2,\ldots,n$,*

$$\begin{aligned}&\left\{\frac{\alpha_i}{\hbar_1}+\frac{\alpha_j}{\hbar_2}+\mathrm{D}\right\}\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2\geq 0}}(-1)^r\sigma_r\mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2-l}(\alpha_j,\hbar_2,q)\\ &\qquad=\left(\frac{1}{\hbar_1}+\frac{1}{\hbar_2}\right)\left\{\sum_{\substack{p_1+p_2+r=n-l\\ p_1,p_2\geq 0}}-\sum_{\substack{p_1+p_2+r=n-l\\ p_1,p_2\leq -1}}\right\}(-1)^r\sigma_r\mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2}(\alpha_j,\hbar_2,q)\\ &\qquad\quad+\frac{1}{\hbar_1}S_{ij}^{(2)}(\hbar_1,\hbar_2,q)+\frac{1}{\hbar_2}S_{ji}^{(2)}(\hbar_2,\hbar_1,q).\end{aligned}$$

Proof. For any $i, j = 1, 2, \ldots, n$, we define

$$\begin{aligned} S_{ij}^{(-)}(\hbar_1, \hbar_2, q) &\equiv \mathcal{Y}_0(\alpha_i, \hbar_1, q) \sum_{\substack{p+r=n-l\\ p\le -1}} (-1)^r \sigma_r \mathcal{Y}_p(\alpha_j, \hbar_2, q), \\ S_{ij}^{(+)}(\hbar_1, \hbar_2, q) &\equiv \mathcal{Y}_0(\alpha_i, \hbar_1, q) \sum_{\substack{p+r=n-l\\ p\ge 0}} (-1)^r \sigma_r \mathcal{Y}_p(\alpha_j, \hbar_2, q) = -S_{ij}^{(-)}(\hbar_1, \hbar_2, q); \end{aligned} \tag{5.4}$$

the equality above follows from (3.5) and (3.11). By Lemma 5.1 and the $p=-1$ case of (5.2),

$$\begin{aligned} &\left\{\frac{\alpha_i}{\hbar_1} + \frac{\alpha_j}{\hbar_2} + \mathrm{D}\right\} \sum_{\substack{p_1+p_2+r=n-1-l\\ p_1,p_2\le -1}} (-1)^r \sigma_r \mathcal{Y}_{p_1}(\alpha_i, \hbar_1, q) \mathcal{Y}_{p_2}(\alpha_j, \hbar_2, q) \\ &\qquad = \left(\frac{1}{\hbar_1} + \frac{1}{\hbar_2}\right) \sum_{\substack{p_1+p_2+r=n-l\\ p_1,p_2\le -1}} (-1)^r \sigma_r \mathcal{Y}_{p_1}(\alpha_i, \hbar_1, q) \mathcal{Y}_{p_2}(\alpha_j, \hbar_2, q) \\ &\qquad\quad + \frac{I_0(q)}{\hbar_1} S_{ij}^{(-)}(\hbar_1, \hbar_2, q) + \frac{I_0(q)}{\hbar_2} S_{ji}^{(-)}(\hbar_2, \hbar_1, q). \end{aligned} \tag{5.5}$$

Similarly,

$$\begin{aligned} &\left\{\frac{\alpha_i}{\hbar_1} + \frac{\alpha_j}{\hbar_2} + \mathrm{D}\right\} \sum_{\substack{p_1+p_2+r=n-1-l\\ p_1,p_2\ge 0}} (-1)^r \sigma_r \mathcal{Y}_{p_1}(\alpha_i, \hbar_1, q) \mathcal{Y}_{p_2}(\alpha_j, \hbar_2, q) \\ &= \left(\frac{1}{\hbar_1} + \frac{1}{\hbar_2}\right) \sum_{\substack{p_1+p_2+r=n-l\\ p_1,p_2\ge 0}} (-1)^r \sigma_r \mathcal{Y}_{p_1}(\alpha_i, \hbar_1, q) \mathcal{Y}_{p_2}(\alpha_j, \hbar_2, q) \\ &\quad + \frac{1}{\hbar_1} S_{ij}^{(2)}(\hbar_1, \hbar_2, q) + \frac{1}{\hbar_2} S_{ji}^{(2)}(\hbar_2, \hbar_1, q) - \frac{I_0(q)}{\hbar_1} S_{ij}^{(+)}(\hbar_1, \hbar_2, q) - \frac{I_0(q)}{\hbar_2} S_{ji}^{(+)}(\hbar_2, \hbar_1, q). \end{aligned} \tag{5.6}$$

Since

$$\begin{aligned} \sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2\ge 0}} (-1)^r \sigma_r \mathcal{Y}_{p_1}(\alpha_i, \hbar_1, q) \mathcal{Y}_{p_2-l}(\alpha_j, \hbar_2, q) &= \sum_{\substack{p_1+p_2+r=n-1-l\\ p_1,p_2\ge 0}} (-1)^r \sigma_r \mathcal{Y}_{p_1}(\alpha_i, \hbar_1, q) \mathcal{Y}_{p_2}(\alpha_j, \hbar_2, q) \\ &\quad - \sum_{\substack{p_1+p_2+r=n-1-l\\ p_1,p_2\le -1}} (-1)^r \sigma_r \mathcal{Y}_{p_1}(\alpha_i, \hbar_1, q) \mathcal{Y}_{p_2}(\alpha_j, \hbar_2, q) \end{aligned}$$

by (3.5) and (3.11), the claim now follows from (5.5), (5.6), and the equality in (5.4). □

Corollary 5.3. *If* $p_1, p_2 \ge 0$, *then*

$$\sum_{r=0}^{p} (-1)^r \sigma_r B_{p-1-r+p_1,1}^{(p-r)}(q) = \sum_{r=0}^{p} (-1)^r \sigma_r B_{p-1-r+p_2,1}^{(p-r)}(q),$$

where $p=n-l-(p_1+p_2)$.

Proof. By (3.10) and (3.11),

$$\hbar\cdot\left\{\frac{\alpha_i}{\hbar_1}+\frac{\alpha_j}{\hbar_2}+\mathrm{D}\right\}\sum_{\substack{p_1+p_2+r=n-1\\p_1,p_2\ge0}}(-1)^r\sigma_r\mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2-l}(\alpha_j,\hbar_2,q)\bigg|_{\substack{\hbar_1=\hbar\\\hbar_2=-\hbar}}=0.$$

In light of Lemma 5.2, this implies that

$$S^{(2)}_{ij}(\hbar,-\hbar,q)=S^{(2)}_{ji}(-\hbar,\hbar,q).$$

Using (5.3) and (5.1), we then obtain

$$\begin{aligned}\alpha_i^l\alpha_j^l\sum_{\substack{p_1+p_2+r=n-l\\p_1,p_2\ge0}}(-1)^r\sigma_r\left[\sum_{r'=0}^{p_1}\mathrm{D}B^{(r')}_{p_1-1,1}(q)\alpha_i^{p_1-r'}\right]\alpha_j^{p_2}&\\ =\alpha_i^l\alpha_j^l\sum_{\substack{p_1+p_2+r=n-l\\p_1,p_2\ge0}}(-1)^r\sigma_r\left[\sum_{r'=0}^{p_1}\mathrm{D}B^{(r')}_{p_1-1,1}(q)\alpha_j^{p_1-r'}\right]\alpha_i^{p_2}.&\end{aligned}\tag{5.7}$$

Comparing the coefficients of $\alpha_i^{l+p_1}\alpha_j^{l+p_2}$ with $p_1,p_2\ge0$ and $p_1+p_2\le n-l$, we obtain

$$\mathrm{D}\sum_{r=0}^{p}(-1)^r\sigma_rB^{(p-r)}_{p-1-r+p_1,1}(q)=\mathrm{D}\sum_{r=0}^{p}(-1)^r\sigma_rB^{(p-r)}_{p-1-r+p_2,1}(q)\,,$$

with p as in the statement of the corollary. Since $B^{(p-r)}_{p-1-r+p_2,1}(q)\in q\cdot\mathbb{Q}_\alpha[[q]]$, this proves the claim. □

Corollary 5.4. *For all* $i,j=1,2,\ldots,n$,

$$S^{(2)}_{ij}(\hbar_1,\hbar_2,q)=S^{(2)}_{ji}(\hbar_2,\hbar_1,q).$$

Proof. By the same reasoning as in the proof of (3.10) on page 20, it is sufficient to verify this identity modulo $\hbar_1^{-1}$ and $\hbar_2^{-1}$. By (5.3) and (5.1),

$$S^{(2)}_{ij}(\hbar_1,\hbar_2,q)\cong\alpha_i^l\alpha_j^l\sum_{\substack{p_1+p_2+r=n-l\\p_1,p_2\ge0}}(-1)^r\sigma_r\left[\sum_{r'=0}^{p_1}\mathrm{D}B^{(r')}_{p_1-1,1}(q)\alpha_i^{p_1-r'}\right]\alpha_j^{p_2}\,,$$

$$S^{(2)}_{ji}(\hbar_2,\hbar_1,q)\cong\alpha_i^l\alpha_j^l\sum_{\substack{p_1+p_2+r=n-l\\p_1,p_2\ge0}}(-1)^r\sigma_r\left[\sum_{r'=0}^{p_1}\mathrm{D}B^{(r')}_{p_1-1,1}(q)\alpha_j^{p_1-r'}\right]\alpha_i^{p_2}\,.$$

The two expressions on the right-hand sides are the same by (5.7). □

Corollary 5.5. *For all* $i,j=1,2,\ldots,n$,

$$\frac{\hbar_1\hbar_2}{\hbar_1+\hbar_2}\left\{\frac{\alpha_i}{\hbar_1}+\frac{\alpha_j}{\hbar_2}+\mathrm{D}\right\}\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2\ge 0}}(-1)^r\sigma_r\mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2-l}(\alpha_j,\hbar_2,q)$$

$$=\left\{\sum_{\substack{p_1+p_2+r=n-l\\ p_1,p_2\ge 0}}-\sum_{\substack{p_1+p_2+r=n-l\\ p_1,p_2\le -1}}\right\}(-1)^r\sigma_r I_{p_1}(q)\mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2}(\alpha_j,\hbar_2,q)$$

$$+\sum_{\substack{p_1+p_2+r+r'=n-1-l\\ p_1,p_2,r'\ge 0}}(-1)^r\sigma_r\mathrm{D}B^{(r'+1)}_{p_1+r',1}(q)\mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2}(\alpha_j,\hbar_2,q),$$

where $I_{p_1}(q)\equiv 1$ *if* $p_1<0$.

Proof. By Lemma 5.2 and Corollary 5.4,

$$\frac{\hbar_1\hbar_2}{\hbar_1+\hbar_2}\left\{\frac{\alpha_i}{\hbar_1}+\frac{\alpha_j}{\hbar_2}+\mathrm{D}\right\}\sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2\ge 0}}(-1)^r\sigma_r\mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2-l}(\alpha_j,\hbar_2,q)$$

$$=\left\{\sum_{\substack{p_1+p_2+r=n-l\\ p_1,p_2\ge 0}}-\sum_{\substack{p_1+p_2+r=n-l\\ p_1,p_2\le -1}}\right\}(-1)^r\sigma_r\mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2}(\alpha_j,\hbar_2,q)+S^{(2)}_{ij}(\hbar_1,\hbar_2,q).$$

By (5.3) and (5.2),

$$S^{(2)}_{ij}(\hbar_1,\hbar_2,q)=\sum_{\substack{p_1+p_2+r=n-l\\ p_1,p_2\ge 0}}(-1)^r\sigma_r\big(I_{p_1}(q)-1\big)\mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2}(\alpha_j,\hbar_2,q)$$

$$+\sum_{\substack{p_1+p_2+r=n-l\\ p_1,p_2\ge 0}}(-1)^r\sigma_r\left[\sum_{r'=1}^{p_1}\mathrm{D}B^{(r')}_{p_1-1,1}(q)\mathcal{Y}_{p_1-r'}(\alpha_i,\hbar_1,q)\right]\mathcal{Y}_{p_2}(\alpha_j,\hbar_2,q).$$

The claim is obtained by combining these two identities. □

Lemma 5.6. *The coefficients* $\widetilde{\mathcal{C}}^{(r)}_{p,s}$ *of Theorem 6 satisfy*

$$I_p(q)\widetilde{\mathcal{C}}^{(r)}_{p,s}=\mathrm{D}\widetilde{\mathcal{C}}^{(r)}_{p-1,s}+I_s(q)\widetilde{\mathcal{C}}^{(r)}_{p-1,s-1}-\sum_{r'=1}^{\min(p,r)}\mathrm{D}B^{(r')}_{p-1,1}(q)\widetilde{\mathcal{C}}^{(r-r')}_{p-r',s}(q)\qquad\forall\,p\in\mathbb{Z}^+,$$

with $\widetilde{\mathcal{C}}^{(r)}_{p-1,-1},\widetilde{\mathcal{C}}^{(r)}_{p-1,p-r}\equiv 0$.

Proof. By the proof of Theorem 6, there is a unique choice of the coefficients $\widetilde{\mathcal{C}}^{(r)}_{p,s}$ so that (4.6) holds. Thus, similarly to the proof of Theorem 6, it is sufficient to show that

$$\sum_{r=0}^{p}\sum_{s=0}^{p-r}\left(\mathrm{D}\widetilde{\mathcal{C}}^{(r)}_{p-1,s}+I_s(q)\widetilde{\mathcal{C}}^{(r)}_{p-1,s-1}-\sum_{r'=1}^{\min(p,r)}\mathrm{D}B^{(r')}_{p-1,1}(q)\widetilde{\mathcal{C}}^{(r-r')}_{p-r',s}(q)\right)\hbar^{p-r-s}\mathfrak{D}^s\mathcal{Y}_0(x,\hbar,q)$$

is congruent to $I_p(q)x^{l+p}$ modulo $\hbar^{-1}$ whenever $x=\alpha_1,\ldots,\alpha_n$. By the p–1 case of (3.12), $\widetilde{\mathcal{C}}^{(0)}_{p,s}=\delta_{p,s}$, and the product rule,

$$\big\{x+\hbar\mathrm{D}\big\}\mathcal{Y}_{p-1}(x,\hbar,q)=\sum_{r=0}^{p}\sum_{s=0}^{p-r}\Big(\mathrm{D}\widetilde{\mathcal{C}}^{(r)}_{p-1,s}+I_s(q)\widetilde{\mathcal{C}}^{(r)}_{p-1,s-1}\Big)\hbar^{p-r-s}\mathfrak{D}^s\mathcal{Y}_0(x,\hbar,q).$$

By Lemma 5.1, (5.2), and (3.12), the left-hand side of this identity is congruent modulo $\hbar^{-1}$ to

$$\begin{aligned}I_p(q)x^{l+p}&+\sum_{r'=1}^{p}\mathrm{D}B^{(r')}_{p-1,1}(q)\mathcal{Y}_{p-r'}(x,\hbar,q)\\&\cong I_p(q)x^{l+p}+\sum_{r=1}^{p}\sum_{s=0}^{p-r}\sum_{r'=1}^{\min(p,r)}\mathrm{D}B^{(r')}_{p-1,1}(q)\widetilde{\mathcal{C}}^{(r-r')}_{p-r',s}(q)\hbar^{p-r-s}\mathfrak{D}^s\mathcal{Y}_0(x,\hbar,q)\,.\end{aligned}$$

This implies the desired congruence. □

Corollary 5.7. *For every* $p=1,2,\ldots,n-l$,

$$\begin{aligned}\widetilde{\mathcal{C}}^{(p)}_{p,0}+\widetilde{\mathcal{C}}^{(p)}_{n-l-1,n-l-1-p}&=(-1)^p\sigma_p\bigg(I_0\prod_{s=0}^{n-l-1-p}I_s-1\bigg)\\&\quad-\frac{1}{I_p}\sum_{r=1}^{p-1}\big(\mathrm{D}B^{(r)}_{p-1,1}\widetilde{\mathcal{C}}^{(p-r)}_{p-r,0}-\mathrm{D}B^{(r)}_{n-l-1,1}\widetilde{\mathcal{C}}^{(p-r)}_{n-l-r,n-l-p}\big)\\&\quad+\frac{1}{I_p}\sum_{r=1}^{p-1}(-1)^r\sigma_r\big(\mathrm{D}B^{(p-r)}_{p-1-r,1}-\mathrm{D}B^{(p-r)}_{n-l-1-r,1}-I_0\widetilde{\mathcal{C}}^{(p-r)}_{n-l-r,n-l-p}\big).\end{aligned}$$

Proof. Using (3.17) and $\widetilde{\mathcal{C}}^{(0)}_{p,s}=\delta_{p,s}$ first and then Lemma 5.6, we obtain

$$\begin{aligned}I_{n-l}\widetilde{\mathcal{C}}^{(p)}_{n-l,n-l-p}&=I_{n-l}\Bigg((-1)^p\sigma_p\bigg(\prod_{s=0}^{n-l-p}I_s-1\bigg)-\sum_{r=1}^{p-1}(-1)^r\sigma_r\widetilde{\mathcal{C}}^{(p-r)}_{n-l-r,n-l-p}\Bigg)\\&=I_{n-l-p}\widetilde{\mathcal{C}}^{(p)}_{n-l-1,n-l-1-p}-\sum_{r=1}^{p}\mathrm{D}B^{(r)}_{n-l-1,1}\cdot\widetilde{\mathcal{C}}^{(p-r)}_{n-l-r,n-l-p}.\end{aligned}$$

Since $I_{n-l-p}=I_p$ by [Po, (4.8)], this gives

$$\begin{aligned}I_p\widetilde{\mathcal{C}}^{(p)}_{n-l-1,n-l-1-p}&=(-1)^p\sigma_p\bigg(I_0\prod_{s=0}^{n-l-p}I_s-I_0\bigg)+\mathrm{D}B^{(p)}_{n-l-1,1}\\&\quad+\sum_{r=1}^{p-1}\big(\mathrm{D}B^{(r)}_{n-l-1,1}\cdot\widetilde{\mathcal{C}}^{(p-r)}_{n-l-r,n-l-p}-(-1)^r\sigma_rI_0\widetilde{\mathcal{C}}^{(p-r)}_{n-l-r,n-l-p}\big).\end{aligned}\tag{5.8}$$

Lemma 5.6 and $\widetilde{\mathcal{C}}^{(0)}_{0,0}=1$ give

$$I_p\widetilde{\mathcal{C}}^{(p)}_{p,0}=-\mathrm{D}B^{(p)}_{p-1,1}-\sum_{r=1}^{p-1}\mathrm{D}B^{(r)}_{p-1,1}\cdot\widetilde{\mathcal{C}}^{(p-r)}_{p-r,0}\,.\tag{5.9}$$

Adding up (5.8) and (5.9) and using Corollary 5.3 with $(p_1,p_2)=(n\text{–}l\text{–}p,0)$, (5.2), and $I_{n-l-p}=I_p$, we obtain the claim. □

5.2 Proof of (3.17)

Since both sides of (3.17) are symmetric polynomials in $\alpha_1,\ldots,\alpha_n$ of degree $n-l$, it is sufficient to verify (3.17) with $\alpha_p=0$ for all $p>n-l$. Thus, the $n-l$ statements in (3.17) are equivalent to

$$\sum_{p=1}^{n-l}\left(\sum_{r=0}^{p}(-1)^r\sigma_r\frac{\widetilde{\mathcal{C}}^{(p-r)}_{n-l-r,n-l-p}(q)}{\prod\limits_{s=0}^{n-l-p}I_s(q)}-(-1)^p\sigma_p\right)x^{n-l-p}=0 \qquad \begin{array}{c}\forall\ x=\alpha_1,\alpha_2,\ldots,\alpha_{n-l}\\ \alpha_p=0\ \forall\, p>n-l.\end{array} \tag{5.10}$$

Let $\mathrm{D}=q\frac{\mathrm{d}}{\mathrm{d}q}$ as before.

Proof of (5.10). Let $\mathrm{D}_{x,\hbar}=x+\hbar\mathrm{D}$; so,

$$\mathfrak{D}^p\mathcal{Y}_0=x^l\left\{\frac{\mathrm{D}_{x,\hbar}}{I_p}\right\}\left\{\frac{\mathrm{D}_{x,\hbar}}{I_{p-1}}\right\}\ldots\left\{\frac{\mathrm{D}_{x,\hbar}}{I_1}\right\}\left(\frac{\mathcal{Y}}{I_0}\right).$$

Throughout this argument, we assume that the conditions on x and α in (5.10) are satisfied. Thus,

$$\left\{\prod_{k=1}^{n-l}(\mathrm{D}_{x,\hbar}-\alpha_k)-q\langle\mathbf{a}\rangle\prod_{k=1}^{l}\prod_{r=1}^{a_k-1}\big(a_k\mathrm{D}_{x,\hbar}+r\hbar\big)\right\}\mathcal{Y}(x,\hbar,q)=0, \tag{5.11}$$

$$\mathcal{Y}_{n-l}(x,\hbar,q)-\sigma_1\mathcal{Y}_{n-l-1}(x,\hbar,q)+\ldots+(-1)^{n-l}\sigma_{n-l}\mathcal{Y}_0(x,\hbar,q)=0; \tag{5.12}$$

the first identity follows directly from (3.6), while the second from the $p=n-l$ case of (3.5) and (3.11). Subtracting (5.11) from $1/x^l$ times (5.12) and using (3.12) and the product rule, we obtain

$$\sum_{s=0}^{n-l}\sum_{p=s}^{n-l}\mathcal{A}_{s;p}(q)\hbar^s\mathrm{D}^{n-l-p}_{x,\hbar}\mathcal{Y}(x,\hbar,q)=0 \tag{5.13}$$

for some $\mathcal{A}_{s;p}(q)\in\mathbb{Q}[\alpha][[q]]$ with

$$\begin{aligned}\mathcal{A}_{0;p}(q)&=\sum_{r=0}^{p}(-1)^r\sigma_r\frac{\widetilde{\mathcal{C}}^{(p-r)}_{n-l-r,n-l-p}(q)}{\prod\limits_{s=0}^{n-l-p}I_s(q)}-(-1)^p\sigma_p+\delta_{p,0}\mathbf{a}^{\mathbf{a}}q\,,\\ \mathcal{A}_{n-l;n-l}(q)&=\left\{\frac{\mathrm{D}}{I_{n-l}(q)}\right\}\left\{\frac{\mathrm{D}}{I_{n-l-1}(q)}\right\}\ldots\left\{\frac{\mathrm{D}}{I_1(q)}\right\}\left(\frac{1}{I_0(q)}\right)+\mathbf{a}!\cdot q.\end{aligned} \tag{5.14}$$

By (3.6),

$$\mathrm{D}^p_{x,\hbar}\mathcal{Y}(x,\hbar,q)=x^p+\left(\prod_{k=1}^{l}\prod_{r=1}^{a_k-1}(a_kx+r\hbar)\right)\mathcal{F}_p(x,\hbar,q) \tag{5.15}$$

for some $\mathcal{F}_p\in q\cdot\mathbb{Q}_\alpha(x,\hbar)[[q]]$ which is regular at $\hbar=-a_kx/r$ for all $r\in[a_k-1]$. Since $\mathcal{A}_{n-l;n-l}=0$ by (5.14) and Lemma 5.8 below, it follows from (5.13) and (5.15) that

$$\sum_{s=0}^{n-l-1}\sum_{p=s}^{n-l}\mathcal{A}_{s;p}(q)x^{n-l-p}\hbar^s=0 \qquad \forall\,\hbar=-a_kx/r,\ r\in[a_k-1],\ k\in[l]. \tag{5.16}$$

Since the left-hand side of (5.16) is a polynomial in $\hbar$ of degree $n-l-1$ with $n-l$ zeros (counted with multiplicity)[11], it vanishes identically. On the other hand, since $\mathcal{A}_{0;0}=0$ by (5.14), $\widetilde{\mathcal{C}}_{p,s}^{(0)}=\delta_{p,s}$, and [Po, (4.9)], the coefficient of $\hbar^0$ on the left-hand side of (5.16) is precisely the left-hand side of (5.10). $\square$

Lemma 5.8. *If* $|\mathbf{a}|=n$, *then*

$$\left\{\frac{\mathrm{D}}{I_{n-l}(q)}\right\}\left\{\frac{\mathrm{D}}{I_{n-l-1}(q)}\right\}\cdots\left\{\frac{\mathrm{D}}{I_1(q)}\right\}\left(\frac{1}{I_0(q)}\right)=-\mathbf{a}!\cdot q.$$

Proof. Let $\mathrm{D}_w=w+\mathrm{D}$; so,

$$w^p\mathbf{M}^pF(w,q)=I_p(q)\left\{\frac{\mathrm{D}_w}{I_p}\right\}\left\{\frac{\mathrm{D}_w}{I_{p-1}}\right\}\cdots\left\{\frac{\mathrm{D}_w}{I_1}\right\}\left(\frac{F}{I_0}\right).$$

The series $F(w,q)$ of (1.1) satisfies the differential equations

$$\left\{\mathrm{D}_w^{n-l}-q\langle\mathbf{a}\rangle\prod_{k=1}^{l}\prod_{r=1}^{a_k-1}(a_k\mathrm{D}_w+r)\right\}F=w^{n-l}, \tag{5.17}$$

$$\mathbf{M}^{n-l}F(w,q)=I_{n-l}(q); \tag{5.18}$$

the first identity follows directly from (1.1), while the second is proved in [Po, Section 4.1]. Subtracting (5.17) from $w^{n-l}/I_{n-l}(q)$ times (5.18) and using the product rule, we obtain

$$\sum_{p=0}^{n-l}A_p(q)\mathrm{D}_w^{n-l-p}F(w,q)=0 \tag{5.19}$$

for some $A_p(q)\in\mathbb{Q}[[q]]$ with

$$\begin{aligned} A_0(q)&=\frac{1}{\prod\limits_{s=0}^{n-l}I_s(q)}-1+\mathbf{a}^{\mathbf{a}}q\,,\\ A_{n-l}(q)&=\left\{\frac{\mathrm{D}}{I_{n-l}(q)}\right\}\left\{\frac{\mathrm{D}}{I_{n-l-1}(q)}\right\}\cdots\left\{\frac{\mathrm{D}}{I_1(q)}\right\}\left(\frac{1}{I_0(q)}\right)+\mathbf{a}!\cdot q. \end{aligned} \tag{5.20}$$

By (1.1),

$$\mathrm{D}_w^pF(w,q)=w^p+\left(\prod_{k=1}^{l}\prod_{r=1}^{a_k-1}(a_kw+r)\right)H_p(w,q) \tag{5.21}$$

for some $H_p\in q\cdot\mathbb{Q}(w)[[q]]$ which is regular at $w=-\frac{r}{a_k}$ for all $r\in[a_k-1]$. Since $A_0=0$ by (5.20) and [Po, (4.9)], it follows from (5.19) and (5.21) that

$$\sum_{p=1}^{n-l}A_p(q)w^{n-l-p}=0 \qquad \forall\, w=-r/a_k,\ r\in[a_k-1],\ k\in[l]. \tag{5.22}$$

Since the left-hand side of (5.22) is a polynomial in w of degree $n-l-1$ with $n-l$ zeros (counted with multiplicity), it vanishes identically; in particular, $A_{n-l}(q)=0$. The claim now follows from the second identity in (5.20). $\square$

[11] This is one of the two places in the proof where the Calabi-Yau condition, $|\mathbf{a}|=n$, is used; the other place is the $p=0$ case of (5.14).

6 Annulus GW-Invariants of CY CI Threefolds

It remains to establish Theorems 3 and 4 concerning the annulus and Klein bottle invariants of $(X_{\mathbf{a}},\Omega)$, where $\Omega:\mathbb{P}^{n-1}\longrightarrow\mathbb{P}^{n-1}$ is the anti-symplectic involution given by (1.9) and $X_{\mathbf{a}}\subset\mathbb{P}^{n-1}$ is a smooth CY CI threefold of multi-degree $\mathbf{a}$ such that $\Omega(X_{\mathbf{a}})=X_{\mathbf{a}}$. For the remainder of the paper, $X_{\mathbf{a}}$ will denote such a threefold. Furthermore, all statements involving the $\mathbb{T}^n$-weights $\alpha=(\alpha_1,\ldots,\alpha_n)$ will be taken to mean that they hold when restricted to the subtorus $\mathbb{T}^m$ of Section 3.2; see (3.19).

6.1 Description and graph-sum definition

A degree d annulus doubled map to $\mathbb{P}^{n-1}$ is an Ω-invariant map $(\tilde{C},\tau,\tilde{f})$ to $\mathbb{P}^{n-1}$ such that τ has two fixed components. An example of the restriction of τ to the principal component $\tilde{C}_0$ of $\tilde{C}$ in this case is given by

$$\begin{gathered}\tilde{C}_0=(\mathbb{Z}_{2k}\times\mathbb{P}^1/\sim),\quad\text{where}\quad (s,[0,1])\sim(s+1,[1,0])\quad\forall\,s\in\mathbb{Z}_{2k}\,,\\ \tau:\tilde{C}_0\longrightarrow\tilde{C}_0,\quad \tau\big(s,[z_1,z_2]\big)=\big(-s,[\bar{z}_2,\bar{z}_1]\big).\end{gathered}\tag{6.1}$$

Since $\tilde{C}/\tau$ can be naturally identified with two subspaces of $\tilde{C}$, a degree d annulus doubled map $\tilde{f}:\tilde{C}\longrightarrow\mathbb{P}^{n-1}$ corresponds to a pair of maps $f_1,f_2:\tilde{C}/\tau\longrightarrow\mathbb{P}^{n-1}$ that differ by the composition with Ω; these maps will be referred to as degree d annulus maps. This is the analogue of the definition for disk maps; see [PSoW, Section 1.3.3]. The moduli space of stable degree d doubled annulus maps to a CY CI threefold $X_{\mathbf{a}}\subset\mathbb{P}^{n-1}$,

$$\overline{\mathfrak{M}}_1(X_{\mathbf{a}},d)^{\Omega}\equiv\overline{\mathfrak{M}}_{1,2,0}(X_{\mathbf{a}},\Omega,d),$$

is expected to have a well-defined virtual degree A_d, with $A_d=0$ if d is odd.[12] A combinatorial description of A_d for d even, due to [W], is recalled and motivated below.

If $1\le i\le 2m$ and $\gamma\in\mathbb{Z}^+$, let

$$\tilde{f}_{i,\gamma}:(\mathbb{P}^1,0)\longrightarrow\big(\ell_{i\bar{i}},P_i\big)\subset(\mathbb{P}^{n-1},P_i)$$

denote the equivalence class of the degree γ cover $\mathbb{P}^1\longrightarrow\ell_{i\bar{i}}$ branched over P_i and $P_{\bar{i}}$ only and taking the marked point $0\in\mathbb{P}^1$ to P_i. Denote by

$$\overline{\mathfrak{M}}_1(\mathbb{P}^{n-1},d)^{\Omega;\mathbb{T}^m}\subset\overline{\mathfrak{M}}_1(\mathbb{P}^{n-1},d)^{\Omega}$$

the subspace of the $\mathbb{T}^m$-fixed equivalence classes of stable doubled annulus maps.

Based on the disk case fully treated in [So], [PSoW], and [Sh], one would expect that

$$A_{2d}=-2\sum_{F\subset\overline{\mathfrak{M}}_1(\mathbb{P}^{n-1},2d)^{\Omega;\mathbb{T}^m}}\int_F\frac{\mathbf{e}(\mathcal{V}_{1,2d}^{\Omega})|_F}{\mathbf{e}(NF)}\,,\tag{6.2}$$

where the sum is taken over the connected components F of $\overline{\mathfrak{M}}_1(\mathbb{P}^{n-1},2d)^{\Omega;\mathbb{T}^m}$, $\mathbf{e}(NF)$ is the equivariant Euler class of the normal "bundle" NF of F in $\overline{\mathfrak{M}}_1(\mathbb{P}^{n-1},2d)^{\Omega}$, and $\mathbf{e}(\mathcal{V}_{1,2d}^{\Omega})$ is the equivariant Euler class of the obstruction "bundle"

$$\mathcal{V}_{1,2d}^{\Omega}\equiv\overline{\mathfrak{M}}_1(\mathcal{L},2d)^{\Omega}\longrightarrow\overline{\mathfrak{M}}_1(\mathbb{P}^{n-1},2d)^{\Omega}\,,$$

[12] This is consistent with the vector bundle $\mathcal{V}_{1,2d}^{\Omega}$ in (6.2) having a direct summand of odd real rank; thus, its Euler class is zero.

with $\mathcal{L}$ given by (3.18). The factor of 2 in (6.2) is due to the fact that a doubled annulus map corresponds to 2 annulus maps, while the choice of sign [W, (3.15),(3.23)] is due to delicate physical considerations. While $\overline{\mathfrak{M}}_1(\mathbb{P}^{n-1},2d)^\Omega$ and $\mathcal{V}_{1,2d}^\Omega$ may be singular near doubled maps $(\tilde{C},\tau,\tilde{f})$ with contracted principle component $\tilde{C}_0$ or a τ-fixed node on $\tilde{C}$, the restriction of $\mathcal{V}_{1,2d}^\Omega$ to the fixed loci F consisting of such maps contains a topologically trivial subbundle of weight 0 and thus does not contribute to (6.2). For the same reason, there is no genus 0 correction to A_{2d}, which in the closed genus 1 case arises from the stable maps with contracted principle component; see [LiZ, (1.5)] and [Z09a, Theorem 1.1].[13]

Thus, by Corollary 3.2, all nonzero contributions to (6.2) come from fixed loci F consisting of stable doubled maps $(\tilde{C},\tau,\tilde{f})$ such that the principal component $\tilde{C}_0\subset\tilde{C}$ is a circle of spheres as in (6.1) and $\tilde{f}|_{\tilde{C}_0}$ is not constant. In this case, $\tilde{C}_0$ contains two components, $\tilde{C}_{0,1}$ and $\tilde{C}_{0,2}$, each of which is mapped by τ into itself; so, $\tilde{C}_{0,1}/\tau$ and $\tilde{C}_{0,2}/\tau$ are disks. Similarly to the disk case treated in [PSoW] and [Sh], the fixed-point loci F containing maps $(\tilde{C},\tau,\tilde{f})$ with $\tilde{f}|_{\tilde{C}_{0,1}}$ or $\tilde{f}|_{\tilde{C}_{0,2}}$ of even degree should not contribute to (6.2). Thus, the fixed-point loci F contributing to (6.2) consist of stable maps $(\tilde{C},\tau,\tilde{f})$ such that $\tilde{f}|_{\tilde{C}_{0,r}}$ is the unique cover of a line $\ell_{i_r\bar{i}_r}$ with $1\le i_r\le 2m$ of odd degree γ_r branched at the nodes of $\tilde{C}_{0,r}$ and over P_{i_r} and $P_{\bar{i}_r}$ only, for each $r=1,2$.

Any such annulus $\mathbb{T}^m$-fixed doubled map $(\tilde{C},\tau,\tilde{f})$ corresponds to a map from the quotient $\tilde{C}/\tau$, which is either a wedge of two disks, $C_{0,1}$ and $C_{0,2}$, or a tree of spheres with two disks attached. As an equivalence class of maps from a disk with 1 marked point (the node), $\tilde{f}|_{C_{0,r}}$ equals to the restriction of the map $\tilde{f}_{i_r,\gamma_r}$ with $1\le i_r\le 2m$ and $\gamma_r\in\mathbb{Z}^+$ odd to a disk containing 0, for each $r=1,2$. If $\tilde{C}/\tau$ contains a nonempty tree of spheres C', $\tilde{f}|_{C'}$ corresponds to a $\mathbb{T}^m$-fixed genus 0 stable map with two marked points (which are mapped to P_{i_1} and P_{i_2}, respectively), i.e. an element of $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},r)^{\mathbb{T}^m}$. Thus, a fixed locus F contributing to (6.2) corresponds to a tuple (i,γ_1,γ_2) in the first case and a tuple $(i_1,\gamma_1;i_2,\gamma_2;F')$ in the second case, where

- $1\le i,i_1,i_2\le 2m$ describe the fixed points $P_k\in\mathbb{P}^{n-1}$ to which the nodes of the two disks are mapped by $\tilde{f}$;
- $\gamma_1,\gamma_2\in\mathbb{Z}^+$ are odd and describe the degrees of the restrictions of $\tilde{f}$ to the doubled disks, $\tilde{C}_{0,1}$ and $\tilde{C}_{0,2}$, and in the first case $\gamma_1+\gamma_2=2d$;
- $F'\subset\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},r)^{\mathbb{T}^m}$ is a component of the fixed locus and $\gamma_1+\gamma_2+2r=2d$ in the second case.

It is convenient to view the first case, when the tree of spheres is empty, as the $r=0$ extension of the second case, with $i_1=i_2=i$. The two cases are illustrated in Figure 1.

Denote by $NF'\longrightarrow F'$ the normal bundle of F' in $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},r)$ and by $T_{0,\gamma}^\Omega|_{\tilde{f}_{i,\gamma}}$ the tangent space at $\tilde{f}_{i,\gamma}$ of the space of maps from disks with 1 marked point. Doubled annulus maps close to one of the above fixed loci F can be obtained by deforming the disk components of a fixed map $(\tilde{C},\tau,\tilde{f})$ and its component lying in F', while keeping them joined at the two nodes; the last

[13] Such a correction would be a multiple of the degree $2d$ disk invariant, which vanishes.

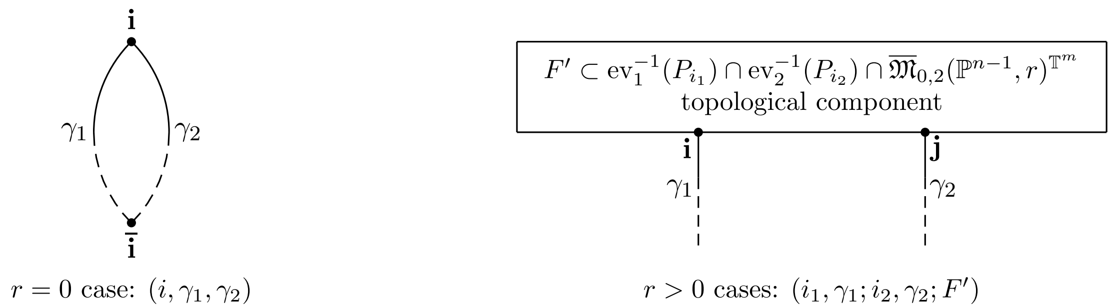

Figure 1: $\mathbb{T}^m$-fixed loci contributing to $A_{\gamma_1+\gamma_2+2r}$, depicted as half-graphs; half-dotted edges correspond to the two doubled disks.

deformation takes place in $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},r)$. Since the disk nodes can also be smoothed out,

$$\begin{aligned}\mathbf{e}(NF) &= \frac{\mathbf{e}(T^{\Omega}_{0,\gamma_1}|_{\tilde f_{i_1,\gamma_1}})\cdot \mathbf{e}(NF')\cdot \mathbf{e}(T^{\Omega}_{0,\gamma_2}|_{\tilde f_{i_2,\gamma_2}})}{\mathbf{e}(T_{P_{i_1}}\mathbb{P}^{n-1})\mathbf{e}(T_{P_{i_2}}\mathbb{P}^{n-1})}\cdot(\hbar_1-\psi_1)\cdot(\hbar_2-\psi_2)\\ &= \frac{\mathbf{e}(T^{\Omega}_{0,\gamma_1}|_{\tilde f_{i_1,\gamma_1}})\cdot \mathbf{e}(T^{\Omega}_{0,\gamma_2}|_{\tilde f_{i_2,\gamma_2}})\cdot \mathbf{e}(NF')}{\phi_{i_1}|_{P_{i_1}}\phi_{i_2}|_{P_{i_2}}}\cdot(\hbar_1-\psi_1)\cdot(\hbar_2-\psi_2)\,,\end{aligned}\tag{6.3}$$

where $\hbar_r$ is the equivariant Euler class of the tangent space at the node of the r-th disk, ψ_1,ψ_2 denote the restrictions of the ψ-classes on $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},r)$ to F' if $r>0$, and

$$\frac{\mathbf{e}(NF')}{\phi_{i_2}|_{P_{i_2}}}(\hbar_1-\psi_1)\cdot(\hbar_2-\psi_2)\equiv \hbar_1+\hbar_2 \qquad \text{if } \ r=0.\tag{6.4}$$

The restriction of $\mathcal{V}^{\Omega}_{1,d}$ to F is the subbundle of the direct sum of the corresponding bundles over the components F consisting of sections that agree at the nodes of the disks. Thus,

$$\begin{aligned}\mathbf{e}(\mathcal{V}^{\Omega}_{1,2d})\big|_F &= \frac{\mathbf{e}(\mathcal{V}^{\Omega}_{0,\gamma_1}|_{\tilde f_{i_1,\gamma_1}})\cdot \mathbf{e}(\mathcal{V}_{0,r})|_{F'}\cdot \mathbf{e}(\mathcal{V}^{\Omega}_{0,\gamma_2}|_{\tilde f_{i_2,\gamma_2}})}{\mathbf{e}(\mathcal{L})|_{P_{i_1}}\mathbf{e}(\mathcal{L})|_{P_{i_2}}}\\ &= \frac{\mathbf{e}(\mathcal{V}^{\Omega}_{0,\gamma_1}|_{\tilde f_{i_1,\gamma_1}})\cdot \mathbf{e}(\mathcal{V}^{\Omega}_{0,\gamma_2}|_{\tilde f_{i_2,\gamma_2}})\cdot \mathbf{e}(\mathcal{V}_{0,r})|_{F'}}{\langle\mathbf{a}\rangle^2\alpha^l_{i_1}\alpha^l_{i_2}}\,,\end{aligned}\tag{6.5}$$

where $\mathcal{V}_{0,r}$ denotes the bundle $\mathcal{V}_{\mathbf{a}}\longrightarrow\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},r)$ of Section 2 if $r>0$, $\mathbf{e}(\mathcal{V}_{0,0})|_{F'}\equiv\langle\mathbf{a}\rangle\alpha^l_{i_1}$, and $\mathcal{V}^{\Omega}_{0,\gamma_r}|_{f_{i_r,\gamma_r}}$ is the vector space of maps $(\tilde C_{0,r},\tau)\longrightarrow(\mathcal{L},\Omega)$ lifting $\tilde f_{i_r,\gamma_r}$.

By (6.3)-(6.5),

$$\int_F\frac{\mathbf{e}(\mathcal{V}^{\Omega}_{1,2d})|_F}{\mathbf{e}(NF)}=\frac{1}{\langle\mathbf{a}\rangle^2\alpha^l_{i_1}\alpha^l_{i_2}}\cdot\frac{\mathcal{D}_{i_1,\gamma_1}}{\hbar_1}\cdot\frac{\mathcal{D}_{i_2,\gamma_2}}{\hbar_2}\cdot\begin{cases}\frac{\langle\mathbf{a}\rangle\alpha^l_{i_1}}{\hbar_1+\hbar_2}\prod\limits_{k\neq i_1}(\alpha_{i_2}-\alpha_k), & \text{if } r=0,\\ \int_{F'}\frac{\mathbf{e}(\mathcal{V}_{0,r})(\mathrm{ev}_1^*\phi_{i_1})(\mathrm{ev}_2^*\phi_{i_2})}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}\Big|_{F'}\frac{1}{\mathbf{e}(NF')}, & \text{if } r>0,\end{cases}\tag{6.6}$$

where $\mathcal{D}_{i,\gamma}$ is the contribution of the half-edge disk map $\tilde f_{i,\gamma}$ without a marked point to the degree γ

disk invariant. This contribution is computed in [PSoW, Lemma 6] and in [Sh] to be

$$\begin{aligned}\mathcal{D}_{i,\gamma} &= (-1)^{\frac{\gamma-1}{2}} \frac{\prod_{k=1}^{l}(a_k\gamma)!!}{2^{\gamma-1}\gamma^{\frac{(n-2)\gamma+l+4}{2}}\gamma!} \frac{\alpha_i^{\frac{(n-2)\gamma+l+2}{2}}}{\prod\limits_{\substack{1\le k\le n\\ k\neq i,\bar{i}}}\prod\limits_{s=0}^{(\gamma-1)/2}\left(\frac{\gamma-2s}{\gamma}\alpha_i-\alpha_k\right)} \\ &= 2\frac{\prod_{k=1}^{l}(a_k\gamma)!!}{\gamma\prod\limits_{\substack{k=1\\(k,s)\neq(i,\gamma)}}^{n}\prod\limits_{\substack{1\le s\le\gamma\\ s\,\text{odd}}}\left(\frac{s}{\gamma}\alpha_i-\alpha_k\right)}\left(\frac{\alpha_i}{\gamma}\right)^{\frac{n\gamma+l+2}{2}} \qquad \forall\ \gamma\ \text{odd}\end{aligned} \tag{6.7}$$

whenever every component a_k of $\mathbf{a}$ is odd; otherwise, $\mathcal{D}_{i,\gamma}=0$.

By the classical Localization Theorem [ABo], if $r>0$

$$\sum_{F'}\int_{F'}\frac{\mathbf{e}(\mathcal{V}_{0,r})(\mathrm{ev}_1^*\phi_{i_1})(\mathrm{ev}_2^*\phi_{i_2})}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}\bigg|_{F'}\frac{1}{\mathbf{e}(NF')} = \int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},r)}\frac{\mathbf{e}(\mathcal{V}_{\mathbf{a}})(\mathrm{ev}_1^*\phi_{i_1})(\mathrm{ev}_2^*\phi_{i_2})}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}, \tag{6.8}$$

where the sum is taken over all fixed loci $F'\subset\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},r)$. The $r=0$ case of (6.6) gives the coefficient of Q^0 in the power series $\mathcal{Z}$ in (4.7). Since

$$\hbar_r = \frac{\alpha_{i_r}-\alpha_{\bar{i}_r}}{\gamma_r} = \frac{2\alpha_{i_r}}{\gamma_r} \qquad \forall\, r=1,2$$

and we have made 4 choices in describing each fixed locus F (choosing an embedding $\tilde{C}/\tau\subset\tilde{C}$ and ordering the resulting disks), plugging in (6.6) and (6.8) into (6.2) leads to the following compact reformulation of the combinatorial description of A_{2d} given in [W].

Definition 6.1. *The degree* $2d$ *annulus invariant* A_{2d} *of* $(X_{\mathbf{a}},\Omega)$ *is given by*

$$\sum_{d=1}^{\infty}Q^dA_{2d} = -\frac{1}{2\langle\mathbf{a}\rangle^2}\sum_{\substack{1\le i,j\le 2m\\ \gamma,\delta\in\mathbb{Z}^+\ \text{odd}}} Q^{\frac{\gamma+\delta}{2}}\frac{\mathcal{D}_{i,\gamma}\mathcal{D}_{j,\delta}}{\alpha_i^l\alpha_j^l}\frac{1}{\hbar_1\hbar_2}\mathcal{Z}\big(\alpha_i,\alpha_j,\hbar_1,\hbar_2,Q\big)\bigg|_{\substack{\hbar_1=\frac{2\alpha_i}{\gamma}\\ \hbar_2=\frac{2\alpha_j}{\delta}}}. \tag{6.9}$$

It is shown in [Sh], as well as in [PSoW] in the $\mathbf{a}=(5)$ case, that

$$Z_{disk}(Q) = \sum_{\substack{1\le i\le 2m\\ \gamma\in\mathbb{Z}^+\ \text{odd}}} Q^{\frac{\gamma}{2}}\frac{\mathcal{D}_{i,\gamma}}{\alpha_i^l}\mathcal{Z}_0(\alpha_i,\hbar,Q)\bigg|_{\hbar=\frac{2\alpha_i}{\gamma}}, \tag{6.10}$$

where $Z_{disk}(Q)$ is the disk potential as in (1.11). Definition 6.1 for the annulus invariants is analogous to this property for the disk invariants. Since $\sigma_1=0$ by (3.19), (6.10) and (3.11) give

$$Z_{disk}(Q) = \sum_{\substack{1\le i\le 2m\\ \gamma\in\mathbb{Z}^+\ \text{odd}}} q^{\frac{\alpha_i}{\hbar}}\frac{\mathcal{D}_{i,\gamma}}{\alpha_i^l}\mathcal{Y}_0(\alpha_i,\hbar,q)\bigg|_{\hbar=\frac{2\alpha_i}{\gamma}}, \tag{6.11}$$

with Q and q related by the mirror map (1.4). This formula is used in [PSoW] and [Sh] to obtain (1.12) whenever all components of $\mathbf{a}$ are odd; in the other cases, $Z_{disk}(Q)=0$ by (6.11), since $\mathcal{D}_{i,\gamma}=0$.

6.2 Proof of Theorem 3

If any component a_k of $\mathbf{a}$ is even, Theorem 3 is meaningless, simply stating that $0=0$. In the remaining cases, Theorem 3 is proved in this section.

By (6.9), (3.10), (3.11), and (4.7),

$$\sum_{d=1}^{\infty} Q^d A_{2d} = -\frac{1}{2\langle \mathbf{a}\rangle} \sum_{\substack{1\le i,j\le 2m\\ \gamma,\delta\in\mathbb{Z}^+ \text{ odd}}} \Bigg\{ \frac{1}{(\hbar_1+\hbar_2)\,\hbar_1\hbar_2} \frac{\mathcal{D}_{i,\gamma}\mathcal{D}_{j,\delta}}{\alpha_i^l\alpha_j^l} \times q^{\frac{\alpha_i}{\hbar_1}+\frac{\alpha_j}{\hbar_2}} \sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2\ge 0}} (-1)^r\sigma_r \mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2-l}(\alpha_j,\hbar_2,q) \Bigg\}\Bigg|_{\substack{\hbar_1=\frac{2\alpha_i}{\gamma}\\ \hbar_2=\frac{2\alpha_j}{\delta}}} \tag{6.12}$$

with $\mathcal{Y}_p(x,\hbar,q)$ given by (3.12), (3.14), and (3.15). Since $Q\frac{\mathrm{d}}{\mathrm{d}Q}=\frac{q}{I_1(q)}\frac{\mathrm{d}}{\mathrm{d}q}$, by Corollary 5.5

$$\begin{aligned}
&\frac{\hbar_1\hbar_2}{\hbar_1+\hbar_2} Q\frac{\mathrm{d}}{\mathrm{d}Q}\Bigg\{ q^{\frac{\alpha_i}{\hbar_1}+\frac{\alpha_j}{\hbar_2}} \sum_{\substack{p_1+p_2+r=n-1\\ p_1,p_2\ge 0}} (-1)^r\sigma_r \mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2-l}(\alpha_j,\hbar_2,q)\Bigg\}\\
&= \frac{1}{I_1(q)} q^{\frac{\alpha_i}{\hbar_1}+\frac{\alpha_j}{\hbar_2}} \Bigg[I_2(q)\mathcal{Y}_2(\alpha_i,\hbar_1,q)\mathcal{Y}_2(\alpha_j,\hbar_2,q) + \sum_{\substack{p_1,p_2\ge -l\\ p_1<2 \text{ or } p_2<2}} A_{p_1p_2}(q)\mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2}(\alpha_j,\hbar_2,q)\Bigg],
\end{aligned}$$

for some $A_{p_1p_2}(q)\in\mathbb{Q}_\alpha[[q]]$. By the $-l\le p<b=2$ case of Corollary 6.3 below,

$$\sum_{\substack{p_1,p_2\ge -l\\ p_1<2 \text{ or } p_2<2}} A_{p_1p_2}(q) \sum_{\substack{1\le i,j\le 2m\\ \gamma,\delta\in\mathbb{Z}^+ \text{ odd}}} \frac{1}{\hbar_1^2\hbar_2^2}\frac{\mathcal{D}_{i,\gamma}\mathcal{D}_{j,\delta}}{\alpha_i^l\alpha_j^l} q^{\frac{\alpha_i}{\hbar_1}+\frac{\alpha_j}{\hbar_2}} \mathcal{Y}_{p_1}(\alpha_i,\hbar_1,q)\mathcal{Y}_{p_2}(\alpha_j,\hbar_2,q)\Bigg|_{\substack{\hbar_1=\frac{2\alpha_i}{\gamma}\\ \hbar_2=\frac{2\alpha_j}{\delta}}} = 0.$$

By the $p=b=2$ case of Corollary 6.3 and (6.11),

$$\begin{aligned}
&\frac{I_2(q)}{I_1(q)} \sum_{\substack{1\le i,j\le 2m\\ \gamma,\delta\in\mathbb{Z}^+ \text{ odd}}} \frac{1}{\hbar_1^2\hbar_2^2}\frac{\mathcal{D}_{i,\gamma}\mathcal{D}_{j,\delta}}{\alpha_i^l\alpha_j^l} q^{\frac{\alpha_i}{\hbar_1}+\frac{\alpha_j}{\hbar_2}} \mathcal{Y}_2(\alpha_i,\hbar_1,q)\mathcal{Y}_2(\alpha_j,\hbar_2,q)\Bigg|_{\substack{\hbar_1=\frac{2\alpha_i}{\gamma}\\ \hbar_2=\frac{2\alpha_j}{\delta}}}\\
&= \frac{I_2(q)}{I_1(q)}\left[\sum_{\substack{1\le i\le 2m\\ \gamma\in\mathbb{Z}^+ \text{ odd}}} \frac{\mathcal{D}_{i,\gamma}}{\alpha_i^l} q^{\frac{\alpha_i}{\hbar}} \frac{1}{I_2(q)}\left\{\frac{\alpha_i}{\hbar}+q\frac{\mathrm{d}}{\mathrm{d}q}\right\}\left(\frac{1}{I_1(q)}\left\{\frac{\alpha_i}{\hbar}+q\frac{\mathrm{d}}{\mathrm{d}q}\right\}\mathcal{Y}_0(\alpha_i,\hbar,q)\right)\Bigg|_{\hbar=\frac{2\alpha_i}{\gamma}}\right]^2\\
&= \frac{1}{I_1(q)I_2(q)}\left[q\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{1}{I_1(q)}q\frac{\mathrm{d}}{\mathrm{d}q}\sum_{\substack{1\le i\le 2m\\ \gamma\in\mathbb{Z}^+ \text{ odd}}} q^{\frac{\alpha_i}{\hbar}}\frac{\mathcal{D}_{i,\gamma}}{\alpha_i^l}\mathcal{Y}_0(\alpha_i,\hbar,q)\Bigg|_{\hbar=\frac{2\alpha_i}{\gamma}}\right)\right]^2\\
&= \frac{I_1(q)}{I_2(q)}\left[\left\{Q\frac{\mathrm{d}}{\mathrm{d}Q}\right\}^2 Z_{disk}(Q)\right]^2.
\end{aligned}$$

The identity in Theorem 3 follows immediately from (6.12) and the last three equations.

Lemma 6.2. *If* $\mathbf{a}$ *is an* l*-tuple of positive integers with* $n\equiv|\mathbf{a}|=l+4$*,* $b,p\in\mathbb{Z}$ *with* $-l\le p\le\min(b,0)$ *and* $b-p\le l+2$*, and* $\mathcal{D}_{i,\gamma}$ *is given by (6.7), then*

$$\sum_{\substack{1\le i\le 2m\\ \gamma\in\mathbb{Z}^+\text{odd}}}\frac{1}{\hbar^b}\frac{\mathcal{D}_{i,\gamma}}{\alpha_i^l}q^{\frac{\alpha_i}{\hbar}}\mathfrak{D}^p\mathcal{Y}_0(\alpha_i,\hbar,q)\Big|_{\hbar=\frac{2\alpha_i}{\gamma}}=\begin{cases}0, & \text{if } p<b;\\ \frac{1}{2^{b-1}I_p(q)}\sum\limits_{d\in\mathbb{Z}^+\text{odd}}d^pq^{\frac{d}{2}}\frac{\prod_{k=1}^{l}(a_kd)!!}{(d!!)^n}, & \text{if } p=b.\end{cases}$$

Proof. By (3.8) and (3.9),

$$\mathfrak{D}^p\mathcal{Y}_0(\alpha_i,\hbar,q)\Big|_{\hbar=\frac{2\alpha_i}{\gamma}}=\frac{\alpha_i^l}{I_p(q)}\sum_{d=0}^{\infty}q^d(\gamma+2d)^p\left(\frac{\alpha_i}{\gamma}\right)^{nd+p}\frac{\prod_{k=1}^{l}\frac{(a_k(\gamma+2d))!!}{(a_k\gamma)!!}}{\prod_{k=1}^{n}\prod_{\substack{\gamma+2\le s\le\gamma+2d\\ s\text{ odd}}}\left(s\frac{\alpha_i}{\gamma}-\alpha_k\right)},$$

where $I_p(q)\equiv1$ if $p<0$. Thus, by (6.7) and the Residue Theorem on S^2,

$$2^{b-1}I_p(q)\sum_{\substack{1\le i\le 2m\\ \gamma\in\mathbb{Z}^+\text{odd}}}\frac{1}{\hbar^b}\frac{\mathcal{D}_{i,\gamma}}{\alpha_i^l}q^{\frac{\alpha_i}{\hbar}}\mathfrak{D}^p\mathcal{Y}_0(\alpha_i,\hbar,q)\Big|_{\hbar=\frac{2\alpha_i}{\gamma}}=\sum_{\substack{1\le i\le 2m\\ \gamma\in\mathbb{Z}^+\text{odd}}}\sum_{\substack{t\in\mathbb{Z}^+\text{odd}\\ t\ge\gamma}}t^pq^{\frac{t}{2}}\frac{\left(\frac{\alpha_i}{\gamma}\right)^{\frac{nt+l+2}{2}+p-b}\prod_{k=1}^{l}(a_kt)!!}{\gamma\prod_{k=1}^{n}\prod_{\substack{1\le s\le t\\ s\text{ odd}\\ (k,s)\neq(i,\gamma)}}\left(s\frac{\alpha_i}{\gamma}-\alpha_k\right)}$$

$$=\sum_{t\in\mathbb{Z}^+\text{odd}}t^pq^{\frac{t}{2}}\sum_{\substack{1\le i\le 2m\\ \gamma\in\mathbb{Z}^+\text{odd}\\ \gamma\le t}}\mathfrak{R}_{z=\frac{\alpha_i}{\gamma}}\left\{\frac{z^{\frac{nt+l+2}{2}+p-b}\prod_{k=1}^{l}(a_kt)!!}{\prod_{k=1}^{n}\prod_{\substack{1\le s\le t\\ s\text{ odd}}}(sz-\alpha_k)}\right\}=\sum_{t\in\mathbb{Z}^+\text{odd}}t^pq^{\frac{t}{2}}\mathfrak{R}_{w=0}\left\{\frac{w^{b-1-p}\prod_{k=1}^{l}(a_kt)!!}{\prod_{k=1}^{n}\prod_{\substack{1\le s\le t\\ s\text{ odd}}}(s-\alpha_kw)}\right\}.$$

The last expression above gives the right-hand side of the formula in Lemma 6.2. Since

$$\frac{nt+l+2}{2}+p-b-\frac{t+1}{2}\ge0$$

with our assumptions, there is no residue at $z=0$. ∎

Corollary 6.3. *If* $\mathbf{a}$ *is an* l*-tuple of positive integers with* $n\equiv|\mathbf{a}|=l+4$*,* $b,p\in\mathbb{Z}$ *with* $-l\le p\le b+1$ *and* $b-p\le l+2$*, and* $\mathcal{D}_{i,\gamma}$ *is given by (6.7), then*

$$\sum_{\substack{1\le i\le 2m\\ \gamma\in\mathbb{Z}^+\text{odd}}}\frac{1}{\hbar^b}\frac{\mathcal{D}_{i,\gamma}}{\alpha_i^l}q^{\frac{\alpha_i}{\hbar}}\mathfrak{D}^p\mathcal{Y}_0(\alpha_i,\hbar,q)\Big|_{\hbar=\frac{2\alpha_i}{\gamma}}=0\qquad \text{if } p<b;$$

$$\sum_{\substack{1\le i\le 2m\\ \gamma\in\mathbb{Z}^+\text{odd}}}\frac{1}{\hbar^b}\frac{\mathcal{D}_{i,\gamma}}{\alpha_i^l}q^{\frac{\alpha_i}{\hbar}}\mathcal{Y}_p(\alpha_i,\hbar,q)\Big|_{\hbar=\frac{2\alpha_i}{\gamma}}=\sum_{\substack{1\le i\le 2m\\ \gamma\in\mathbb{Z}^+\text{odd}}}\frac{1}{\hbar^b}\frac{\mathcal{D}_{i,\gamma}}{\alpha_i^l}q^{\frac{\alpha_i}{\hbar}}\mathfrak{D}^p\mathcal{Y}_0(\alpha_i,\hbar,q)\Big|_{\hbar=\frac{2\alpha_i}{\gamma}}\qquad \text{if } b\le l+2.$$

Proof. The first claim follows from (3.8), Lemma 6.2, and the product rule by induction on p. The $p<0$ cases of the second claim are immediate from (3.9). The remaining cases follow from (3.12) together with $\widetilde{\mathcal{C}}^{(1)}_{p,s}=0$ and the first claim. ∎

7 Klein Bottle GW-Invariants of CY CI Threefolds

We now consider the one-point analogue of the graph-sum description of the Klein bottle invariants in [W]. We show that the resulting sums are weight-independent; so the invariants are well-defined for a fixed CY CI threefold $X_{\mathbf{a}}$. Furthermore, they satisfy the one-point analogue of the Klein bottle mirror symmetry prediction in [W]; see Theorem 4. Once Klein bottle invariants are defined intrinsically and shown to satisfy some sort of hyperplane relation, the one-point version of the Klein bottle prediction of [W] will become equivalent to the original one, due to the divisor relation.

7.1 Description and graph-sum definition

A degree d one-marked Klein bottle doubled map to $\mathbb{P}^{n-1}$ is a tuple $(\tilde{C},\tau,y,\tilde{f})$ such that $(\tilde{C},\tau,\tilde{f})$ is an Ω-invariant map to $\mathbb{P}^{n-1}$, τ is a fixed-point free involution, and $y\in\tilde{C}^*$ is a smooth point. An example of the restriction of τ to the principal component $\tilde{C}_0$ of $\tilde{C}$ in this case is given by

$$\begin{gathered}\tilde{C}_0=(\mathbb{Z}_{2k}\times\mathbb{P}^1/\sim),\quad\text{where}\quad(s,[0,1])\sim(s+1,[1,0])\ \ \forall\, s\in\mathbb{Z}_{2k}\,,\\ \tau\colon\tilde{C}_0\longrightarrow\tilde{C}_0,\quad \tau\big(s,[z_1,z_2]\big)=\big(s+k,[\bar{z}_1,\bar{z}_2]\big).\end{gathered}\tag{7.1}$$

The moduli space of stable degree d one-marked Klein bottle doubled maps to a CY CI threefold $X_{\mathbf{a}}\subset\mathbb{P}^{n-1}$,

$$\overline{\mathfrak{M}}_{1,1}(X_{\mathbf{a}},d)^{\Omega}\equiv\overline{\mathfrak{M}}_{1,0,1}(X_{\mathbf{a}},\Omega,d),$$

is expected to have a well-defined virtual class so that the number

$$\tilde{K}_d\equiv\int_{\left[\overline{\mathfrak{M}}_{1,1}(X_{\mathbf{a}},d)^{\Omega}\right]^{vir}}\mathrm{ev}_1^*\mathrm{H}\in\mathbb{Q}$$

is well-defined, with $\tilde{K}_d=0$ if d is odd.[14] A combinatorial description of $\tilde{K}_d$ for d even, in the spirit of [W], is motivated below.

Denote by

$$\overline{\mathfrak{M}}_{1,1}(\mathbb{P}^{n-1},d)^{\Omega;\mathbb{T}^m}\subset\overline{\mathfrak{M}}_{1,1}(\mathbb{P}^{n-1},d)^{\Omega}$$

the subspace of the $\mathbb{T}^m$-fixed equivalence classes of stable one-marked Klein bottle doubled maps to $\mathbb{P}^{n-1}$. As in the disk and annulus cases, one would expect that

$$\tilde{K}_{2d}=-\sum_{F\subset\overline{\mathfrak{M}}_{1,1}(\mathbb{P}^{n-1},2d)^{\Omega;\mathbb{T}^m}}\int_F\frac{\mathbf{e}(\mathcal{V}_{1,1,2d}^{\Omega})(\mathrm{ev}_1^*x)|_F}{\mathbf{e}(NF)}\,,\tag{7.2}$$

where the sum is taken over the connected components F of $\overline{\mathfrak{M}}_{1,1}(\mathbb{P}^{n-1},2d)^{\Omega;\mathbb{T}^m}$, $\mathbf{e}(NF)$ is the equivariant Euler class of the normal "bundle" NF of F in $\overline{\mathfrak{M}}_{1,1}(\mathbb{P}^{n-1},2d)^{\Omega}$, and $\mathbf{e}(\mathcal{V}_{1,1,2d}^{\Omega})$ is the equivariant Euler class of the obstruction "bundle"

$$\mathcal{V}_{1,1,2d}^{\Omega}\equiv\overline{\mathfrak{M}}_{1,1}(\mathcal{L},2d)^{\Omega}\longrightarrow\overline{\mathfrak{M}}_{1,1}(\mathbb{P}^{n-1},2d)^{\Omega}\,,$$

with $\mathcal{L}$ given by (3.18). Similarly to the annulus case, $\overline{\mathfrak{M}}_{1,1}(\mathbb{P}^{n-1},2d)^{\Omega}$ and $\mathcal{V}_{1,1,2d}^{\Omega}$ may be singular near maps $(\tilde{C},\tau,y,\tilde{f})$ with contracted principle component $\tilde{C}_0$ or such that $\tilde{C}/\tau$ contains a copy

[14] This is consistent with the vector bundle $\mathcal{V}_{1,1,2d}^{\Omega}$ in (7.2) having a direct summand of odd real rank; thus, its Euler class is zero.

of $\mathbb{R}P^2$. However, the restriction of $\mathcal{V}_{1,1,2d}^{\Omega}$ to the fixed loci F consisting of such maps again contains a topologically trivial subbundle of weight 0 and thus does not contribute to (7.2). The same is the case for the fixed loci consisting of maps $(\tilde{C},\tau,y,\tilde{f})$ such that $\tilde{f}$ takes the point in the principal component $\tilde{C}_0\subset\tilde{C}$ closest to y to P_{2m+1} (if n is odd). There is also no genus 0 correction to $\tilde{K}_{2d}$.

Thus, by Corollary 3.2, all nonzero contributions to (7.2) come from fixed loci F consisting of stable maps $(\tilde{C},\tau,y,\tilde{f})$ such that the principal component $\tilde{C}_0\subset\tilde{C}$ is a circle of spheres as in (7.1), $\tilde{f}|_{\tilde{C}_0}$ is not constant, and the irreducible component of $\tilde{C}_0$ closest to y is mapped to a fixed point P_i with $i\le 2m$. In this case, $\tilde{C}_0$ breaks in a unique way into four connected subsets:

(c1) $\tilde{C}_{0;c1}$, the maximum connected union of irreducible components of $\tilde{C}_0$ which is contracted by $\tilde{f}$ and contains the irreducible component of $\tilde{C}_0$ closest to y;

(c2) $\tilde{C}_{0;c2}=\tau(\tilde{C}_{0;c1})$, the maximum connected union of irreducible components of $\tilde{C}_0$ which is contracted by $\tilde{f}$ and contains the irreducible component of $\tilde{C}_0$ closest to $\tau(y)$;

(e1) $\tilde{C}_{0;e1}$, a chain of spheres running from $\tilde{C}_{0;c1}$ to $\tilde{C}_{0;c2}$;

(e2) $\tilde{C}_{0;e2}=\tau(\tilde{C}_{0;e1})$, the other chain of spheres running from $\tilde{C}_{0;c1}$ to $\tilde{C}_{0;c2}$.[15]

The restrictions $\tilde{f}_{\tilde{C}|_{0;e1}}$ and $\tilde{f}|_{\tilde{C}_{0;e2}}$ are distinguished in the counting scheme of [W]. Let $\tilde{C}_{e1}\subset\tilde{C}$ be the maximal connected union of irreducible components containing $\tilde{C}_{0;e1}$, but not any of the irreducible components of $\tilde{C}_{0;c1}$ or $\tilde{C}_{0;c2}$. The restriction $\tilde{f}|_{\tilde{C}_{e1}}$ determines an element

$$[\tilde{f}_{e1}]\in \mathrm{ev}_1^{-1}(P_i)\cap \mathrm{ev}_2^{-1}(P_{\bar{i}})\cap\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},r)^{\mathbb{T}^m}$$

for some $r\in\mathbb{Z}^+$ such that the restrictions of $\tilde{f}_{e1}$ to the irreducible components of $\tilde{C}_{e1}$ containing the two marked points are not constant. Denote the set of such maps by $F_{r;i\bar{i}}$. Since $F_{r;i\bar{i}}$ is a union of topological components of $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},r)^{\mathbb{T}^m}$, it is smooth and so has a well-defined normal bundle, $NF_{r;i\bar{i}}$. In addition to the nodes $\tilde{C}_{0;c1}\cap\tilde{C}_{0;e1}$ and $\tilde{C}_{0;c1}\cap\tilde{C}_{0;e2}$, $\tilde{C}_{0;c1}$ carries either the marked point y or the node that separates $\tilde{C}_{0;c1}$ and y as well as $B\in\mathbb{Z}^{\ge0}$ additional nodes. If $y\notin\tilde{C}_{0;c1}$, the restriction of $\tilde{f}$ to the maximal connected union $\tilde{C}_y$ of irreducible components of $\tilde{C}$ containing y, but not any of the irreducible components of $\tilde{C}_{0;c1}$, determines an element

$$[\tilde{f}_y]\in \mathrm{ev}_1^{-1}(P_i)\cap\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},r^*)^{\mathbb{T}^m},$$

for some $r^*\in\mathbb{Z}^+$, such that the restriction of $\tilde{f}_y$ to the irreducible component of $\tilde{C}_y$ containing the first marked point is not constant (y corresponds to the second marked point). Denote the set of such maps by $F_{r^*;i}$. Since $F_{r^*;i}$ is a union of topological components of $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},r^*)^{\mathbb{T}^m}$, it is smooth and so has a well-defined normal bundle, $NF_{r^*;i}$. Finally, for each $b=4,\ldots,B+3$, the restriction of $\tilde{f}$ to $\tilde{C}_b$, the maximal connected union of irreducible components of $\tilde{C}$ containing the additional node b, but not any of the irreducible components of $\tilde{C}_{0;c1}$, determines an element

$$[\tilde{f}_b]\in \mathrm{ev}_1^{-1}(P_i)\cap\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^{n-1},r_b)^{\mathbb{T}^m}$$

for some $r_b\in\mathbb{Z}^+$ such that the restriction of $\tilde{f}_b$ to the irreducible component of $\tilde{C}_b$ containing the marked point is not constant. Denote the set of such maps by $F'_{r_b;i}$. Since $F'_{r_b;i}$ is a union of topological components of $\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^{n-1},r_b)^{\mathbb{T}^m}$, it is smooth and has a well-defined normal bundle, $NF'_{r_b;i}$.

[15] e stands for *effective*, as $\tilde{f}|_{\tilde{C}_{0;e1}}$ and $\tilde{f}|_{\tilde{C}_{0;e2}}$ are not constant

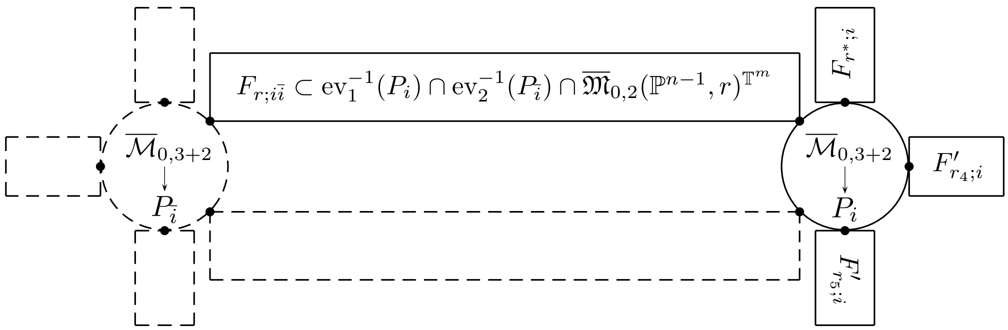

Figure 2: $\mathbb{T}^m$-fixed locus contributing to $\tilde{K}_{2(r+r^*+r_4+r_5)}$; the dotted portion is the reflection of the solid portion under the composition with Ω and τ

In summary, a $\mathbb{T}^m$-fixed locus F of $\overline{\mathfrak{M}}_{1,1}(\mathbb{P}^{n-1}, d)^\Omega$ contributing to (7.2) admits a decomposition

$$F \approx \big(\overline{\mathcal{M}}_{0,3+B} \times F_{r;i\bar{i}} \times F_{r^*;i} \times \prod_{b=4}^{B+3} F'_{r_b;i}\big)/S_B, \tag{7.3}$$

where

- $B \in \mathbb{Z}^{\geq 0}$, $1 \leq i \leq 2m$, and $\overline{\mathcal{M}}_{0,3+B}$ is the moduli space of stable rational curves with $3+B$ marked points;
- $r \in \mathbb{Z}^+$ and $F_{r;i\bar{i}} \subset \mathrm{ev}_1^{-1}(P_i) \cap \mathrm{ev}_2^{-1}(P_{\bar{i}}) \cap \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, r)^{\mathbb{T}^m}$ is the subset of maps with non-trivial restrictions to the irreducible components of the domain containing the two marked points;
- $r^* \in \mathbb{Z}^{\geq 0}$ and $F_{r^*;i} \subset \mathrm{ev}_1^{-1}(P_i) \cap \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, r^*)^{\mathbb{T}^m}$ is the subset of maps with non-trivial restriction to the irreducible component of the domain containing the first marked point if $r^* > 0$, while $F_{0;i} \equiv \{pt\}$;
- $r_b \in \mathbb{Z}^+$ and $F'_{r_b;i} \subset \mathrm{ev}_1^{-1}(P_i) \cap \overline{\mathfrak{M}}_{0,1}(\mathbb{P}^{n-1}, r_b)^{\mathbb{T}^m}$ is the subset of maps with non-trivial restriction to the irreducible component of the domain containing the marked point;
- S_B is the symmetric group on B elements that acts on $\overline{\mathcal{M}}_{0,3+B}$ by permuting the last B marked points and on the B-fold product in (7.3) by permuting its factors.

This is illustrated in Figure 1, with $B=2$.

The composition of maps with τ on the right and Ω on the left determines an automorphism

$$\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, r) \longrightarrow \overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, r).$$

Let $\bar{\psi}_i \in H^*(\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, r))$ denote the pull-back of the usual i-th ψ-class by this automorphism. With respect to the decomposition (7.3), the normal bundle of the fixed locus $F \subset \overline{\mathfrak{M}}_{1,1}(\mathbb{P}^{n-1}, d)^\Omega$ is described by

$$\begin{aligned}\frac{\mathbf{e}(NF)}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})} &= \frac{\mathbf{e}(NF_{r;i\bar{i}})(\hbar_1-\psi_1)(\hbar_2-\bar{\psi}_2)}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})\mathbf{e}(T_{P_{\bar{i}}}\mathbb{P}^{n-1})} \cdot \frac{\mathbf{e}(NF_{r^*;i})(\hbar_3-\psi_3)}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})} \cdot \prod_{b=4}^{B+3} \frac{\mathbf{e}(NF'_{r_b;i})(\hbar_b-\psi_b)}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})} \\ &= \frac{\mathbf{e}(NF_{r;i\bar{i}})(\hbar_1-\psi_1)(\hbar_2-\bar{\psi}_2)}{\phi_i|_{P_i}\,\phi_{\bar{i}}|_{P_{\bar{i}}}} \cdot \frac{\mathbf{e}(NF_{r^*;i})(\hbar_3-\psi_3)}{\phi_i|_{P_i}} \cdot \prod_{b=4}^{B+3} \frac{\mathbf{e}(NF'_{r_b;i})(\hbar_b-\psi_b)}{\phi_i|_{P_i}},\end{aligned}$$

where $\hbar_b \equiv c_1(L_b) \in H^*(\overline{\mathcal{M}}_{0,3+B})$ is the first Chern class of the universal tangent line bundle at the b-th marked point, $\psi_b \in H^*\big(\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^{n-1}, r_b)\big)$ is the usual ψ-class for $b \geq 4$, and ψ_3 denotes the pull-back of $\psi_1 \in H^*(\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, r^*))$ by the third component projection map in (7.3), with the convention that

$$\frac{\mathbf{e}(NF_{r^*;i})(\hbar_3 - \psi_3)}{\phi_i|_{P_i}} \equiv 1 \qquad \text{if} \quad r^* = 0.$$

Similarly, the restriction of the numerator in (7.2) splits as

$$\begin{aligned}\frac{(\mathrm{ev}_1^* x)\mathbf{e}(\mathcal{V}^{\Omega}_{1,1,2d})}{\mathbf{e}(\mathcal{L})|_{P_i}}\bigg|_F &= \frac{\mathbf{e}(\mathcal{V}_{0,r})}{\mathbf{e}(\mathcal{L})|_{P_i}\mathbf{e}(\mathcal{L})|_{P_{\bar{i}}}} \cdot \frac{(\mathrm{ev}_2^* x)\mathbf{e}(\mathcal{V}_{0,r^*})}{\mathbf{e}(\mathcal{L})|_{P_i}} \cdot \prod_{b=4}^{B+3} \frac{\mathbf{e}(\mathcal{V}_{0,r_b})}{\mathbf{e}(\mathcal{L})|_{P_i}} \\ &= \frac{\mathbf{e}(\mathcal{V}_{0,r})}{\langle \mathbf{a} \rangle^2 \alpha_i^l \alpha_{\bar{i}}^l} \cdot \frac{\mathbf{e}(\mathcal{V}''_{0,r^*})(\mathrm{ev}_2^* x^{l+1})}{\alpha_i^l} \cdot \prod_{b=4}^{B+3} \mathbf{e}(\mathcal{V}'_{0,r_b}),\end{aligned}$$

where $\mathcal{V}''_{0,r^*}, \mathcal{V}'_{0,r_b}$ are as in Section 2 if $r > 0$ and

$$\frac{\mathbf{e}(\mathcal{V}''_{0,r^*})(\mathrm{ev}_2^* x^{l+1})}{\alpha_i^l} \equiv \alpha_i \qquad \text{if} \quad r^* = 0.$$

Putting this all together, we conclude that

$$\begin{aligned}\int_F \frac{\mathbf{e}(\mathcal{V}^{\Omega}_{1,1,2d})(\mathrm{ev}_1^* x)|_F}{\mathbf{e}(NF)} &= \frac{(-1)^{l+1}}{\langle \mathbf{a} \rangle \alpha_i^{2l} \prod_{k \neq i} (\alpha_i - \alpha_k)} \sum_{\substack{p_1 + \ldots + p_{B+3} = B \\ p_1, \ldots, p_{B+3} \geq 0}} \\ \Bigg(\bigg(\frac{\psi_1^{-(p_1+1)} \bar{\psi}_2^{-(p_2+1)}}{p_1! p_2!} &\int_{F_{r;i\bar{i}}} \frac{\mathbf{e}(\mathcal{V}_{0,r})(\mathrm{ev}_1^* \phi_i)(\mathrm{ev}_2^* \phi_{\bar{i}})}{\mathbf{e}(NF_{b;i\bar{i}})} \bigg) \\ \times \bigg(\frac{\psi_3^{-(p_3+1)}}{p_3!} \int_{F_{r^*;i}} \frac{\mathbf{e}(\mathcal{V}''_{0,r^*})(\mathrm{ev}_1^* \phi_i)(\mathrm{ev}_2^* x^{l+1})}{\mathbf{e}(NF_{r^*;i})} \bigg) &\prod_{b=4}^{B+3} \bigg(\frac{\psi_b^{-(p_b+1)}}{p_b!} \int_{F'_{r_b;i}} \frac{\mathbf{e}(\mathcal{V}'_{0,r_b})(\mathrm{ev}_1^* \phi_i)}{\mathbf{e}(NF'_{r_b;i})} \bigg) \Bigg),\end{aligned} \tag{7.4}$$

with

$$\bigg(\frac{\psi_3^{-(p_3+1)}}{p_3!} \int_{F_{r^*;i}} \frac{\mathbf{e}(\mathcal{V}''_{0,r^*})(\mathrm{ev}_1^* \phi_i)(\mathrm{ev}_2^* x^{l+1})}{\mathbf{e}(NF_{b;i})} \bigg) \equiv -\delta_{p_3,0} \alpha_i^{l+1} \qquad \text{if} \quad r^* = 0. \tag{7.5}$$

Re-writing the right-hand side of (7.4) explicitly in terms of the vertices and edges of the graph encoding F gives the one-marked version of the long formula in [W, Section 3.4].

We will instead use (7.4) to re-write (7.2) in terms of the generating functions $\mathcal{Z}$ of (4.7) and (3.4) and $\mathcal{Z}_p$ of (3.3). If $1 \leq i \neq j \leq n$ and $d \in \mathbb{Z}^+$, let

$$\tilde{\mathfrak{C}}_i^j(d) = \frac{\prod\limits_{k=1}^{l} \prod\limits_{r=1}^{a_k d} \Big(a_k \alpha_i + r \, \frac{\alpha_j - \alpha_i}{d} \Big)}{d \prod\limits_{r=1}^{d} \prod\limits_{\substack{k=1 \\ (r,k) \neq (d,j)}}^{n} \Big(\alpha_i - \alpha_k + r \, \frac{\alpha_j - \alpha_i}{d} \Big)} \in \mathbb{Q}_\alpha. \tag{7.6}$$

This is undefined if n is odd, d is even, and $j=\bar{i}$. In such a case, we cancel the $(k,r)=(n,d/2)$ factor in the denominator with the $(k,r)=(1,a_1d/2)$ factor in the numerator to define

$$\tilde{\mathfrak{C}}_i^{\bar{i}}(d)=\delta_{l,1}\frac{n\prod\limits_{r=1}^{nd/2-1}\Big(r\,\frac{\alpha_i}{d/2}\Big)\cdot\prod\limits_{r=1}^{nd/2}\Big(r\,\frac{\alpha_{\bar{i}}}{d/2}\Big)}{d\prod\limits_{r=1}^{d/2-1}\prod\limits_{k=1}^{n}\big(\frac{2r}{d}\alpha_i-\alpha_k\big)\cdot\prod\limits_{k=1}^{2m}(-\alpha_k)\cdot\prod\limits_{\substack{r=1\\(r,k)\neq(d/2,\bar{i})}}^{d/2}\prod\limits_{k=1}^{n}\big(\frac{2r}{d}\alpha_{\bar{i}}-\alpha_k\big)}. \tag{7.7}$$

With $\mathrm{ev}_1:\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^{n-1},d)\longrightarrow\mathbb{P}^{n-1}$ as before, let

$$\mathcal{Z}'(x,\hbar,Q)\equiv\hbar+\sum_{d=1}^{\infty}Q^d\mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a}})}{\hbar-\psi_1}\right]\in\big(H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\big)\big[\big[\hbar^{-1},Q\big]\big].$$

By the string relation [MirSym, Section 26.3],

$$\mathcal{Z}'(x,\hbar,Q)=\hbar\mathcal{Z}(x,\hbar,Q) \tag{7.8}$$

and thus is described by (4.3). In particular,

$$\underset{\hbar=\frac{\alpha_j-\alpha_i}{d}}{\mathfrak{R}}\big\{\hbar^{-(p+1)}\mathcal{Z}'(\alpha_i,\hbar,Q)\big\}=\left(\frac{\alpha_j-\alpha_i}{d}\right)^{-(p+1)}\tilde{\mathfrak{C}}_i^j(d)Q^d\mathcal{Z}'(\alpha_j,(\alpha_j-\alpha_i)/d,Q), \tag{7.9}$$

if $1\le i\neq j\le n$, $d\in\mathbb{Z}^+$, and $p\in\mathbb{Z}$.

If n is even, the restriction of an element of $F'_{r_b;i}$ to the component of its domain containing the marked point is the degree d_b cover of the line ℓ_{ij_b} branched over only P_i and P_{j_b}, for some $d_b\in\mathbb{Z}^+$ and $j_b\in[n]-i$. Similarly to [MirSym, Section 30.1] and [Z09b, Section 2.2],

$$\begin{split}\sum_{\substack{F'_{r_b;i}\\ d_b=d,j_b=j}}Q^{r_b}\left(\psi_b^{-(p+1)}\int_{F'_{r_b;i}}\frac{\mathbf{e}(\mathcal{V}'_{0,r_b})(\mathrm{ev}_1^*\phi_i)}{\mathbf{e}(NF'_{r_b;i})}\right)\\ =Q^d\left(\frac{\alpha_j-\alpha_i}{d}\right)^{-(p+1)}\tilde{\mathfrak{C}}_i^j(d)\mathcal{Z}'(\alpha_j,(\alpha_j-\alpha_i)/d,Q)\end{split} \tag{7.10}$$

for all $j\in[n]-i$ and $d\in\mathbb{Z}^+$. If n is odd, but $j\neq\bar{i}$ or d is odd, (7.10) holds by the same reasoning. If n is odd, $j=\bar{i}$, and d is even, the first component $F'^{(1)}_{r_b;i}$ of the locus $F'_{r_b;i}$ consists of the $d/2$-fold covers of the conics $\mathcal{T}_i(a,b)$, with $[a,b]\in\mathbb{P}^1$, branched over only P_i and $P_{\bar{i}}$. The contribution of this component to the integral in (7.10) can be computed by turning the action of the last component of $\mathbb{T}^n$ back on (making $\alpha_n\neq0$). This action on $F'^{(1)}_{r_b;i}$ has two fixed points: the d-fold cover of the line $\ell_{i\bar{i}}$ branched over only P_i and $P_{\bar{i}}$ and the $d/2$-fold cover of $\ell_{in}\cup\ell_{n\bar{i}}$, whose restriction to each of the two components of the domain is branched over only P_n and either P_i or $P_{\bar{i}}$. The contribution of the former to the integral in (7.10) vanishes.[16] The contribution of the latter is given by the

[16] As in [MirSym, Section 30.1] and [Z09b, Section 2.2], the Euler class of the restriction of the bundle $\mathcal{V}'_{\mathbf{a}}$ to this d-fold cover is given by the numerator in (7.6), but its $(k,r)=(1,a_1d/2)$ factor vanishes in this case.

right-hand side of (7.10) with $\mathfrak{C}_i^j(d)=\mathfrak{C}_i^{\bar{i}}(d)$ given by (7.7).[17] Thus, (7.10) holds in all cases.

By (7.10), (7.9), and (7.8),

$$\sum_{F'_{r_b;i}} Q^{r_b}\left(\psi_b^{-(p+1)}\int_{F'_{r_b;i}}\frac{\mathbf{e}(\mathcal{V}'_{0,r_b})(\mathrm{ev}_1^*\phi_i)}{\mathbf{e}(NF'_{r_b;i})}\right)=\sum_{d=1}^{\infty}\sum_{j\neq i}\underset{\hbar=\frac{\alpha_j-\alpha_i}{d}}{\mathfrak{R}}\big\{\hbar^{-p}\mathcal{Z}(\alpha_i,\hbar,Q)\big\}, \tag{7.11}$$

where the first sum is taken over all possibilities for $F'_{r_b;i}\subset\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^{n-1},r_b)^{\mathbb{T}^m}$ as described above. By (4.3), $\mathcal{Z}(\alpha_i,\hbar,Q)$ is a power series in Q with coefficients in $\mathbb{Q}_\alpha(\hbar)$ that are regular away from $\hbar=(\alpha_j-\alpha_i)/d$ with $i,j\in[n]$ and $d\in\mathbb{Z}^+$. Furthermore, by (4.1),

$$\underset{\hbar=\infty}{\mathfrak{R}}\big\{\hbar^{-p}\mathcal{Z}(\alpha_i,\hbar,Q)\big\}=-\delta_{p,1}\qquad\forall\,p\in\mathbb{Z}^{\ge0}.$$

Thus, by (7.11) and the Residue Theorem on S^2,

$$\sum_{F'_{r_b;i}} Q^{r_b}\left(\psi_b^{-(p+1)}\int_{F'_{r_b;i}}\frac{\mathbf{e}(\mathcal{V}'_{0,r_b})(\mathrm{ev}_1^*\phi_i)}{\mathbf{e}(NF'_{r_b;i})}\right)=-\underset{\hbar=0}{\mathfrak{R}}\big\{\hbar^{-p}\mathcal{Z}^*(\alpha_i,\hbar,Q)\big\}\qquad\forall\,p\in\mathbb{Z}^{\ge0}, \tag{7.12}$$

where

$$\mathcal{Z}^*(x,\hbar,Q)\equiv\mathcal{Z}(x,\hbar,Q)-1\in Q\cdot\mathbb{Q}_\alpha(\hbar)\big[\big[Q\big]\big].$$

By the same reasoning,

$$\sum_{F_{r^*;i}} Q^{r^*}\left(\psi_3^{-p}\int_{F_{r^*;i}}\frac{\mathbf{e}(\mathcal{V}''_{0,r^*})(\mathrm{ev}_1^*\phi_i)(\mathrm{ev}_2^*x^{l+1})}{\mathbf{e}(NF_{r^*;i})}\right)=-\underset{\hbar=0,\infty}{\mathfrak{R}}\big\{\hbar^{-p}\mathcal{Z}_1(\alpha_i,\hbar,Q)\big\}\qquad\forall\,p\in\mathbb{Z}^+. \tag{7.13}$$

The analogue of (7.10) holds with $\tilde{\mathfrak{C}}_i^j(d)$ replaced by the constant $\mathfrak{C}_i^j(d)$ defined in (4.14) and its reduction similarly to (7.7), while

$$\underset{\hbar=\infty}{\mathfrak{R}}\big\{\hbar^{-p}\mathcal{Z}_1(\alpha_i,\hbar,Q)\big\}=-\delta_{p,1}\alpha_i^{l+1}\qquad\forall\,p\in\mathbb{Z}^+.$$

This extra correction (which appears with the opposite sign) is off-set by the $r^*=0$ case; see (7.5). Finally,

$$\begin{aligned}\sum_{F_{r;i\bar{i}}} Q^{r}&\left(\psi_1^{-p_1}\bar{\psi}_2^{-p_2}\int_{F_{r;i\bar{i}}}\frac{\mathbf{e}(\mathcal{V}_{0,r})(\mathrm{ev}_1^*\phi_i)(\mathrm{ev}_2^*\phi_{\bar{i}})}{\mathbf{e}(NF_{b;i\bar{i}})}\right)\\&=(-1)^{p_2}\sum_{F_{r;i\bar{i}}} Q^{r}\left(\psi_1^{-p_1}\psi_2^{-p_2}\int_{F_{r;i\bar{i}}}\frac{\mathbf{e}(\mathcal{V}_{0,r})(\mathrm{ev}_1^*\phi_i)(\mathrm{ev}_2^*\phi_{\bar{i}})}{\mathbf{e}(NF_{b;i\bar{i}})}\right)\\&=(-1)^{p_2}\underset{\hbar_1=0}{\mathfrak{R}}\Big\{\underset{\hbar_2=0}{\mathfrak{R}}\big\{\hbar_1^{-p_1}\hbar_2^{-p_2}\mathcal{Z}(\alpha_i,\alpha_{\bar{i}},\hbar_1,\hbar_2,Q)\big\}\Big\}\qquad\forall\,p_1,p_2\in\mathbb{Z}^+.\end{aligned} \tag{7.14}$$

[17] By [MirSym, Section 27.4], the normal bundle to $F'^{(1)}_{r_b;i}$ contains the line bundle of smoothings of the node of the domain of the cover of $\ell_{in}\cup\ell_{n\bar{i}}$; its first Chern class is

$$\frac{\alpha_n-\alpha_i}{d/2}+\frac{\alpha_n-\alpha_{\bar{i}}}{d/2}=\frac{4\alpha_n}{d}\,.$$

By [MirSym, Section 27.2], the restriction of $\mathcal{V}_{\mathbf{a}}$ contains line bundles with first Chern classes $a_k\alpha_n$ with $k\in[n]$. Thus, the only factor of α_n in the denominator can be canceled out with a factor in the numerator, before α_n is set to 0.

The analogue of (7.10) now holds for $\hbar_1$ and $\hbar_2$ separately, with $\tilde{\mathfrak{C}}_i^j(d)$ replaced by $\mathfrak{C}_i^j(d)$, as above. The coefficient of Q^0 in (4.7) has no effect on the residue in this case.

Combining (7.2), (7.4), and (7.12)-(7.14), we obtain the following re-formulation of the combinatorial definition of the Klein bottle invariants in [W].

Definition 7.1. *The degree $2d$ one-point Klein bottle invariant $\tilde{K}_{2d}$ of $(X_{\mathbf{a}},\Omega)$ is given by*

$$\begin{split}\sum_{d=1}^{\infty} Q^d \tilde{K}_{2d} = \frac{(-1)^{l+1}}{\langle \mathbf{a}\rangle}\sum_{i=1}^{2m}\frac{1}{\alpha_i^{2l}\prod\limits_{k\neq i}(\alpha_i-\alpha_k)}\sum_{B=0}^{\infty}\sum_{\substack{p_1+\ldots+p_{B+3}=B\\ p_1,\ldots,p_{B+3}\ge 0}}\\ \left(\mathfrak{R}_{\hbar_1=0}\left\{\mathfrak{R}_{\hbar_2=0}\left\{\frac{(-1)^{p_1}}{p_1!p_2!\,\hbar_1^{p_1+1}\hbar_2^{p_2+1}}\mathcal{Z}(\alpha_i,\alpha_{\bar{i}},\hbar_1,\hbar_2,Q)\right\}\right\}\right.\\ \left.\times\mathfrak{R}_{\hbar=0}\left\{\frac{(-1)^{p_3}}{p_3!\,\hbar^{p_3+1}}\mathcal{Z}_1(\alpha_i,\hbar,Q)\right\}\prod_{b=4}^{B+3}\mathfrak{R}_{\hbar=0}\left\{\frac{(-1)^{p_b}}{p_b!\,\hbar^{p_b}}\mathcal{Z}^*(\alpha_i,\hbar,Q)\right\}\right).\end{split} \tag{7.15}$$

7.2 Some preliminaries

The last factor in (7.15) can be readily summed up over all possibilities for $(p_4,p_5,\ldots)$ if the power series $\mathcal{Z}(\alpha_i,\hbar,Q)\in\mathbb{Q}_\alpha(\hbar)[[Q]]$ admits an expansion of a certain form; see Lemma 7.2 below. By Lemma 7.4 below and (4.3), $\mathcal{Z}(\alpha_i,\hbar,Q)$ does admit such an expansion; this lemma also provides the two coefficients for this expansion that are relevant for our purposes.

For $r\in\mathbb{Z}^{\ge 0}$, denote by $\sigma_r(z)$ the r-th elementary symmetric polynomial in $\{z-\alpha_k\}$. Define

$$\begin{split} L(x,q)\in x+x^nq\cdot\mathbb{Q}[\alpha_1,\ldots,\alpha_n,x,\sigma_{n-1}(x)^{-1}]\big[\big[x^{n-1}q\big]\big],\\ \text{by}\quad \sigma_n\big(L(x,q)\big)-q\,\mathbf{a}^{\mathbf{a}}L(x,q)^n=\sigma_n(x).\end{split} \tag{7.16}$$

Lemma 7.2 ([Z09b, Lemma 2.2(ii)][18]). *Let $R\supset\mathbb{Q}$ be any field. If $\mathcal{F}^*\in R(\hbar)[[q]]$ admits an expansion around $\hbar=0$ of the form*

$$1+\mathcal{F}^*(\hbar,q)=\mathrm{e}^{\zeta(q)/\hbar}\sum_{s=0}^{\infty}\Psi_s(q)\hbar^s$$

with $\xi,\Psi_1,\Psi_2,\ldots\in qR[[q]]$ and $\Psi_0\in 1+qR[[q]]$, then

$$\sum_{B=0}^{\infty}\sum_{\substack{p_1+\ldots+p_B=B-p\\ p_1,\ldots,p_B\ge 0}}\prod_{b=1}^{B}\mathfrak{R}_{\hbar=0}\left\{\frac{(-1)^{p_b}}{p_b!\,\hbar^{p_b}}\mathcal{F}^*(\hbar,q)\right\}=\frac{\zeta(q)^p}{\Psi_0(q)}\qquad\forall\,p\in\mathbb{Z}^{\ge 0}.$$

Lemma 7.3. *The power series*

$$\widetilde{\mathcal{Y}}_{-l}(x,\hbar,q)\equiv\sum_{d=0}^{\infty}q^d\frac{\prod\limits_{k=1}^{l}\prod\limits_{r=0}^{a_kd-1}(a_kx+r\hbar)}{\prod\limits_{r=1}^{d}\big(\sigma_n(x+r\hbar)-\sigma_n(x)\big)}\in\mathbb{Q}_\alpha(x,\hbar)\big[\big[q\big]\big]$$

[18] [Z09b, Lemma 2.2(ii)] is stated only for $R=\mathbb{Q}_\alpha$, but the argument applies to any field containing $\mathbb{Q}$.

admits an expansion around $\hbar=0$ *of the form*

$$\widetilde{\mathcal{Y}}_{-l}(x,\hbar,q)=\mathrm{e}^{\xi(x,q)/\hbar}\sum_{s=0}^{\infty}\Phi_{-l;s}(x,q)\hbar^s$$

with $\xi,\Phi_{-l;1},\Phi_{-l;2},\ldots\in q\cdot\mathbb{Q}_\alpha(x)[[q]]$ *and* $\Phi_{-l;0}\in 1+q\cdot\mathbb{Q}_\alpha(x)[[q]]$.

Proof. Since $\widetilde{\mathcal{Y}}_{-l}\in 1+q\cdot\mathbb{Q}_\alpha(x,\hbar)[[q]]$, there is an expansion

$$\ln\widetilde{\mathcal{Y}}_{-l}(x,\hbar,q)=\sum_{d=1}^{\infty}\sum_{s=s_{\min}(d)}^{\infty}C_{d,s}(x)\hbar^s q^d \tag{7.17}$$

around $\hbar=0$, with $C_{d,s}(x)\in\mathbb{Q}_\alpha(x)$; we can assume that $C_{d,s_{\min}(d)}\neq 0$ if $s_{\min}(d)<0$. The claim of Lemma 7.3 is equivalent to the statement $s_{\min}(d)\geq -1$ for all $d\in\mathbb{Z}^+$; in such a case

$$\xi(x,q)=\sum_{d=1}^{\infty}C_{d,-1}(x)q^d\,.$$

Suppose instead $s_{\min}(d)<-1$ for some $d\in\mathbb{Z}^+$. Let

$$d^*\equiv\min\big\{d\in\mathbb{Z}^+:\ s_{\min}(d)<-1\big\}\geq 1,\qquad s^*\equiv s_{\min}(d^*)\leq -2. \tag{7.18}$$

The power series $\widetilde{\mathcal{Y}}_{-l}$ satisfies the differential equation

$$\left\{\sigma_n(x+\hbar\mathrm{D})-q\prod_{k=1}^{l}\prod_{r=0}^{a_k-1}\big(a_kx+a_k\hbar\mathrm{D}+r\hbar\big)\right\}\widetilde{\mathcal{Y}}_{-l}(x,\hbar,q)=\sigma_n(x)\widetilde{\mathcal{Y}}_{-l}(x,\hbar,q), \tag{7.19}$$

where $\mathrm{D}=q\frac{\mathrm{d}}{\mathrm{d}q}$ as before. By (7.17), (7.18), and induction on the number of derivatives taken,

$$\begin{aligned}\frac{\{\sigma_n(x+\hbar\mathrm{D})\}\,\widetilde{\mathcal{Y}}_{-l}(x,\hbar,q)}{\sigma_n(x)\cdot\widetilde{\mathcal{Y}}_{-l}(x,\hbar,q)}&=1+\sum_{k=1}^{n}\frac{d^*C_{d^*,s^*}}{x-\alpha_k}\hbar^{s^*+1}q^{d^*}+A(x,\hbar,q)\,,\\ q\,\frac{\left\{\prod\limits_{k=1}^{l}\prod\limits_{r=0}^{a_k-1}\big(a_kx+a_k\hbar\mathrm{D}+r\hbar\big)\right\}\widetilde{\mathcal{Y}}_{-l}(x,\hbar,q)}{\prod\limits_{k=1}^{l}\prod\limits_{r=0}^{a_k-1}(a_kx+r\hbar)\cdot\widetilde{\mathcal{Y}}_{-l}(x,\hbar,q)}&=B(x,\hbar,q)\,,\end{aligned} \tag{7.20}$$

for some

$$A,B\in q\cdot\mathbb{Q}_\alpha(x,\hbar)_0\big[[q]\big]+q^{d^*}\hbar^{s^*+2}\cdot\mathbb{Q}_\alpha(x,\hbar)_0\big[[q]\big]+q^{d^*+1}\cdot\mathbb{Q}_\alpha(x,\hbar)\big[[q]\big],$$

where $\mathbb{Q}_\alpha(x,\hbar)_0\subset\mathbb{Q}_\alpha(x,\hbar)$ is the subring of rational functions in α, x, and $\hbar$ that are regular at $\hbar=0$. Combining (7.19) and (7.20), we conclude that $C_{d^*,s^*}=0$, contrary to the assumption. □

Lemma 7.4. *The power series*

$$\widetilde{\mathcal{Y}}(x,\hbar,q)\equiv\sum_{d=0}^{\infty}q^d\frac{\prod\limits_{k=1}^{l}\prod\limits_{r=1}^{a_kd}(a_kx+r\hbar)}{\prod\limits_{r=1}^{d}\big(\sigma_n(x+r\hbar)-\sigma_n(x)\big)}\in\mathbb{Q}_\alpha(x,\hbar)\big[[q]\big]$$

admits an expansion around $\hbar = 0$ of the form

$$\widetilde{\mathcal{Y}}(x,\hbar,q) = \mathrm{e}^{\xi(x,q)/\hbar} \sum_{s=0}^{\infty} \Phi_s(x,q)\hbar^s \tag{7.21}$$

with $\xi, \Phi_1, \Phi_2, \ldots \in q \cdot \mathbb{Q}_\alpha(x)[[q]]$ and $\Phi_0 \in 1 + q \cdot \mathbb{Q}_\alpha(x)[[q]]$ such that

$$x + \mathrm{D}\xi(x,q) = L(x,q), \tag{7.22}$$

$$\Phi_0(x,q) = \left(\frac{x \cdot \sigma_{n-1}(x)}{L(x,q)\,\sigma_{n-1}(L(x,q)) - n(\sigma_n(L(x,q)) - \sigma_n(x))}\right)^{1/2} \left(\frac{L(x,q)}{x}\right)^{(l+1)/2}. \tag{7.23}$$

Proof. Since the power series $\widetilde{\mathcal{Y}}_{-l}$ admits an expansion of the form (7.21) by Lemma 7.3 and

$$\widetilde{\mathcal{Y}}(x,\hbar,q) = \left\{1 + \frac{\hbar}{x} D\right\}^l \widetilde{\mathcal{Y}}_{-l}(x,\hbar,q)\,, \tag{7.24}$$

the power series $\widetilde{\mathcal{Y}}$ also admits an expansion of the form (7.21), with the same power series ξ. Let

$$\overline{\mathcal{Y}}(x,\hbar,q) \equiv \mathrm{e}^{-\xi(x,q)/\hbar} \widetilde{\mathcal{Y}}(x,\hbar,q) = \sum_{s=0}^{\infty} \Phi_s(x,q)\hbar^s\,.$$

Since the power series $\widetilde{\mathcal{Y}}(x,\hbar,q)$ satisfies

$$\left\{\sigma_n(x+\hbar\mathrm{D}) - q\prod_{k=1}^{l}\prod_{r=1}^{a_k}\big(a_k x + a_k\hbar\mathrm{D} + r\hbar\big)\right\}\widetilde{\mathcal{Y}}(x,\hbar,q) = \sigma_n(x)\widetilde{\mathcal{Y}}(x,\hbar,q),$$

the power series $\overline{\mathcal{Y}}(x,\hbar,q)$ satisfies

$$\left\{\sigma_n(x+\mathrm{D}\xi+\hbar\mathrm{D}) - q\prod_{k=1}^{l}\prod_{r=1}^{a_k}\big(a_k(x+\mathrm{D}\xi) + a_k\hbar\mathrm{D} + r\hbar\big) - \sigma_n(x)\right\}\overline{\mathcal{Y}}(x,\hbar,q) = 0. \tag{7.25}$$

Considering the coefficient of $\hbar^0$ in this equation, we obtain

$$\big\{\sigma_n(x+\mathrm{D}\xi) - q\mathbf{a}^{\mathbf{a}}(x+\mathrm{D}\xi)^n - \sigma_n(x)\big\}\Phi_0(x,q) = 0.$$

Since $\Phi_0(x,0) = 1$, this gives (7.22) by (7.16). Note that

$$\begin{aligned}\frac{\mathrm{D}L(x,q)}{L(x,q)} &= \frac{\sigma_n(L(x,q)) - \sigma_n(x)}{L(x,q)\sigma_{n-1}\big(L(x,q)\big) - n\big(\sigma_n(L(x,q)) - \sigma_n(x)\big)} \\ &= \frac{q\mathbf{a}^{\mathbf{a}}L(x,q)^n}{L(x,q)\sigma_{n-1}\big(L(x,q)\big) - n\big(\sigma_n(L(x,q)) - \sigma_n(x)\big)}\end{aligned} \tag{7.26}$$

by (7.16). Substituting L for $x+\mathrm{D}\xi$ in (7.25), taking the coefficient of $\hbar^1$ of this equation, and using (7.16) and (7.26), we obtain

$$\mathrm{D}\left\{\left(\frac{x \cdot \sigma_{n-1}(x)}{L(x,q)\,\sigma_{n-1}(L(x,q)) - n(\sigma_n(L(x,q)) - \sigma_n(x))}\right)^{-\frac{1}{2}} \left(\frac{L(x,q)}{x}\right)^{-\frac{l+1}{2}} \Phi_0\right\} = 0.$$

Along with $\Phi_0(x,0) = 1$, this gives (7.23). □

7.3 Proof of Theorem 4

With ξ as in Lemma 7.4, let $\eta(x,q)=\xi(x,q)-J(q)x$. Since

$$\mathfrak{R}_{\hbar=0}\left\{\frac{1}{\hbar^p}\mathcal{Y}(\alpha_i,\hbar,q)\right\}=\mathfrak{R}_{\hbar=0}\left\{\frac{1}{\hbar^p}\widetilde{\mathcal{Y}}(\alpha_i,\hbar,q)\right\}=\left(\mathfrak{R}_{\hbar=0}\left\{\frac{1}{\hbar^p}\widetilde{\mathcal{Y}}(x,\hbar,q)\right\}\right)\bigg|_{x=\alpha_i}\qquad\forall\, i\in[n],$$

by (4.3) and Lemmas 7.2 and 7.4

$$\sum_{B=0}^{\infty}\sum_{\substack{p_4+\ldots+p_{B+3}=B-p\\ p_4,\ldots,p_{B+3}\ge0}}\prod_{b=4}^{B+3}\mathfrak{R}_{\hbar=0}\left\{\frac{(-1)^{p_b}}{p_b!\,\hbar^{p_b}}\mathcal{Z}^*(x,\hbar,Q)\right\}\bigg|_{x=\alpha_i}=\frac{\eta(x,q)^p}{\Phi_0(x,q)/I_0(q)}\bigg|_{x=\alpha_i}\qquad\forall\, p\in\mathbb{Z}^{\ge0}.\tag{7.27}$$

By (7.16), $L(-x,q)=-L(x,q)$; since $\alpha_{\bar{i}}=-\alpha_i$, $\eta(\alpha_{\bar{i}},q)=-\eta(\alpha_i,q)$ by (7.22). Thus, by (7.15) and (7.27),

$$\begin{split}\sum_{d=1}^{\infty}Q^d\tilde{K}_{2d}=\frac{(-1)^{l+1}}{\langle\mathbf{a}\rangle}\sum_{i=1}^{2m}\Bigg(\frac{I_0(q)/\Phi_0(x,q)}{x^{2l}\sigma_{n-1}(x)}\,\mathfrak{R}_{\hbar=0}\left\{\frac{\mathrm{e}^{-\frac{\eta(x,q)}{\hbar}}\mathcal{Z}_1(x,\hbar,Q)}{\hbar}\right\}\\ \times\mathfrak{R}_{\hbar_1=0}\left\{\mathfrak{R}_{\hbar_2=0}\left\{\frac{\mathrm{e}^{-\frac{\eta(x,q)}{\hbar_1}-\frac{\eta(\bar{x},q)}{\hbar_2}}\mathcal{Z}(x,\bar{x},\hbar_1,\hbar_2,Q)}{\hbar_1\hbar_2}\right\}\right\}\Bigg)\Bigg|_{x=\alpha_i}.\end{split}\tag{7.28}$$

By (3.11), (3.12), Lemmas 7.3 and 7.4, and (7.24),

$$\mathfrak{R}_{\hbar=0}\left\{\frac{\mathrm{e}^{-\frac{\eta(x,q)}{\hbar}}\mathcal{Z}_p(x,\hbar,Q)}{\hbar}\right\}=x^l\Phi_0(x,q)L(x,q)^p\sum_{r=0}^{|p|}\frac{\widetilde{\mathcal{C}}^{(r)}_{p,p-r}(q)}{L(x,q)^r\prod\limits_{s=0}^{p-r}I_s(q)}\,,\tag{7.29}$$

$$\begin{split}\mathfrak{R}_{\hbar=0}\left\{\frac{\mathrm{e}^{-\frac{\eta(x,q)}{\hbar}}\mathcal{Z}_p(x,\hbar,Q)}{\hbar^2}\right\}=x^l\Phi_0(x,q)L(x,q)^p\Bigg(\sum_{r=2}^{p-1}\frac{\widetilde{\mathcal{C}}^{(r)}_{p,p-r-1}(q)}{L(x,q)^{r+1}\prod\limits_{s=0}^{p-r-1}I_s(q)}\\ +\sum_{r=0}^{|p|}\frac{\widetilde{\mathcal{C}}^{(r)}_{p,p-r}(q)}{L(x,q)^r\prod\limits_{s=0}^{p-r}I_s(q)}\left(\frac{\Phi_1(x,q)}{\Phi_0(x,q)}+A_{p-r}(x,q)\right)\Bigg),\end{split}\tag{7.30}$$

with $\widetilde{\mathcal{C}}^{(r)}_{p,s}\equiv\delta_{p,s}\delta_{r,0}$ for $p<0$ or $s<0$ and

$$A_p=\frac{1}{L}\left(p\frac{\mathrm{D}\Phi_0}{\Phi_0}+\frac{p(p-1)}{2}\frac{\mathrm{D}L}{L}-\sum_{s=0}^{p}(p-s)\frac{\mathrm{D}I_s}{I_s}\right).$$

In this case, $n-l=4$. Thus, by (3.5), (3.10), (4.7), and $\sigma_r=0$ for $r\in\mathbb{Z}$ odd,

$$\mathcal{Z}(x,\bar{x},\hbar_1,\hbar_2,Q)=\frac{\langle\mathbf{a}\rangle}{\hbar_1+\hbar_2}\left\{\sum_{\substack{p_1+p_2+2r=3\\ p_1,p_2\ge0}}-\sum_{\substack{p_1+p_2+2r=3\\ -l\le p_1,p_2<0}}\right\}\sigma_{2r}\mathcal{Z}_{p_1}(x,\hbar_1,Q)\mathcal{Z}_{p_2}(\bar{x},\hbar_2,Q).\tag{7.31}$$

By the $p=2$ case of Corollary 5.7 and $\sigma_1, B^{(1)}_{s,1}=0$,

$$\widetilde{\mathcal{C}}^{(2)}_{2,0}+\widetilde{\mathcal{C}}^{(2)}_{3,1}+\sigma_2=I_0^2I_1\sigma_2\,. \tag{7.32}$$

Since $L(-x,q)=-L(x,q)$ by (7.16), $\Phi_0(-x,q)=\Phi_0(x,q)$ by (7.23). By [Po, (4.8)], $I_3(q)=I_1(q)$. Thus, by (7.31), (7.29), (7.30), (7.32), and $\widetilde{\mathcal{C}}^{(0)}_{p,p}=1$,

$$\begin{split}\frac{(-1)^l}{\langle\mathbf{a}\rangle x^{2l}}\underset{\hbar_1=0}{\mathfrak{R}}\bigg\{\underset{\hbar_2=0}{\mathfrak{R}}\bigg\{\frac{\mathrm{e}^{-\frac{\eta(x,q)}{\hbar_1}-\frac{\eta(\bar{x},q)}{\hbar_2}}\mathcal{Z}(x,\bar{x},\hbar_1,\hbar_2,Q)}{\hbar_1\hbar_2}\bigg\}\bigg\}=\Phi_0(x,q)^2\bigg(\frac{\widetilde{\mathcal{C}}^{(2)}_{3,0}(q)}{I_0(q)^2}-\sigma_2\frac{\mathrm{D}I_0(q)}{I_0(q)}\bigg)\\+\frac{L(x,q)^2\Phi_0(x,q)^2}{I_0(q)^2I_1(q)^2I_2(q)}\bigg(2\frac{\mathrm{D}\Phi_0(x,q)}{\Phi_0(x,q)}+2\frac{\mathrm{D}L(x,q)}{L(x,q)}-2\frac{\mathrm{D}I_0(q)}{I_0(q)}-\frac{\mathrm{D}I_1(q)}{I_1(q)}-\frac{\mathrm{D}I_2(q)}{I_2(q)}\bigg)\\-\sum_{r=1}^{m}(r-2)\frac{\sigma_{2r}\Phi_0(x,q)^2}{L(x,q)^{2r-2}}\bigg(\frac{\mathrm{D}\Phi_0(x,q)}{\Phi_0(x,q)}-(r-1)\frac{\mathrm{D}L(x,q)}{L(x,q)}\bigg).\end{split} \tag{7.33}$$

We note that the identities (7.29)-(7.33) above hold in $H^*_{\mathbb{T}^m}(\mathbb{P}^{n-1})$, or equivalently with $x=\alpha_i$ for each $i\in[n]$ and α_i as in (3.19).

Let

$$\begin{split}\Psi(x,q)&=L(x,q)\sigma_{n-1}\big(L(x,q)\big)-n\big(\sigma_n(L(x,q))-\sigma_n(x)\big),\\ \dot{\Psi}(x,q)&\equiv L(x,q)\frac{\mathrm{d}\Psi}{\mathrm{d}L}(x,q)=2L(x,q)^2\sigma_{n-2}\big(L(x,q)\big)-(n-1)L(x,q)\sigma_{n-1}\big(L(x,q)\big).\end{split}$$

By [Po, (4.8),(4.9)],

$$I_0(q)^2I_1(q)^2I_2(q)=\big(1-\mathbf{a}^{\mathbf{a}}q\big)^{-1}.$$

Thus, by (7.28), (7.33), the $p=1$ case of (7.29), and (7.23),

$$\begin{split}\sum_{d=1}^{\infty}Q^d\tilde{K}_{2d}=\frac{1}{I_1(q)}\sum_{i=1}^{2m}\bigg\{\frac{L(x,q)^n}{\Psi(x,q)}\bigg[\frac{1}{2}\sum_{r=0}^{m}\frac{(r-2)\sigma_{2r}}{L(x,q)^{2r}}\bigg(n-1-2r-\frac{\dot{\Psi}(x,q)}{\Psi(x,q)}\bigg)\frac{\mathrm{D}L(x,q)}{L(x,q)}\\+\mathbf{a}^{\mathbf{a}}q\bigg(n-1-\frac{\dot{\Psi}(x,q)}{\Psi(x,q)}\bigg)\frac{\mathrm{D}L(x,q)}{L(x,q)}+\bigg(\mathbf{a}^{\mathbf{a}}q-(1-\mathbf{a}^{\mathbf{a}}q)\frac{\mathrm{D}I_1(q)}{I_1(q)}\bigg)\\-\frac{1}{L(x,q)^2}\bigg(\frac{\widetilde{\mathcal{C}}^{(2)}_{3,0}(q)}{I_0(q)^2}-\sigma_2\frac{\mathrm{D}I_0(q)}{I_0(q)}\bigg)\bigg]\bigg\}\bigg|_{x=\alpha_i}.\end{split} \tag{7.34}$$

By (7.16) and (7.26),

$$\frac{L(x,q)^n}{\Psi(x,q)}\sum_{r=0}^{m}\frac{(r-2)\sigma_{2r}}{L(x,q)^{2r}}=-\frac{1}{2}-2\frac{\mathrm{D}L(x,q)}{L(x,q)}+\frac{n-4}{2}\cdot\frac{\sigma_n(x)}{\Psi(x,q)}\,.$$

This identity and (7.26) give

$$\frac{L(x,q)^n}{\Psi(x,q)}\bigg[\frac{1}{2}\sum_{r=0}^{m}\frac{(r-2)\sigma_{2r}}{L(x,q)^{2r}}\bigg(n-1-2r-\frac{\dot{\Psi}(x,q)}{\Psi(x,q)}\bigg)+\mathbf{a}^{\mathbf{a}}q\bigg(n-1-\frac{\dot{\Psi}(x,q)}{\Psi(x,q)}\bigg)\bigg]=-\frac{3}{4}$$

for $x=\alpha_i$ with $i\in[n]$. Combining this with (7.34) and using (7.26) again, we obtain

$$\begin{aligned}\sum_{d=1}^{\infty}Q^d\tilde{K}_{2d}=\frac{1}{I_1(q)}\Bigg(\bigg(\frac{1}{4}-\frac{(1-\mathbf{a}^{\mathbf{a}}q)}{\mathbf{a}^{\mathbf{a}}q}\frac{\mathrm{D}I_1(q)}{I_1(q)}\bigg)\mathrm{D}\sum_{i=1}^{2m}\ln L(\alpha_i,q)\\ +\frac{1}{2\mathbf{a}^{\mathbf{a}}q}\bigg(\frac{\widetilde{\mathcal{C}}^{(2)}_{3,0}(q)}{I_0(q)^2}-\sigma_2\frac{\mathrm{D}I_0(q)}{I_0(q)}\bigg)\mathrm{D}\sum_{i=1}^{2m}\frac{1}{L(\alpha_i,q)^2}\Bigg).\end{aligned}\tag{7.35}$$

Since $\{L(\alpha_i,q)\}$ with $i\in[2m]$ are the roots of the polynomial equation $\sigma_{2m}(z)=\mathbf{a}^{\mathbf{a}}qz^{2m}$,

$$\mathrm{D}\ln\prod_{i=1}^{2m}L(\alpha_i,q)=\mathrm{D}\ln\big((1-\mathbf{a}^{\mathbf{a}}q)^{-1}\sigma_{2m}\big)=\frac{\mathbf{a}^{\mathbf{a}}q}{1-\mathbf{a}^{\mathbf{a}}q}\,,$$

$$\mathrm{D}\sum_{i=1}^{2m}\frac{1}{L(\alpha_i,q)^2}=\mathrm{D}\bigg(-2\frac{\sigma_{2m-2}}{\sigma_{2m}}\bigg)=0\quad\forall\,m\geq 2.$$

Plugging these two identities into (7.35) and using $Q\frac{\mathrm{d}}{\mathrm{d}Q}=\frac{1}{I_1(q)}\mathrm{D}$, we obtain (1.13).

A Proof of (2.9)

We assume that $\nu_{\mathbf{a}}>0$ and denote $\mathbf{a}^{\mathbf{a}}$ by A throughout this section. Let $\mathfrak{R}$ be as defined in (4.15) and (4.16). For each $p\in\mathbb{Z}$, let $\hat{p}=n-1-l-p$.

The $d=0$ case of (2.9) follows immediately from the first equation in (2.7). Lemma A.1 is the key observation in the proof of the remaining cases of (2.9). Since $\widetilde{\mathrm{c}}^{(1)}_{p,s}=-\mathrm{c}^{(1)}_{p,s}$ whenever $s\leq p-\nu_{\mathbf{a}}$, the $(d,s)=(1,p-\nu_{\mathbf{a}})$ case of this lemma is precisely the $d=1$ case of (2.9).

Lemma A.1. *If $d\in\mathbb{Z}^+$ and $p,s\in\mathbb{Z}^{\geq 0}$ are such that $p,s\leq n-1-l$ and $p-s\leq\nu_{\mathbf{a}}d$, then*

$$\mathrm{c}^{(d)}_{p,s}=\delta_{p-s,\nu_{\mathbf{a}}d}A^d-(-1)^{\nu_{\mathbf{a}}d+p+s}\mathrm{c}^{(d)}_{\hat{s},\hat{p}}-\sum_{\substack{d_1+d_2=d\\ d_1,d_2\geq 1}}\sum_{t=0}^{n-1-l}(-1)^{\nu_{\mathbf{a}}d_2+s+t}\mathrm{c}^{(d_1)}_{p,t}\mathrm{c}^{(d_2)}_{\hat{s},\hat{t}}\,.$$

Proof. If d,p,s are as above,

$$\begin{aligned}\mathrm{c}^{(d)}_{p,s}=\mathfrak{R}_{w=0}\left\{\frac{(w+d)^p\prod\limits_{k=1}^{l}\prod\limits_{r=1}^{a_kd}(a_kw+r)}{w^{s+1}\prod\limits_{r=1}^{d}(w+r)^n}\right\}=\langle\mathbf{a}\rangle\mathfrak{R}_{w=0}\left\{\frac{\prod\limits_{k=1}^{l}\prod\limits_{r=1}^{a_kd-1}(a_kw+r)}{w^{s+1}(w+d)^{n-l-p}\prod\limits_{r=1}^{d-1}(w+r)^n}\right\}\\ =(-1)^{\nu_{\mathbf{a}}d+p+s}\langle\mathbf{a}\rangle\mathfrak{R}_{w=-d}\left\{\frac{\prod\limits_{k=1}^{l}\prod\limits_{r=1}^{a_kd-1}(a_kw+r)}{(w+d)^{s+1}w^{n-l-p}\prod\limits_{r=1}^{d-1}(w+r)^n}\right\};\end{aligned}\tag{A.1}$$

the last equality is obtained by substituting $-w-d$ for w. If in addition $d_2\in\mathbb{Z}^+$ is such that $d_2<d$,

$$
\begin{aligned}
&\underset{w=-d_2}{\mathfrak{R}}\left\{\frac{(w+d)^p\prod\limits_{k=1}^{l}\prod\limits_{r=1}^{a_kd}(a_kw+r)}{w^{s+1}\prod\limits_{r=1}^{d}(w+r)^n}\right\}\\
&= \langle\mathbf{a}\rangle\sum_{t=0}^{n-1-l}\underset{w=-d_2}{\mathfrak{R}}\left\{\frac{\prod\limits_{k=1}^{l}\prod\limits_{r=1}^{a_kd_2-1}(a_kw+r)}{w^{s+1}(w+d_2)^{\hat{t}+1}\prod\limits_{r=1}^{d_2-1}(w+r)^n}\right\}\underset{w=-d_2}{\mathfrak{R}}\left\{\frac{(w+d)^p\prod\limits_{k=1}^{l}\prod\limits_{r=a_kd_2+1}^{a_kd}(a_kw+r)}{(w+d_2)^{t+1}\prod\limits_{r=d_2+1}^{d}(w+r)^n}\right\}\\
&= \sum_{t=0}^{n-1-l}(-1)^{\nu_{\mathbf{a}}d_2+s+t}\mathrm{c}_{\hat{s},\hat{t}}^{(d_2)}\mathrm{c}_{p,t}^{(d-d_2)}\,;
\end{aligned}
$$

the last equality follows from (A.1) with (d,p,s) replaced by $(d_2,\hat{s},\hat{t})$. Thus, by the Residue Theorem on S^2,

$$
\begin{aligned}
\mathrm{c}_{p,s}^{(d)} &= \underset{w=0}{\mathfrak{R}}\left\{\frac{(w+d)^p\prod\limits_{k=1}^{l}\prod\limits_{r=1}^{a_kd}(a_kw+r)}{w^{s+1}\prod\limits_{r=1}^{d}(w+r)^n}\right\} = -\underset{w=\infty,-1,-2,\ldots,-d}{\mathfrak{R}}\left\{\frac{(w+d)^p\prod\limits_{k=1}^{l}\prod\limits_{r=1}^{a_kd}(a_kw+r)}{w^{s+1}\prod\limits_{r=1}^{d}(w+r)^n}\right\}\\
&= \delta_{p-s,\nu_{\mathbf{a}}d}A^d-(-1)^{\nu_{\mathbf{a}}d+p+s}\mathrm{c}_{\hat{s},\hat{p}}^{(d)}-\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}}\sum_{t=0}^{n-1-l}(-1)^{\nu_{\mathbf{a}}d_2+s+t}\mathrm{c}_{p,t}^{(d_1)}\mathrm{c}_{\hat{s},\hat{t}}^{(d_2)}\,,
\end{aligned}
$$

as claimed. □

For any k-tuple $\mathbf{d}\equiv(d_i)_{i\in[k]}\in(\mathbb{Z}^+)^k$, let

$$\ell(\mathbf{d})\equiv k,\qquad |\mathbf{d})\equiv d_k\,.$$

If in addition $p\in\mathbb{Z}^{\ge0}$ and $\mathbf{r}\equiv(r_i)_{i\in[k]}\in(\mathbb{Z}^{\ge0})^k$ is another k-tuple, let

$$\mathrm{c}_{p\mathbf{r}}^{(\mathbf{d})}\equiv(-1)^k\cdot\mathrm{c}_{p,r_1}^{(d_1)}\cdot\prod_{i=2}^{k}\mathrm{c}_{r_{i-1},r_i}^{(d_i)}\,.$$

If $d\in\mathbb{Z}^+$ and $p\in\mathbb{Z}^{\ge0}$, let

$$
\begin{aligned}
\mathcal{S}_p(d)&\equiv\Big\{(\mathbf{d},\mathbf{r})\in(\mathbb{Z}^+)^k\times(\mathbb{Z}^{\ge0})^{k-1}\colon\ k\in\mathbb{Z}^+,\ \sum_{i=1}^{k}d_i=d,\ r_i\le p-\nu_{\mathbf{a}}\sum_{j=1}^{i}d_j\ \ \forall\,i\in[k-1]\Big\},\\
\mathcal{S}_p^*(d)&\equiv\Big\{(\mathbf{d},\mathbf{r})\in\mathcal{S}_p(d)\colon\ r_i<p-\nu_{\mathbf{a}}\sum_{j=1}^{i}d_j\ \ \forall\,i\in[k-1]\Big\}.
\end{aligned}
$$

By (2.6), if in addition $s\le p-\nu_{\mathbf{a}}d$, then

$$\widetilde{\mathrm{c}}_{p,s}^{(d)}=\sum_{(\mathbf{d},\mathbf{r})\in\mathcal{S}_p(d)}\mathrm{c}_{p\mathbf{r}s}^{(\mathbf{d})}\,,\qquad\text{where}\quad\mathrm{c}_{p\mathbf{r}s}^{(\mathbf{d})}\equiv\mathrm{c}_{ps}^{(d)}\equiv-\mathrm{c}_{p,s}^{(d)}\quad\text{if}\ \ \ell(\mathbf{d})=1.\tag{A.2}$$

Corollary A.2. *If* $d \in \mathbb{Z}^+$ *and* $p, s \in \mathbb{Z}^{\geq 0}$ *are such that* $p, s \leq n-1-l$ *and* $p-s \leq \nu_{\mathbf{a}} d$, *then*

$$\mathrm{c}_{p,s}^{(d)} = \delta_{p-s,\nu_{\mathbf{a}}d} A^d + (-1)^{\nu_{\mathbf{a}}d+p+s} \sum_{(\mathbf{d},\mathbf{r}) \in \mathcal{S}_{\hat{s}}^*(d)} \mathrm{c}_{\hat{s}\mathbf{r}\hat{p}}^{(\mathbf{d})} + \sum_{\substack{d_1+d_2=d \\ d_1,d_2 \geq 1}} \sum_{(\mathbf{d},\mathbf{r}) \in \mathcal{S}_{\hat{s}}^*(d_2)} \sum_{t=0}^{s+\nu_{\mathbf{a}}d_2} (-1)^{\nu_{\mathbf{a}}d_2+s+t} \mathrm{c}_{p,t}^{(d_1)} \mathrm{c}_{\hat{s}\mathbf{r}\hat{t}}^{(\mathbf{d})}.$$

Proof. By Lemma A.1 and induction on K,

$$\begin{aligned} \mathrm{c}_{p,s}^{(d)} = \delta_{p-s,\nu_{\mathbf{a}}d} A^d &+ (-1)^{\nu_{\mathbf{a}}d+p+s} \sum_{\substack{(\mathbf{d},\mathbf{r}) \in \mathcal{S}_{\hat{s}}^*(d) \\ \ell(\mathbf{d}) \leq K}} \mathrm{c}_{\hat{s}\mathbf{r}\hat{p}}^{(\mathbf{d})} + \sum_{\substack{d_1+d_2=d \\ d_1,d_2 \geq 1}} \sum_{\substack{(\mathbf{d},\mathbf{r}) \in \mathcal{S}_{\hat{s}}^*(d_2) \\ \ell(\mathbf{d}) \leq K}} \sum_{t=0}^{s+\nu_{\mathbf{a}}d_2} (-1)^{\nu_{\mathbf{a}}d_2+s+t} \mathrm{c}_{p,t}^{(d_1)} \mathrm{c}_{\hat{s}\mathbf{r}\hat{t}}^{(\mathbf{d})} \\ &+ \sum_{\substack{d_1+d_2=d \\ d_1,d_2 \geq 1}} \sum_{\substack{(\mathbf{d},\mathbf{r}) \in \mathcal{S}_{\hat{s}}^*(d_2) \\ \ell(\mathbf{d}) = K}} \sum_{t=s+\nu_{\mathbf{a}}d_2+1}^{n-1-l} (-1)^{\nu_{\mathbf{a}}d_2+s+t} \mathrm{c}_{p,t}^{(d_1)} \mathrm{c}_{\hat{s}\mathbf{r}\hat{t}}^{(\mathbf{d})} \end{aligned}$$

for all $K \geq 1$.[19] Setting $K = d$, we obtain the claim. □

Corollary A.3. *If* $d, p \in \mathbb{Z}$ *are such that* $d \geq 2$ *and* $\nu_{\mathbf{a}} d \leq p \leq n-1-l$, *then*

$$\sum_{(\mathbf{d},\mathbf{r}) \in \mathcal{S}_p(d)} \mathrm{c}_{p\mathbf{r}(p-\nu_{\mathbf{a}}d)}^{(\mathbf{d})} = -\sum_{(\mathbf{d},\mathbf{r}) \in \mathcal{S}_{\hat{p}+\nu_{\mathbf{a}}d}^*(d)} \mathrm{c}_{(\hat{p}+\nu_{\mathbf{a}}d)\mathbf{r}\hat{p}}^{(\mathbf{d})} + A \sum_{(\mathbf{d},\mathbf{r}) \in \mathcal{S}_{\hat{p}+\nu_{\mathbf{a}}(d-1)}^*(d-1)} \mathrm{c}_{(\hat{p}+\nu_{\mathbf{a}}(d-1))\mathbf{r}\hat{p}}^{(\mathbf{d})}.$$

Proof. By Corollary A.2 and induction on d_2^*,

$$\begin{aligned} \sum_{\substack{(\mathbf{d},\mathbf{r}) \in \mathcal{S}_p(d) \\ |\mathbf{d}| \leq d_2^*}} \mathrm{c}_{p\mathbf{r}(p-\nu_{\mathbf{a}}d)}^{(\mathbf{d})} = &-A \sum_{\substack{(\mathbf{d},\mathbf{r}) \in \mathcal{S}_p(d-1) \\ |\mathbf{d}| \geq d_2^*}} \mathrm{c}_{p\mathbf{r}(p-\nu_{\mathbf{a}}(d-1))}^{(\mathbf{d})} \\ &- \sum_{\substack{d_1+d_2=d \\ 1 \leq d_2 \leq d_2^*}} \sum_{\substack{(\mathbf{d}',\mathbf{r}') \in \mathcal{S}_p(d_1) \\ |\mathbf{d}'| > d_2^*-d_2}} \sum_{(\mathbf{d}'',\mathbf{r}'') \in \mathcal{S}_{\hat{p}+\nu_{\mathbf{a}}d}^*(d_2)} \sum_{t=0}^{p-\nu_{\mathbf{a}}d_1} (-1)^{p-\nu_{\mathbf{a}}d_1+t} \mathrm{c}_{p\mathbf{r}'t}^{(\mathbf{d}')} \mathrm{c}_{(\hat{p}+\nu_{\mathbf{a}}d)\mathbf{r}''\hat{t}}^{(\mathbf{d}'')} \\ &+ A \sum_{\substack{d_1+d_2=d \\ 2 \leq d_2 \leq d_2^*}} \sum_{\substack{(\mathbf{d}',\mathbf{r}') \in \mathcal{S}_p(d_1) \\ |\mathbf{d}'| > d_2^*-d_2}} \sum_{(\mathbf{d}'',\mathbf{r}'') \in \mathcal{S}_{\hat{p}+\nu_{\mathbf{a}}(d-1)}^*(d_2-1)} \sum_{t=0}^{p-\nu_{\mathbf{a}}d_1} (-1)^{p-\nu_{\mathbf{a}}d_1+t} \mathrm{c}_{p\mathbf{r}'t}^{(\mathbf{d}')} \mathrm{c}_{(\hat{p}+\nu_{\mathbf{a}}(d-1))\mathbf{r}''\hat{t}}^{(\mathbf{d}'')} \end{aligned}$$

for all $d_2^* = 1, \ldots, d-1$.[20] Combining the $d_2^* = d-1$ case of this identity and Corollary A.2 with (d, s) replaced by $(d, p-\nu_{\mathbf{a}}d)$ and $(d-1, p-\nu_{\mathbf{a}}(d-1))$, we obtain the claim. □

[19] The $K=1$ case of this identity is Lemma A.1. The inductive step is carried out by applying Lemma A.1 to the factor $\mathrm{c}_{p,t}^{(d_1)}$ on the last line of this identity and noting that the assumptions imply that $p-t < \nu_{\mathbf{a}} d_1$.

[20] The $d_2^* = 1$ case of this identity is obtained from Corollary A.2 with (d, p, s) replaced by $(1, r_{\ell(\mathbf{d})-1}, p-\nu_{\mathbf{a}}d)$. The inductive step is carried out by using Corollary A.2 with (d, p, s) replaced by $(d_2^*+1, r_{\ell(\mathbf{d})-1}, p-\nu_{\mathbf{a}}d)$ and $(d_2^*, r_{\ell(\mathbf{d})-1}, p-\nu_{\mathbf{a}}(d-1))$.

We now verify (2.9) for $d\geq 2$. By (A.2) and Corollary A.3,

$$\begin{aligned}\widetilde{c}^{(d)}_{p,p-\nu_{\mathbf{a}}d} &= -\Bigg(\widetilde{c}^{(d)}_{\hat{p}+\nu_{\mathbf{a}}d,\hat{p}} - \sum_{\substack{d_1+d_2=d\\ d_1,d_2\geq 1}} \sum_{\substack{(\mathbf{d}',\mathbf{r}')\in\mathcal{S}^*_{\hat{p}+\nu_{\mathbf{a}}d}(d_1)\\ (\mathbf{d}'',\mathbf{r}'')\in\mathcal{S}_{\hat{p}+\nu_{\mathbf{a}}d_2}(d_2)}} c^{(\mathbf{d}')}_{(\hat{p}+\nu_{\mathbf{a}}d)\mathbf{r}'(\hat{p}+\nu_{\mathbf{a}}d_2)} c^{(\mathbf{d}'')}_{(\hat{p}+\nu_{\mathbf{a}}d_2)\mathbf{r}''\hat{p}}\Bigg)\\
&\qquad + A\Bigg(\widetilde{c}^{(d-1)}_{\hat{p}+\nu_{\mathbf{a}}(d-1),\hat{p}} - \sum_{\substack{d_1+d_2=d\\ 2\leq d_1\leq d-1}} \sum_{\substack{(\mathbf{d}',\mathbf{r}')\in\mathcal{S}^*_{\hat{p}+\nu_{\mathbf{a}}(d-1)}(d_1-1)\\ (\mathbf{d}'',\mathbf{r}'')\in\mathcal{S}_{\hat{p}+\nu_{\mathbf{a}}d_2}(d_2)}} c^{(\mathbf{d}')}_{(\hat{p}+\nu_{\mathbf{a}}(d-1))\mathbf{r}'(\hat{p}+\nu_{\mathbf{a}}d_2)} c^{(\mathbf{d}'')}_{(\hat{p}+\nu_{\mathbf{a}}d_2)\mathbf{r}''\hat{p}}\Bigg)\\
&= -\widetilde{c}^{(d)}_{\hat{p}+\nu_{\mathbf{a}}d,\hat{p}} + \widetilde{c}^{(1)}_{\hat{p}+\nu_{\mathbf{a}}d,\hat{p}+\nu_{\mathbf{a}}(d-1)} \widetilde{c}^{(d-1)}_{\hat{p}+\nu_{\mathbf{a}}(d-1),\hat{p}} + A\widetilde{c}^{(d-1)}_{\hat{p}+\nu_{\mathbf{a}}(d-1),\hat{p}} - \sum_{\substack{d_1+d_2=d\\ 2\leq d_1\leq d-1}} \widetilde{c}^{(d_1)}_{p-\nu_{\mathbf{a}}d_2,p-\nu_{\mathbf{a}}d}\widetilde{c}^{(d_2)}_{\hat{p}+\nu_{\mathbf{a}}d_2,\hat{p}}\,.\end{aligned}$$

Combining this with the first equation in (2.7), we obtain

$$\sum_{\substack{d_1+d_2=d\\ d_1,d_2\geq 0}} \widetilde{c}^{(d_1)}_{p-\nu_{\mathbf{a}}d_2,p-\nu_{\mathbf{a}}d}\widetilde{c}^{(d_2)}_{\hat{p}+\nu_{\mathbf{a}}d_2,\hat{p}} = \big(\widetilde{c}^{(1)}_{p-\nu_{\mathbf{a}}(d-1),p-\nu_{\mathbf{a}}d} + \widetilde{c}^{(1)}_{\hat{p}+\nu_{\mathbf{a}}d,\hat{p}+\nu_{\mathbf{a}}(d-1)} + A\big)\widetilde{c}^{(d-1)}_{\hat{p}+\nu_{\mathbf{a}}(d-1),\hat{p}} = 0;$$

the last equality is the $d=1$ case of (2.9) with p replaced by $p-\nu_{\mathbf{a}}(d-1)$.

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