On Symplectic Sum Formulas in Gromov-Witten Theory

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Abstract

This manuscript describes in detail the symplectic sum formulas in Gromov-Witten theory and related topological and analytic issues. In particular, we analyze and compare the two analytic approaches to these formulas. The Ionel-Parker formula contains two unique features, rim tori refinements of relative invariants and the so-called S-matrix, which disappear in all computed applications and have been a mystery in Gromov-Witten theory over the past decade. The former is aimed at addressing a delicate topological deficiency with symplectic sum decompositions obscured in all other approaches to symplectic sum formulas. We show that unfortunately this deficiency is unavoidable and that the Ionel-Parker work concludes otherwise due to imprecise definitions and arguments. However, we also extract some additional qualitative information in certain cases from their insight. We also show that the S-matrix, which appears due to incomplete reasoning in the transition from analysis to geometry, in fact acts as the identity in the Ionel-Parker symplectic sum formula anyway. Furthermore, the key gluing argument in their paper contains several highly technical, but crucial, mistakes. The idea behind the Li-Ruan approach is to adapt the SFT type stretching of the target. This has the potential of avoiding many issues with the degeneration of the metric on the target occurring in the Ionel-Parker approach, which we expect to realize in a forthcoming paper. Unfortunately, the implementation of this idea in the Li-Ruan paper does not contain even an attempt at a complete proof of any major statement, such the compactness of the moduli space of relative maps or the bijectivity of the gluing construction. In fact, the Li-Ruan paper does not contain even a reasonably precise definition of relative stable map or a complete symplectic sum formula. The only technical arguments in this paper concern fairly minor points and are either incorrect or add unnecessary complications. Neither of the two papers even considers gluing stable maps with extra rubber structure, which is necessary to do for defining the relevant invariants outside of the relatively narrow collection of "semi-positive" cases. In this manuscript, we re-formulate the (numerical) symplectic sum formula correctly, describe the issues arising in both approaches, and explain how the Li-Ruan SFT type idea can be used to address them.

Contents

1 Introduction

Gromov-Witten invariants of symplectic manifolds, which include nonsingular projective varieties, are certain counts of pseudo-holomorphic curves that play prominent roles in symplectic topology, algebraic geometry, and string theory. The decomposition formulas of [IP5, Lj2, LR], known as symplectic sum formulas in symplectic topology and degeneration formulas in algebraic geometry, are one of the main tools used to compute Gromov-Witten invariants. They relate Gromov-Witten invariants of a target symplectic manifold to Gromov-Witten invariants of simpler symplectic manifolds; in many cases, these formulas completely determine the former in terms of the latter. The main formula of [IP5] contains two features not present in the formulas of [Lj2] and [LR]: a rim tori refinement of relative invariants and the so-called S-matrix; we explain why neither should appear. We also point out several highly technical, but crucial, mistakes in the key gluing argument of [IP5]. On the other hand, the symplectic sum formula of [LR], which is spread out across multiple statements, contains several errors, while the formula of [Lj2] is not as sharp, even in the algebraic category. The idea of [LR] to adapt the SFT type stretching of the target beautifully captures the degeneration of both the domain and the target and has a great potential of avoiding many analytic difficulties caused by the degeneration of the latter arising in [IP5]. Unfortunately, the implementation of this idea does not contain even an attempt at a complete proof of any major statement, such the compactness of the moduli space of relative maps or the bijectivity of the gluing construction; there is not even a reasonably precise definition of relative stable map in [LR]. The only technical arguments in this paper concern fairly minor points and are either incorrect or add unnecessary complications. Neither [IP4, IP5] nor [LR] even considers gluing stable maps with extra rubber structure, which is necessary to do for defining the relevant invariants outside of the relatively narrow collection of "semi-positive" cases. Section 2 summarizes our understanding of the issues with [IP4, IP5] and [LR] and directs to places in this manuscript where they are described in more detail; considerations related to [Lj1, Lj2] appear in [AF, Ch, GS]. Throughout this manuscript, we generally follow the reasoning and notation in [IP5] closely, but also use some of the statements from [LR] and [Lj2].

We denote by (X,ω_X) and (Y,ω_Y) compact symplectic manifolds, of the same dimension and without boundary. A compact submanifold V of (X,ω_X) is a symplectic hypersurface if the real codimension of V in X is 2 and $\omega_X|_V$ is a nondegenerate two-form on V. The normal bundle of a symplectic hypersurface V in X,

$$\mathcal{N}_X V \equiv TX|_V/TV \approx TV^{\omega_X} \equiv \big\{v\in T_xV:\ x\in V,\, \omega_X(v,w)=0\ \forall\, w\in T_xV\big\},$$

then inherits a symplectic structure from ω_X and thus a complex structure up to homotopy. If

$$e(\mathcal{N}_X V) = -e(\mathcal{N}_Y V) \in H^2(V;\mathbb{Z}), \tag{1.1}$$

there exists an isomorphism

$$\mathcal{N}_X V \otimes \mathcal{N}_Y V \approx V\times\mathbb{C} \tag{1.2}$$

of complex line bundles. As recalled in Section 3.3, a symplectic sum of symplectic manifolds (X,ω_X) and (Y,ω_Y) with a common symplectic divisor V such that (1.1) holds is a symplectic manifold $(Z,\omega_Z)=(X\#_V Y,\omega_\#)$ obtained from X and Y by gluing the complements of tubular neighborhoods of V in X and Y along their common boundary as directed by the isomorphism (1.2). In fact, the symplectic sum construction of [Gf, MW] produces a symplectic fibration $\pi\colon \mathcal{Z}\longrightarrow\Delta$ with central fiber $\mathcal{Z}_0=X\cup_V Y$, where $\Delta\subset\mathbb{C}$ is a disk centered at the origin and $\mathcal{Z}$ is a symplectic manifold with symplectic form $\omega_{\mathcal{Z}}$ such that

- π is surjective and is a submersion over $\Delta^*\equiv\Delta-0$,
- the restriction ω_λ of $\omega_{\mathcal{Z}}$ to $\mathcal{Z}_\lambda\equiv\pi^{-1}(\lambda)$ is nondegenerate for every $\lambda\in\Delta^*$,
- $\omega_{\mathcal{Z}}|_X=\omega_X$, $\omega_{\mathcal{Z}}|_Y=\omega_Y$.

The symplectic manifolds $(\mathcal{Z}_\lambda,\omega_\lambda)$ with $\lambda\in\Delta^*$ are then symplectically deformation equivalent to each other and denoted $(X\#_V Y,\omega_\#)$. However, different homotopy classes of the isomorphisms (1.2) give rise to generally different topological manifolds; see [Gf0]. There is also a retraction $q\colon\mathcal{Z}\longrightarrow\mathcal{Z}_0$ such that $q_\lambda\equiv q|_{\mathcal{Z}_\lambda}$ restricts to a diffeomorphism

$$\mathcal{Z}_\lambda - q_\lambda^{-1}(V) \longrightarrow \mathcal{Z}_0 - V$$

and to an S^1-fiber bundle $q_\lambda^{-1}(V)\longrightarrow V$, whenever $\lambda\in\Delta^*$. We denote by $q_\#\colon X\#_V Y\longrightarrow X\cup_V Y$ a typical collapsing map q_λ. In the algebraic setting of [Lj2], $\pi\colon\mathcal{Z}\longrightarrow\Delta$ is a holomorphic map from a Kahler manifold $\mathcal{Z}$ with an ample line bundle $\mathcal{L}\longrightarrow\mathcal{Z}$; the curvature form of a suitably chosen metric on $\mathcal{L}$ gives rise to a symplectic form $\omega_{\mathcal{Z}}$ on $\mathcal{Z}$, as in [GH, Section 1.2].

If $g,k\in\mathbb{Z}^{\ge0}$, $\chi\in\mathbb{Z}$, $A\in H_2(X;\mathbb{Z})$, and J is an ω_X-compatible almost complex structure on X, let $\overline{\mathcal{M}}_{g,k}(X,A)$ and $\widetilde{\mathcal{M}}_{\chi,k}(X,A)$ denote the moduli spaces of stable J-holomorphic k-marked maps from connected nodal curves of genus g and from (possibly) disconnected nodal curves of euler characteristic χ, respectively; the latter moduli spaces are quotients of disjoint unions of products of the former moduli spaces. If $V\subset X$ is a symplectic divisor, $\mathbf{s}\equiv(s_1,\ldots,s_\ell)$ is an ℓ-tuple of positive integers such that

$$s_1+\ldots+s_\ell = A\cdot V, \tag{1.3}$$

and J restricts to a complex structure on V, let $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A)$ and $\widetilde{\mathcal{M}}^V_{\chi,k;\mathbf{s}}(X,A)$ denote the moduli spaces of stable J-holomorphic $(k+\ell)$-marked maps from connected nodal curves of genus g and from (possibly) disconnected nodal curves of euler characteristic χ, respectively, that have contact with V at the last ℓ marked points of orders $s_1,\ldots,s_\ell$. These moduli spaces are introduced in [IP4, Lj1, LR] under certain assumptions on J and reviewed in Section 4.2.

There are natural evaluation morphisms

$$\begin{aligned}\mathrm{ev}\equiv\mathrm{ev}_1\times\ldots\times\mathrm{ev}_k\colon \widetilde{\mathcal{M}}_{\chi,k}(X,A),\widetilde{\mathcal{M}}^V_{\chi,k;\mathbf{s}}(X,A)&\longrightarrow X^k,\\ \mathrm{ev}^V\equiv\mathrm{ev}_{k+1}\times\ldots\times\mathrm{ev}_{k+\ell}\colon \widetilde{\mathcal{M}}^V_{\chi,k;\mathbf{s}}(X,A)&\longrightarrow V_{\mathbf{s}}\equiv V^\ell,\end{aligned} \tag{1.4}$$

sending each element to the values of the map at the marked points. We denote the restrictions of these maps to

$$\overline{\mathcal{M}}_{g,k}(X,A)\subset\widetilde{\mathcal{M}}_{2-2g,k}(X,A) \qquad\text{and}\qquad \overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A)\subset\widetilde{\mathcal{M}}^V_{2-2g,k;\mathbf{s}}(X,A)$$

by the same symbols. Along with the virtual class for $\overline{\mathcal{M}}_{g,k}(X,A)$, constructed in [RT2] in the semi-positive case, in [BF] in the algebraic case, and in [FO, LT] in the general case, the morphisms (1.4) with $V=\emptyset$ give rise to the (absolute) Gromov-Witten and Gromov-Taubes invariants of (X,ω_X),

$$\begin{aligned}\mathrm{GW}_{X,A,g}\colon\mathbb{T}^*(X)\longrightarrow\mathbb{Q},\quad \mathrm{GW}_{X,A,g}(\alpha)&=\sum_{k=0}^{\infty}\big\langle\mathrm{ev}^*\alpha,[\overline{\mathcal{M}}_{g,k}(X,A)]^{\mathrm{vir}}\big\rangle,\\ \mathrm{GT}_{X,A,\chi}\colon\mathbb{T}^*(X)\longrightarrow\mathbb{Q},\quad \mathrm{GT}_{X,A,\chi}(\alpha)&=\sum_{k=0}^{\infty}\big\langle\mathrm{ev}^*\alpha,[\widetilde{\mathcal{M}}_{\chi,k}(X,A)]^{\mathrm{vir}}\big\rangle,\end{aligned}$$

where

$$\mathbb{T}^*(X)\equiv\bigoplus_{k=0}^{\infty}H^{2*}(X)^{\otimes k}\subset\bigoplus_{k=0}^{\infty}H^{2*}(X^k)$$

is the tensor algebra of $H^{2*}(X)\equiv H^{2*}(X;\mathbb{Q})$.[1] Along with the virtual class for $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A)$, the morphisms (1.4) give rise to the relative Gromov-Witten and Gromov-Taubes invariants of (X,V,ω_X),

$$\begin{aligned}\mathrm{GW}^V_{X,A,g;\mathbf{s}}\colon\mathbb{T}^*(X)\longrightarrow H_*(V_{\mathbf{s}}),\quad \mathrm{GW}^V_{X,A,g;\mathbf{s}}(\alpha)&=\sum_{k=0}^{\infty}\mathrm{ev}^V_*\big(\mathrm{ev}^*\alpha\cap\big[\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A)\big]^{\mathrm{vir}}\big),\\ \mathrm{GT}^V_{X,A,\chi;\mathbf{s}}\colon\mathbb{T}^*(X)\longrightarrow H_*(V_{\mathbf{s}}),\quad \mathrm{GT}^V_{X,A,\chi;\mathbf{s}}(\alpha)&=\sum_{k=0}^{\infty}\mathrm{ev}^V_*\big(\mathrm{ev}^*\alpha\cap\big[\widetilde{\mathcal{M}}^V_{\chi,k;\mathbf{s}}(X,A)\big]^{\mathrm{vir}}\big),\end{aligned}$$

[1] Odd cohomology classes can be considered as well, but at the cost of introducing suitable signs into the symplectic sum formulas.

where $H_*(V_{\mathbf{s}})\equiv H_*(V_{\mathbf{s}};\mathbb{Q})$. Such a virtual class is constructed in [IP4] in the semi-positive case and in [Lj1] in the algebraic case and is used in [LR] in the general case; see Section 4.3 for more details. While the homomorphisms $\mathrm{GW}_{X,A,g}$ and $\mathrm{GW}^V_{X,A,g;\mathbf{s}}$ completely determine the homomorphisms $\mathrm{GT}_{X,A,\chi}$ and $\mathrm{GT}^V_{X,A,\chi;\mathbf{s}}$, the latter lead to more streamlined decomposition formulas for (primary) GW-invariants, as noticed in [IP5].

The symplectic sum formulas relate the absolute GW-invariants of $X\#_V Y$ to the relative GW-invariants of the pairs (X,V) and (Y,V). Let

$$H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z})=\big\{(A_X,A_Y)\in H_2(X;\mathbb{Z})\times H_2(Y;\mathbb{Z})\colon\ A_X\cdot_X V=A_Y\cdot_Y V\big\},$$

where $\cdot_X$ and $\cdot_Y$ denote the homology intersection pairings in X and Y, e.g.

$$A_X\cdot_X V=\big\langle \mathrm{PD}_X A_X\cup \mathrm{PD}_X[V],[X]\big\rangle\in\mathbb{Z}.$$

As described in Section 3.1, there is a natural homomorphism

$$H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z})\longrightarrow H_2(X\#_V Y;\mathbb{Z})/\mathcal{R}^V_{X,Y},\qquad (A_X,A_Y)\longrightarrow A_X\#_V A_Y,\tag{1.5}$$

where

$$\mathcal{R}^V_{X,Y}=\ker\big\{q_{\#*}\colon H_2(X\#_V Y;\mathbb{Z})\longrightarrow H_2(X\cup_V Y;\mathbb{Z})\big\}.\tag{1.6}$$

We arrange the GW-invariants of $X\#_V Y$ into the formal power series

$$\mathrm{GT}_{X\#_V Y}=\sum_{\chi\in\mathbb{Z}}\ \sum_{\eta\in H_2(X\#_V Y;\mathbb{Z})/\mathcal{R}^V_{X,Y}}\ \sum_{C\in\eta}\mathrm{GT}_{X\#_V Y,C,\chi}\,t_\eta\lambda^\chi.\tag{1.7}$$

By Gromov's Compactness Theorem for J-holomorphic curves, only finitely many distinct elements $C\in\eta$ can be represented by J-holomorphic curves of a given genus, since $\omega_\#$ vanishes on $\mathcal{R}^V_{X,Y}$. Thus, the coefficient of each $t_\eta\lambda^\chi$ in $\mathrm{GT}_{X\#_V Y}$ is finite.

For a tuple $\mathbf{s}=(s_1,\ldots,s_\ell)\in(\mathbb{Z}^+)^\ell$, let

$$\ell(\mathbf{s})=\ell,\qquad |\mathbf{s}|=s_1+\ldots+s_\ell,\qquad \langle\mathbf{s}\rangle=s_1\cdot\ldots\cdot s_\ell.$$

We arrange the GW-invariants of (X,V) and (Y,V) into the formal power series

$$\mathrm{GT}^V_M=\sum_{\chi\in\mathbb{Z}}\ \sum_{A\in H_2(M;\mathbb{Z})}\ \sum_{\ell=0}^{\infty}\ \sum_{\substack{\mathbf{s}\in(\mathbb{Z}^+)^\ell\\ |\mathbf{s}|=A\cdot_M V}}\mathrm{GT}^V_{M,A,\chi;\mathbf{s}}\,t_A\lambda^\chi,\tag{1.8}$$

where $M=X,Y$. Let

$$V_\infty=\bigsqcup_{\ell=0}^{\infty}\ \bigsqcup_{\mathbf{s}\in(\mathbb{Z}^+)^\ell}V_{\mathbf{s}}\,.$$

We define a pairing $\star\colon H_*(V_\infty)\otimes H_*(V_\infty)\longrightarrow\mathbb{Q}[\lambda^{-1}]$ by

$$Z_X\star Z_Y=\begin{cases}\frac{\langle\mathbf{s}\rangle}{\ell(\mathbf{s})!}\lambda^{-2\ell(\mathbf{s})}Z_X\cdot_{V_{\mathbf{s}}}Z_Y, & \text{if } Z_X,Z_Y\in H_*(V_{\mathbf{s}});\\ 0, & \text{if } Z_X\in H_*(V_{\mathbf{s}}),\ Z_Y\in H_*(V_{\mathbf{s}'}),\ \mathbf{s}\neq\mathbf{s}'.\end{cases}\tag{1.9}$$

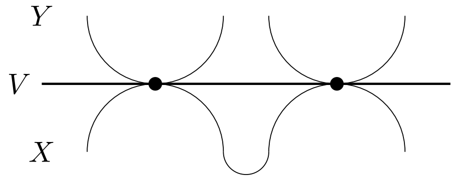

Figure 1: A possible limit of connected curves.

For homomorphisms $L_X : \mathbb{T}^*(X) \longrightarrow H_*(V_\infty)$ and $L_Y : \mathbb{T}^*(Y) \longrightarrow H_*(V_\infty)$, define

$$L_X \star L_Y : \mathbb{T}^*(X) \otimes \mathbb{T}^*(Y) \longrightarrow \mathbb{Q}[\lambda^{-1}] \quad \text{by} \quad \{L_X \star L_Y\}(\alpha_X \otimes \alpha_Y) = L_X(\alpha_X) \star L_Y(\alpha_Y). \tag{1.10}$$

If in addition $(A_X, A_Y) \in H_2(X;\mathbb{Z}) \times_V H_2(Y;\mathbb{Z})$ and $\chi_X, \chi_Y \in \mathbb{Z}$, let

$$L_X t_{A_X} \lambda^{\chi_X} \star L_Y t_{A_Y} \lambda^{\chi_Y} = L_X \star L_Y \, t_{A_X \#_V A_Y} \lambda^{\chi_X + \chi_Y}. \tag{1.11}$$

Theorem 1.1. *Let (X, ω_X) and (Y, ω_Y) be symplectic manifolds and $V \subset X, Y$ be a symplectic hypersurface satisfying (1.1). If $q_\# : X \#_V Y \longrightarrow X \cup_V Y$ is a collapsing map for an associated symplectic sum fibration and $q_\sqcup : X \sqcup Y \longrightarrow X \cup_V Y$ is the quotient map,*

$$\mathrm{GT}_{X\#_V Y}(q_\#^* \alpha) = \{\mathrm{GT}_X^V \star \mathrm{GT}_Y^V\}(q_\sqcup^* \alpha) \qquad \forall\, \alpha \in \mathbb{T}^*(X \cup_V Y). \tag{1.12}$$

The motivation behind (1.12), as well as all other symplectic sum formulas for GW-invariants, is the following. The curves in the smooth fibers $\mathcal{Z}_\lambda = X\#_V Y$ of the fibration $\pi : \mathcal{Z} \longrightarrow \Delta$ that contribute to the left-hand side of (1.12) degenerate, as $\lambda \longrightarrow 0$, to curves in the singular fiber $\mathcal{Z}_0 = X \cup_V Y$. Each of the irreducible components of a limiting curve lies in either X or Y. Furthermore, the union of the irreducible components of each limiting curve that map to X meet $V \subset X$ at the same points with the same multiplicity as the union of the irreducible components of each limiting curve that map to Y; see Figure 1. Such curves contribute to the right-hand side of (1.12). The contact conditions with V are encoded by a tuple $\mathbf{s}$ as above. For the reasons outlined in [IP5, p938], each limiting curve of type $\mathbf{s}$ arises as a limit of $\langle \mathbf{s} \rangle$ distinct families of curves into smooth fibers, requiring the factor of $\langle \mathbf{s} \rangle$ in (1.9); see also Section 5.1. The factor of $\ell(\mathbf{s})!$ in (1.9) arises due to the fact that the contact points with V are not á priori ordered, while the factor of $\lambda^{-2\ell(\mathbf{s})}$ accounts for the difference between the geometric and algebraic euler characteristics of the limiting curve. Since connected curves can limit to disconnected curves, it is more natural to formulate decomposition formulas for GW-invariants in terms of counts of disconnected curves, i.e. the GT-invariants, as done in [IP5].

Theorem 1.1 is a basic decomposition formula for GW-invariants, presented in the succinct style of [IP5]. However, it is not in any of the three standard symplectic sum papers and is not directly implied by any formula in these papers. The primary inputs $q_\#^* \alpha$ on the $X\#_V Y$ side of (1.12) are of the same type as in [IP5, Lj2, LR]. A characterization of which cohomology classes on $X\#_V Y$ are of the form $q_\#^* \alpha$ is provided in [IP5]; see Lemma 3.7 below. The identity (1.12) is equivalent to the intended symplectic sum formula in [LR]; unfortunately, it is spread out across several statements in [LR] and contains some misstatements, as described in Section 4.1. The symplectic sum formula in [IP5], even in the basic case of primary invariants, mistakenly contains two distinct features, the S-matrix and rim tori refinements of relative invariants, described in more detail below. Even ignoring these two features, the main symplectic sum statements in [IP5], (0.2) and (10.14), do not reduce to (1.12), in part because of definitions that do not make sense; see Section 4.1. The only one of the three standard symplectic sum papers which contains a correct (or even nearly correct) version of the symplectic sum formula (even in the basic case of primary inputs) is [Lj2]. Unfortunately, the main decomposition formulas in [Lj2], the two formulas at the bottom of page 201, often yield less sharp versions of (1.1), as their left-hand sides combine GW-invariants in the homology classes whose difference lies in a submodule of $H_2(X;\mathbb{Z})$ containing (often strictly) $\mathcal{R}^V_{X,Y}$.

The general symplectic sum formulas, considered in [IP5, Lj2] and mentioned in [LR], involve descendant classes. These classes effectively impose an order on the combined set of marked points of the limiting curve, which has to be taken into account by the pairing (1.11). This is done in [Lj2] by summing over rules of assignment I (ϑ in the notation of Section 4.1). It is stated in [LR] that the symplectic sum formula extends to descendant invariants, without any mention of some kind of rule of assignment. It is fairly clear what the symplectic sum formula for GW-invariants should be, but it is also important to formulate it correctly so that it can be easily used. In Theorem 4.1, we give a general symplectic sum formula summing the GT-type formulas in the style of [IP5] over the rules of assignments of [Lj2]. It seems impossible to condense the general symplectic sum formula into the format of the formulas (0.2) and (10.14) in [IP5], i.e. the attempted formulation of the symplectic sum formulas in [IP5] is a beautiful idea which unfortunately does not work as well beyond the case of primary invariants.

A deficiency of the decomposition formula (1.12) is that it expresses sums of GW-invariants of $X\#_V Y$ over homology classes differing by elements of $\mathcal{R}^V_{X,Y}$ in terms of relative GW-invariants of (X,V) and (Y,V); it would of course be preferable to express GW-invariants of $X\#_V Y$ in each homology class in terms of relative GW-invariants of (X,V) and (Y,V). Rim tori are introduced in [IP4, Section 5] with the aim of defining sufficiently fine relative GW-invariants to rectify this deficiency; they also provide a concrete way of understanding this deficiency. Unfortunately, the construction of the refined relative invariants in [IP4] is only sketched; as explained in Section 3.2, some version of this sketch can be realized if V is connected, but not in general. The usual relative invariants, defined in [LR] and [Lj1], factor through the relative invariants of [IP4] whenever they can be defined, and so the latter are thus indeed refinements (though not necessarily strict refinements) of the former. However, these refinements are insufficient to resolve the aforementioned deficiency of (1.12), except in rare cases, and the dependence of the topological type of $X\#Y$ on the homotopy class of the isomorphism (1.2) suggest that one should not expect otherwise. In most cases, it is impossible to address the relevant issue essentially for the reasons indicated in [IP4, Section 5]; see Section 3.2 for more details. As explained in Section 4.1, the use of these invariants in the statement of the symplectic sum formula in [IP5] causes further problems, including with the definitions of the GT power series in [IP5, Section 1] and of the key convolution product in [IP5, Section 10]; the latter can be resolved in general only at the cost of dropping any refinements to the usual relative GW-invariants arising from rim tori. However, our analysis of the general situation in Section 3.2 establishes the following application of the refined relative invariants of [IP4] as a corollary of the approach in [IP5].

Theorem 1.2. *Let (X,ω_X) and (Y,ω_Y) be symplectic manifolds, $V\subset X,Y$ be a connected symplectic hypersurface satisfying (1.1), and (A_X,A_Y) be an element of $H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z})$. If the kernels of the homomorphisms*

$$H_2(X-V;\mathbb{Z})\longrightarrow H_2(X;\mathbb{Z}) \qquad \textit{and} \qquad H_2(Y-V;\mathbb{Z})\longrightarrow H_2(Y;\mathbb{Z})$$

induced by the inclusions $X-V\longrightarrow X$ and $Y-V\longrightarrow Y$, respectively, are finite and their orders are prime relative to $A_X\cdot_X V=A_Y\cdot_Y V$, then

$$\mathrm{GT}_{X\#_V Y,C_1,\chi}=\mathrm{GT}_{X\#_V Y,C_2,\chi} \qquad \forall\ C_1,C_2\in A_X\#_V A_Y,\ \chi\in\mathbb{Z}\,.$$

This identity also applies to the descendant invariants of $(X\#_V Y,\omega_\#)$ described in Section 4.1.

Remark 1.3. The conclusions of Theorem 1.2 also hold if $V\subset X,Y$ is virtually connected in the sense of Definition 3.10 and for every connected component V_r of V the subsets

$$\iota^{X-V}_{S_XV*}\Delta_{X,V}\big(H_1(V_r;\mathbb{Z})\big)\subset H_2(X-V;\mathbb{Z}) \qquad \text{and} \qquad \iota^{Y-V}_{S_YV*}\Delta_{Y,V}\big(H_1(V_r;\mathbb{Z})\big)\subset H_2(Y-V;\mathbb{Z})$$

are finite and their orders are prime relative to $A_X\cdot_X V_r=A_Y\cdot_Y V_r$; see the beginning of Section 3.1 for the notation.

Finally, the deficiency in question is at most minor in the Kahler category, as stated in Proposition 3.17, and possibly even in the symplectic category, as indicated by the following conjecture.

Conjecture 1.4. Let (X,ω_X) and (Y,ω_X) be symplectic manifolds and $(Z,\omega_Z)=(X\#_V Y,\omega_\#)$ be their symplectic sum along a symplectic hypersurface $V\subset X,Y$ satisfying (1.1). If $C_1,C_2\in H_2(Z;\mathbb{Z})$ are such that $C_1-C_2\in\mathcal{R}^V_{X,Y}$ and $\mathrm{GW}_{Z,g_1,C_1},\mathrm{GW}_{Z,g_2,C_2}\neq 0$ for some $g_1,g_2\in\mathbb{Z}^{\ge 0}$, then C_1-C_2 is a torsion class.

Families of curves in the smooth fibers $\mathcal{Z}_\lambda=X\#_V Y$ of the fibration $\pi\colon\mathcal{Z}\longrightarrow\Delta$ can limit, as $\lambda\longrightarrow 0$, to a curve in the singular fiber $\mathcal{Z}_0=X\cup_V Y$ with some components contained in the divisor V. The S-matrix in the symplectic sum formula of [IP5] is intended to account for such components of the limiting curves by viewing them as curves in the "rubber", a union of a finite number of copies of

$$\mathbb{P}_X V\equiv\mathbb{P}(\mathcal{N}_X V\oplus\mathcal{O}_V)\approx\mathbb{P}(\mathcal{O}_V\oplus\mathcal{N}_Y V)\equiv\mathbb{P}_Y V, \tag{1.13}$$

where $\mathcal{O}_V\longrightarrow V$ is the trivial complex line bundle. Such curves also appear as limits of relative maps into (X,V) in [IP4], but only up to the natural action of $\mathbb{C}^*$ on each $\mathbb{P}_X V$; for this reason, moduli spaces of such limits have lower (virtual) dimensions than the corresponding moduli space of smooth relative maps, after a suitable regularization, and thus do not contribute to the relative invariants of (X,V). By the same reasoning as in [IP4], the components of limits of curves in $\mathcal{Z}$ that map to V should be viewed as $\mathbb{C}^*$-equivalence classes of curves in $\mathbb{P}_X V$; moduli spaces of such limits have lower (virtual) dimensions than the corresponding moduli space of maps without irreducible components contained in V, after a suitable regularization, and thus have no effect on the symplectic sum formula. Even without taking the $\mathbb{C}^*$-equivalence classes, the effect of the spaces of maps with non-trivial rubber components on the action of the S-matrix in the main decomposition formulas in [IP5], (0.2) and (10.4), is to produce 0-dimensional sets (after cutting down by all possible constraints) on which $\mathbb{C}^*$ acts non-trivially; these sets are thus empty. It follows that the maps with rubber components have no effect on the action of the S-matrix in the symplectic sum formulas in [IP5] and so the S-matrix acts as if it were the identity; this is not observed in [IP5] either. We discuss the situation with the S-matrix in more detail in Section 5.5.

Section 2 summarizes the key issues with [IP4, IP5] and [LR] and directs the reader to the portions of this manuscript where they are analyzed in detail. Section 3 deals with topological preliminaries related to the symplectic sum formulas in GW-theory. In particular, we describe changes in the topology of manifolds under surgery, analyze obstructions to constructing rim tori refinements of relative invariants, and finally review the symplectic sum construction of [Gf, MW]. A symplectic sum formula is formulated in Section 4.1; see Theorem 4.1. The notions of relative stable maps and relative invariants are reviewed in Sections 4.2 and 4.3, respectively. Section 5 reviews the arguments of [IP5] and [LR] that are intended to establish symplectic sum formulas and outline how to complete them. The power of these formulas for Gromov-Witten invariants is illustrated in Section 6, based on the applications described in [IP5]. For the reader's convenience, we include detailed lists of typos/misstatements in [IP4, IP5] and [LR]. The references in this manuscript are labeled as in [IP5], whenever possible.

2 Summary of issues with [IP4, IP5] and [LR]

In this section, we summarize our understanding of the key problems with the arguments in [IP4, IP5] and [LR] and direct the reader to the portions of this manuscript where they are analyzed in detail. The problems in [IP4, IP5] and [LR] are of very different nature. The arguments in [IP4, IP5] are generally very concrete, often highly technical, and aim to completely address all relevant issues, but go wrong in several specific places and in particular do not deal correctly with the crucial gluing issues (see (IPa6)-(IPa11) below), which were the main problems that needed to be addressed. In contrast, [LR] introduces the beautiful idea of stretching the target in the normal direction to the divisor V, which had been previously used in contact geometry by others and fits naturally with the relevant gluing issues in the symplectic sum setting. Unfortunately, [LR] makes hardly any reasonably precise statements, either when defining the key objects, specifying the questions to be addressed, or proving the key claims, even in special cases (to which many sketches of the arguments in [LR] are restricted).

A notable exception from the generally precise nature of [IP4, IP5] is the abstract, the long summary, and the main theorems in [IP5], i.e. *Symplectic Sum Theorem* and Theorems 10.6 and 12.3. These suggest that the symplectic sum formulas in [IP5] are proved without any restrictions on X, Y, V, but the arguments are clearly restricted to the "semi-positive" case; see Section 4.3 for more details. The remaining issues with [IP4, IP5] are either of topological flavor or of technical, analytic nature; we describe them below.

We begin with problems of topological flavor in [IP4, IP5].

(IPt1) The refined relative invariants of (X, V) are obtained in [IP4] by lifting the relative evaluation morphism ev^V in (1.4) over a covering $\mathcal{H}_{X;\mathbf{s}}^V$ of $V_{\mathbf{s}}$. Such a covering is described set-theoretically in [IP4, Section 5] without formally specifying a topology on $\mathcal{H}_{X;\mathbf{s}}^V$ or showing that ev^V actually lifts. The standard way of doing both is to specify subgroups of the fundamental groups of the topological components of $V_{\mathbf{s}}$ and to compare them with the appropriate images of the fundamental groups of the components of the domain of ev^V. The informal sketch at the end of [IP4, Section 5] captures the situation reasonably well when V is connected. However, we show in Section 3.2 that ev^V generally does not lift to any non-trivial cover of $V_{\mathbf{s}}$ if V is not connected.

(IPt2) Even if ev^V lifts to $\mathcal{H}^V_{X;\mathbf{s}}$, the lift is not unique. If the greatest common divisor of the components of $\mathbf{s}$ is 1, the different lifts cannot be distinguished homologically. This implies that the refined relative invariants, even if defined, cannot be used to homologically distinguish the topological components $\mathcal{H}^{V;C}_{X,Y;\mathbf{s}}$ of the fiber product $\mathcal{H}^V_{X,Y;\mathbf{s}}$ of $\mathcal{H}^V_{X;\mathbf{s}}$ and $\mathcal{H}^V_{Y;\mathbf{s}}$. Thus, in general, a symplectic sum formula cannot be used to solve for the GW-invariants of $X\#_V Y$ completely in terms of the relative invariants of (X,V) and (Y,V). The product formulas in the middle of page 993 of [IP5], which are needed to separate contributions to GW-invariants of $X\#_V Y$ in classes differing by rim tori, require taking Poincare duals in possibly non-compact manifolds and thus are not really defined; see Section 3.2 for more details.

(IPt3) As explained in the summary and in Section 12 in [IP5], the S-matrix appears in the main formulas (0.2) and (12.7) of [IP5] due to components of limiting maps sinking into V. As we explain in Section 5.5, such components correspond to maps into $\mathbb{P}_X V = \mathbb{P}_Y V$ only up to the $\mathbb{C}^*$-action on the target, just as happens in the relative maps setting of [IP4, Section 7]. This action, which is forgotten in the imprecise limiting argument of [IP5, Section 12], implies that such limits do not contribute to the GW-invariants of $X\#_V Y$ for dimensional reasons, and so the S-matrix should not appear in any symplectic sum formula of [IP5]. As we also show in Section 5.5, the S-matrix does not matter anyway because it *acts as* the identity in all cases and not just in the cases considered in [IP5, Sections 14,15], when the S-matrix *is* the identity.

(IPt4) The main symplectic sum formulas in [IP5] involve generating series defined by exponentiating homology classes on $\overline{\mathcal{M}}_{g,n} \times \mathcal{H}^V_{X;\mathbf{s}}$ without an explanation of how these exponentials are defined. The use of $\mathcal{H}^V_{X;\mathbf{s}}$ in place of $V_{\mathbf{s}}$ makes defining such exponentials particularly difficult, even in the case of primary insertions (as in Theorem 1.1). If descendant insertions are also used (as in Theorem 4.1), a symplectic sum formula must incorporate some version of rules of assignment of [Lj2]. Finally, the normalizations of the generating series for the absolute and relative GW-invariants in [IP5] are not the same, which makes them incompatible with the stated symplectic sum formulas. These issues are discussed in detail in Section 4.1.

(IPt5) The extension of the symplectic sum formula to arbitrary cohomology insertions in [IP5, Section 13] is not well-defined; see Section 4.1.

(IPt6) While it is not stated in the assumptions for [IP5, (0.2),(10.14),(12.7)], the proof of these main formulas in [IP5] is restricted to a fairly narrow range of semi-positive cases, which are never correctly specified; see Section 4.3 and in particular the paragraph before Remark 4.9.

We next list problems of analytic nature in [IP4, IP5]; these concern fairly technical, but at the same time very specific, points.

(IPa1) The index of the linearization of the $\bar{\partial}$-operator at a V-regular map u described below [IP4, (6.2)] is lower than the desired index, given by [IP4, (6.2)], while the index of the linearization described at the beginning of [IP5, Section 7] is typically higher than the desired one. As a result, a transverse claim is made about a wrong bundle section in [IP4, Section 6]; see Remark 5.12 for more details.

(IPa2) The rescaling arguments of [IP5, Sections 6,7] do not involve adding new components to the domain of a map to X. They cannot lead to limiting maps such that some of the component maps into a rubber level are stable and some are unstable; see Remark 4.4 for more details.

(IPa3) The gluing constructions of [IP5, Sections 6-9] claim uniform estimates along each stratum, which are not established even when restricting to δ-flat maps. The first failure of uniformity occurs on the level of curves, essentially because the construction above [IP5, Remark 4.1] need not extend outside of the open strata $\mathcal{N}_\ell$; see Remark 5.2 for more details. The second failure occurs on the level of maps because the extra bubbling can occur away from the nodes on the divisor and because the construction requires stabilizing the domains as in [IP5, Remark 1.1], which can be done only locally. The statement about the linearized operator being Fredholm for a generic δ in the second paragraph of page 976 in [IP5] pretty much rules out any possibility for uniform estimates across whole strata. However, such uniform estimates along entire strata are not necessary and seem unrealistic especially in situations requiring a virtual fundamental class construction, while uniform estimates along compact subsets of open strata are much easier to establish. This implies that the top arrow in [IP5, (10.3)] is defined only after restricting to the preimage of a compact subset K and for λ sufficiently small (depending on K); see Remarks 5.2 and 5.10 for more details.

(IPa4) The uniform control of the C^0-norm by the L^p_1-norm claimed in [IP5, Remark 6.6] requires a justification because the domains C_μ change (which is not an issue) and the metric on the targets $\mathcal{Z}_\lambda$ degenerates; see Section 5.2 for more details.

(IPa5) The proof of [IP5, Lemma 6.9] ignores two of the three components of the map $F-f$ as in (6.14). The actual estimate is weaker, but good enough; see Section 5.2 for more details.

(IPa6) The operator in [IP5, (7.5)] is not the adjoint of the operator in [IP5, (7.4)] with respect to any inner-product, because the first component of its image does not satisfy the average condition. This ruins the argument regarding the linearized operators being uniformly invertible, which is the main point of the analytic part of [IP5], at the start; see Section 5.4 for more details.

(IPa7) Gauss's relation for curvatures, [IP5, (8.7)], is written in a rather peculiar way, resulting in a sign error. This appears to be what is referred to as a Bochner formula on page 939 of [IP5]. The sign error in [IP5, (8.7)] is crucial to establishing a uniform bound on the incorrect adjoint operator in [IP5, (7.5)]; see Remark 5.13 for more details.

(IPa8) The argument at the bottom of page 984 in [IP5] implicitly presupposes that the limiting element η lies in the Sobolev space $L^{1,2}_{\mathbf{s}}$; see Section 5.3 for more details.

(IPa9) The justification for the uniform elliptic estimate in [IP5, Lemma 8.5] indicates why the degeneration of the domains does not cause a problem, but makes no comment about the degeneration of the target. It is unclear that it is in fact uniform; see Section 5.3 for more details.

(IPa10) The map Φ_λ in [IP5, Proposition 9.1] appears to be non-injective because the metrics on the target $\mathcal{Z}_\lambda$ collapse in the normal direction to the divisor V as $\lambda \longrightarrow 0$. The wording of the second-to-last paragraph on page 938 suggests that the norms are weighted to account for this collapse and the convergence estimate of [IP5, Lemma 5.4] could accommodate norms weighted heavier in the vertical direction, but the rather light weights in the norms of [IP5, Definition 6.5] appear far from sufficient. We discuss this issue in Section 5.4.

(IPa11) Neither the summary of [IP5] nor the proof of [IP5, Proposition] makes any mention of whether the quadratic error term in the expansion [IP5, (9.10)] of the $\bar{\partial}$-operator is uniformly bounded. The former mentions only the need for the 0-th and 1-st order terms to be uniform (in (a) and (b) on page 939).

(IPa12) In order to define relative invariants and prove a symplectic sum formula without any semi-positivity restrictions via known techniques, it is necessary to describe a gluing procedure for maps involving rubber components; see Section 4.2. This involves two issues not encountered in gluing rubber-free maps into $X \cup_V Y$:

(RG1) the component maps into each rubber level are defined only up to $\mathbb{C}^*$-action;

(RG2) the natural generalization of the gluing construction for maps to $X \cup_V Y$ would send maps with rubber to an isomorphic, but not identical, space (see Section 5.4).

Since [IP4] and [IP5] are restricted to the semi-positive case, these two issues do not need to arise. However, because of (IPt3), gluing of maps to rubber still needs to be considered, and so the second issue above still arises.

We next summarize our comments on [LR].

(LR1) The symplectic sum formula (for primary invariants only) in [LR] is spread out between three formulas in Section 5, one of which is incorrect as stated; see Section 4.1.

(LR2) Definition 3.14 in [LR] of the key notion of relative stable map is not remotely precise. For example, it is not specific about the relation between the two different domains of the map or the equivalence relation; see Section 4.2.

(LR3) In addition to being imprecise, Definition 3.18 in [LR] of the key notion of stable map to $X \cup_V Y$ ($\overline{M}^+ \cup_D \overline{M}^-$ in the notation of [LR]) is incorrect, as it separates the rubber components into X and Y-parts; see Section 4.2.

(LR4) The proof of [LR, Proposition 3.4] is based on an infinite-dimensional version of the Morse lemma, for which no justification or citation is provided. The desired conclusion of this Morse lemma involves the inner-product [LR, (3.14)] with respect to which the domain $W^1_r(S^1, SV)$ is not even complete.

(LR5) The statement of [LR, Theorem 3.7] is incorrect. It describes the asymptotic behavior of J-holomorphic maps from $\mathbb{C}$, but what is needed to establish compactness in [LR, Section 3.2] and pregluing estimates in [LR, Section 4.1] is its analogue for maps from the punctured disk. The 4-5 page justification of [LR, Theorem 3.7], which is one of only three somewhat technical arguments in the paper, includes [LR, Proposition 3.4] and circular reasoning. The correct, required version can be justified in a few lines and the elaborate sup energy of [H] can be avoided in the present situation; see Section 5.1.

(LR6) The compactness argument of [LR, Section 3.2] is vague on the targets of the relevant sequences of maps and does not even consider marked maps. It also involves one node at a time and thus does not lead to the kinds of maps described in (IPa2) either. Furthermore, the statement of [LR, Lemma 3.12] explicitly rules out "contracted" rubber maps from stable domains with only one puncture/node at one of the divisors. These issues are described in more detail in Section 4.2.

(LR7) The relative gluing issues, (RG1) and (RG2) above, are not addressed in [LR] either, even in the special, one-node, case considered in [LR, Section 4.1]. The gluing construction of [LR, Section 4.1] for relative maps involves a specific representative of a map to the rubber (not up to the $\mathbb{C}^*$-action on the target) and defines the target of the glued map in a way which depends on the gluing parameter. These issues are fundamental to [LR], in contrast to [IP5], because the former does not impose any semi-positivity conditions. We discuss them in more detail in Sections 4.3 and 5.2.

(LR8) Neither the injectivity nor surjectivity of the gluing construction of [LR, Section 4.1] is even mentioned; in light of (LR7), this would be impossible to do. Some version of [IP5, Sections 4,5] is a necessary preliminary to handle these issues. Both properties are implicitly used in the proof of [LR, Proposition 4.10].

(LR9) The proof of [LR, Proposition 4.10] applies the Implicit Function Theorem in an infinite-dimensional setting without any mention of the needed bounds on the 0-th and 1-st order terms and the quadratic correction term. The first two are the subject of the preceding section, but there is no mention of uniform estimates on the last one anywhere in [LR]; see Section 5.4.

(LR10) The VFC approach of [LR] is based on a global regularization of the moduli space using the twisted dualizing sheaf introduced after [LR, Lemma 4.4]. It is treated as a line bundle over the entire moduli space with Sobolev norms on its sections, without any explanation. The 3-4 pages dedicated to this line bundle in [LR, Sections 4.1,4.2] could be avoided by using the local VFC approach of [FO] or [LT].

(LR11) The regularization of maps in [LR, Sections 4.1,4.2] needs to respect the $\mathbb{C}^*$-action on maps to the rubber; this issue is not even mentioned in [LR].

(LR12) The discussion of gluing for maps to $X \cup_V Y$, which is needed to establish a symplectic sum formula, consists of a few lines after [LR, Lemma 5.4]. There is no explanation of the crucial multiplicity coefficient k ($\langle \mathbf{s} \rangle$ in our notation) appearing in [LR, Theorem 5.7]. The domain and target gluing formulas [LR, (4.12)-(4.15)] hint at this coefficient, but barely so even in the case of one node. If the rubber components are present, these multiplicities no longer show up directly; the argument in [Lj2] obtaining them on the level of homology classes (rather than numbers) is pretty delicate and involves passing to a desingularization. Because of the much more limited scope of [IP5], this issue is not relevant for [IP5]. In contrast to [IP5], [LR] does not even clearly describe the general setup. In particular, the one-node case considered in [LR] as supposedly capturing all the issues in the general case cannot be representative of the general case because the target of the glued maps, described by [LR, (4.12),(4.13)], depends on the gluing parameter associated with each node. Thus, these parameters must be chosen systematically, which is done for rubber-free maps in [IP5] and becomes more complicated for general maps; see Section 5.2.

(LR13) The most technical part of [LR], roughly 4 pages, concerns the variation of various operators in Section 4.1 with respect to the norm r of the gluing parameter (r), which is considered without explicitly identifying the domains and targets of these operators. This part is used only to show that the integrals [LR, (4.50)] defining relative invariants converge. However, this is not necessary, since the relevant evaluation morphisms had supposedly been shown to be rational pseudocycles before then (and thus define invariants by intersection as in [MS2, Section 7.1] and [RT1, Section 1]).

3 Topological preliminaries

The symplectic sum construction is a surgery operation along a submanifold of codimension two. We discuss changes in the topology of a manifold under surgeries along arbitrary compact oriented submanifolds in Section 3.1 and specialize to the symplectic sum setting in Section 3.2. Lemma 3.1 contains [IP4, Lemma 5.2] and the corresponding part of the proof of the former is essentially the same as the proof of the latter. Lemma 3.7 contains the precise statements of (a) and (b) on page 996 of [IP5]. The remaining statements in Section 3.1 are not in [IP4] or [IP5]. Some of these statements are useful in determining when the module $\mathcal{R}_{X,Y}^V$ defined in (1.6) vanishes, so that the gluing operation of (1.5) is well-defined on the level of homology. In Section 3.2, we recall the twisting construction of [IP4] aimed at bypassing the issue of the gluing operation not being well-defined in general, correct some statements in [IP4], and include some examples. In Section 3.3, we review the symplectic sum construction from the points of view of [IP5] and [LR].

3.1 Some general observations

Given a smooth manifold X and a smooth submanifold $V\subset X$, denote by $S_XV\subset\mathcal{N}_XV$ the sphere subbundle of the normal bundle of V in X, which we will view as a hypersurface in X, and by

$$\iota_{V*}^X\colon H_*(V;\mathbb{Z})\longrightarrow H_*(X;\mathbb{Z})$$

the homomorphism induced by the inclusion $\iota_V^X : V\longrightarrow X$. If in addition X and V are compact oriented and the codimension of V in X is $\mathfrak{c}$, we define

$$\begin{aligned} \Delta_{X,V}\colon H_*(V;\mathbb{Z})&\longrightarrow H_{*+\mathfrak{c}-1}(S_XV;\mathbb{Z}), &\qquad \Delta_{X,V}(A)&=\mathrm{PD}_{S_XV}\big(q_{X,V}^*(\mathrm{PD}_V(A))\big),\\ \cap V\colon H_*(X;\mathbb{Z})&\longrightarrow H_{*-\mathfrak{c}}(V;\mathbb{Z}), & A\cap V&=\mathrm{PD}_V\big((\mathrm{PD}_XA)|_V\big), \end{aligned} \tag{3.1}$$

where $q_{X,V}\colon S_XV\longrightarrow V$ is the projection map.

Lemma 3.1. *If X is a compact oriented manifold and $V\subset X$ is a compact oriented submanifold of codimension $\mathfrak{c}$, the sequence*

$$\ldots\longrightarrow H_m(X-V;\mathbb{Z})\xrightarrow{\iota_{X-V*}^X} H_m(X;\mathbb{Z})\xrightarrow{\cap V} H_{m-\mathfrak{c}}(V;\mathbb{Z})\xrightarrow{\iota_{S_XV*}^{X-V}\circ\Delta_{X,V}} H_{m-1}(X-V;\mathbb{Z})\longrightarrow\ldots$$

is exact.

Proof. Taking the Poincare dual of the Gysin sequence for $S_XV\longrightarrow V$, we obtain an exact sequence

$$\begin{aligned} \ldots&\xrightarrow{\Delta_{X,V}} H_m(S_XV;\mathbb{Z})\xrightarrow{q_{X,V*}} H_m(V;\mathbb{Z})\xrightarrow{e(\mathcal{N}_XV)\cap} H_{m-\mathfrak{c}}(V;\mathbb{Z})\\ &\xrightarrow{\Delta_{X,V}} H_{m-1}(S_XV;\mathbb{Z})\xrightarrow{q_{X,V*}}\ldots. \end{aligned} \tag{3.2}$$

By the proof of Mayer-Vietoris for $X=(X-V)\cup_{S_XV}V$,

$$\begin{aligned} \ldots &\xrightarrow{\delta_X} H_m(S_XV;\mathbb{Z}) \xrightarrow{(\iota^{X-V}_{S_XV*},-q_{X,V*})} H_m(X-V;\mathbb{Z})\oplus H_m(V;\mathbb{Z}) \xrightarrow{\iota^X_{X-V*}+\iota^X_{V*}} H_m(X;\mathbb{Z}) \\ &\xrightarrow{\delta_X} H_{m-1}(S_XV;\mathbb{Z}) \xrightarrow{(\iota^{X-V}_{S_XV*},-q_{X,V*})} \ldots, \end{aligned} \tag{3.3}$$

the connecting homomorphism δ_X is the composition $\Delta_{X,V}\circ(V\cap\cdot)$. Since

$$\iota^X_{X-V*}(A)\cap V=0 \qquad \forall\, A\in H_m(X-V;\mathbb{Z}),$$

the claim now follows from the observation that

$$e(\mathcal{N}_XV)\cap A=\iota^X_{V*}(A)\cap V \qquad \forall\, A\in H_m(V;\mathbb{Z}),$$

at least up to sign (dependent on one's definitions of cup and cap products and Poincare dual). □

By [Z1, Theorem 1.1], every integral homology class can be represented by a pseudocycle. If V is as in (3.1) and $f\colon Z\longrightarrow V$ is a pseudocycle, then the pseudocycle

$$\pi_2\colon f^*S_XV\equiv\big\{(z,v)\in Z\times S_XV:\ f(z)=q_{X,V}(v)\big\}\longrightarrow S_XV,$$

where $\pi_2\colon Z\times S_XV\longrightarrow S_XV$ is the projection on the second coordinate, represents $\Delta_{X,V}([f])$. If $Z=S^1$, $f^*S_XV\longrightarrow S^1$ is a trivial $S^{\mathfrak{c}-1}$-bundle and thus f lifts to a map

$$\tilde f\colon S^1\times S^{\mathfrak{c}-1}\longrightarrow q^{-1}_{X,V}(f(S^1))\subset S_XV \qquad \text{s.t.}\quad q_{X,V}\circ\tilde f=f\circ\pi_1\,,$$

where $\pi_1\colon S^1\times S^{\mathfrak{c}-1}\longrightarrow S^1$ is the projection on the first component. Thus, the elements of

$$\mathcal{R}^V_X\equiv\ker\big\{\iota^X_{X-V*}\colon H_{\mathfrak{c}}(X-V;\mathbb{Z})\longrightarrow H_{\mathfrak{c}}(X;\mathbb{Z})\big\} \tag{3.4}$$

can be represented by cycles of the form $\iota^{X-V}_{S_XV}(S_XV|_\gamma)$ for loops $\gamma\subset V$, according to Lemma 3.1. In the $\mathfrak{c}=2$ case, these cycles are called rim tori in [IP4] and [IP5].

Example 3.2. Let Z be a compact oriented manifold and $E_1,E_2\longrightarrow Z$ be complex vector bundles of ranks r_1 and r_2, respectively. By Theorem 9 in [Sp, Section 5.7], there is a commutative diagram

$$\begin{array}{ccc} H_*(\mathbb{P}E_2;\mathbb{Z}) & \longrightarrow & H_*(\mathbb{P}(E_1\oplus E_2);\mathbb{Z}) \\ \approx\downarrow & & \downarrow\approx \\ H_*(Z;\mathbb{Z})\otimes H_*(\mathbb{P}^{r_2-1};\mathbb{Z}) & \longrightarrow & H_*(Z;\mathbb{Z})\otimes H_*(\mathbb{P}^{r_1+r_2-1};\mathbb{Z}) \end{array}$$

of homomorphisms of modules. In particular, the homomorphism

$$\iota^{\mathbb{P}(E_1\oplus E_2)}_{\mathbb{P}(E_1\oplus E_2)-\mathbb{P}E_1}\colon H_{2r_2}\big(\mathbb{P}(E_1\oplus E_2)-\mathbb{P}E_1;\mathbb{Z}\big)\approx H_{2r_2}\big(\mathbb{P}E_2;\mathbb{Z}\big)\longrightarrow H_{2r_2}\big(\mathbb{P}(E_1\oplus E_2);\mathbb{Z}\big)$$

is injective. Thus,

$$\mathcal{R}^{\mathbb{P}E_1}_{\mathbb{P}(E_1\oplus E_2)}=\{0\}\,.$$

In the case $r_2=1$, which is the most relevant for our purposes, this statement follows from $\mathbb{P}E_2$ being a section of the fiber bundle $\mathbb{P}(E_1\oplus E_2)\longrightarrow Z$.

If X and Y are smooth manifolds, $V\subset X,Y$ is a smooth submanifold, and $\varphi\colon S_XV\longrightarrow S_YV$ is a diffeomorphism commuting with the projections to V, let $X\#_\varphi Y$ be the smooth manifold obtained by gluing the complements of tubular neighborhoods of V in X and Y by φ along their common boundary. We denote by

$$q_\varphi\colon X\#_\varphi Y\longrightarrow X\cup_V Y$$

a continuous map which restricts to the identity outside of a tubular neighborhood of $S_XV=_\varphi S_YV$, is a diffeomorphism on the complement of $q_\varphi^{-1}(V)$, and restricts to the bundle projection $S_XV\longrightarrow V$. We will call such a map q_φ a collapsing map. If X and Y are oriented and φ is orientation-reversing, then $X\#_\varphi Y$ is oriented as well.

Lemma 3.3. *If X and Y are smooth compact orientable manifolds, $V\subset X,Y$ is a smooth compact orientable submanifold, $\varphi\colon S_XV\longrightarrow S_YV$ is a diffeomorphism commuting with the projections to V, and $q_\varphi\colon X\#_\varphi Y\longrightarrow X\cup_V Y$ is a collapsing map, then*

$$\begin{aligned}\ker\big\{q_{\varphi*}\colon\ &H_m(X\#_\varphi Y;\mathbb{Z})\longrightarrow H_m(X\cup_\varphi Y;\mathbb{Z})\big\}\\ =\big\{&\iota^{X\#_\varphi Y}_{X-V*}(A_{X-V})+\iota^{X\#_\varphi Y}_{Y-V*}(A_{Y-V})\colon\ A_{X-V}\in H_m(X-V;\mathbb{Z}),\ \iota^X_{X-V*}(A_{X-V})=0,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad A_{Y-V}\in H_m(Y-V;\mathbb{Z}),\ \iota^Y_{Y-V*}(A_{Y-V})=0\big\}\end{aligned}\tag{3.5}$$

for all $m\in\mathbb{Z}$.

Proof. Denote the codimension of V in X and Y by $\mathfrak{c}$, $S_XV\approx S_YV$ by SV, and the bundle projection map $SV\longrightarrow V$ by q_V. Mayer-Vietoris for $X\#_\varphi Y=(X-V)\cup_{SV}(Y-V)$ and $X\cup_V Y$ give a commutative pair of long exact sequences

$$\begin{array}{ccccccc}
H_m(SV;\mathbb{Z}) & \longrightarrow & H_m(X-V;\mathbb{Z})\oplus H_m(Y-V;\mathbb{Z}) & \xrightarrow{\iota^{X\#_\varphi Y}_{X-V*}+\iota^{X\#_\varphi Y}_{Y-V*}} & H_m(X\#_\varphi Y;\mathbb{Z}) & \xrightarrow{\delta_\varphi} & H_{m-1}(SV;\mathbb{Z})\\
\downarrow{\scriptstyle q_{V*}} & & \downarrow{\scriptstyle \iota^X_{X-V*}\oplus\iota^Y_{Y-V*}} & & \downarrow{\scriptstyle q_{\varphi*}} & & \downarrow{\scriptstyle q_{V*}}\\
H_m(V;\mathbb{Z}) & \xrightarrow{(\iota^X_{V*},-\iota^Y_{V*})} & H_m(X;\mathbb{Z})\oplus H_m(Y;\mathbb{Z}) & \xrightarrow{\iota^{X\cup_V Y}_{X*}+\iota^{X\cup_V Y}_{Y*}} & H_m(X\cup_V Y;\mathbb{Z}) & \xrightarrow{\delta_\cup} & H_{m-1}(V;\mathbb{Z})
\end{array}$$

The commutativity of the middle square above implies that the right-hand side of (3.5) is contained in the left-hand side.

Suppose $A_\#\in H_m(X\#_V Y;\mathbb{Z})$ and $q_{\varphi*}(A_\#)=0$. By the proof of Mayer-Vietoris for $X\#_\varphi Y=(X-V)\cup_{SV}(Y-V)$, there exist bordered pseudocycles

$$f_X\colon(Z_X,\partial Z_X)\longrightarrow(X-V,SV)\qquad\text{and}\qquad f_Y\colon(Z_Y,\partial Z_Y)\longrightarrow(Y-V,SV),$$

such that $\partial Z_X=-\partial Z_Y$ and

$$f_X\underset{\partial Z_X=-\partial Z_Y}{\cup}f_Y\colon Z_X\underset{\partial Z_X=-\partial Z_Y}{\cup}Z_Y\longrightarrow X\#_\varphi Y$$

represents the homology class $A_\#$. Since

$$q_{V*}\big[f_X|_{\partial Z_X}\big]=q_{V*}\delta_\varphi(A_\#)=\delta_\cup q_{\varphi*}(A_\#)=0,$$

by (3.2) we can choose f_X and f_Y so that $f_X(\partial Z_X)=f_Y(\partial Z_Y)$ equals $SV|_{B_V}$ for some class $B_V\in H_{m-\mathfrak{c}}(V;\mathbb{Z})$. The smooth maps

$$\iota^X_{X-V}\circ f_X\colon\Sigma_X\longrightarrow X\qquad\text{and}\qquad\iota^Y_{Y-V}\circ f_Y\colon\Sigma_Y\longrightarrow Y$$

then determine homology classes A_X on X and A_Y on Y; in the exceptional $\mathfrak{c}=1$ case, the two boundary components of these maps come with opposite signs and thus cancel. By the commutativity of the diagram on the chain level inducing the above diagram in homology,

$$\iota_{X*}^{X\cup_V Y}[A_X] + \iota_{Y*}^{X\cup_V Y}[A_Y] = q_{\varphi*}(A_\#) = 0.$$

Thus, there exists $A_V \in H_m(V;\mathbb{Z})$ such that

$$A_X = \iota_{V*}^X([A_V]), \qquad A_Y = -\iota_{V*}^Y([A_V]).$$

The Mayer-Vietoris sequence (3.3) then gives

$$\big[f_X|_{\partial Z_X}\big] = \delta_X(A_X) = 0 \qquad \Longrightarrow \qquad \delta_\varphi(A_\#) = \big[f_X|_{\partial Z_X}\big] = 0.$$

The above pair of Mayer-Vietoris sequences now implies that

$$A_\# = \iota_{X-V*}^{X\#_\varphi Y}(A_{X-V}) + \iota_{Y-V*}^{X\#_\varphi Y}(A_{Y-V}) \tag{3.6}$$

for some $A_{X-V} \in H_m(X-V;\mathbb{Z})$ and $A_{Y-V} \in H_m(Y-V;\mathbb{Z})$ and

$$\iota_{X-V*}^X(A_{X-V}) = \iota_{V*}^X([A_V]), \qquad \iota_{Y-V*}^Y(A_{Y-V}) = -\iota_{V*}^Y([A_V]) \tag{3.7}$$

for some $A_V \in H_m(V;\mathbb{Z})$. By (3.3), there exist

$$A_{SV} \in H_m(SV;\mathbb{Z}) \qquad \text{s.t.} \qquad \iota_{SV*}^{X-V}(A_{SV}) = A_{X-V}, \quad q_{V*}(A_{SV}) = A_V. \tag{3.8}$$

By Mayer-Vietoris for $X\#_\varphi Y = (X-V)\cup_{SV}(Y-V)$ and (3.6),

$$\begin{aligned} A_\# &= \iota_{X-V*}^{X\#_\varphi Y}\big(A_{X-V} - \iota_{SV*}^{X-V}(A_{SV})\big) + \iota_{Y-V*}^{X\#_\varphi Y}\big(A_{Y-V} + \iota_{SV*}^{Y-V}(A_{SV})\big) \\ &= \iota_{Y-V*}^{X\#_\varphi Y}\big(A_{Y-V} + \iota_{SV*}^{Y-V}(A_{SV})\big). \end{aligned}$$

By the commutativity of the first square in the above diagram, (3.7), and (3.8),

$$\iota_{Y-V*}^Y\big(A_{Y-V} + \iota_{SV*}^{Y-V}(A_{SV})\big) = 0.$$

Thus, the left-hand side of (3.5) is contained in the right-hand side. □

Let $\mathfrak{c}$ be the codimension of V in X and Y as before and

$$\mathcal{R}_{X,Y}^V \equiv \big\{\iota_{X-V*}^{X\#_\varphi Y}(A_{X-V}) + \iota_{Y-V*}^{X\#_\varphi Y}(A_{Y-V}) \colon\ (A_{X-V}, A_{Y-V}) \in \mathcal{R}_X^V \oplus \mathcal{R}_Y^V\big\}. \tag{3.9}$$

By Lemma 3.3, this definition of $\mathcal{R}_{X,Y}^V$ agrees with (1.6) in the $\mathfrak{c}=2$ case.

Suppose X and V are oriented. For each $\mathfrak{c}$-pseudocycle $f\colon Z \longrightarrow X$ and an isolated point $x \in f^{-1}(V)$, we define the order of contact of f with V at x, $\mathrm{ord}_x^V f \in \mathbb{Z}$, as follows. On a small neighborhood of x, f can be homotoped without changing its intersection with V so that it takes a small sphere $S_Z x$ in $T_x Z$ to a small sphere $S_X V|_{f(x)} \subset \mathcal{N}_V X$; the number $\mathrm{ord}_x^V f$ is the degree of this map. This definition agrees with the definition used in the construction of the space of relative stable maps, $\overline{\mathcal{M}}_{g,k;\mathbf{s}}^V(X,A)$, in Section 4.2. If

$$f_X\colon (Z_X, x_1, \ldots, x_\ell) \longrightarrow (X,V) \qquad \text{and} \qquad f'_X\colon (Z'_X, x'_1, \ldots, x'_\ell) \longrightarrow (X,V)$$

are two $\mathfrak{c}$-pseudocycles such that

$$f_X^{-1}(V)=\{x_1,\ldots,x_\ell\},\qquad f_X'^{-1}(V)=\{x_1',\ldots,x_\ell'\},$$
$$f_X(x_i)=f_X'(x_i'),\quad \mathrm{ord}^V_{x_i}f_X=\mathrm{ord}^V_{x_i'}f_X'\qquad \forall\, i=1,2,\ldots,\ell,$$

we can then obtain a smooth map $f_X\#(-f_X')\colon Z_X\#Z_X'\longrightarrow X-V$ by

- removing small balls B_{x_i} and $B_{x_i'}$ around each of the points x_i and x_i' to form manifolds with boundary $\hat{Z}_X$ and $\hat{Z}_X'$,
- forming a smooth oriented manifold $Z_X\#(-Z_X')$ by identifying the i-th boundary components of $\hat{Z}_X$ and $\hat{Z}_X'$ by an orientation-preserving diffeomorphism $\varphi_i\colon(\partial\hat{Z}_X)_i\longrightarrow(\partial\hat{Z}_X')_i$ for each i,
- homotoping f_X and f_X' on small neighborhoods of ∂B_{x_i} and $\partial B_{x_i'}$ within a small ball around $f_X(x_i)=f_X'(x_i')$ in X so that $f_X=f_X'\circ\varphi_i$ for all i.

The last condition is achievable because the degrees of $f_X, f_X'\circ\varphi_i\colon S^{\mathfrak{c}-1}\longrightarrow S^{\mathfrak{c}-1}$ are the same and the degree homomorphism $\pi_{\mathfrak{c}-1}(S^{\mathfrak{c}-1})\longrightarrow\mathbb{Z}$ is an isomorphism if $\mathfrak{c}\geq 2$. This construction of the map

$$f_X\#(-f_X')\colon Z_X\#(-Z_X')\longrightarrow X-V$$

depends only on f_X, f_X', and choices of degree 1 maps from ℓ disjoint copies of $[0,1]\times S^{\mathfrak{c}-1}$ to $[0,1]\times S^{\mathfrak{c}-1}$; thus, f_X and f_X' completely determine the homology class of $f_X\#(-f_X')$. If in addition $[f_X]=[f_X']$ in $H_{\mathfrak{c}}(X;\mathbb{Z})$, $[f_X\#(-f_X')]\in\mathcal{R}^V_X$; see (3.4).

If X, Y, and V are compact and oriented, let

$$H_{\mathfrak{c}}(X;\mathbb{Z})\times_V H_{\mathfrak{c}}(Y;\mathbb{Z})=\big\{(A_X,A_Y)\in H_{\mathfrak{c}}(X;\mathbb{Z})\times H_{\mathfrak{c}}(Y;\mathbb{Z})\colon\ A_X\cdot_X V=A_Y\cdot_Y V\big\},$$

where $\cdot_X$ and $\cdot_Y$ denote the homology intersection pairings in X and Y, respectively. Given an orientation-reversing diffeomorphism $\varphi\colon S_XV\longrightarrow S_YV$ commuting with the projections to V, we next describe an operation gluing $\mathfrak{c}$-cycles in X and Y into $\mathfrak{c}$-cycles in $X\#_\varphi Y$ and inducing a homomorphism

$$H_{\mathfrak{c}}(X;\mathbb{Z})\times_V H_{\mathfrak{c}}(Y;\mathbb{Z})\longrightarrow H_{\mathfrak{c}}(X\#_\varphi Y;\mathbb{Z})/\mathcal{R}^V_{X,Y},\qquad (A_X,A_Y)\longrightarrow A_X\#_\varphi A_Y. \tag{3.10}$$

Suppose

$$f_X\colon(Z_X,x_1,\ldots,x_\ell)\longrightarrow(X,V)\qquad\text{and}\qquad f_Y\colon(Z_Y,y_1,\ldots,y_\ell)\longrightarrow(Y,V)$$

are $\mathfrak{c}$-pseudocycles with boundary disjoint from V such that

$$f_X^{-1}(V)=\{x_1,\ldots,x_\ell\},\qquad f_Y^{-1}(V)=\{y_1,\ldots,y_\ell\},$$
$$f_X(x_i)=f_Y(y_i),\quad \mathrm{ord}^V_{x_i}f_X=\mathrm{ord}^V_{y_i}f_Y\qquad \forall\, i=1,2,\ldots,\ell;$$

for the purposes of this paper, we could take $\mathfrak{c}=2$ and f_X and f_Y to be J-holomorphic maps from Riemann surfaces. We can then obtain a smooth map $f_X\#_\varphi f_Y\colon Z_X\#Z_Y\longrightarrow X\#_\varphi Y$ by

- removing small balls B_{x_i} and B_{y_i} around each of the points x_i and y_i to form manifolds with boundary $\hat{Z}_X$ and $\hat{Z}_Y$,

- forming a smooth oriented manifold $Z_X\#Z_Y$ by identifying the i-th boundary components of $\hat{Z}_X$ and $\hat{Z}_Y$ by an orientation-reversing diffeomorphism $\varphi_i\colon (\partial\hat{Z}_X)_i\longrightarrow(\partial\hat{Z}_Y)_i$ for each i,
- homotoping f_X and f_Y on small neighborhoods of ∂B_{x_i} and ∂B_{y_i} within small balls around $f_X(x_i)$ in X and $f_Y(y_i)$ in Y so that $\varphi\circ f_X=f_Y\circ\varphi_i$ for all i.

The last condition is achievable because the degrees of

$$\varphi\circ f_X\circ\varphi_i^{-1}, f_Y\colon S^{\mathfrak c-1}\longrightarrow S^{\mathfrak c-1}$$

are the same. This construction of the map

$$f_X\#_\varphi f_Y\colon Z_X\#Z_Y\longrightarrow X\#_\varphi Y$$

depends only on f_X, f_Y, and choices of degree -1 maps from ℓ disjoint copies of $[0,1]\times S^{\mathfrak c-1}$ to $[0,1]\times S^{\mathfrak c-1}$; thus, f_X and f_Y completely determine the homology class of $f_X\#_\varphi f_Y$.

If $[f_X]=[f_X']$ in $H_{\mathfrak c}(X;\mathbb Z)$, $[f_X\#(-f_X')]\in\mathcal R_X^V$ by the paragraph above the previous one. Thus, the homology class of $f_X\#_\varphi f_Y$ in $X\#_\varphi Y$ as above is determined by the homology classes of f_X in X and f_Y in Y only up to an element of $\mathcal R_{X,Y}^V$; see (3.9). The rim tori refinement to the usual relative invariants is introduced in [IP4, Section 5] with the aim of dealing with this problem. The following corollary is useful in determining $\mathcal R_{X,Y}^V$.

Corollary 3.4. *Let X and Y be smooth compact oriented manifolds, $V\subset X,Y$ be a smooth compact oriented submanifold of codimension $\mathfrak c$, and $\varphi\colon S_XV\longrightarrow S_YV$ be an orientation-reversing diffeomorphism commuting with the projections to V.*

(1) For every $(A_X,A_Y)\in H_{\mathfrak c}(X;\mathbb Z)\times_V H_{\mathfrak c}(Y;\mathbb Z)$,

$$q_{\varphi*}\big(A_X\#_\varphi A_Y\big)=\iota_{X*}^{X\cup_V Y}(A_X)+\iota_{Y*}^{X\cup_V Y}(A_Y)\in H_{\mathfrak c}(X\cup_\varphi Y;\mathbb Z).$$

(2) The subgroup $\mathcal R_{X,Y}^V\subset H_{\mathfrak c}(X\#_\varphi Y;\mathbb Z)$ is isomorphic to the cokernel of the homomorphism

$$\{q_{V*}\}^{-1}\big(\ker\iota_{V*}^X\cap\ker\iota_{V*}^Y\big)\longrightarrow\mathcal R_X^V\oplus\mathcal R_Y^V,\qquad A_{SV}\longrightarrow\big(\iota_{SV*}^{X-V}(A_{SV}),\iota_{SV*}^{Y-V}(A_{SV})\big),\tag{3.11}$$

where SV is the sphere bundle $S_XV\approx S_YV$ and $q_V\colon SV\longrightarrow V$ is the bundle projection.

(3) If either of the homomorphisms

$$\iota_{V*}^X\colon H_{\mathfrak c}(V;\mathbb Z)\longrightarrow H_{\mathfrak c}(X;\mathbb Z)\qquad\text{or}\qquad\iota_{V*}^Y\colon H_{\mathfrak c}(V;\mathbb Z)\longrightarrow H_{\mathfrak c}(Y;\mathbb Z)$$

is injective, $\mathcal R_{X,Y}^V$ is isomorphic to the cokernel of the homomorphism

$$H_1(V;\mathbb Z)\longrightarrow\mathcal R_X^V\oplus\mathcal R_Y^V,\qquad\gamma\longrightarrow\big(\iota_{SV*}^{X-V}(\Delta_{X,V}(\gamma)),\iota_{SV*}^{Y-V}(\Delta_{Y,V}(\gamma))\big),$$

where $\Delta_{X,V}$ and $\Delta_{Y,V}$ are the homomorphisms as in (3.1).

(4) If either $\mathcal R_X^V=\{0\}$ or $\mathcal R_Y^V=\{0\}$, the homomorphism

$$\#\colon H_{\mathfrak c}(X;\mathbb Z)\times_V H_{\mathfrak c}(Y;\mathbb Z)\longrightarrow H_{\mathfrak c}(X\#_\varphi Y;\mathbb Z),\quad(A_X,A_Y)\longrightarrow A_X\#_\varphi A_Y,\tag{3.12}$$

induced by gluing representatives of homology classes along V, is well-defined.

Proof. (1) This claim is immediate from the construction of $A_X\#_\varphi A_Y$.

(2) By (3.9), this statement is equivalent to

$$\begin{split}\big\{\big(\iota_{SV*}^{X-V}(A_{SV}),\iota_{SV*}^{Y-V}(A_{SV})\big)\colon\ A_{SV}\in H_{\mathfrak c}(SV;\mathbb Z),\ q_{V*}(A_{SV})\in\ker\iota_{V*}^X\cap\ker\iota_{V*}^Y\big\}\\ =\big\{(A_{X-V},A_{Y-V})\in\mathcal R_X^V\oplus\mathcal R_Y^V\colon\ \iota_{X-V*}^{X\#_\varphi Y}(A_{X-V})-\iota_{Y-V*}^{X\#_\varphi Y}(A_{Y-V})=0\big\}.\end{split}\tag{3.13}$$

Let $A_{X-V}\in H_{\mathfrak c}(X-V;\mathbb Z)$ and $A_{Y-V}\in H_{\mathfrak c}(Y-V;\mathbb Z)$. By the Mayer-Vietoris sequence for $X\#_\varphi Y=(X-V)\cup_{SV}(Y-V)$ in the proof of Lemma 3.3,

$$\begin{gathered}\iota_{X-V*}^{X\#_\varphi Y}(A_{X-V})-\iota_{Y-V*}^{X\#_\varphi Y}(A_{Y-V})=0\qquad\Longleftrightarrow\\ (A_{X-V},A_{Y-V})=\big(\iota_{SV*}^{X-V}(A_{SV}),\iota_{SV*}^{Y-V}(A_{SV})\big)\quad\text{for some } A_{SV}\in H_{\mathfrak c}(SV;\mathbb Z).\end{gathered}$$

For any $A_{SV}\in H_{\mathfrak c}(SV;\mathbb Z)$, the commutativity of the first square in the diagram of short exact sequences in the proof of Lemma 3.3 implies that

$$\big(\iota_{SV*}^{X-V}(A_{SV}),\iota_{SV*}^{Y-V}(A_{SV})\big)\in\mathcal R_X^V\oplus\mathcal R_Y^V\qquad\Longleftrightarrow\qquad q_{V*}(A_{SV})\in\ker\iota_{V*}^X\cap\ker\iota_{V*}^Y.$$

The last two statements give (3.13).

(3) If either ι_{V*}^X or ι_{V*}^Y is injective on the k-th homology,

$$\{q_{V*}\}^{-1}\big(\ker\iota_{V*}^X\cap\ker\iota_{V*}^Y\big)=\ker q_{V*}.$$

Thus, the third statement of Corollary 3.4 follows from the second and the exactness of the Poincare dual of the Gysin sequence for $q_V\colon SV\longrightarrow V$ as in (3.2).

(4) By (3.3), for each $A_{X-V}\in\mathcal R_X^V$ there exists $A_{SV}\in H_{\mathfrak c}(SV;\mathbb Z)$ such that

$$A_{X-V}=\iota_{SV*}^{X-V}(A_{SV})\qquad\text{and}\qquad q_{V*}(A_{SV})=0.$$

By the exactness of the Mayer-Vietoris sequence for $Y=(Y-V)\cup_{SV}V$, $\iota_{SV*}^{Y-V}(A_{SV})\in\mathcal R_Y^V$. Thus, if $\mathcal R_Y^V=\{0\}$, the homomorphism (3.11) is surjective, and the last statement of Corollary 3.4 follows from the second. □

Example 3.5. If $X=S^2\times\mathbb T^2$ and $V=\{0,\infty\}\times\mathbb T^2$,

$$\mathcal R_X^V=\ker\big\{H_2(\mathbb C^*\times\mathbb T^2;\mathbb Z)\longrightarrow H_2(S^2\times\mathbb T^2;\mathbb Z)\big\}\approx H_1(\mathbb C^*;\mathbb Z)\otimes H_1(\mathbb T^2;\mathbb Z)\approx\mathbb Z^2.$$

From Corollary 3.4(2), we then conclude that $\mathcal R_{X,X}^V\approx\mathbb Z^2$.

Example 3.6. Suppose X is an oriented manifold and $Z\subset X$ is a compact submanifold so that the normal bundle $\mathcal N_XZ$ admits a complex structure. Fix a complex structure in $\mathcal N_XZ$ and an identification of the unit disk bundle $D(\mathcal N_XZ)$ of $\mathcal N_XZ$ with a neighborhood of Z in X. Let

$$\mathbb P_XZ=\mathbb P(\mathcal N_XZ\times\mathbb C)=\mathbb P(\mathcal N_XZ\oplus Z\times\mathbb C),\qquad V=\mathbb P(\mathcal N_XZ)\subset\mathbb P(\mathcal N_XZ\times\mathbb C),$$

and Bl_ZX be the manifold obtained from X by replacing $D(\mathcal N_XZ)\subset X$ with the disk bundle of the complex tautological line bundle $\gamma\longrightarrow V$ (which has the same boundary consisting of the unit vectors in $\mathcal N_XV$). Thus,

$$\mathcal N_{\mathrm{Bl}_ZX}V=\gamma,\qquad\mathcal N_{\mathbb P_XZ}V=\gamma^*,\quad\text{and}\quad X=\mathrm{Bl}_ZX\#_\varphi\mathbb P_XZ\,,$$

for an orientation-reversing diffeomorphism $\varphi\colon S_{\mathrm{Bl}_Z X}V \longrightarrow S_{\mathbb{P}_X Z}V$ induced by the canonical isomorphism $\gamma\otimes\gamma^* = V\times\mathbb{C}$ (e.g. $\{\varphi(v)\}v=1$ for all $v\in S_{\mathrm{Bl}_Z X}V$). By Corollary 3.4(4) and Example 3.2,

$$\mathcal{R}^V_{\mathrm{Bl}_Z X, \mathbb{P}_X Z} = \{0\},$$

i.e. there are no rim tori in this case. A geometric reasoning for this conclusion is given in the proof of [LR, Lemma 2.11]. If (X,ω) is a symplectic manifold and $Z\subset X$ is a symplectic submanifold, the construction of [MS1, Section 7.1] endows $\mathrm{Bl}_Z X$ with a symplectic form $\omega_{Z,\epsilon}$; $(\mathrm{Bl}_Z X, \omega_{Z,\epsilon})$ is then called a symplectic blowup of (X,ω) along Z.

By Lemma 3.3 and (3.3),

$$\begin{split}\ker\big\{q_{\varphi*}\colon\, H_m(X\#_\varphi Y;\mathbb{Z})\longrightarrow H_m(X\cup_V Y;\mathbb{Z})\big\}\qquad\qquad\qquad\qquad\\ =\big\{\iota^{X\#_\varphi Y}_{SV*}(A_{SV})\colon\, A_{SV}\in\ker\{q_V\colon H_m(SV;\mathbb{Z})\longrightarrow H_m(V;\mathbb{Z})\}\big\}.\end{split}\tag{3.14}$$

Lemma 3.7 below, which describes cohomology classes used as primary inputs for GW-invariants in the symplectic sum formulas, can be seen as the dual of (3.14). The analogue of this lemma with field coefficients, which would be sufficient for the purposes of this paper, follows immediately by dualizing the analogue of (3.14) with coefficients in the same field. Similarly, the proof of Lemma 3.7 can be viewed as the dual version of the proof of Lemma 3.3, but we include it for the sake of completeness; like the proof of Lemma 3.3, it contains a delicate step.

Lemma 3.7. *If X and Y are smooth compact orientable manifolds, $V\subset X,Y$ is a smooth compact orientable submanifold, $\varphi\colon S_X V\longrightarrow S_Y V$ is a diffeomorphism commuting with the projections to V, and $q_\varphi\colon X\#_\varphi Y\longrightarrow X\cup_V Y$ is a collapsing map, then*

$$\big\{q_\varphi^*\alpha_\cup\colon\, \alpha_\cup\in H^*(X\cup_V Y;\mathbb{Z})\big\} = \big\{\alpha_\#\in H^*(X\#_\varphi Y;\mathbb{Z})\colon\, \alpha_\#|_{SV}\in q_V^*(H^*(V;\mathbb{Z}))\big\},\tag{3.15}$$

where $SV\subset X\#_\varphi Y$ is the sphere bundle $S_X V\approx S_Y V$.

Proof. The commutativity of the diagram

$$\begin{array}{ccc} SV & \xrightarrow{\iota^{X\#_\varphi Y}_{SV}} & X\#_\varphi Y\\ \Big\downarrow{\scriptstyle q_V} & & \Big\downarrow{\scriptstyle q_\varphi}\\ V & \xrightarrow{\iota^{X\cup_V Y}_{V}} & X\cup_V Y\end{array}$$

implies that the left-hand side of (3.15) is contained in the right-hand side. Below we confirm the opposite inclusion.

We will use the commutative diagram of the Mayer-Vietoris cohomology sequences for $X\cup_V Y$ and $X\#_\varphi Y = (X-V)\cup_{SV}(Y-V)$,

$$\begin{array}{ccccccc} H^{m-1}(V) & \xrightarrow{\delta^*_\cup} & H^m(X\cup_V Y) & \xrightarrow{(\iota^{X\cup_V Y*}_X,\,\iota^{X\cup_V Y*}_Y)} & H^m(X)\oplus H^m(Y) & \xrightarrow{\iota^{X*}_V-\iota^{Y*}_V} & H^m(V)\\ \Big\downarrow{\scriptstyle q_V^*} & & \Big\downarrow{\scriptstyle q_\varphi^*} & & \Big\downarrow{\scriptstyle \iota^{X*}_{X-V}\oplus\iota^{Y*}_{Y-V}} & & \Big\downarrow{\scriptstyle q_V^*}\\ H^{m-1}(SV) & \xrightarrow{\delta^*_\varphi} & H^m(X\#_\varphi Y) & \xrightarrow{(\iota^{X\#_\varphi Y*}_{X-V},\,\iota^{X\#_\varphi Y*}_{Y-V})} & H^m(X-V)\oplus H^m(Y-V) & \xrightarrow{\iota^{X-V*}_{SV}-\iota^{Y-V*}_{SV}} & H^m(SV)\end{array}$$

where H^* denotes integral cohomology groups. Suppose

$$\alpha_\# \in H^*(X\#_\varphi Y;\mathbb{Z}), \quad \alpha_V \in H^*(V;\mathbb{Z}), \quad \alpha_\#|_{SV} = q_V^*\alpha_V.$$

By Mayer-Vietoris for $M=(M-V)\cup_{SV} V$, where $M=X,Y$,

$$H^m(M;\mathbb{Z}) \xrightarrow{(\iota_{M-V}^{M*},\iota_V^{M*})} H^m(M-V;\mathbb{Z}) \oplus H^m(V;\mathbb{Z}) \xrightarrow{\iota_{SV}^{M-V*}-q_V^*} H^m(SV;\mathbb{Z}),$$

there exist $\alpha_X \in H^m(X;\mathbb{Z})$ and $\alpha_Y \in H^m(Y;\mathbb{Z})$ such that

$$\alpha_X|_{X-V} = \alpha_\#|_{X-V}, \quad \alpha_Y|_{Y-V} = \alpha_\#|_{Y-V}, \quad \alpha_X|_V, \alpha_Y|_V = \alpha_V\,.$$

By the Mayer-Vietoris sequence for $X\cup_V Y$ above, there exists

$$\alpha_\cup \in H^m(X\cup_V Y;\mathbb{Z}) \qquad \text{s.t.} \quad \alpha_\cup|_X = \alpha_X, \qquad \alpha_\cup|_Y = \alpha_Y.$$

The commutativity of the middle square in the above diagram and the exactness of the bottom row then imply that

$$\alpha_\# - q_\varphi^*\alpha_\cup \in \big\{\delta_\varphi^*(\beta_{SV}) \colon \beta_{SV} \in H^{m-1}(SV;\mathbb{Z})\big\}.$$

The claim then follows from the observation that

$$\big\{\delta_\varphi^*(\beta_{SV}) \colon \beta_{SV} \in H^{m-1}(SV;\mathbb{Z})\big\} \subset \big\{q_\varphi^*(\alpha_\cup) \colon \alpha_\cup \in H^m(X\cup_V Y;\mathbb{Z})\big\}, \tag{3.16}$$

which is established below.

Choose an open subset $\tilde{Y}$ of $X\cup_V Y$ consisting of Y and a tubular neighborhood of V in X. Let $\mathcal{S}_\cup^*$ denote the cochain complex of $\mathbb{Z}$-valued homomorphisms on the sub-complex of singular chains generated by simplicies in $X\cup_V Y$ with images in either $X-V$ or $\tilde{Y}$. Similarly, let $\mathcal{S}_\#^*$ denote the cochain complex of $\mathbb{Z}$-valued homomorphisms on the sub-complex of singular chains generated by simplicies in $X\#_\varphi Y$ with images in either $q_\varphi^{-1}(X-V)$ or $q_\varphi^{-1}(\tilde{Y})$. By [Wa, Section 5.32], the restriction homomorphisms from the usual singular cochain complexes,

$$\mathcal{S}^*(X\cup_V Y) \longrightarrow \mathcal{S}_\cup^* \qquad \text{and} \qquad \mathcal{S}^*(X\#_\varphi Y) \longrightarrow \mathcal{S}_\#^*\,,$$

induce isomorphisms in cohomology. Thus, we can replace the domains of these homomorphisms by their targets in order to verify (3.16). Let $V_\cup=(X-V)\cap\tilde{Y}$ and $SV_\#=q_\varphi^{-1}(V_\cup)$.

For any $\eta\in\mathcal{S}^*(SV_\#)$, define

$$\eta_{q_\varphi^{-1}(X-V)} \in \mathcal{S}^*\big(q_\varphi^{-1}(X-V)\big)\,, \qquad \eta_{q_\varphi^{-1}(X-V)}(\sigma) = \begin{cases} \eta(\sigma), & \text{if } \operatorname{Im}\sigma \subset SV_\#;\\ 0, & \text{otherwise;}\end{cases}$$

$$\eta_\# \in \mathcal{S}_\#^*\,, \qquad \eta_\#(\sigma) = \begin{cases} \eta_{q_\varphi^{-1}(X-V)}(\partial\sigma), & \text{if } \operatorname{Im}\sigma \subset q_\varphi^{-1}(X-V);\\ 0, & \text{if } \operatorname{Im}\sigma \subset q_\varphi^{-1}(\tilde{Y});\end{cases}$$

$$\eta_\cup \in \mathcal{S}_\cup^*\,, \qquad \eta_\cup(\sigma) = \begin{cases} \eta_{q_\varphi^{-1}(X-V)}(\partial(q_\varphi^{-1}\circ\sigma)), & \text{if } \operatorname{Im}\sigma \subset X-V;\\ 0, & \text{if } \operatorname{Im}\sigma \subset \tilde{Y};\end{cases}$$

where σ denotes an appropriate singular simplex. The homomorphisms $\eta_\#$ and $\eta_\cup$ are well-defined on the overlaps if $\delta\eta=0$, i.e. η determines an element $[\eta]$ in $H^*(SV_\#)$. In such a case,

$$\delta_\varphi^*[\eta] = [\eta_\#], \qquad q_\varphi^*[\eta_\cup] = [\eta_\#],$$

by the construction of the connecting homomorphism in the Snake Lemma and the definition of pull-back homomorphisms. This establishes (3.16). □

Corollary 3.8. *If X, Y, V, φ, and q_φ are as in Lemma 3.7, $\alpha_\# \in H^*(X\#_\varphi Y;\mathbb{Z})$ is of the form $q_\varphi^*\alpha_\cup$ for some $\alpha_\cup \in H^*(X\cup_V Y;\mathbb{Z})$ if and only if the Poincare dual of $\alpha_\#$ can be represented by a pseudocycle $f_\#\colon Z_\# \longrightarrow X\#_\varphi Y$ transverse to SV such that $f_\#^{-1}(SV)=f_V^*SV$ for some pseudocycle $f_V\colon Z_V \longrightarrow V$ of dimension $\mathfrak{c}$ less, where $\mathfrak{c}$ is the codimension of V in X and Y.*

Proof. Let $f_\#\colon Z_\# \longrightarrow X\#_\varphi Y$ be a pseudocycle representative for the Poincare dual of $\alpha_\#$ transverse to SV. The restriction of $f_\#$ to $f_\#^{-1}(SV)$ then represents the Poincare dual of $\alpha_\#|_{SV}$.

(1) If $f_\#^{-1}(SV)=f_V^*SV$ for some pseudocycle $f_V\colon Z_V \longrightarrow V$ of dimension $\mathfrak{c}$ less, $\alpha_\#|_{SV}=q_V^*\alpha_V$, where $\alpha_V \in H^*(V;\mathbb{Z})$ is the Poincare dual of the class represented by f_V. Lemma 3.7 then implies that $\alpha_\#=q_\varphi^*\alpha_\cup$ for some $\alpha_\cup \in H^*(X\cup_V Y;\mathbb{Z})$.

(2) If $\alpha_\# = q_\varphi^*\alpha_\cup$ for some $\alpha_\cup \in H^*(X\cup_V Y;\mathbb{Z})$, then $\alpha_\#|_{SV} = q_V^*\alpha_V$ for some $\alpha_V \in H^*(V;\mathbb{Z})$; see Lemma 3.7. Let $f_V\colon Z_V \longrightarrow V$ be a pseudocycle representing the Poincare dual of α_V and $\tilde f_V\colon f_V^*SV \longrightarrow SV$ be the induced pseudocycle from the total space of the bundle $SV \longrightarrow V$ pulled back by f_V. Thus, there exists a pseudocycle equivalence $\tilde f\colon \tilde Z \longrightarrow SV$ so that

$$\partial\tilde f = f_\#\big|_{f_\#^{-1}(S_V)} - \tilde f_V\,.$$

Cutting $Z_\#$ along the hypersurface $f_\#^{-1}(SV)$, gluing in $\tilde f$ and $-\tilde f$ along the resulting cuts, identifying $\tilde f$ and $-\tilde f$ along $\tilde f_V$, and moving $\pm\tilde f$ on the complement of $\tilde f_V$ outside SV, we obtain a pseudocycle representative $\hat f_\#\colon \hat Z_\# \longrightarrow X\#_\varphi Y$ for the Poincare dual of $\alpha_\#$ transverse to SV such that $\hat f_\#^{-1}(SV)=f_V^*SV$. □

Remark 3.9. There is a slight misstatement in part (a) at the bottom of page 996 in [IP5] related to the $\mathfrak{c}=2$ case of Lemma 3.7, since the first map in [IP5, (10.13)] is never injective for dimensional reasons. The statement in (a) should instead be that $\alpha \in H^m(Z_\lambda;\mathbb{Z})$ separates if

$$\cup c_1\colon H^{m-1}(V;\mathbb{Z}) \longrightarrow H^{m+1}(V;\mathbb{Z})$$

is injective. In (b), $j^*\colon H^*(Z_\lambda;\mathbb{Z}) \longrightarrow H^*(SV;\mathbb{Z})$ is the restriction map.

3.2 Twisting by rim tori: [IP4, Section 5], [IP5, Section 10]

In this section, we specify the coverings of the manifolds V^ℓ implicitly used in [IP4] to refine the usual relative invariants. The latter is possible to do if V is connected or satisfies the condition of Definition 3.10, but not in general, as can be seen from the proof of Lemma 3.11 and from Example 3.12. If V satisfies the condition of Definition 3.10, the evaluation maps can be lifted systematically over such coverings and fiber products $\mathcal{H}_{X,Y;\mathbf{s}}^V$ of these coverings can be split into unions of topological components $\mathcal{H}_{X,Y;\mathbf{s}}^{V,C}$, as suggested by [IP5, (3.10)]; see Proposition 3.13 and

Corollary 3.14. However, contrary to a key premise of [IP5], the components $\mathcal{H}^{V,C}_{X,Y;\mathbf{s}}$ can rarely be distinguished homologically and used to resolve the rim tori deficiency of the usual symplectic sum formulas, even if V satisfies the condition of Definition 3.10. We end this section by deducing Theorem 1.2 from the approach in [IP5].

Definition 3.10. Let X be a smooth manifold and $V\subset X$ be a compact submanifold of codimension $\mathfrak{c}$ with topological components $V_1,\ldots,V_N$ and an orientable normal bundle $\mathcal{N}_X V$. We call $V\subset X$ virtually connected if

$$\mathcal{R}^V_X = \iota^{X-V}_{S_X V*}\Delta_{X,V}\big(H_1(V_1;\mathbb{Z})\big)\oplus\ldots\oplus\iota^{X-V}_{S_X V*}\Delta_{X,V}\big(H_1(V_N;\mathbb{Z})\big), \tag{3.17}$$

with $\Delta_{X,V}$ and $\iota^{X-V}_{S_X V*}$ as at the beginning of Section 3.1 and $\mathcal{R}^V_X$ as in (3.4).

By Lemma 3.1, the modules on the right-hand side of (3.17) are always contained in $\mathcal{R}^V_X$ and span it. In particular, every connected compact submanifold $V\subset X$ with an orientable normal bundle is virtually connected. In general, if $V_1,\ldots,V_N$ are the topological components of V and $r=1,\ldots,N$, let

$$\mathcal{R}^V_{X;r} = \iota^{X-V}_{S_X V*}\Delta_{X,V}\big(H_1(V_r;\mathbb{Z})\big)\subset\mathcal{R}^V_X\subset H_{\mathfrak{c}}(X-V;\mathbb{Z}). \tag{3.18}$$

If $V\subset X$ is virtually connected, $\mathcal{R}^V_X/\mathcal{R}^V_{X;r}$ is the direct sum of the submodules $\mathcal{R}^V_{X;r'}$ with $r'\neq r$.

Let X be a smooth manifold and $V\subset X$ be a compact submanifold of codimension $\mathfrak{c}$ with connected components $V_1,\ldots,V_N$ and an orientable normal bundle. For a tuple $\mathbf{s}=(s_1,\ldots,s_\ell)\in\mathbb{Z}^\ell$, we define

$$V_{\mathbf{s}}=V^\ell\,,\qquad \sigma_{\mathbf{s}}\colon H_{\mathfrak{c}}(X-V;\mathbb{Z})^{\oplus\ell}\longrightarrow H_{\mathfrak{c}}(X-V;\mathbb{Z}),\quad \sigma_{\mathbf{s}}(A_1,\ldots,A_\ell)=s_1A_1+\ldots+s_\ell A_\ell\,.$$

For each $r=1,\ldots,N$, let

$$\begin{aligned}\Phi_{r;\mathbf{s}}\colon \pi_1\big((V_r)_{\mathbf{s}}\big)\longrightarrow H_1\big((V_r)_{\mathbf{s}};\mathbb{Z}\big)=H_1(V_r;\mathbb{Z})^{\oplus\ell}\hookrightarrow H_1(V;\mathbb{Z})^{\oplus\ell}&\\ \xrightarrow{(\iota^{X-V}_{S_X V*}\circ\Delta_{X,V})^{\oplus\ell}} H_{\mathfrak{c}}(X-V;\mathbb{Z})^{\oplus\ell}\xrightarrow{\sigma_{\mathbf{s}}} H_{\mathfrak{c}}(X-V;\mathbb{Z}),&\end{aligned}$$

where the first arrow is the Hurewicz homomorphism [Sp, Section 7.4], denote the composition map. Let

$$\pi'^V_{X;r;\mathbf{s}}\colon\mathcal{H}'^V_{X;r;\mathbf{s}}\longrightarrow (V_r)_{\mathbf{s}} \tag{3.19}$$

be the covering projection corresponding to the normal subgroup $\ker\Phi_{r;\mathbf{s}}$ of $\pi_1((V_r)_{\mathbf{s}})$; see [Mu, Theorems 79.4, 82.1]. By (3.18), the fiber of (3.19) is

$$\frac{\pi_1((V_r)_{\mathbf{s}})}{\ker\Phi_{r;\mathbf{s}}}\approx \gcd(\mathbf{s})\,\mathcal{R}^V_{X;r}\,,$$

where $\gcd(\mathbf{s})$ is the greatest common divisor of $s_1,\ldots,s_\ell$. Let

$$\pi^V_{X;r;\mathbf{s}}\colon\mathcal{H}^V_{X;r;\mathbf{s}}\equiv\mathcal{R}^V_{X;r}\big/\gcd(\mathbf{s})\,\mathcal{R}^V_{X;r}\times\mathcal{H}'^V_{X;r;\mathbf{s}}\longrightarrow (V_r)_{\mathbf{s}}, \tag{3.20}$$

where $\pi^V_{X;r;\mathbf{s}}$ is the composition of the map (3.19) with the projection onto the second component. If $\mathbf{s}_r\in\mathbb{Z}^{\ell_r}$ for $r=1,\ldots,N$, we define

$$\begin{aligned}\pi^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\equiv\pi^V_{X;1;\mathbf{s}_1}\times\ldots\times\pi^V_{X;N;\mathbf{s}_N}\colon\mathcal{H}^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\equiv\mathcal{H}^V_{X;1;\mathbf{s}_1}\times\ldots\times\mathcal{H}^V_{X;N;\mathbf{s}_N}&\\ \longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N}\equiv(V_1)_{\mathbf{s}_1}\times\ldots\times(V_N)_{\mathbf{s}_N}.&\end{aligned} \tag{3.21}$$

The fibers of this projection are $\mathcal{R}_X^V$ if V is virtually connected.

If Σ is a smooth compact oriented m-dimensional manifold, $A\in H_m(X;\mathbb{Z})$, $k\in\mathbb{Z}^{\ge 0}$, and $p>m$, let $\mathfrak{X}_{\Sigma,k}(X,A)$ be the space of tuples $(z_1,\ldots,z_k,f)$ such that $f\in L^p_1(\Sigma;X)$, $f_*[\Sigma]=A$, and $z_1,\ldots,z_k\in\Sigma$ are distinct points. If in addition $m=\mathfrak{c}$, $V_1,\ldots,V_N$ and $\mathbf{s}_1,\ldots,\mathbf{s}_N$ are as before, and (1.3) holds for each $(V,\mathbf{s})=(V_r,\mathbf{s}_r)$, let

$$\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}(X,A)\subset\mathfrak{X}_{\Sigma,k+\ell_1+\ldots+\ell_N}(X,A)$$

be the subspace of tuples $(z_1,\ldots,z_{k+\ell_1+\ldots+\ell_N},f)$ such that

$$\begin{aligned} &f^{-1}(V_r)=\big\{z_{k+\ell_1+\ldots,\ell_{r-1}+1},\ldots,z_{k+\ell_1+\ldots+\ell_r}\big\} && \forall\, r=1,\ldots,N,\\ &\mathrm{ord}^{V_r}_{z_{k+\ell_1+\ldots,\ell_{r-1}+i}}f=s_{r;i} && \forall\, i=1,2,\ldots,\ell_r,\ r=1,\ldots,N. \end{aligned}$$

For each $r=1,\ldots,N$, we denote by

$$\mathrm{ev}^{V_r}=\mathrm{ev}_{k+\ell_1+\ldots+\ell_{r-1}+1}\times\ldots\times\mathrm{ev}_{k+\ell_1+\ldots+\ell_r}\colon\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}(X,A)\longrightarrow(V_r)_{\mathbf{s}_r} \tag{3.22}$$

the total evaluation map to the r-th topological component of V. Let

$$\mathrm{ev}_{X,V}=\mathrm{ev}^{V_1}\times\ldots\times\mathrm{ev}^{V_N}\colon\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}(X,A)\longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N}\equiv(V_1)_{\mathbf{s}_1}\times\ldots\times(V_N)_{\mathbf{s}_N}\,. \tag{3.23}$$

The conclusion of the $\mathfrak{c}=2$ case of the next lemma is essentially the desired outcome of [IP4, Section 5].

Lemma 3.11. *Suppose X is a smooth manifold, $V\subset X$ is a virtually connected submanifold of codimension $\mathfrak{c}$, and $A\in H_{\mathfrak{c}}(X;\mathbb{Z})$. Let $V_1,\ldots,V_N$ be the connected components of V and $\mathbf{s}_r\in\mathbb{Z}^{\ell_r}$ for $r=1,\ldots,N$ so that (1.3) holds for each $(V,\mathbf{s})=(V_r,\mathbf{s}_r)$. If Σ is a smooth compact oriented $\mathfrak{c}$-dimensional manifold, $k\in\mathbb{Z}^{\ge0}$, and $r=1,\ldots,N$, then the total evaluation map (3.22) lifts over the covering (3.19) to a continuous map*

$$\widetilde{\mathrm{ev}}^V_{X;r}\colon\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}(X,A)\longrightarrow\mathcal{H}'^V_{X;r;\mathbf{s}_r}\,.$$

Proof. Let

$$\gamma\colon[0,1]\longrightarrow\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}(X,A),\qquad t\longrightarrow(z_{t;1},\ldots,z_{t;k+\ell_1+\ldots+\ell_N},f_t),$$

be a loop. For each $r=1,\ldots,N$ and $i=1,\ldots,\ell_r$,

$$\gamma_{r;i}\equiv\mathrm{ev}_{k+\ell_1+\ldots+\ell_{r-1}+i}\circ\gamma\colon[0,1]\longrightarrow V_r$$

is also a loop. For each $i=1,\ldots,\ell_1+\ldots+\ell_N$, let $B_{z_{k+i}}\subset\Sigma$ be a small ball around z_{k+i}. As in the construction of $f_X\#_\varphi f_Y$ in Section 3.1, it can be assumed that

$$f_t\colon\partial B_{z_{k+\ell_1+\ldots+\ell_{r-1}+i}}\longrightarrow S_XV|_{f_t(z_{k+\ell_1+\ldots+\ell_{r-1}+i})}\,.$$

Since γ is a loop,

$$0=[f_0\#(-f_1)]=\sum_{r=1}^N\sum_{i=1}^{\ell_r}\iota^{X-V}_{S_XV*}\big(\Delta_{X,V}(s_{r;i}\gamma_{r;i})\big)=\sum_{r=1}^N\Phi_{r;\mathbf{s}_r}(\mathrm{ev}^{V_r}\circ\gamma)\in\mathcal{R}^V_X=\bigoplus_{r=1}^N\mathcal{R}^V_{X;r};$$

the last equality holds becomes $V\subset X$ is virtually connected. It follows that $\Phi_{r;\mathbf{s}_r}(\mathrm{ev}^{V_r}\circ\gamma)=0$ for every $r=1,\ldots,N$. Thus, the image of $\pi_1(\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}(X,A))$ in $\pi_1((V_r)_{\mathbf{s}_r})$ under $\mathrm{ev}^{V_r}_*$ lies in the image of $\pi_1(\mathcal{H}'^V_{X;r;\mathbf{s}_r})$ under (3.19). By [Mu, Lemma 79.1], this implies the claim. $\square$

Example 3.12. Continuing with Example 3.5, we take

$$X = S^2\times\mathbb{T}^2, \quad V_1 = \{0\}\times\mathbb{T}^2, \quad V_2 = \{\infty\}\times\mathbb{T}^2, \quad V = V_1\sqcup V_2, \quad \text{and} \quad A = [S^2\times\{\text{pt}\}],$$

where $\text{pt}\in\mathbb{T}^2$ is a fixed point. If $\alpha\colon S^1\longrightarrow\mathbb{T}^2$ is a loop based at pt and

$$\mathbf{f} = (z_1, z_2, f) \in \mathfrak{X}^{\{0\},\{\infty\}}_{\Sigma,0;(1),(1)}(S^2, [S^2]),$$

then $\gamma = (z_1, z_2, f\times\alpha)$ is a loop in $\mathfrak{X}^{V_1,V_2}_{\Sigma,0;(1),(1)}(X, A)$ such that $\text{ev}_{1*}\gamma = \alpha$. Thus, ev_{1*} is a surjective homomorphism between the fundamental groups of $\mathfrak{X}^{V_1,V_2}_{\Sigma,0;(1),(1)}(X, A)$ and V_1. In light of [Mu, Lemma 79.1], this implies that ev_1 does not lift to any non-trivial connected cover of V_1.

We will next choose the lifts in Lemma 3.11 in a consistent way.

Proposition 3.13. *Suppose X is a smooth manifold, $V\subset X$ is a virtually connected submanifold of codimension $\mathfrak{c}$, and $A\in H_{\mathfrak{c}}(X;\mathbb{Z})$. Let $V_1, \ldots, V_N$ be the connected components of V and $\mathbf{s}_r\in\mathbb{Z}^{\ell_r}$ for $r = 1, \ldots, N$ so that (1.3) holds for each $(V,\mathbf{s}) = (V_r,\mathbf{s}_r)$. If Σ is a smooth compact oriented $\mathfrak{c}$-dimensional manifold and $k\in\mathbb{Z}^{\ge 0}$, there exists a lift*

$$\widetilde{\text{ev}}_{X,V}\colon \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}(X, A) \longrightarrow \mathcal{H}^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \mathcal{H}^V_{X;1;\mathbf{s}_1}\times\ldots\times\mathcal{H}^V_{X;N;\mathbf{s}_N}$$

of the evaluation map $\text{ev}_{X,V}$ in (3.23) over the covering $\pi^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$ in (3.21) such that $\widetilde{\text{ev}}_{X,V}(\mathbf{f}) = \widetilde{\text{ev}}_{X,V}(\mathbf{f}')$ if and only if $f\#(-f')$ is defined and vanishes in $H_{\mathfrak{c}}(X-V;\mathbb{Z})$, where f and f' are the map components of $\mathbf{f}$ and $\mathbf{f}'$, respectively.

Proof. For each topological component V_r of V, choose a base point $\tilde{x}_r\in\mathcal{H}'^V_{X;r;\mathbf{s}_r}$. For each coset in $\mathcal{R}^V_{X;r}/\gcd(\mathbf{s}_r)\mathcal{R}^V_{X;r}$, choose a representative $T_{r;j}\in\mathcal{R}^V_{X;r}$; we will denote its coset by $[T_{r;j}]$. Choose an element

$$f_0 \equiv (z_{0;1}, \ldots, z_{0;k+\ell_1+\ldots+\ell_N}, f_0) \in \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}(X, A)$$

so that $\text{ev}^{V_r}(\mathbf{f}_0) = \pi'^V_{X;r;\mathbf{s}_r}(\tilde{x}_r)$ for every $r = 1, \ldots, N$. Choose a base point

$$\mathbf{f}_m \equiv (z_{m;1}, \ldots, z_{m;k+\ell_1+\ldots+\ell_N}, f_m)$$

for each topological component of $\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}(X, A)$ so that $\text{ev}^{V_r}(\mathbf{f}_m) = \pi'^V_{X;r;\mathbf{s}_r}(\tilde{x}_r)$ for every $r = 1, \ldots, N$ and

$$[f_m\#(-f_0)] = T_{1;j_1(m)} + \ldots + T_{N;j_N(m)} \in H_{\mathfrak{c}}(X-V;\mathbb{Z}) \tag{3.24}$$

for some $j_r = j_r(m)$ with $r = 1, \ldots, N$. We define

$$\widetilde{\text{ev}}^V_{X;r}\colon \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}(X, A) \longrightarrow \mathcal{H}^V_{X;r;\mathbf{s}_r} \equiv \mathcal{R}^V_{X;r}/\gcd(\mathbf{s}_r)\mathcal{R}^V_{X;r}\times\mathcal{H}'^V_{X;r;\mathbf{s}_r}$$

to be the lift of ev^{V_r} over $\pi^V_{X;r;\mathbf{s}_r}$ so that

$$\widetilde{\text{ev}}^V_{X;r}(\mathbf{f}_m) = \big([T_{r;j_r(m)}], \tilde{x}_r\big) \qquad \forall\, m, r;$$

see [Mu, Lemma 79.1]. Let

$$\widetilde{\text{ev}}_{X,V} \equiv \widetilde{\text{ev}}^V_{X;1}\times\ldots\times\widetilde{\text{ev}}^V_{X;N}\colon \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}(X, A) \longrightarrow \mathcal{H}^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \mathcal{H}^V_{X;1;\mathbf{s}_1}\times\ldots\times\mathcal{H}^V_{X;N;\mathbf{s}_N}.$$

It remains to verify that this lift has the claimed property for any $\mathbf{f},\mathbf{f}'\in\mathfrak{X}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}^{V_1,\ldots,V_N}(X,A)$ such that $\mathrm{ev}_{X,V}(\mathbf{f})=\mathrm{ev}_{X,V}(\mathbf{f}')$.

Let $\mathbf{f}_m$ and $\mathbf{f}_{m'}$ be the base points of the topological components of $\mathfrak{X}_{\Sigma,k;\mathbf{s}_1,\ldots,\mathbf{s}_N}^{V_1,\ldots,V_N}(X,A)$ containing $\mathbf{f}$ and $\mathbf{f}'$, respectively, and γ and γ' be paths from $\mathbf{f}_m$ to $\mathbf{f}$ and from $\mathbf{f}_{m'}$ to $\mathbf{f}'$, respectively. Since $\mathrm{ev}_{X,V}(\mathbf{f}_m)=\mathrm{ev}_{X,V}(\mathbf{f}_{m'})$ and $\mathrm{ev}_{X,V}(\mathbf{f})=\mathrm{ev}_{X,V}(\mathbf{f}')$,

$$\gamma_{r;i}\equiv\big(-\mathrm{ev}_{k+\ell_1+\ldots+\ell_{r-1}+i}\circ\gamma\big)*\mathrm{ev}_{k+\ell_1+\ldots+\ell_{r-1}+i}\circ\gamma'\colon[0,1]\longrightarrow V_r$$

is a well-defined loop for every $r=1,\ldots,N$ and $i=1,\ldots,\ell_r$ and

$$\begin{aligned}[][f_m\#(-f_{m'})]-[f\#(-f')]&=\sum_{r=1}^N\sum_{i=1}^{\ell_r}\iota_{S_XV*}^{X-V}\big(\Delta_{X,V}(s_{r;i}\gamma_{r;i})\big)\\&=\sum_{r=1}^N\Phi_{r;\mathbf{s}_r}\big(\gamma_{r;1},\ldots,\gamma_{r;\ell_r}\big)\in\bigoplus_{r=1}^N\mathcal{R}_{X;r}^V\subset H_{\mathfrak{c}}(X-V;\mathbb{Z})\,.\end{aligned}\tag{3.25}$$

If $\widetilde{\mathrm{ev}}_{X,V}(\mathbf{f})=\widetilde{\mathrm{ev}}_{X,V}(\mathbf{f}')$, then

$$\big([T_{r;j_r(m)}],\tilde{x}_r\big)=\widetilde{\mathrm{ev}}_{X;r}^V(\mathbf{f}_m)=\widetilde{\mathrm{ev}}_{X;r}^V(\mathbf{f}_{m'})=\big([T_{r;j_r(m')}],\tilde{x}_r\big)\qquad\forall\ r=1,\ldots,N\tag{3.26}$$

and the paths $\mathrm{ev}^{V_r}\circ\gamma$ and $\mathrm{ev}^{V_r}\circ\gamma'$ lift to paths in $\mathcal{H}'^V_{X;r;\mathbf{s}_r}$ from $\tilde{x}_r$ with the same end points. By (3.24) and (3.26), $[f_m\#(-f_{m'})]=0$. Since the paths $\mathrm{ev}^{V_r}\circ\gamma$ and $\mathrm{ev}^{V_r}\circ\gamma'$ lift to paths in $\mathcal{H}'^V_{X;r;\mathbf{s}_r}$ from $\tilde{x}_r$ with the same end points, $(\gamma_{r;1},\ldots,\gamma_{r;\ell_r})$ is in the image of $\pi_1(\mathcal{H}'^V_{X;r;\mathbf{s}_r})$ and so lies in $\ker\Phi_{r;s_r}$. From (3.25), we then conclude that $[f\#(-f')]=0$. Conversely, if $[f\#(-f')]=0$, then

$$\sum_{r=1}^N\big(T_{r;j_r(m)}-T_{r;j_r(m')}\big)=[f_m\#(-f_{m'})]=\sum_{r=1}^N\Phi_{r;\mathbf{s}_r}\big(\gamma_{r;1},\ldots,\gamma_{r;\ell_r}\big)\in\bigoplus_{r=1}^N\mathcal{R}_{X;r}^V\,.$$

Since $T_{r;j_r(m)}$ and $T_{r;j_r(m')}$ are representative of cosets $\mathcal{R}_{X;r}^V$ modulo the image of $\Phi_{r;\mathbf{s}_r}$, it follows that

$$\widetilde{\mathrm{ev}}_{X;r}^V(\mathbf{f}_m)=\big([T_{r;j_r(m)}],\tilde{x}_r\big)=\big([T_{r;j_r(m')}],\tilde{x}_r\big)=\widetilde{\mathrm{ev}}_{X;r}^V(\mathbf{f}_{m'})\qquad\forall\ r=1,\ldots,N$$

and $(\gamma_{r;1},\ldots,\gamma_{r;\ell_r})$ lies in $\ker\Phi_{r;s_r}$. By the latter, the paths $\mathrm{ev}^{V_r}\circ\gamma$ and $\mathrm{ev}^{V_r}\circ\gamma'$ lift to paths in $\mathcal{H}'^V_{X;r;\mathbf{s}_r}$ from $\tilde{x}_r$ with the same end points. Thus, $\widetilde{\mathrm{ev}}_{X;r}^V(\mathbf{f})=\widetilde{\mathrm{ev}}_{X;r}^V(\mathbf{f}')$. $\square$

Suppose X and Y are oriented manifolds, $V\subset X,Y$ is a virtually connected submanifold of codimension $\mathfrak{c}$, $\varphi\colon S_XV\longrightarrow S_YV$ is an orientation-reversing diffeomorphism commuting with the projections to V, and $X\#_\varphi Y$ is the smooth manifold obtained by gluing the complements of tubular neighborhoods of V in X and Y by φ along their common boundary. If $V_1,\ldots,V_N$ and $\mathbf{s}_1,\ldots,\mathbf{s}_N$ are as above, denote by

$$\Delta_{\mathbf{s}_1\ldots\mathbf{s}_N}^V\subset V_{\mathbf{s}_1\ldots\mathbf{s}_N}\times V_{\mathbf{s}_1\ldots\mathbf{s}_N}$$

the diagonal and let

$$\mathcal{H}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}^V=\mathcal{H}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V\times_{V_{\mathbf{s}_1\ldots\mathbf{s}_N}}\mathcal{H}_{Y;\mathbf{s}_1\ldots\mathbf{s}_N}^V\equiv\big\{\pi_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V\times\pi_{Y,;\mathbf{s}_1\ldots\mathbf{s}_N}^V\big\}^{-1}\big(\Delta_{\mathbf{s}_1\ldots\mathbf{s}_N}^V\big).$$

If Σ_X, Σ_Y are smooth compact oriented $\mathfrak{c}$-dimensional manifolds, $k_X, k_Y \in \mathbb{Z}^{\ge 0}$, and (A_X, A_Y) is an element of $H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z})$, let

$$\begin{split}\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_X,k_X;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A_X)\times_{V_{\mathbf{s}_1\ldots\mathbf{s}_N}}\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_Y,k_Y;\mathbf{s}_1\ldots\mathbf{s}_N}(Y,A_Y) &= \big\{\widetilde{\mathrm{ev}}_{X;V}\times\widetilde{\mathrm{ev}}_{Y;V}\big\}^{-1}(\mathcal{H}^V_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N})\\ &= \big\{\mathrm{ev}_{X;V}\times\mathrm{ev}_{Y;V}\big\}^{-1}(\Delta^V_{\mathbf{s}_1\ldots\mathbf{s}_N}),\end{split} \tag{3.27}$$

where $\widetilde{\mathrm{ev}}_{X,V}$ and $\widetilde{\mathrm{ev}}_{Y,V}$ are the lifts of $\mathrm{ev}_{X,V}$ and $\mathrm{ev}_{Y,V}$, respectively, provided by Proposition 3.13. The gluing construction of Section 3.1 defines a continuous map

$$\#_\varphi\colon \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_X,k_X;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A_X)\times_{V_{\mathbf{s}_1\ldots\mathbf{s}_N}}\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_Y,k_Y;\mathbf{s}_1\ldots\mathbf{s}_N}(Y,A_Y)\longrightarrow \bigsqcup_{C\in A_X\#_\varphi A_Y}\mathfrak{X}_{\Sigma_X\#\Sigma_Y,k_X+k_Y}(X\#_\varphi Y,C)$$

and thus a continuous map

$$\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_X,k_X;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A_X)\times_{V_{\mathbf{s}_1\ldots\mathbf{s}_N}}\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_Y,k_Y;\mathbf{s}_1\ldots\mathbf{s}_N}(Y,A_Y)\longrightarrow H_{\mathfrak{c}}(X\#_\varphi Y;\mathbb{Z}). \tag{3.28}$$

Corollary 3.14. *Suppose X and Y are oriented manifolds, $V\subset X,Y$ is a virtually connected submanifold of codimension $\mathfrak{c}$, $\varphi\colon S_XV\longrightarrow S_YV$ is a orientation-reversing diffeomorphism commuting with the projections to V, and $(A_X,A_Y)\in H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z})$. If $\widetilde{\mathrm{ev}}_{X,V}$ and $\widetilde{\mathrm{ev}}_{Y,V}$ are as in Proposition 3.13, there exists a continuous map*

$$g_{A_X,A_Y}\colon \mathcal{H}^V_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}\longrightarrow H_{\mathfrak{c}}(X\#_\varphi Y;\mathbb{Z}) \tag{3.29}$$

such that $g_{A_X,A_Y}\circ\{\widetilde{\mathrm{ev}}_{X,V}\times\widetilde{\mathrm{ev}}_{Y,V}\}$ restricts to $\#_\varphi$ over the common domain.

Proof. For every element (A_X,A_Y) in $H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z})$, the space (3.27) decomposes into disjoint unions of topological components

$$\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_X,k_X;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A_X)\times_C\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_Y,k_Y;\mathbf{s}_1\ldots\mathbf{s}_N}(Y,A_Y)\equiv\#_\varphi^{-1}\big(\mathfrak{X}_{\Sigma_X\#\Sigma_Y,k_X+k_Y}(X\#_\varphi Y,C)\big).$$

By Proposition 3.13,

$$\begin{split}\big\{\widetilde{\mathrm{ev}}_{X,V}\times\widetilde{\mathrm{ev}}_{Y,V}\big\}\big(\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_X,k_X;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A_X)\times_{C_1}\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_Y,k_Y;\mathbf{s}_1\ldots\mathbf{s}_N}(Y,A_Y)\big)&\\ \cap\big\{\widetilde{\mathrm{ev}}_{X,V}\times\widetilde{\mathrm{ev}}_{Y,V}\big\}\big(\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_X,k_X;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A_X)\times_{C_2}\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_Y,k_Y;\mathbf{s}_1\ldots\mathbf{s}_N}(Y,A_Y)\big)&=\emptyset\end{split}$$

if $C_1\neq C_2$. This implies the claim. □

The map (3.28) depends on the liftings $\widetilde{\mathrm{ev}}_{X,V}$ and $\widetilde{\mathrm{ev}}_{Y,V}$. For each $C\in A_X\#A_Y$, let

$$\mathcal{H}^{V,C}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}=g^{-1}_{A_X,A_Y}(C).$$

These submanifolds decompose $\mathcal{H}^V_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}$ into disjoint unions of topological components and

$$\widetilde{\mathrm{ev}}_{X,V}\times\widetilde{\mathrm{ev}}_{Y,V}\colon\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_X,k_X;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A_X)\times_C\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_Y,k_Y;\mathbf{s}_1\ldots\mathbf{s}_N}(Y,A_Y)\longrightarrow\mathcal{H}^{V,C}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N},$$

by the definition of g_{A_X,A_Y}.

An unfortunate deficiency of the symplectic sum formulas of Theorems 1.1 and 4.1 is that generally they describe combinations of GW-invariants, rather than individual GW-invariants, of a symplectic sum $(X\#_VY,\omega_\#)$ of (X,ω_X) and (Y,ω_Y) in terms of relative GW-invariants of (X,ω_X,V)

and (Y, ω_Y, V). The rim tori refinement of [IP4] to the usual relative invariants is used in [IP5] with the aim of resolving this deficiency. In order to do so, each $\mathcal{H}^{V,C}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}$ needs to be described as the dual to some cohomology class on $\mathcal{H}^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\times\mathcal{H}^V_{Y;\mathbf{s}_1\ldots\mathbf{s}_N}$; in turn, this would describe a pairing on $H_*(\mathcal{H}^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N})\otimes H_*(\mathcal{H}^V_{Y;\mathbf{s}_1\ldots\mathbf{s}_N})$. An indication of such a pairing is given in the middle of page 993 in [IP5]. The usage of g_* in this formula is inconsistent with the definition of g in [IP5, (3.10)], but the identifications used in this formula are partially clarified on the following page, making the last equality essentially the definition of this pairing. It involves taking an intersection number of a homology class $Z_X\times Z_Y$ on $\mathcal{H}^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\times\mathcal{H}^V_{Y;\mathbf{s}_1\ldots\mathbf{s}_N}$ with $\mathcal{H}^{V,C}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}$. This number is

$$\big\langle \mathrm{PD}_{\mathcal{H}^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\times\mathcal{H}^V_{Y;\mathbf{s}_1\ldots\mathbf{s}_N}}\big(\mathcal{H}^{V,C}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}\big), Z_X\times Z_Y\big\rangle \tag{3.30}$$

if $\mathcal{H}^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$ and $\mathcal{H}^V_{Y;\mathbf{s}_1\ldots\mathbf{s}_N}$ are compact (i.e. if $\mathcal{R}^V_X$ and $\mathcal{R}^V_Y$ are finite).

However, it appears that no definition of an intersection number as above could possibly exist in general. The assignment $C\longrightarrow\mathcal{H}^{V,C}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}$ is defined only up to the action of $\mathcal{R}^V_X\times\mathcal{R}^V_Y$ corresponding to different choices of lifts in Lemma 3.11. If

$$\mathcal{R}^V_{X;r}=\gcd(\mathbf{s}_r)\mathcal{R}^V_{X;r} \quad\text{and}\quad \mathcal{R}^V_{Y;r}=\gcd(\mathbf{s}_r)\mathcal{R}^V_{Y;r} \qquad \forall\ r=1,\ldots,N\,, \tag{3.31}$$

the coverings (3.20) are connected and the subspaces

$$\mathcal{H}^{V,C}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}\subset\mathcal{H}^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\times\mathcal{H}^V_{Y;\mathbf{s}_1\ldots\mathbf{s}_N}$$

with various $C\in A_X\#A_Y$ are isotopic; in general, they are isomorphic submanifolds of isomorphic manifolds. If there were a meaningful notion of intersection number of $Z_X\times Z_Y$ with $\mathcal{H}^{V,C}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}$, the resulting number would be independent of $C\in A_X\#_\varphi A_Y$. In the symplectic sum setting, this would imply that the contribution of $\mathcal{H}^{V,C}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}$ to $\mathrm{GT}_{X\#_VY,\chi,C}$ is independent of $C\in A_X\#_VA_Y$ whenever (3.31) holds. In the general case, the contribution of each $\mathcal{H}^{V,C}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}$ may depend on the topological components of $\mathcal{H}^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\times\mathcal{H}^V_{Y;\mathbf{s}_1\ldots\mathbf{s}_N}$ containing it, as the refined relative "invariants" depend on the choices of the lifts in Lemma 3.11. By Corollary 3.4(2) and the above construction, the contributions for $C_1,C_2\in A_X\#_VA_Y$ are the same if

$$C_1-C_2=\sum_{r=1}^N\gcd(\mathbf{s}_r)\big(\iota^{X\#_VY}_{X-V*}\iota^{X-V}_{S_XV*}\Delta_{X,V}(\alpha_{r;X})+\iota^{X\#_VY}_{Y-V*}\iota^{Y-V}_{S_YV*}\Delta_{Y,V}(\alpha_{r;Y})\big)$$

for some $\alpha_{r;X},\alpha_{r;Y}\in H_1(V;\mathbb{Z})$ with $r=1,\ldots,N$. This equivalence relation breaks each $\mathcal{R}^V_{X,Y}$-coset $A_X\#_VA_Y$ of $H_2(X\#_VY;\mathbb{Z})$ into finitely many subsets of the same cardinality. If there were a meaningful notion of an intersection product with $\mathcal{H}^{V,C}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}$, the contribution of $\mathcal{H}^{V,C}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}$ to $\mathrm{GT}_{X\#_VY,\chi,C}$ would be independent of C inside of each of these subsets. If $\mathcal{R}^V_{X,Y}$ were infinite and $\mathrm{GT}_{X\#_VY,\chi,C}$ were nonzero for some C, there would thus be an infinite subset of $H_2(X\#_VY;\mathbb{Z})$ of classes C' differing by rim tori such that $\mathrm{GT}_{X\#_VY,\chi,C'}\neq 0$. However, this would contradict Gromov's Compactness Theorem, since $\omega_\#(C')$ is the same for all such classes; see Section 3.3.

By the above, the pairing (3.30) is well-defined under the finiteness assumptions of Theorem 1.2 and Remark 1.3. By the relatively prime assumptions of Theorem 1.2 and Remark 1.3, (3.31) holds for all relevant tuples $\mathbf{s}_1,\ldots,\mathbf{s}_N$. Thus, the pairing (3.30) is independent of the choice of $C\in A_X\#A_Y$. The claims of Theorem 1.2 and Remark 1.3 now follow from the main argument in [IP5].

Remark 3.15. Since the purpose of [IP4] is to define relative invariants, [IP4] does not explicitly describe the actual motivation behind the introduction of the rim tori refinement to these invariants (i.e. replacing $V_{\mathbf{s}}$ with $\mathcal{H}_{X;\mathbf{s}}^V$ as the target of the relative evaluation map). Two, essentially identical (not just equivalent), descriptions of the set $\mathcal{H}_{X;\mathbf{s}}^V$ are given in [IP4], neither of which formally specifies a topology on $\mathcal{H}_{X;\mathbf{s}}^V$. In particular, contrary to the sentence below [IP4, (5.6)], the topology on $\hat{X}$ is not changed, but the inclusion map $S^*\longrightarrow\hat{X}$ is still continuous and induces precisely the same inclusion of chain complexes as in the first description of $\mathcal{H}_{X;\mathbf{s}}^V$. A hands-on description of the topology of $\mathcal{H}_{X;\mathbf{s}}^V$, focusing on the $\mathbf{s}=(1)$ case, is given at the end of [IP4, Section 5]; our definition of $\mathcal{H}_{X;\mathbf{s}}^V$ via the homomorphisms $\Phi_{r;\mathbf{s}}$ formalizes this description whenever possible. In particular, contrary to what one might infer from the description at the end of [IP4, Section 5], the relevant covering $\mathcal{H}_{X;\mathbf{s}}^V\longrightarrow V_{\mathbf{s}}$ does not always exist and may be disconnected when it does exist. The map [IP5, (3.10)] is not specified; it is provided by Corollary 3.14 whenever possible. The typos in this part of [IP4] include:

p66, (5.2): $\mathcal{M}_{g,n}^V(X,A) \longrightarrow \mathcal{M}_{g,n}^V(X)$;
p66, line -2: $\mathcal{H} \longrightarrow \mathcal{H}_X^V$;
p67, line 20: $\mathcal{H}$ is never used.

There are also some typos in the related parts of [IP5]:

p992, after (10.5): (10.5) already involves disconnected domains;
p993, line 13: $\mathcal{H}_Y^V\times\mathcal{H}_Y^V \longrightarrow \mathcal{H}_X^V\times\mathcal{H}_Y^V$;
p994, line 7: $\sum\sum \longrightarrow \bigoplus\bigoplus$.

3.3 The symplectic sum: [IP5, Section 2], [LR, Sections 2,3.0]

In this section, we first recall the symplectic sum construction of [Gf, MW] from the point of view in [IP5] and then translate it to the point of view in [LR]. The former is more geometric and leads to a simpler description of the key notions of relative stable map and relative moduli space; our description of this setup is only a slight variation of [IP5, Section 2]. On the other hand, the latter fits better with the analytic issues that need to be addressed in proving symplectic sum formulas; unfortunately, [LR, Sections 2,3.0] do not actually specify a symplectic sum, but instead provide plenty of related examples of symplectic quotients. The symplectic manifolds $(\overline{M}_-,\omega_-)$, $(\overline{M}_+,\omega_+)$, and (M,ω) in [LR] correspond to (X,ω_X), (Y,ω_Y), and $(Z,\omega_\#)$, respectively, in our notation (which is similar to that in [IP5]); the hypersurface $\widetilde{M}\subset M$ along which M is split into its parts is denoted by SV below.

Let (X,ω_X) and (Y,ω_Y) be compact symplectic manifolds and $V\subset X,Y$ be a symplectic hypersurface so that (1.1) holds. Fix compatible complex structures on the symplectic bundles

$$\mathcal{N}_XV,\mathcal{N}_YV\longrightarrow V$$

and an isomorphism as in (1.2). We denote by ω_V the symplectic form $\omega_X|_V=\omega_Y|_V$. Choose a hermitian metric $g_{X,V}$ and connection on $\mathcal{N}_XV$; they induce a hermitian metric $g_{Y,V}$ and connection on $\mathcal{N}_YV$ via the above isomorphism (1.2). Let

$$\begin{gathered}\mathcal{Z}_X=\Delta\times(X-V),\qquad \mathcal{Z}_Y=\Delta\times(Y-V),\\ \mathcal{Z}_{\mathrm{neck}}=\big\{(x,y)\in\mathcal{N}_XV\oplus\mathcal{N}_YV\colon\ |x|,|y|<2,\ \ |xy|<\delta\big\},\end{gathered}\tag{3.32}$$

for some $\delta\in\mathbb{R}^+$ to be chosen later. We will glue these three manifolds to construct a smooth manifold $\mathcal{Z}$ so that the map $\pi\colon\mathcal{Z}\longrightarrow\Delta$ given by

$$\mathcal{Z}_X\longrightarrow\Delta,\ \ (\lambda,x)\longrightarrow\lambda,\quad \mathcal{Z}_Y\longrightarrow\Delta,\ \ (\lambda,y)\longrightarrow\lambda,\quad \mathcal{Z}_{\mathrm{neck}}\longrightarrow\Delta,\ \ (x,y)\longrightarrow xy, \tag{3.33}$$

is well-defined. Let $\pi_X:\mathcal{Z}_{\mathrm{neck}}\longrightarrow\mathcal{N}_XV$ and $\pi_Y:\mathcal{Z}_{\mathrm{neck}}\longrightarrow\mathcal{N}_YV$ be the component projection maps.

Let $q_{X,V}\colon S_XV\longrightarrow V$ and $q_{Y,V}\colon S_YV\longrightarrow V$ again be the sphere (circle) bundles of $\mathcal{N}_XV$ and $\mathcal{N}_YV$, respectively. Denote by

$$T^{\mathrm{vrt}}(S_XV)\equiv\ker\mathrm{d}q_{X,V}\subset T(S_XV)\qquad\text{and}\qquad T^{\mathrm{vrt}}(S_YV)\equiv\ker\mathrm{d}q_{Y,V}\subset T(S_YV)$$

the corresponding vertical tangent bundles. The connections in $\mathcal{N}_XV$ and $\mathcal{N}_YV$ induce splittings of the exact sequences

$$0\longrightarrow T^{\mathrm{vrt}}(S_XV)\longrightarrow T(S_XV)\stackrel{\mathrm{d}q_{X,V}}{\longrightarrow}q_{X,V}^*TV\longrightarrow0\qquad\text{and}$$
$$0\longrightarrow T^{\mathrm{vrt}}(S_YV)\longrightarrow T(S_YV)\stackrel{\mathrm{d}q_{Y,V}}{\longrightarrow}q_{Y,V}^*TV\longrightarrow0$$

of vector bundles over S_XV and S_YV, respectively; see [Z2, Lemma 1.1]. Denote by α_X and α_Y the 1-forms on S_XV and S_YV vanishing on the images of $q_{X,V}^*TV$ in $T(S_XV)$ and of $q_{Y,V}^*TV$ in $T(S_YV)$ corresponding to these splittings such that

$$\alpha_X\bigg(\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{e}^{\mathrm{i}\theta}x\bigg|_{\theta=0}\bigg)=1\quad\forall\,x\in S_XV,\qquad \alpha_Y\bigg(\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{e}^{\mathrm{i}\theta}y\bigg|_{\theta=0}\bigg)=1\quad\forall\,y\in S_YV\,.$$

Let $\gamma(t)$ be a path in V, $\tilde\gamma_X(t)$ be a horizontal lift of γ to S_XV, and $\tilde\gamma_Y(t)$ be a horizontal lift of γ to S_YV. Since the connection in $\mathcal{N}_YV$ is induced from the connection in $\mathcal{N}_XV$ via the isomorphism (1.2), $\tilde\gamma_X(t)\tilde\gamma_Y(t)$ is a horizontal lift of $\gamma(t)$ to $V\times\mathbb{C}$ with the trivial connection, and so $\pi(\tilde\gamma_X(t),\tilde\gamma_Y(t))$ is a constant function. Thus,

$$\pi^*\mathrm{d}\theta=\pi_X^*\alpha_X+\pi_Y^*\alpha_Y \tag{3.34}$$

on $S_XV\times_VS_YV$; this identity can also be verified using local coordinates.

We extend α_X and α_Y to 1-forms on $\mathcal{N}_XV-X$ and $\mathcal{N}_YV-Y$ by pulling back by the retractions

$$\mathcal{N}_XV-X\longrightarrow S_XV,\quad x\longrightarrow\frac{x}{|x|}\,,\qquad \mathcal{N}_YV-Y\longrightarrow S_YV,\quad y\longrightarrow\frac{y}{|y|}\,.$$

Denote by $\pi_{X,V}\colon\mathcal{N}_XV\longrightarrow V$ and $\pi_{Y,V}\colon\mathcal{N}_YV\longrightarrow V$ the bundle projections. Let

$$\rho_X\colon\mathcal{N}_XV\longrightarrow\mathbb{R},\quad \rho_X(x)=|x|^2,\qquad \rho_Y\colon\mathcal{N}_YV\longrightarrow\mathbb{R},\quad \rho_Y(y)=|y|^2\,;$$

we will denote the smooth functions $\rho_X\circ\pi_X$ and $\rho_Y\circ\pi_Y$ on $\mathcal{Z}_{\mathrm{neck}}$ also by ρ_X and ρ_Y, respectively. The closed forms

$$\omega_{X,V}=\pi_{X,V}^*\omega_V+\mathrm{d}(\rho_X\alpha_X)\qquad\text{and}\qquad \omega_{Y,V}=\pi_{Y,V}^*\omega_V+\mathrm{d}(\rho_Y\alpha_Y)$$

on the total spaces of $\mathcal{N}_XV$ and $\mathcal{N}_YV$ restrict to $\omega_X|_V$ and $\omega_Y|_V$ along V. By the Symplectic Neighborhood Theorem [MS1, Theorem 3.30], we can thus symplectically identify

$$D_X^{\epsilon}V\equiv\big\{x\in\mathcal{N}_XV\colon\ |x|<\epsilon\big\}\qquad\text{and}\qquad D_Y^{\epsilon}V\equiv\big\{y\in\mathcal{N}_YV\colon\ |y|<\epsilon\big\} \tag{3.35}$$

with tubular neighborhoods of V in X and Y, respectively, for some $\epsilon\in\mathbb{R}^+$ sufficiently small. By rescaling the metric on $\mathcal{N}_XY$ and the isomorphism (1.2), we can assume that $\epsilon=2$. Let $\mathcal{Z}$ be the smooth manifold obtained by gluing $\mathcal{Z}_X$, $\mathcal{Z}_Y$, and $\mathcal{Z}_{\text{neck}}$ by the open maps

$$\begin{aligned}\psi_X\colon \mathcal{Z}_{\text{neck}}-\mathcal{N}_YV &\longrightarrow \mathcal{Z}_X, \qquad (x,y)\longrightarrow(xy,x),\\ \psi_Y\colon \mathcal{Z}_{\text{neck}}-\mathcal{N}_XV &\longrightarrow \mathcal{Z}_Y, \qquad (x,y)\longrightarrow(xy,y).\end{aligned}$$

The projection map $\pi:\mathcal{Z}\longrightarrow\Delta$ described by (3.33) is then well-defined. By the Symplectic Neighborhood Theorem identification above,

$$\psi_X^*\omega_X=\pi_V^*\omega_V+\pi_X^*\mathrm{d}(\rho_X\alpha_X) \qquad\text{and}\qquad \psi_Y^*\omega_Y=\pi_V^*\omega_V+\pi_Y^*\mathrm{d}(\rho_Y\alpha_Y)\,, \tag{3.36}$$

where $\pi_V:\mathcal{Z}_{\text{neck}}\longrightarrow V$ is the bundle projection map.

We next define a symplectic structure $\omega_{\mathcal{Z}}$ on $\mathcal{Z}$. By (3.34),

$$\pi^*\omega_{\mathbb{C}}=\frac{1}{2}\mathrm{d}\big(\rho_X\rho_Y\pi^*\mathrm{d}\theta\big)=\frac{1}{2}\mathrm{d}\big(\rho_X\rho_Y(\pi_X^*\alpha_X+\pi_Y^*\alpha_Y)\big), \tag{3.37}$$

where $\omega_{\mathbb{C}}=\frac{1}{2}\mathrm{d}r^2\wedge\mathrm{d}\theta$ is the standard symplectic form on $\mathbb{C}$. We assume that $4\delta\le1$ from now on. Let $\eta\colon\mathbb{R}\longrightarrow[0,1]$ be a smooth function such that

$$\eta(r)=\begin{cases}0, & \text{if } r\le1-\delta;\\ 1, & \text{if } r\ge1;\end{cases} \qquad \delta|\eta'(r)|\le2.$$

By (3.36) and (3.37), the restrictions of the closed two-form

$$\begin{aligned}\omega_{\text{neck}}\equiv\pi_V^*\omega_V+\mathrm{d}\Big(&(1-\eta\circ\rho_Y)\pi_X^*(\rho_X\alpha_X)+\big(1-\eta\circ\rho_X\big)\pi_Y^*(\rho_Y\alpha_Y)\\ &+\frac{1}{2}\big(\eta\circ\rho_X+\eta\circ\rho_Y\big)\rho_X\rho_Y\big(\pi_X^*\alpha_X+\pi_Y^*\alpha_Y\big)\Big)\end{aligned}$$

on $\mathcal{Z}_{\text{neck}}$ to the regions $|x|\ge1$ and $|y|\ge1$ are

$$\begin{aligned}\pi_V^*\omega_V+\mathrm{d}\Big(\pi_X^*(\rho_X\alpha_X)+\frac{1}{2}\rho_X\rho_Y\big(\pi_X^*\alpha_X+\pi_Y^*\alpha_Y\big)\Big)&=\psi_X^*\omega_X+\pi^*\omega_{\mathbb{C}}=\psi_X^*(\omega_X+\pi^*\omega_{\mathbb{C}}) \quad\text{and}\\ \pi_V^*\omega_V+\mathrm{d}\Big(\pi_Y^*(\rho_Y\alpha_Y)+\frac{1}{2}\rho_X\rho_Y\big(\pi_X^*\alpha_X+\pi_Y^*\alpha_Y\big)\Big)&=\psi_Y^*\omega_Y+\pi^*\omega_{\mathbb{C}}=\psi_Y^*(\omega_Y+\pi^*\omega_{\mathbb{C}}),\end{aligned}$$

respectively. Thus, along with the two-forms $\omega_X+\pi^*\omega_{\mathbb{C}}$ on $\mathcal{Z}_X$ and $\omega_Y+\pi^*\omega_{\mathbb{C}}$ on $\mathcal{Z}_Y$, ω_{neck} induces a closed two-form $\omega_{\mathcal{Z}}$ on $\mathcal{Z}$. The restriction of ω_{neck} to the region $|x|,|y|\le(1-\delta)^{1/2}$ is

$$\begin{aligned}&\pi_V^*\omega_V+\pi_X^*\mathrm{d}(\rho_X\alpha_X)+\pi_Y^*\mathrm{d}(\rho_Y\alpha_Y)\\ &\qquad=\pi_V^*\omega_V+(\rho_X-\rho_Y)\pi_V^*D\alpha_X+\pi_X^*(\mathrm{d}\rho_X\wedge\alpha_X)+\pi_Y^*(\mathrm{d}\rho_Y\wedge\alpha_Y),\end{aligned} \tag{3.38}$$

where $D\alpha_X\in\Omega^2(V)$ is the curvature of α_X. This restriction is thus symplectic if $D\alpha_X$ is sufficiently small, which can be achieved by rescaling $\mathcal{N}_XV$. The restriction of ω_{neck} to the region $(1-\delta)^{1/2}\le|x|\le1$ is

$$\begin{aligned}\pi_V^*\omega_V&+\Big(1+\frac{1}{2}(\eta\circ\rho_X)\rho_Y\Big)\pi_X^*\mathrm{d}(\rho_X\alpha_X)+\Big((1-\eta\circ\rho_X)+\frac{1}{2}(\eta\circ\rho_X)\rho_X\Big)\pi_Y^*\mathrm{d}(\rho_Y\alpha_Y)\\ &+\frac{1}{2}\Big((\rho_X-2)\mathrm{d}(\eta\circ\rho_X)+(\eta\circ\rho_X)\mathrm{d}\rho_X\Big)\wedge\pi_Y^*(\rho_Y\alpha_Y)\\ &+\frac{1}{2}\Big(\rho_Y\mathrm{d}(\eta\circ\rho_X)+(\eta\circ\rho_X)\mathrm{d}\rho_Y\Big)\wedge\pi_X^*(\rho_X\alpha_X)\,.\end{aligned}$$

The first line describes a non-degenerate two-form on a compact subset of $\mathcal{N}_X V \oplus \mathcal{N}_Y V$ containing the above region (provided $D\alpha_X$ is sufficiently small). The two-form ϖ on the second line is of bounded norm, since $|y| \le \delta$ in this region. Its addition to the first line keeps the form non-degenerate, since the two-form on the first line vanishes on a complement to the subspace

$$\big\{v \in T_{(x,y)}\mathcal{Z}_{\rm neck}\colon\ \varpi(v,\cdot)=0\big\}.$$

Finally, the expression on the last line above is bounded by $C\delta$. This implies that $\omega_{\rm neck}$ is symplectic in the region $(1-\delta)^{1/2} \le |x| \le 1$ if δ is sufficiently small. For the same reason, $\omega_{\rm neck}$ is symplectic in the region $(1-\delta)^{1/2} \le |y| \le 1$ if δ is sufficiently small. Thus, the two-form $\omega_{\mathcal{Z}}$ is symplectic everywhere on $\mathcal{Z}$ if δ is sufficiently small.

We now define an $\omega_{\mathcal{Z}}$-compatible almost complex structure on $\mathcal{Z}$ which preserves the fibers of the fibration $\pi\colon \mathcal{Z} \longrightarrow \Delta$. The connection in $\mathcal{N}_X V$ induces a splitting of the exact sequence

$$0 \longrightarrow \pi_V^*(\mathcal{N}_X V \oplus \mathcal{N}_Y V) \longrightarrow T\mathcal{Z}_{\rm neck} \stackrel{{\rm d}\pi_V}{\longrightarrow} \pi_V^* TV \longrightarrow 0 \tag{3.39}$$

of vector bundles over $\mathcal{Z}_{\rm neck}$. The image of $\pi_V^* TV$ corresponding to this splitting is

$$\ker {\rm d}\rho_X \cap \ker \pi_X^*\alpha_X \cap \ker {\rm d}\rho_Y \cap \ker \pi_Y^*\alpha_Y \subset T\mathcal{Z}_{\rm neck}$$

outside of V, as can be seen from [Z2, Lemma 1.1], for example. By [IP4, Appendix], there exist $C>0$ and a smooth family $J_{V;\rho}$ with $\rho \in (-1,1)$ of almost complex structures on V such that $J_{V;\rho}$ is compatible with the symplectic form $\omega_V + \rho D\alpha_X$ and

$$\big\|J_{V;\rho} - J_{V;0}\big\| \le C\rho \qquad \forall\ \rho \in (-1,1). \tag{3.40}$$

We denote by $J_{\rm neck}$ the complex structure on $T_{(x,y)}\mathcal{Z}_{\rm neck}$ induced by the complex structure in the fibers of π_V and $J_{V;\rho_X(x)-\rho_Y(y)}$ on V via the splitting (3.39) and by $J_{{\rm neck};0}$ the complex structure induced by the former and $J_{V;0}$. The almost complex structure $J_{\rm neck}$ on $\mathcal{Z}_{\rm neck}$ is $\omega_{\mathcal{Z}}$-compatible. On the other hand, the restriction of $J_{{\rm neck};0}$ to $\pi_V^{-1}(\Sigma)$ is Kahler for every (real) surface $\Sigma \subset V$ preserved by $J_{V;0}$; see [Z2, Lemma 2.4]. Along with (3.40), this implies that

$$N_{J_{\rm neck}}(v,w) \in T_z V \qquad \forall\ v \in T_z V,\ w \in T_z\mathcal{Z},\ z \in V,$$

where $N_{J_{\rm neck}}$ is the Nijenhuis tensor of $J_{\rm neck}$.

The metric $\omega_{\mathcal{Z}}(\cdot, J_{\rm neck}\cdot)$ agrees with the product metric $g_{X,V} \oplus g_{Y,V} \oplus \omega_V(\cdot, J_{V;0}\cdot)$ in a trivialization of $\mathcal{N}_X V \oplus \mathcal{N}_Y V$ over an open subset of V to the second order in (x,y), since the splitting of (3.39) is $\omega_{\mathcal{Z}}$-orthogonal. Thus, the second fundamental form ${\rm II}_V$ of V with respect to the metric $\omega_{\mathcal{Z}}(\cdot, J_{\rm neck}\cdot)$ vanishes. For each $\lambda \in \Delta$,

$$\pi^{-1}(\lambda) \cap \mathcal{Z}_{\rm neck} = \big\{(x,y) \in \mathcal{N}_X V \oplus \mathcal{N}_Y V\colon\ xy = \lambda\big\} \longrightarrow V \tag{3.41}$$

is a fibration with fibers preserved by $J_{\rm neck}$. Thus, $J_{\rm neck}$ preserves $\pi_V^{-1}(\lambda)$. By [IP4, Lemma A.1], an ω_Z-compatible almost complex structure J_Z on a symplectic manifold (Z,ω_Z) preserves the tangent space of a symplectic submanifold $W \subset Z$ if and only if the ω_Z-orthogonal complement of TW is also orthogonal to TW with respect to the metric $\omega_Z(\cdot, J_Z\cdot)$. Since the almost complex structure $J_{\rm neck}$ on $\mathcal{Z}_{\rm neck}$ and product almost complex structures on $\mathcal{Z}_X$ and $\mathcal{Z}_Y$ preserve $\mathcal{Z}_\lambda \equiv \pi^{-1}(\lambda)$ for all $\lambda \in \Delta$, the metrics induced by them and $\omega_{\mathcal{Z}}$ can be patched together over the regions $(1-\delta)^{1/2}\le|x|\le1$ and $(1-\delta)^{1/2}\le|y|\le1$ in $\mathcal{Z}_{\rm neck}$ into a metric $g_{\mathcal{Z}}$ compatible with $\omega_{\mathcal{Z}}$ so that the corresponding almost complex structure $J_{\mathcal{Z}}$ preserves $\mathcal{Z}_\lambda$ everywhere.

We next compare the first chern class of $(\mathcal{Z}_\lambda,\omega|_{\mathcal{Z}_\lambda})$ for $\lambda\in\Delta^*$ with the first chern classes of (X,ω_X) and (Y,ω_Y). By the gluing construction of Section 3.1, 2-pseudocycles

$$f_X\colon (Z_X,x_1,\ldots,x_\ell)\longrightarrow(X,V) \qquad\text{and}\qquad f_Y\colon(Z_Y,y_1,\ldots,y_\ell)\longrightarrow(Y,V)$$

with boundary disjoint from V such that

$$\begin{gathered} f_X^{-1}(V)=\{x_1,\ldots,x_\ell\}, \qquad f_Y^{-1}(V)=\{y_1,\ldots,y_\ell\},\\ f_X(x_i)=f_Y(y_i), \quad \mathrm{ord}^V_{x_i}f_X=\mathrm{ord}^V_{y_i}f_Y \qquad \forall\, i=1,2,\ldots,\ell, \end{gathered}$$

determine a 2-pseudocycle $f_X\#_\lambda f_Y: Z_X\#Z_Y\longrightarrow\mathcal{Z}_\lambda$. By the symplectic sum construction above, the complex normal bundles of $X,Y\subset\mathcal{Z}$ are given by

$$\begin{aligned} \mathcal{N}_{\mathcal{Z}}X &\approx \big((X-V)\times\mathbb{C}\sqcup\pi^*_{X,V}\mathcal{N}_YV\big)/\!\sim, \qquad (x,xy)\sim(x,y) \quad \forall\,(x,y)\in\mathcal{N}_XV\oplus\mathcal{N}_YV,\\ \mathcal{N}_{\mathcal{Z}}Y &\approx \big((Y-V)\times\mathbb{C}\sqcup\pi^*_{Y,V}\mathcal{N}_XV\big)/\!\sim, \qquad (y,xy)\sim(y,x) \quad \forall\,(y,x)\in\mathcal{N}_YV\oplus\mathcal{N}_XV\,. \end{aligned}$$

Since the canonical meromorphic sections of these line bundles are nowhere zero and have polar divisors V,

$$\big\langle c_1(\mathcal{N}_{\mathcal{Z}}X),A\big\rangle=-A\cdot_XV \quad \forall\,A\in H_2(X;\mathbb{Z}), \quad \big\langle c_1(\mathcal{N}_{\mathcal{Z}}Y),B\big\rangle=-B\cdot_XV \quad \forall\,B\in H_2(Y;\mathbb{Z}).$$

On the other hand, the normal bundle of $\mathcal{Z}_\lambda$ with $\lambda\in\Delta^*$ is trivial. Since the homology class of $f_X\#_\lambda f_Y$ in $\mathcal{Z}$ is the sum of the homology classes of f_X and f_Y, it follows that

$$\begin{aligned} \big\langle c_1(T\mathcal{Z}_\lambda),[f_X\#_\lambda f_Y]\big\rangle &= \big\langle c_1(T\mathcal{Z}),[f_X]\big\rangle+\big\langle c_1(T\mathcal{Z}),[f_Y]\big\rangle\\ &= \big(\big\langle c_1(TX),[f_X]\big\rangle-[f_X]\cdot_XV\big)+\big(\big\langle c_1(TY),[f_Y]\big\rangle-[f_Y]\cdot_YV\big). \end{aligned}$$

In particular, the left-hand side of this expression depends only on the homology classes of $[f_X]$ in X and $[f_Y]$ in Y. Thus,

$$\big\langle c_1(T\mathcal{Z}_\lambda),A\#_\lambda B\big\rangle\in\mathbb{Z}$$

is well-defined for all $(A,B)\in H_2(X;\mathbb{Z})\times_VH_2(Y;\mathbb{Z})$. We have thus established the following.

Proposition 3.16. *Let (X,ω_X) and (Y,ω_Y) be compact symplectic manifolds and $V\subset X,Y$ be a symplectic hypersurface so that*

$$e(\mathcal{N}_XV)=-e(\mathcal{N}_YV)\in H^2(V;\mathbb{Z})\,.$$

There exist a symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$, a smooth map $\pi\colon\mathcal{Z}\longrightarrow\Delta$, and an $\omega_{\mathcal{Z}}$-compatible almost complex structure $J_{\mathcal{Z}}$ on $\mathcal{Z}$ such that

- *π is surjective and is a submersion over $\Delta^*\equiv\Delta-0$,*
- *the restriction ω_λ of $\omega_{\mathcal{Z}}$ to $\mathcal{Z}_\lambda\equiv\pi^{-1}(\lambda)$ is non-degenerate for every $\lambda\in\Delta^*$,*
- *$\mathcal{Z}_0=X\cup_VY$, $\omega_{\mathcal{Z}}|_X=\omega_X$, $\omega_{\mathcal{Z}}|_Y=\omega_Y$,*

- *$J_{\mathcal{Z}}$ preserves $T\mathcal{Z}_\lambda$ for every $\lambda\in\Delta^*$,*
- *$N_{J_{\mathcal{Z}}}(v,w)\in T_zV$ for all $v\in T_zV$, $w\in T_z\mathcal{Z}$, $z\in V$, and*
- *the second fundamental form II_V of V with respect to the metric $\omega_{\mathcal{Z}}(\cdot,J_{\mathcal{Z}}\cdot)$ vanishes.*

Furthermore,

$$\begin{aligned}\big\langle\omega_\lambda,A\#_\lambda B\big\rangle&=\langle\omega_X,A\rangle+\langle\omega_Y,B\rangle,\\ \big\langle c_1(T\mathcal{Z}_\lambda),A\#_\lambda B\big\rangle&=\big\langle c_1(TX),A\big\rangle+\big\langle c_1(TY),B\big\rangle-2A\cdot_X V\end{aligned}\tag{3.42}$$

for all $\lambda\in\Delta^*$ and $(A,B)\in H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z})$.

The algebraic approach of [Lj2] considers only Kahler fibrations $\pi\colon\mathcal{Z}\longrightarrow\Delta$ which come with an ample line bundle $\mathcal{L}\longrightarrow\mathcal{Z}$. Since every element of $\mathcal{R}^V_{X,Y}$ in $\mathcal{Z}_\lambda$ can be represented by a totally real submanifold, its homology intersection with every complex hyperplane in $\mathcal{Z}_\lambda$ is zero. Thus, by the Lefschetz Theorem on $(1,1)$-classes [GH, p163], an element of $\mathcal{R}^V_{X,Y}$ in $\mathcal{Z}_\lambda$ determines a class in $H^{n-2,n}(\mathcal{Z}_\lambda)\oplus H^{n,n-2}(\mathcal{Z}_\lambda)$, where n is the complex dimension of $\mathcal{Z}_\lambda$, X, and Y. In particular, curve classes in $H_2(\mathcal{Z}_\lambda;\mathbb{Z})$ differing by an element of $\mathcal{R}^V_{X,Y}$ differ by a torsion class; this observation establishes the following.

Proposition 3.17. *Let $\pi:\mathcal{Z}\longrightarrow\Delta$ be a one-dimensional family of projective varieties with a polarization $\mathcal{L}\longrightarrow\mathcal{Z}$ such that the central fiber $\mathcal{Z}_0$ consists of two smooth irreducible components X and Y intersecting transversally along a smooth divisor $V\subset X,Y$. Let $Z=\mathcal{Z}_\lambda$ for some $\lambda\in\Delta^*$. If $C_1,C_2\in H_2(Z;\mathbb{Z})$ are such that $C_1-C_2\in\mathcal{R}^V_{X,Y}$ and $\mathrm{GW}_{Z,g_1,C_1},\mathrm{GW}_{Z,g_2,C_2}\neq0$ for some $g_1,g_2\in\mathbb{Z}^{\ge0}$, then C_1-C_2 is a torsion class.*

Thus, the symplectic sum formula in the algebraic setting in fact expresses sums of Gromov-Witten invariants of $\mathcal{Z}_\lambda=X\#Y$ in degrees differing by torsion in terms of Gromov-Witten invariants of X and Y; so the deficiency of this formula described in the second halves of Sections 1 and 3.2 is at most minor in these cases.

Remark 3.18. The above reasoning does not apply outside of the Kahler setting. For example, let $X=S^2\times\mathbb{T}^2$, $V=\{0,\infty\}\times\mathbb{T}^2$,

$$f_1,f_2\colon\mathbb{T}^2\longrightarrow X-V,\qquad f_1(\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2})=(2,\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2}),\quad f_2(\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2})=(\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2}).$$

Since the images of the embeddings f_1 and f_2 are disjoint symplectic submanifolds of X, we can choose an almost complex structure J_X on X which is standard around V and makes these images J_X-holomorphic. The two maps f_1 and f_2 differ by a rim torus. Since both maps miss V, they induce J_Z-holomorphic maps into $Z=X\#_V X$, which differ by a non-trivial element of $\mathcal{R}^V_{X,X}\approx\mathbb{Z}^2$; see Example 3.5.

Remark 3.19. The moment maps appearing in [IP5, Section 2] play no role in the symplectic sum construction described there. The justification of (3.37) in [IP5, Section 2] is incomplete and refers to $\pi_X^*\alpha_X+\pi_Y^*\alpha_Y$ as a connection one-form on $\mathcal{N}_XV\oplus\mathcal{N}_YV$, which differs from the standard usage of connection one-form. The second identity in (3.42) is [IP5, Lemma 2.4]. Our proof of this identity add details to the proof in [IP5] and in particular formally extends over X and Y the key bundles that are described only over neighborhoods of V in [IP5].

The symplectic sum construction is approached in [LR, Section 3.0] from the opposite direction by cutting $(M,\omega)=(X\#_V Y,\omega_\#)$ into two pieces M^- and M^+ along a compact hypersurface $\widetilde{M}$. This hypersurface is the preimage of a regular value of a Hamiltonian H on a neighborhood U of $\widetilde{M}$ generating a free S^1-action on $\widetilde{M}$. By the Mardsen-Weinstein construction [MS1, Section 5.4], the quotient $V=\widetilde{M}/S^1$ is then a smooth manifold with a symplectic form ω_V such that $\pi^*\omega_V=\omega|_{\widetilde{M}}$, where $\pi:\widetilde{M}\longrightarrow V$ is the projection (in [LR], (V,ω_V) is denoted by (Z,τ_0)). The symplectic cutting construction of [Ler] collapses the ends of M^- and M^+ and produces symplectic manifolds $(\overline{M}^-,\omega_-)$ and $(\overline{M}^+,\omega_+)$ containing (V,ω_V) as a symplectic hypersurface with dual normal bundles. In the description of [IP5, Section 2] recalled above, $\widetilde{M}$ corresponds to the hypersurface

$$SV_\lambda \equiv \big\{(x,y)\in\mathcal{Z}_{\mathrm{neck}}:\ xy=\lambda,\ |x|=|y|\big\}\subset\mathcal{Z}_\lambda\,,$$

with $\lambda\in\mathbb{C}^*$ small. The symplectic manifolds $(\overline{M}^-,\omega_-)$ and $(\overline{M}^+,\omega_+)$ obtained in this way are symplectically deformation equivalent to (X,ω_X) and (Y,ω_Y). We will identify SV_λ with the sphere (circle) bundle $SV\equiv S_XV$ of $\mathcal{N}_XV$ and use the isomorphism (1.2) to identify S_YV with SV, i.e.

$$S_YV\ni y\longleftrightarrow x\in SV=S_XV \qquad \text{if}\quad xy=1\in\mathbb{C}. \tag{3.43}$$

In particular, we use the complex structure on $\mathcal{N}_XV$ to induce an S^1-action on SV for the purposes of the approach in [LR]; the complex structure on $\mathcal{N}_YV$ would induce the inverse S^1-action on SV. The restriction of the Hamiltonian vector field ζ_H, denoted X_H in [LR, Section 3.0], to $\widetilde{M}$ then corresponds to the characteristic vector field of the S^1-action on SV, i.e. $\frac{\mathrm{d}}{\mathrm{d}\vartheta}(\mathrm{e}^{\mathrm{i}\vartheta}x)\big|_{\vartheta=0}$ at each $x\in SV$. Let $\alpha=\alpha_X$ be a connection one-form on SV as before (denoted by λ in [LR]).

The family of complex structures $\tilde{J}_\lambda$ on $\mathcal{Z}_\lambda$ used in [LR] is more restrictive than in [IP5] on the necks. Given $\delta'\in(0,\frac{1}{4})$, let $\tilde{\eta}:\mathbb{R}\longrightarrow[0,1]$ be a smooth function such that

$$\tilde{\eta}(r)=\begin{cases}0, & \text{if } r\le\delta';\\ 1, & \text{if } r\ge 2\delta'.\end{cases}$$

With $J_{V;\rho}$ as in (3.40), let

$$J_{V;\rho_+,\rho_-}=J_{V;\tilde{\eta}(\rho_++\rho_-)(\rho_+-\rho_-)} \qquad \forall\ \rho_+,\rho_-\in[0,1].$$

If δ' is sufficiently small, this almost complex structure on V is tamed by the symplectic form $\omega_V+(\rho_+-\rho_-)D\alpha$. We denote by $\tilde{J}_{\mathrm{neck}}$ the complex structure on $T_{(x,y)}\mathcal{Z}_{\mathrm{neck}}$ induced by the complex structure in the fibers of $\pi_V:\mathcal{Z}_{\mathrm{neck}}\longrightarrow V$ and $J_{V;\rho_X(x),\rho_Y(y)}$ on V via the splitting (3.39) described above. The almost complex structure $\tilde{J}_{\mathrm{neck}}$ again preserves the fibers of (3.41) for each $\lambda\in\Delta$. It is tamed by the symplectic form $\omega_{\mathcal{Z}}$ constructed above everywhere on $\mathcal{Z}_{\mathrm{neck}}$ and compatible with $\omega_{\mathcal{Z}}$ over the regions $(1-\delta)^{1/2}\le|x|\le1$ and $(1-\delta)^{1/2}\le|y|\le1$. Thus, as below (3.41), we can patch the metrics induced by $\omega_{\mathcal{Z}}$, $\tilde{J}_{\mathrm{neck}}$, and the product almost complex structures on $\mathcal{Z}_X$ and $\mathcal{Z}_Y$ over these two regions into a metric $\hat{g}_{\mathcal{Z}}$ compatible with $\omega_{\mathcal{Z}}$ on the two regions so that the corresponding almost complex structure $\tilde{J}_{\mathcal{Z}}$ preserves $\mathcal{Z}_\lambda$ everywhere. We denote the restrictions of $(\tilde{J}_{\mathcal{Z}},\hat{g}_{\mathcal{Z}})$ to X, Y, and $\mathcal{Z}_\lambda$ with $\lambda\in\Delta^*$ small by $(\tilde{J}_X,\hat{g}_X)$, $(\tilde{J}_Y,\hat{g}_Y)$, and $(\tilde{J}_\lambda,\hat{g}_\lambda)$, respectively.

The stretching construction of [LR] presents the complements of V in tubular neighborhoods in X and Y as bundles over V whose fibers are infinite cylinders. In the notation of [IP5], the "height" coordinates can be taken to be

$$a_X(x) = \ln|x| \qquad \text{and} \qquad a_Y(y) = -\ln|y|$$

on X and Y, respectively. Thus,

$$\mathbb{R}^-\times SV \subset \mathring{X}_V \equiv X-V, \quad \mathbb{R}^+\times SV \subset \mathring{Y}_V \equiv Y-V, \tag{3.44}$$

X and Y are quotients of the manifolds with boundary

$$\widehat{X}_V \equiv \big(\mathring{X}_V \cup_{\mathbb{R}^-\times SV} [-\infty,0)\times V\big)/\sim \quad \text{and} \quad \widehat{Y}_V \equiv \big(\mathring{Y}_V \cup_{\mathbb{R}^+\times SV} (0,\infty]\times V\big)/\sim, \tag{3.45}$$

respectively, with $V\subset X,Y$ being the quotient of $\{\mp\infty\}\times SV$ by the S^1-action. For each $a\in(0,\infty]$, let

$$X_a = \mathring{X}_V - \big\{(a_X,x)\in\mathbb{R}^-\times SV\colon\ a_X\le -\frac{3}{4}a\big\}, \qquad Y_a = \mathring{Y}_V - \big\{(a_Y,y)\in\mathbb{R}^+\times SV\colon\ a_Y\ge \frac{3}{4}a\big\}.$$

In the approach of [LR], the symplectic sum $\mathcal{Z}_\lambda$ of [IP5] is viewed as

$$\mathcal{Z}_{a,\vartheta} = \big(X_a\sqcup Y_a\big)/\sim, \quad X_a-\overline{X}_{\frac{a}{3}} \ni (a_X,x) \sim (a_X+a, \mathrm{e}^{\mathrm{i}\vartheta}x) \in Y_a-\overline{Y}_{\frac{a}{3}} \tag{3.46}$$

if $\lambda^{-1}=\mathrm{e}^{a+\mathrm{i}\vartheta}$; in the notation of [LR, (4.11,4.12)], $(a,\vartheta)=(4kr,\theta_0)$ and $(a_X,a_Y)=(a_2,a_1)$. For any $\epsilon\in(0,1]$, let

$$\begin{aligned} \mathcal{Z}_{a,\vartheta;\epsilon} &= \big\{(x,y)\in\mathcal{Z}_{\mathrm{neck}}\colon\ xy=\lambda,\ |x|,|y|\le\epsilon^{1/2}\big\} \\ &= \big\{(a_X,x)\in\mathbb{R}^-\times SV\colon\ \frac{\ln\epsilon}{2}\ge a_X\ge -\frac{3a}{4}\big\} \cup \big\{(a_Y,y)\in\mathbb{R}^+\times SV\colon -\frac{\ln\epsilon}{2}\le a_Y\le \frac{3a}{4}\big\}, \end{aligned} \tag{3.47}$$

with the union on the second line taken inside of $\mathcal{Z}_{a,\vartheta}$. Denote by $\frac{\partial}{\partial a_\lambda}$ the vector field on $\mathcal{Z}_{\lambda;1}$ restricting to $\frac{\partial}{\partial a_X}$ on the intersection with X_a and to $\frac{\partial}{\partial a_Y}$ on the intersection with Y_a. The almost complex structure $\tilde{J}$ of the previous paragraph satisfies

$$\tilde{J}_X\frac{\partial}{\partial a_X} = \zeta_H \ \text{ on } X_a-X_{-\frac{2}{3}\ln\delta'}, \quad \tilde{J}_Y\frac{\partial}{\partial a_Y} = \zeta_H \ \text{ on } Y_a-Y_{-\frac{2}{3}\ln\delta'}, \quad \tilde{J}_\lambda\frac{\partial}{\partial a_\lambda} = \zeta_H \ \text{ on } \mathcal{Z}_{a,\vartheta;\delta'}\,.$$

It restricts to the pull-back of J_V on $\ker\alpha\subset T(SV)\subset T\mathcal{Z}_{a,\vartheta}$ and differs slightly from the initially fixed almost complex structures J_X and J_Y over

$$\Big\{(a_X,x)\in\mathbb{R}^-\times SV\colon\ a_X\ge\frac{\ln\delta'}{2}\Big\}\subset X \qquad \text{and} \qquad \Big\{(a_Y,y)\in\mathbb{R}^+\times SV\colon\ a_Y\le -\frac{\ln\delta'}{2}\Big\}\subset Y\,,$$

respectively, in a way depending by λ.

Finally, we specify complete metrics $\tilde{g}_X$, $\tilde{g}_Y$, and $\tilde{g}_{a,\vartheta}$ on $\mathring{X}_V$, $\mathring{Y}_V$, and $\mathcal{Z}_{a,\vartheta}$, respectively. Let $\hat{\eta}(r)=\tilde{\eta}(16r)$. Denote by g_{cyl} the metric on $\mathbb{R}\times SV$ given by

$$g_{\mathrm{cyl}}\big((a_1,v_1),(a_2,v_2)\big) = a_1a_2 + \alpha(v_1)\alpha(v_2) + q_V^*g_V(v_1,v_2),$$

where $g_V(\cdot,\cdot)=\omega_V(\cdot,J_V\cdot)$ is the metric on V induced by J_V. Following [LR, (3.7),(3.8)], we define the metrics $\tilde{g}_X$ on $X-V$ and $\tilde{g}_Y$ on $Y-V$ by

$$\tilde{g}_X|_x=\begin{cases}\hat{g}_X|_x, & \text{if } x\in\mathring{X}_V-(-\infty,-1)\times SV;\\ \hat{\eta}(\rho_X(x))\hat{g}_X|_x+(1-\hat{\eta}(\rho_X(x)))g_{\mathrm{cyl}}|_x, & \text{if } x\in\mathbb{R}^-\times SV;\end{cases}$$

$$\tilde{g}_Y|_y=\begin{cases}\hat{g}_Y|_y, & \text{if } y\in\mathring{Y}_V-(1,\infty)\times SV;\\ \hat{\eta}(\rho_Y(y))\hat{g}_Y|_y+(1-\hat{\eta}(\rho_y(y)))g_{\mathrm{cyl}}|_y, & \text{if } y\in\mathbb{R}^+\times SV.\end{cases}$$

For each $a\in\mathbb{R}+$ sufficiently large, we similarly define

$$\tilde{g}_{a,\vartheta}|_x=\begin{cases}\hat{g}_\lambda|_x, & \text{if } x\in\mathcal{Z}_{a,\vartheta}-\mathcal{Z}_{a,\vartheta;1};\\ \hat{\eta}(\rho_X(x)+\rho_Y(x))\hat{g}_{\mathrm{e}^{-(a+\mathrm{i}\vartheta)}}|_x+(1-\hat{\eta}(\rho_X(x)+\rho_Y(x)))g_{\mathrm{cyl}}|_x, & \text{if } x\in\mathcal{Z}_{a,\vartheta;2}.\end{cases}$$

This metric agrees with the cylindrical metric on $\mathcal{Z}_{a,\vartheta;\delta'/32}$. Its injectivity radius is uniformly (independently of (a,ϑ)) bounded below and the norm of its Riemannian curvature tensor is uniformly bounded above.

Remark 3.20. The review of the symplectic sum and cutting constructions in [LR] consists of [LR, Examples 2.6-2.8]. In particular, the symplectic form $\omega_{\mathcal{Z}}|_{\mathcal{Z}_\lambda}$ on the glued manifold in [LR, (4.11,4.12)] is not specified; as indicated above, constructing such a form is not trivial. The symplectic form ω_0 on $\mathbb{C}^n$ in [LR, Example 2.6] is not specified. The second set on the RHS of [LR, (2.6)] can be easily absorbed into the first; it would perhaps be clearer to describe $\mu^{-1}(0)$ as $|z|^2+|w|^2=\epsilon$. Since z is a vector, the expression $z\mathrm{d}\bar{z}$ in [LR, (2.8)] does not make sense; the intended meaning is presumably as in [LR, (2.16)]. The formula [LR, (2.8)] does not seem to appear in [MS1]. The S^1-action for the Hamiltonian in [LR, (2.10)] with respect to the symplectic form in [LR, (2.9)] is given by the multiplication by $\mathrm{e}^{-\mathrm{i}t/\epsilon}$, not as in [LR, (2.11)]. The wording of [LR, Lemma 2.5] is incorrect; there should be a homotopy of the maps φ as well. The third sentence on page 165 in [LR] is vague. The wording of the paragraph in [LR] containing this sentence suggests that the symplectic blowup construction involves almost complex structures, which is not the case; it is described explicitly on pages 239-250 in [MS1]. A direct connection of this paragraph to [LR, Proposition 2.10] is also unclear. Other, fairly minor misstatements in [LR, Sections 2,3.0] include

- p161, Ex 2.2: $\mathrm{e}^{\mathrm{i}\theta}\longrightarrow\mathrm{e}^{\mathrm{i}\theta/2}$;
- p161, Dfn 2.3: concave dfn is correct only if N is connected;
- p161, bottom: not by (1.10); *The map (1.10) induces a homomorphism...*
- p162, top: [FO] and [LT] do not require integrality;
- p162, line 7: this equality does not hold, as LHS is degenerate along $\pi^{-1}(Z)$;
- p162, after (2.9): on the whole total space, as used above Prop. 2.10;
- p163, lines 4,13: Example 1 $\longrightarrow$ Example 2.6;
- p164, line -17: φ is not specified in (2.13);
- p164, line -6: the antipodal $\longrightarrow$ a conjugation;
- p165, line 12: Example 2 $\longrightarrow$ Example 2.7;
- p165, Lemma 2.11, proof: $\overline{M}^-$ is not a subset of M;
- p166, line 14: M_t has not been defined;
- p168, Section 3: need to require $H^{-1}(0)$ to be compact and the S^1-action to be free; the relation of $\mathrm{d}\lambda$ with the Chern class is irrelevant;
- p168, lines -9,-8,-3: $\{$ and $\}$ should not be here; identifications along $\{\pm\ell\}\times\widetilde{M}$;
- p169, line 1: this sentence does not make sense and is not used here;
- p169, (3.7),(3.8): these metrics need to be patched together;
- p170, line 1: Π can be taken to be $\mathrm{d}\pi$.

4 The symplectic sum formula

4.1 Main Statement: [IP5, Sections 0,1,10-13,16]

For $g,k\in\mathbb{Z}^{\ge 0}$ and $\chi\in\mathbb{Z}$, denote by $\overline{\mathcal{M}}_{g,k}$ and $\widetilde{\mathcal{M}}_{\chi,k}$ the Deligne-Mumford moduli spaces of stable nodal k-marked complex curves with connected domains of genus g and with (possibly) disconnected domains of double holomorphic euler characteristic χ, respectively; in the unstable range, $2g+k<3$ and $k-\chi<1$, we define each of these spaces to be a point. Let

$$\overline{\mathcal{M}}=\bigsqcup_{g,k\in\mathbb{Z}^{\ge 0}}\overline{\mathcal{M}}_{g,k}, \qquad \widetilde{\mathcal{M}}=\bigsqcup_{\chi\in\mathbb{Z},k\in\mathbb{Z}^{\ge 0}}\widetilde{\mathcal{M}}_{\chi,k}\,.$$

A rule of assignment is a bijection

$$\vartheta\colon\{1,\ldots,k_1\}\sqcup\{1,\ldots,k_2\}\longrightarrow\{1,\ldots,k_1+k_2\} \tag{4.1}$$

for some $k_1,k_2\in\mathbb{Z}^{\ge 0}$ preserving the ordering of the elements in each of the two subsets of the domain. Let RA denote the set of all rules of assignment. If in addition $\ell\in\mathbb{Z}^{\ge 0}$, let

$$\xi_{\ell,\vartheta}\colon\widetilde{\mathcal{M}}_{\chi_1,k_1+\ell}\times\widetilde{\mathcal{M}}_{\chi_2,k_2+\ell}\longrightarrow\widetilde{\mathcal{M}}_{\chi_1+\chi_2-2\ell,k_1+k_2} \tag{4.2}$$

be the morphism obtained by identifying the (k_1+i)-th point on the first curve with the (k_2+i)-th point on the second curve for $i=1,\ldots,\ell$ and ordering the remaining points by the bijection ϑ.

Let (X,ω) be a compact symplectic manifold, $V\subset X$ be a closed symplectic hypersurface, J be an ω-compatible almost complex structure, such that $J(TV)=TV$, and $A\in H_2(X;\mathbb{Z})$. There are natural stabilization morphisms

$$\mathrm{st}\colon\widetilde{\mathcal{M}}_{\chi,k}(X,A)\longrightarrow\widetilde{\mathcal{M}}_{\chi,k}, \qquad \mathrm{st}\colon\widetilde{\mathcal{M}}^V_{\chi,k;\mathbf{s}}(X,A)\longrightarrow\widetilde{\mathcal{M}}_{\chi,k+\ell}, \tag{4.3}$$

forgetting the map and contracting the unstable components of the domain. We denote the restrictions of these maps to

$$\overline{\mathcal{M}}_{g,k}(X,A)\subset\widetilde{\mathcal{M}}_{2-2g,k}(X,A) \qquad\text{and}\qquad \overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A)\subset\widetilde{\mathcal{M}}^V_{2-2g,k;\mathbf{s}}(X,A)$$

by the same symbols. The morphisms (1.4) and (4.3) give rise to the (absolute) Gromov-Witten and Gromov-Taubes invariants of (X,ω_X) with descendants,

$$\mathrm{GW}_{X,A,g}\colon\mathbb{T}^*(X)\longrightarrow H_*(\overline{\mathcal{M}}), \quad \mathrm{GW}_{X,A,g}(\alpha)=\sum_{k=0}^{\infty}\mathrm{st}_*\big(\mathrm{ev}^*\alpha\cap\big[\overline{\mathcal{M}}_{g,k}(X,A)\big]^{\mathrm{vir}}\big), \tag{4.4}$$
$$\mathrm{GT}_{X,A,\chi}\colon\mathbb{T}^*(X)\longrightarrow H_*(\widetilde{\mathcal{M}}), \quad \mathrm{GT}_{X,A,\chi}(\alpha)=\sum_{k=0}^{\infty}\mathrm{st}_*\big(\mathrm{ev}^*\alpha\cap\big[\widetilde{\mathcal{M}}_{\chi,k}(X,A)\big]^{\mathrm{vir}}\big),$$

where H_* denotes the homology with $\mathbb{Q}$-coefficients. They also give rise to the relative Gromov-Witten and Gromov-Taubes invariants of (X,V,ω),

$$\mathrm{GW}^V_{X,A,g;\mathbf{s}}\colon\mathbb{T}^*(X)\longrightarrow H_*(\overline{\mathcal{M}}\times V_{\mathbf{s}}), \quad \mathrm{GW}^V_{X,A,g;\mathbf{s}}(\alpha)=\sum_{k=0}^{\infty}\{\mathrm{st}\times\mathrm{ev}^V\}_*\big(\mathrm{ev}^*\alpha\cap\big[\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A)\big]^{\mathrm{vir}}\big),$$
$$\mathrm{GT}^V_{X,A,\chi;\mathbf{s}}\colon\mathbb{T}^*(X)\longrightarrow H_*(\widetilde{\mathcal{M}}\times V_{\mathbf{s}}), \quad \mathrm{GT}^V_{X,A,\chi;\mathbf{s}}(\alpha)=\sum_{k=0}^{\infty}\{\mathrm{st}\times\mathrm{ev}^V\}_*\big(\mathrm{ev}^*\alpha\cap\big[\widetilde{\mathcal{M}}^V_{\chi,k;\mathbf{s}}(X,A)\big]^{\mathrm{vir}}\big).$$

We assemble the homomorphisms $\mathrm{GT}_{X\#_V Y,C,\chi}$ and $\mathrm{GT}^V_{M,A,\chi;\mathbf{s}}$ into generating functions as in (1.7) and (1.8):

$$\mathrm{GT}_{X\#_V Y} = \sum_{\chi\in\mathbb{Z}} \sum_{\eta\in H_2(X\#_V Y;\mathbb{Z})/\mathcal{R}^V_{X,Y}} \sum_{C\in\eta} \mathrm{GT}_{X\#_V Y,C,\chi}\, t_\eta \lambda^\chi, \tag{4.5}$$

$$\mathrm{GT}^V_M = \sum_{\chi\in\mathbb{Z}} \sum_{A\in H_2(M;\mathbb{Z})} \sum_{\ell=0}^{\infty} \sum_{\substack{\mathbf{s}\in(\mathbb{Z}^+)^\ell \\ |\mathbf{s}|=A\cdot_M V}} \mathrm{GT}^V_{M,A,\chi;\mathbf{s}}\, t_A \lambda^\chi. \tag{4.6}$$

The generating functions in (1.7) and (1.8) are the sums of the terms in the generating functions in (4.5) and (4.6), respectively, that are of $\widetilde{\mathcal{M}}$-degree 0.

If ϑ is a rule of assignment as in (4.1) and

$$\alpha \equiv (\alpha_{1;X}, \alpha_{1;Y}) \otimes \ldots \otimes (\alpha_{k;X}, \alpha_{k;Y}) \in H^{2*}(X \sqcup_V Y)^{\otimes k},$$

we define

$$\alpha_{\vartheta;X} = \alpha_{\vartheta(1,1);X} \otimes \ldots \otimes \alpha_{\vartheta(1,k_1);X} \in \mathbb{T}^*(X), \quad \alpha_{\vartheta;Y} = \alpha_{\vartheta(2,1);Y} \otimes \ldots \otimes \alpha_{\vartheta(2,k_2);Y} \in \mathbb{T}^*(Y),$$
$$\text{and} \qquad \alpha_\vartheta = \alpha_{\vartheta;X} \otimes \alpha_{\vartheta;Y} \in \mathbb{T}^*(X) \otimes \mathbb{T}^*(Y)$$

if $k_1+k_2=k$ and $\alpha_\vartheta = 0$ otherwise. Using the pairing $\star$ of (1.9), we define the pairing

$$\star_\vartheta \colon H_*(\widetilde{\mathcal{M}}\times V_\infty) \otimes H_*(\widetilde{\mathcal{M}}\times V_\infty) \longrightarrow H_*(\widetilde{\mathcal{M}})[\lambda^{-1}]$$

to be given by the composition

$$H_*(\widetilde{\mathcal{M}}_{\chi_1,k_1+\ell(\mathbf{s})}\times V_{\mathbf{s}}) \otimes H_*(\widetilde{\mathcal{M}}_{\chi_2,k_2+\ell(\mathbf{s})}\times V_{\mathbf{s}}) = H_*\big(\widetilde{\mathcal{M}}_{\chi_1,k_1+\ell(\mathbf{s})} \times \widetilde{\mathcal{M}}_{\chi_2,k_2+\ell(\mathbf{s})}\big) \otimes H_*(V_{\mathbf{s}}) \otimes H_*(V_{\mathbf{s}})$$
$$\xrightarrow{\xi_{\ell(\mathbf{s}),\vartheta*}\otimes\star} H_*(\widetilde{\mathcal{M}}) \otimes \mathbb{Q}[\lambda^{-1}] = H_*(\widetilde{\mathcal{M}})[\lambda^{-1}]$$

on the specified summands and be 0 on the remaining summands. This pairing induces a pairing

$$\star_\vartheta \colon \mathrm{Hom}\big(\mathbb{T}^*(X), H_*(\widetilde{\mathcal{M}}\times V_\infty)\big) \otimes \mathrm{Hom}\big(\mathbb{T}^*(Y), H_*(\widetilde{\mathcal{M}}\times V_\infty)\big)$$
$$\longrightarrow \mathrm{Hom}\big(\mathbb{T}^*(X)\otimes\mathbb{T}^*(Y), H_*(\widetilde{\mathcal{M}}\times V_\infty)\big)\big[\lambda^{-1}\big]$$

as in (1.10), which we extend as in (1.11), replacing $\star$ by $\star_\vartheta$.

Theorem 4.1. *Let (X,ω_X) and (Y,ω_Y) be symplectic manifolds and $V\subset X,Y$ be a symplectic hypersurface satisfying (1.1). If $q_\# : X\#_V Y \longrightarrow X\cup_V Y$ is a collapsing map for an associated symplectic sum fibration and $q_\sqcup \colon X\sqcup Y \longrightarrow X\cup_V Y$ is the quotient map, then*

$$\mathrm{GT}_{X\#_V Y}(q_\#^*\alpha) = \sum_{\vartheta\in\mathrm{RA}} \big\{\mathrm{GT}^V_X \star_\vartheta \mathrm{GT}^V_Y\big\}\big((q_\sqcup^*\alpha)_\vartheta\big) \qquad \forall\, \alpha\in\mathbb{T}^*(X\cup_V Y). \tag{4.7}$$

The identity (4.7) readily extends to cover descendant invariants (ψ-classes). Furthermore, it is not necessary to assume that X and Y are different manifolds: the reasoning behind Theorem 4.1 readily applies to symplectic manifolds obtained by gluing along two disjoint hypersurfaces V_1 and V_2 in X which have dual normal bundles.

In [IP5, Section 1], the absolute GW/GT-invariants of X are defined as cycles in a space involving a Cartesian product of copies of X, while the relative GW/GT-invariants of (X,V) are defined as homomorphisms on $\mathbb{T}^*(X)$ and $\mathbb{T}^*(Y)$. The former is inconsistent with the main symplectic sum formulas in [IP5], i.e. (0.2), (10.14), and (12.17). The GT-invariants are formally defined as exponentials of the GW-invariants. According to [IP5, p944],

$$\mathrm{GT}_X = 1+\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\substack{A_1,\ldots,A_m\in H_2(X;\mathbb{Z})\\ g_1,\ldots,g_m\ge 0\\ k_1,\ldots,k_m\ge 0}}\frac{\mathrm{GW}_{X,A_1,g_1,k_1}\cdot\ldots\cdot\mathrm{GW}_{X,A_m,g_m,k_m}}{k_1!\ldots k_m!}t_{A_1+\ldots+A_m}\lambda^{2(m-g_1-\ldots-g_m)}\,,$$

where $\mathrm{GW}_{X,A,g,k}$ is the homomorphism corresponding to the k-th summand in (4.4) and $\cdot$ is some (unspecified) product on $\mathrm{Hom}(\mathbb{T}^*(X),H_*(\widetilde{\mathcal{M}}))$. The wording at the top of page 948 in [IP5] is somewhat misleading, as [IP5, (1.24)] is the definition of GW_X^V in [IP5], not a consequence of another definition. With this interpretation, [IP5, (1.25)] gives

$$\mathrm{GT}_X^V = 1+\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\substack{A_1,\ldots,A_m\in H_2(X;\mathbb{Z})\\ g_1,\ldots,g_m\in\mathbb{Z}^{\ge 0}\\ \mathbf{s}_1,\ldots,\mathbf{s}_m}}\frac{\mathrm{GW}^V_{X,A_1,g_1;\mathbf{s}_1}\cdot\ldots\cdot\mathrm{GW}^V_{X,A_m,g_m;\mathbf{s}_m}}{\ell(\mathbf{s}_1)!\ldots\ell(\mathbf{s}_m)!}t_{A_1+\ldots+A_m}\lambda^{2(m-g_1-\ldots-g_m)}\,, \tag{4.8}$$

where $\cdot$ is some (unspecified) product on $\mathrm{Hom}(\mathbb{T}^*(X),H_*(\widetilde{\mathcal{M}}\times\mathcal{H}_X^V))$ and

$$\mathcal{H}_X^V=\bigsqcup_{\ell=0}^{\infty}\bigsqcup_{\mathbf{s}\in(\mathbb{Z}^+)^\ell}\mathcal{H}_{X;\mathbf{s}}^V\,.$$

In particular, the normalizations of GT_X and GT_X^V with respect to the absolute marked points in [IP5] are inconsistent. Thus, the symplectic sum formulas of [IP5], even without the rim tori and the S-matrix features, do not recover (1.12).

In [IP5, Section 16], the usual (without rim tori refinement) relative GT-invariants of (X,V) are described in terms of counts of disconnected curves. If $\{\gamma_i\}$ is a basis for $H^*(V)\equiv H^*(V;\mathbb{Q})$ and $\{\gamma_i^\vee\}$ is the dual basis for $H_*(V)$, then

$$\begin{aligned} C_{\mathbf{s},\mathbf{I}}&\equiv\gamma_{i_1}\otimes\ldots\otimes\gamma_{i_\ell}\in H^*(V)^{\otimes\ell}\approx H^*(V_{\mathbf{s}}), &&\text{with}\quad \mathbf{I}=(i_1,\ldots,i_\ell),\\ C^\vee_{\mathbf{s},\mathbf{I}}&\equiv\gamma^\vee_{i_1}\otimes\ldots\otimes\gamma^\vee_{i_\ell}\in H_*(V)^{\otimes\ell}\approx H_*(V_{\mathbf{s}}), &&\text{with}\quad \mathbf{I}=(i_1,\ldots,i_\ell),\end{aligned}$$

are dual bases for $H^*(V_{\mathbf{s}})$ and $H_*(V_{\mathbf{s}})$, respectively, for compatible choices of the above isomorphisms. According to [IP5, (A.3)],

$$\begin{aligned}\mathrm{GT}_X^V(\kappa,\alpha)&\equiv\kappa\cap\mathrm{GT}_X^V(\alpha)\\ &=\sum_{A,\chi}\sum_{\ell(\mathbf{s})=\ell(\mathbf{I})=\ell}\frac{\mathrm{GT}^V_{X,A,\chi}(\kappa,\alpha;C_{\mathbf{s},\mathbf{I}})}{\ell!}C^\vee_{\mathbf{s},\mathbf{I}}t_A\lambda^\chi\quad\forall\ \kappa\in H^*(\widetilde{\mathcal{M}}_{\chi,k+\ell}),\ \alpha\in H^*(X)^{\otimes k},\end{aligned} \tag{4.9}$$

where $\alpha=\alpha_1\otimes\ldots\otimes\alpha_k$ and $\mathrm{GT}^V_{X,A,\chi}(\kappa,\alpha;C_{\mathbf{s},\mathbf{I}})$ is the number of (J,ν)-holomorphic maps u, for a generic (J,ν), from a possibly disconnected, marked curve $(\Sigma,z_1,\ldots,z_{k+\ell})$ such that

- $(\Sigma,z_1,\ldots,z_{k+\ell})\in K$ for a fixed generic representative K for $\mathrm{PD}_{\widetilde{\mathcal{M}}_{\chi,k+\ell}}\kappa$,
- for each $i=1,\ldots,k$, $u(z_i)\in Z_i$ for a fixed generic representative Z_i for $\mathrm{PD}_X\alpha_i$, and
- for each $j=1,\ldots,\ell(\mathbf{s})$, $\mathrm{ord}^V_{z_{k+j}}u=s_j$ and $u(z_{k+j})\in\Gamma_j$ for a fixed generic representative Γ_j for $\mathrm{PD}_V\gamma_{i_j}$.

A comparison of (4.8) and (4.9) suggests that the product $\cdot$ on $\mathrm{Hom}(\mathbb{T}^*(X),H_*(\widetilde{\mathcal{M}}\times V_\infty))$ not explicitly specified in [IP5] would have to involve rather elaborate coefficients in order to obtain [IP5, (A.3)] from [IP5, (1.24)].

The alternative description of the relative GT generating series in the last paragraph of [IP5, Section 16] does not make sense on several levels. Let N be the dimension of $H^*(V)$, i.e. the number of elements in the set $\{\gamma_i\}$ above. For each

$$\mathbf{m}\equiv(m_{a,i})_{a,i}\colon\mathbb{Z}^+\times\{1,\ldots,N\}\longrightarrow\mathbb{Z}^+$$

with finitely many nonzero entries (such a matrix $\mathbf{m}$ is called a sequence in [IP5]), let $(\mathbf{s_m},\mathbf{I_m})$ be a pair of tuples with $m_{a,i}$ entries of the form (a,γ_i) for each (a,i) and set

$$\ell(\mathbf{m})\equiv\ell(\mathbf{s_m})=\sum_{a,i}m_{a,i}\,,\quad \mathbf{m}!\equiv\mathbf{s_m}!=\prod_{a,i}m_{a,i}!\,,\quad \mathbf{C_m}=\prod_{a,i}(a,\gamma_i)^{m_{a,i}}\,,\quad \mathbf{z^m}=(a,\gamma_i^\vee)^{m_{a,i}}\,.$$

According to [IP5, (A.6)],

$$\mathrm{GT}^V_X(\kappa,\alpha)=\sum_{A,\chi}\sum_{\ell(\mathbf{m})=\ell}\frac{\mathrm{GT}^V_{X,A,\chi}(\kappa,\alpha;\mathbf{C_m})}{\mathbf{m}!}\mathbf{z^m}t_A\lambda^\chi\quad\forall\;\kappa\in H^*(\widetilde{\mathcal{M}}_{\chi,k+\ell}),\;\alpha\in H^*(X)^{\otimes k},$$

for some unspecified numbers $\mathrm{GT}^V_{X,A,\chi}(\kappa,\alpha;\mathbf{C_m})$. According to [IP5], the collection $\{\mathbf{C_m}\}$ is a basis replacing the above basis $\{C_{\mathbf{s},\mathbf{I}}\}$, but these collections are subsets of different vector spaces (with the former generating a symmetrization of the vector space generated by the latter). The formal variable $z_{a,i}=(a,\gamma_i)$ is described as an element of *the dual basis*, without specifying of dual to what. According to [IP5], these formal variables generate a super-commutative polynomial algebra; presumably the same should apply to the variables (a,γ_i). This makes $\mathbf{z^m}$ and $\mathbf{C_m}$ undefined if there is more than one class γ_i of odd degree. Even if all $\mathbf{z^m}$ are defined, they generate a symmetrization of the vector space generated by $\{C_{\mathbf{s},\mathbf{I}}\}$. Thus, the right-hand sides of [IP5, (A.3)] and [IP5, (A.6)] lie in different vector spaces, even though both are supposed to be $\mathrm{GT}^V_X(\kappa,\alpha)$. Furthermore, the numbers $\mathrm{GT}^V_{X,A,\chi}(\kappa,\alpha;C_{\mathbf{s},\mathbf{I}})$ are symmetric in the inputs pulled back from V if κ is symmetric in the relative marked points, but not in general. If there is at most one odd class γ_i and κ is symmetric in the relative marked points, [IP5, (A.6)] can be made sense of by viewing its left-hand side as the projection of $\mathrm{GT}^V_X(\kappa,\alpha)$ to the symmetrization of $H_*(V_\infty)$ over the permutations of components of each tuple $\mathbf{s}$. Comparing with [IP5, (A.3)] and summing over all permutations of pairs of components of $(\mathbf{s_m},\mathbf{I_m})$, we then find that

$$\mathrm{GT}^V_{X,A,\chi}(\kappa,\alpha;\mathbf{C_m})=\mathrm{GT}^V_{X,A,\chi}\big(\kappa,\alpha;C_{\mathbf{s_m},\mathbf{I_m}}\big)\,.$$

However, this is inconsistent with [IP5, Section 15.2], in particular the equation after [IP5, (15.2)], in which the relative contacts are unordered. The number $\mathrm{GT}^V_{X,A,\chi}(\kappa,\alpha;\mathbf{C_m})$ obtained as above from [IP5, (A.3),(A.6)] would count curves with unordered relative contacts if $\ell(\mathbf{s})!$ were dropped from [IP5, (A.3)], i.e. with our choices of the normalizations for the GW/GT generating series.

With our choices of the normalizations of the GW/GT generating functions, the relationship

$$\mathrm{GT}^V_X = \mathrm{e}^{\mathrm{GW}^V_X}\,, \tag{4.10}$$

which is not crucial for the symplectic sum formulas, holds for a product on the vector space $\mathrm{Hom}(\mathbb{T}^*(X), H_*(\widetilde{\mathcal{M}}\times V_\infty))$ with the simplest possible coefficients. Specifically, every pair of tuples $\mathbf{s}_1$ and $\mathbf{s}_2$ of nonnegative integers and every rule of assignment

$$\begin{gathered}\vartheta\colon \{1\}\times\{1,\ldots,k_1+\ell(\mathbf{s}_1)\}\sqcup\{2\}\times\{1,\ldots,k_2+\ell(\mathbf{s}_2)\}\longrightarrow\{1,\ldots,k_1+k_2+\ell(\mathbf{s}_2)\}\qquad\text{s.t.}\\ \vartheta(i_1),\vartheta(i_2)\le k_1+k_2\quad\forall\, i_1\in\{1\}\times\{1,\ldots,k_1\},\ i_2\in\{2\}\times\{1,\ldots,k_2\}\end{gathered} \tag{4.11}$$

determine a tuple $\mathbf{s}_1\wedge_\vartheta\mathbf{s}_2\in(\mathbb{Z}^+)^{\ell(\mathbf{s}_1)+\ell(\mathbf{s}_2)}$, assembled from $\mathbf{s}_1$ and $\mathbf{s}_2$ according to the action of ϑ on the last $\ell(\mathbf{s}_1)$ points in the first tuple above and the last $\ell(\mathbf{s}_2)$ points in the second tuple. Thus, ϑ defines an embedding

$$\widetilde{\mathcal{M}}_{\chi_1,k_1+\ell(\mathbf{s}_1)}\times V_{\mathbf{s}_1}\times\widetilde{\mathcal{M}}_{\chi_2,k_2+\ell(\mathbf{s}_2)}\times V_{\mathbf{s}_2}\longrightarrow\widetilde{\mathcal{M}}_{\chi_1+\chi_2,k_1+k_2+\ell(\mathbf{s}_1\wedge_\vartheta\mathbf{s}_2)}\times V_{\mathbf{s}_1\wedge_\vartheta\mathbf{s}_2}.$$

We denote by

$$\begin{aligned}\vartheta_*\colon H_*\big(\widetilde{\mathcal{M}}_{\chi_1,k_1+\ell(\mathbf{s}_1)}\times V_{\mathbf{s}_1}\big)\otimes H_*\big(\widetilde{\mathcal{M}}_{\chi_2,k_2+\ell(\mathbf{s}_2)}\times V_{\mathbf{s}_2}\big)&\approx H_*\big(\widetilde{\mathcal{M}}_{\chi_1,k_1+\ell(\mathbf{s}_1)}\times V_{\mathbf{s}_1}\times\widetilde{\mathcal{M}}_{\chi_2,k_2+\ell(\mathbf{s}_2)}\times V_{\mathbf{s}_2}\big)\\ &\longrightarrow H_*\big(\widetilde{\mathcal{M}}_{\chi_1+\chi_2,k_1+k_2+\ell(\mathbf{s}_1\wedge_\vartheta\mathbf{s}_2)}\times V_{\mathbf{s}_1\wedge_\vartheta\mathbf{s}_2}\big)\subset H_*\big(\widetilde{\mathcal{M}}\times V_\infty\big)\end{aligned}$$

the induced homomorphism. If in addition $\alpha=\alpha_1\otimes\ldots\otimes\alpha_{k_1+k_2}\in H^*(X)^{\otimes(k_1+k_2)}$, let

$$\alpha_{\vartheta;i}=\alpha_{\vartheta(i,1)}\otimes\ldots\otimes\alpha_{\vartheta(i,k_i)}\in H^*(X)^{\otimes k_i}\qquad i=1,2.$$

For $L_i\colon H^*(X)^{\otimes k_i}\longrightarrow H_*(\widetilde{\mathcal{M}}_{\chi_i,k_i+\ell(\mathbf{s}_i)}\times V_{\mathbf{s}_i})$ with $i=1,2$, we define

$$\begin{gathered}L_1\cdot L_2\colon H^*(X)^{\otimes(k_1+k_2)}\longrightarrow H_*(\widetilde{\mathcal{M}}\times V_\infty)\qquad\text{by}\\ \{L_1\cdot L_2\}(\alpha)=\sum_\vartheta\vartheta_*\big(L_1(\alpha_{\vartheta;1})\otimes L_2(\alpha_{\vartheta;2})\big)\quad\forall\,\alpha\in H^*(X)^{\otimes(k_1+k_2)},\end{gathered}$$

where the sum is taken over all rules of assignment ϑ satisfying (4.11). Combining our definitions of the GW/GT generating functions with this definition, we obtain (4.10).

The relative invariants of [IP4, IP5] are refinements of the usual relative invariants and take values in the coverings $\mathcal{H}^V_{X;\mathbf{s}}$ and $\mathcal{H}^V_{Y;\mathbf{s}}$ of $V_\mathbf{s}$ described in Section 3.2 of this paper, instead of $V_\mathbf{s}$. Their use causes additional difficulty with exponentiating the GW-invariants, even in the case of primary constraints, since one must also specify a product

$$H_*(\mathcal{H}^V_{X;\mathbf{s}_1})\otimes H_*(\mathcal{H}^V_{X;\mathbf{s}_2})\longrightarrow H_*(\mathcal{H}^V_{X;\mathbf{s}_1\mathbf{s}_2})$$

lifting the Kunneth product

$$H_*(V_{\mathbf{s}_1})\otimes H_*(V_{\mathbf{s}_2})\longrightarrow H_*(V_{\mathbf{s}_1\mathbf{s}_2})=H_*(V_{\mathbf{s}_1}\times V_{\mathbf{s}_2})\,.$$

Such a lifting does exist, since the natural map

$$V_{\mathbf{s}_1}\times V_{\mathbf{s}_2}\longrightarrow V_{\mathbf{s}_1\mathbf{s}_2}$$

lifts to a map on the covering spaces, but this lifting is not unique. The use of the refined relative invariants in [IP5] also causes problems with defining a suitable product $\star$ of relative invariants in [IP5, Section 10]; see the second half of Section 3.2 for details. Another notable feature of the symplectic sum formulas in [IP5] is the presence of the so-called S-matrix, which is shown to be trivial in many cases. As we explain in Section 5.5, it appears due to an oversight in [IP5, Section 11] and its action is always trivial, essentially due to the nature of this oversight.

Remark 4.2. The meaning of $C_{\mathbf{s},\mathbf{I}}$ in [IP5, (A.2)] is not specified. The entire collection $\{C_{\mathbf{s},\mathbf{I}}\}$, over all pairs $(\mathbf{s},\mathbf{I})$ of tuples of the same length, is described as a basis for the tensor algebra on $\mathbb{N}\times H^*(V)$, which is not even a vector space over $\mathbb{R}$, while the collection $\{C^\vee_{\mathbf{s},\mathbf{I}}\}$ is described as the dual basis. In fact, $\{C_{\mathbf{s},\mathbf{I}}\}$ and $\{C^\vee_{\mathbf{s},\mathbf{I}}\}$ are bases for

$$\bigoplus_{\ell=0}^{\infty}\bigoplus_{\mathbf{s}\in(\mathbb{Z}^+)^\ell}H^*(V_{\mathbf{s}})\qquad\text{and}\qquad H_*(V_\infty)=\bigoplus_{\ell=0}^{\infty}\bigoplus_{\mathbf{s}\in(\mathbb{Z}^+)^\ell}H_*(V_{\mathbf{s}}),$$

respectively; these two vector spaces are not duals of each other. The summation indices in [IP5, (A.3)] are described incorrectly and the two appearances of $\overline{\mathcal{M}}$ in this paragraph refer to $\widetilde{\mathcal{M}}$. The description of the number $\mathrm{GT}^V_{X,A,\chi}(\kappa,\alpha;C_{\mathbf{s},\mathbf{I}})$ is incorrect, even with the proper normalizations of the relevant power series, since the j-th relative marked point should be mapped to a generic representative for $\mathrm{PD}_V\alpha_{i_j}$, not for $\mathrm{PD}_V\alpha_j$, and these representatives should be different for $j_1\neq j_2$, even if $i_{j_1}=i_{j_2}$. Other, fairly minor misstatements in the related parts of [IP5] include:

p935, middle: the finiteness holds only under ideal circumstances;
p938, top: $x(z)$ and $y(w)$ are expansions in the normal directions to V as explained in Section 5;
p940, bottom: $\mathbb{T}^*(Z)$ is defined only on p944;
p946, (1.17),(1.18): $\mathcal{M}^V_{\chi,n,s}(X,A)$ and $\overline{\mathcal{M}}^V_{\chi,n,s}(X,A)$ refer to disconnected domains here;
p946, after (1.17): each unstable $\mathbb{P}^1$ needs to have at least one marked point to insure compactness; χ is twice the holomorphic Euler characteristic, not the usual EC;
p994, line 13: the domain of g is the union of these Δ_s;
p994, line 15: $\cup\ \longrightarrow\cap$; this defines LHS;
p994, line 19: $Q^V_{p,q}$ needs to be the inverse of the intersection form for the first equality;
p994, line 21: (A.4) is not a basis for $H^*(V_\infty)$; neither is (A.2);
p994, (10.7): last product does not make sense with conventions as on p1023;
p996, (10.12): $\oplus\ \longrightarrow\otimes$;
p997, line 9: $(\alpha_X,\alpha_Y)\ \longrightarrow\alpha$;
p997, (10.15): $\mathrm{GT}_Z(\alpha_X,\alpha_Y)\ \longrightarrow\mathrm{GT}_Z(\alpha)$;
p997, line -5, and p998, line 2: $\mathrm{GT}_{\chi,A,Z}(\alpha_X,\alpha_Y)\ \longrightarrow\mathrm{GT}_{Z,A,\chi}(\alpha)$;
p997, line -4: $\mathrm{GT}^V_{\chi_2,A_2,Y}(C_{\mathbf{m}^*})\ \longrightarrow\mathrm{GT}^V_{Y,A_2,\chi_2}(\alpha_Y;C_{\mathbf{m}^*};\alpha_Y)$;
p998, line 1: (A.6) also involves κ;
p998, line 3: it is unclear how the relative constraints enter in the notation;
p1024, (A.6): $g\ \longrightarrow\chi$; same on line 6 (twice).

The intended symplectic sum formula for primary invariants in [LR], i.e. as in Theorem 1.1 in this paper, is split between equations (5.4), (5.7), and (5.9). The first of these is vague on the set $\mathcal{C}_{g,m}^{J,[A]}$ indexing the summands, while the last is vague on the relation between α and $\alpha^{\pm}$. The key set $\mathcal{C}_{g,m}^{J,[A]}$ is independent of J, but is generally infinite, contrary to [LR, Lemma 5.4], in part because its elements are not restricted to the classes that can be represented by J-holomorphic maps. Taken together, the three formulas are at least missing the factor of $\ell(\mathbf{k})!$ in the denominator corresponding to the reorderings of the contact points.

The symplectic sum formulas in [Lj2], in the bottom half of page 201, involve triples Γ *consisting of the genus, the number of marked points and the degree of the stable morphisms*; see [Lj2, p200, middle]). This is written as $\Gamma=(g,k,A)$ at the bottom of page 200, suggesting that A is a second homology class. The degree becomes d at the bottom of page 202, suggesting that $d\in\mathbb{Z}$ is the degree with respect to some ample line bundle, as at the bottom of page 547 in [Lj1]; the line bundle is finally mentioned as being implicitly chosen at the bottom of page 226 in [Lj2]. On the other hand, A becomes b at the top of page 215, suggesting again that this is a second homology class, as in the middle of page 512 in [Lj1]. The correct interpretation of A for the purposes of these formulas is that it is the degree with respect to an ample line bundle $\mathcal{L}$ over the total space $\mathcal{Z}\longrightarrow\Delta$. Thus, the set $\mathcal{R}_{X,Y}^V$ in (4.5) is essentially replaced in [Lj2] by the (generally) larger subset of second homology classes of $X\#_V Y$ vanishing on the first chern class of $\mathcal{L}$. Different ample line bundles $\mathcal{L}$ give different formulas; so effectively, the approach of [Lj2] replaces $\mathcal{R}_{X,Y}^V$ in (4.5) by the set of second homology classes of $X\#_V Y$ vanishing on the first chern classes of all ample line bundles $\mathcal{L}$. The last set can still be larger than $\mathcal{R}_{X,Y}^V$, since the chern class of every ample line bundle vanishes over torsion classes. Thus, the numerical decomposition formula for primary invariants on page 201 of [Lj2] is weaker than Theorem 1.1, even when restricting to the algebraic category. This weakness is fully addressed in [AF], according to the authors.

The (stronger) analogue of (4.7) in [Lj2] is an immediate consequence of the decomposition formula for virtual fundamental classes (VFCs) at the bottom of page 201 in [Lj2]. The latter requires constructing a VFC for (absolute) stable maps to the singular target $X\sqcup_V Y$, showing that it equals to the VFCs for stable maps to $X\#_V Y$ in a suitable sense (a priori they lie in homology groups of different spaces), and decomposing the former into VFCs for relative maps into (X,V) and (Y,V). The last step in particular is not even a priori intuitive because the stable maps into $X\#_V Y$ generally do not split uniquely into relative maps to (X,V) and to (Y,V); see the end of Section 4.2. As pointed out in [AF, Remark 3.2.11], the constructions in [Lj1, Lj2] involve some delicate issues; these are further elaborated on in [GS, Ch].

The argument in [LR] considers only primary insertions, as in Theorem 1.1, while the argument in [IP5] considers only primary insertions and constraints that are pulled back from the Deligne-Mumford space, as in Theorem 4.1. There are brief statements in both papers that the arguments apply to descendants (ψ-classes), but neither paper contains a symplectic sum formula involving descendants. As illustrated by the appearance of rules of assignment in the symplectic sum formula in [Lj2], stating such a formula requires a bit of care. Furthermore, descendants do not even fit with the approach in [IP4, IP5], as it is based on defining invariants by intersecting with classes in X^k and the Deligne-Mumford space (such intersections do not directly cover the ψ-classes).

The stated symplectic formulas of [Lj2, IP5, LR] involve cohomology insertions of the form $q_{\#}^*\alpha$, as in Theorem 4.1. Arbitrary cohomology insertions are considered in [IP5, Section 13], as follows. Let $\hat{X}$ be the compactification of $X-V$ obtained by removing an open tubular neighborhood of V or equivalently by replacing V with SV in X; see [IP4, Section 5] and the end of Section 3.3. Let

$$q\colon (\hat{X},\partial\hat{X}) \longrightarrow (X,V)$$

be the natural projection map. Given a pseudocycle representative $\phi\colon (P,\partial P)\longrightarrow(\hat{X},SV)$ for a class $B\in H_*(\hat{X},\partial X)$ and $i=1,\ldots,k$, let

$$\widetilde{\mathcal{M}}_{\chi,k;\mathbf{s}}(X,A)\times_i\phi = \big\{(u,x)\in\widetilde{\mathcal{M}}_{\chi,k;\mathbf{s}}(X,A)\times P\colon\ \mathrm{ev}_i(u)=q(\phi(x))\big\}\,.$$

Intersecting with other pseudocycle representatives in a similar way, we obtain a virtual orbifold with boundary and evaluation map ev^V to $V_{\mathbf{s}}$, which can then be used to define "extended" relative invariants of (X,V). In general, these invariants depend on the choice of the almost complex structure J, deformation ν, and the representatives ϕ for classes B. By [IP5, Lemma 13.1], this dependence disappears whenever

$$\partial B\in\ker\big\{q_{V*}\colon H_{*-1}(SV)\longrightarrow H_{*-1}(V)\big\}\,. \tag{4.12}$$

By Corollary 3.8, these are precisely the cases obtained from cutting a Poincare dual in $X\#_V Y$ of a cohomology insertion as in Theorem 4.1. The dependence on the representative ϕ for B, but not on (J,ν), is analyzed in [IP5, Lemma 13.2]. Unfortunately, the intended meaning of [IP5, (13.4)] is unclear: it involves $\mathrm{GT}_F^{VV}(\phi')$, which is not defined, as well as some convolution product of GT_X^V and $\mathrm{GT}_F^{VV}(\phi')$; no proof of this lemma is provided either. The intention of [IP5, Lemma 13.2] is to extend the definition of relative invariants to homology insertions $B\in H_*(\hat{X},\partial\hat{X})$ by defining such numbers for a fixed J, ν, and ϕ and then to use them in an extended symplectic sum formula, which is not stated. Even if this were possible to do, it is not apparent that the resulting relative invariants could be readily computed, especially given their dependence on J, ν, and ϕ; so such an extended symplectic sum formula may not be particularly useful.

Remark 4.3. The definition of $\widetilde{\mathcal{M}}_{\chi,k;\mathbf{s}}(X,A)\times_i\phi$ in the displayed equation above [IP5, (13.1)] as an intersection does not make sense, since the two sets being intersected lie in different spaces. By Lemma 3.1, every class B as in (4.12) is the boundary of a pseudocycle into a closed tubular neighborhood of V in X. Thus, for such a class B, the cut-down moduli space $\widetilde{\mathcal{M}}_{\chi,k;\mathbf{s}}(X,A)\times_i\phi$ is a boundary as well. This implies that the GT-invariants for classes $B\in H_*(X-V)$ depend only on their images in $H_*(X)$, contrary to the suggestion at the top of [IP5, p1006]. The conclusion after [IP5, Lemma 13.2] is that extended relative invariants can be defined by choosing pseudocycle representative ϕ_β as above for each

$$\beta\in\ker\big\{H_{*-1}(SV)\longrightarrow H_{*-1}(X)\big\}$$

such that $[\partial\phi]=\beta$; as just indicated, this would not provide any additional information. If the intended meaning in [IP5] were to fix a representative ϕ_B for each $B\in H_*(\hat{X},X)$ as in (4.12), this would still cover only the insertions in Theorem 4.1. Other, fairly minor misstatements in [IP5, Section 13] include

p1005, Section 13, line 3: constraints not of the form $q_{\#}^*\alpha_\cup$ as in Corollary 3.8;
p1005, Section 13, line 15: $g\ \longrightarrow f$;
p1005, Section 13, line 17: $\phi\colon P\longrightarrow\hat{X}$;
p1006, line 9: real codimension one in cut-down of (13.2);
p1006, Lemma 13.1, line 1: $\mathrm{GT}_{X,A,\mathbf{s}}^V(\phi)$ in [IP5, (13.1)] is not a number;
p1006, Lemma 13.2, line 2: PD is not defined;
p1006, Lemma 13.2, line 5; p1006, line -2: $H_*(X,V)\ \longrightarrow H_*(\hat{X},\partial\hat{X})$.

4.2 Relative stable maps: [IP4, Section 7], [LR, Sections 3.2,3.3]

In this section, we first recall the notion of relative stable map of [IP4, Definitions 7.1,7.2], then define the notion of stable map into the singular space $X\sqcup_V Y$ in the same style, and finally formulate both notions as intended in [LR, Definitions 3.14,3.18], but not properly done. The definitions of these notions in the style of [IP4, IP5] are simpler, but the approach of [LR] fits better with the analytic issues arising in the proof of the symplectic sum formula. The correspondence between our notation and that in [LR] is described in the first paragraph of Section 3.3.

Let (X,ω_X) be a compact symplectic manifold, $V\subset X$ be a closed symplectic hypersurface, and J_X be an ω_X-compatible almost complex structure, such that $J_X(TV)=TV$. If $u\colon(\Sigma,\mathfrak{j})\longrightarrow(X,J_X)$ is a smooth map from a Riemann surface, let

$$\bar\partial_{J_X,\mathfrak{j}}u=\frac{1}{2}\big(\mathrm{d}u+\{u^*J_X\}\circ\mathrm{d}u\circ\mathfrak{j}\big)\in\Gamma^{0,1}_{J_X,\mathfrak{j}}(\Sigma;u^*TX)\equiv\Gamma\big(\Sigma;(T^*\Sigma)^{0,1}\otimes_{\mathbb{C}}u^*TX\big).$$

We denote by ∇ the Levi-Civita connection of the metric $\omega_X(\cdot,J_X\cdot)$ on X and by $\widetilde\nabla$ the corresponding J_X-linear connection; see [MS2, p41]. If $u:(\Sigma,\mathfrak{j})\longrightarrow(X,J_X)$ is $(J_X,\mathfrak{j})$-holomorphic, i.e. $\bar\partial_{J_X,\mathfrak{j}}u=0$, the linearization of the $\bar\partial_{J_X,\mathfrak{j}}$-operator at u is given by

$$\begin{aligned}D_u\colon\Gamma(\Sigma;u^*TX)&\longrightarrow\Gamma^{0,1}_{J_X,\mathfrak{j}}(\Sigma;u^*TX),\\ D_u\xi&=\frac{1}{2}\big(\widetilde\nabla^u\xi+\{u^*J_X\}\circ\widetilde\nabla^u\xi\circ\mathfrak{j}\big)+\frac{1}{4}N^u_{J_X}(\xi,\mathrm{d}u),\end{aligned}\tag{4.13}$$

where $\widetilde\nabla^u$ and $N^u_{J_X}$ are the pull-backs of the connection $\widetilde\nabla$ and of the Nijenhuis tensor N_{J_X} of J_X normalized as in [MS2, p18], respectively, by u; see [MS2, (3.1.6)]. If in addition $u(\Sigma)\subset V$,

$$D_u\big(\Gamma(\Sigma;u^*TV)\big)\subset\Gamma^{0,1}_{J_X,\mathfrak{j}}(\Sigma;u^*TV),$$

because the restriction of D_u to $\Gamma(\Sigma;u^*TV)$ is the linearization of the $\bar\partial_{J_X,\mathfrak{j}}$-operator at u for the space of maps to V. Thus, D_u descends to a first-order differential operator

$$D_u^{\mathcal{N}_XV}\colon\Gamma(\Sigma;u^*\mathcal{N}_XV)\longrightarrow\Gamma^{0,1}_{J_X,\mathfrak{j}}(\Sigma;u^*\mathcal{N}_XV),\tag{4.14}$$

which plays a central role in compactifying the moduli space of relative maps to (X,V).

Since $J_X(TV)=TV$, J_X induces a complex structure $\mathfrak{i}_{X,V}$ on (the fibers of) the normal bundle

$$\pi_{X,V}\colon\mathcal{N}_XV\equiv TX|_V/TV\longrightarrow V.$$

A connection $\nabla^{\mathcal{N}_XV}$ in $(\mathcal{N}_XV,\mathfrak{i}_{X,V})$ induces a splitting of the exact sequence

$$0\longrightarrow\pi^*_{X,V}\mathcal{N}_XV\longrightarrow T(\mathcal{N}_XV)\stackrel{\mathrm{d}\pi_{X,V}}{\longrightarrow}\pi^*_{X,V}TV\longrightarrow0\tag{4.15}$$

of vector bundles over $\mathcal{N}_XV$ which restricts to the canonical splitting over the zero section and is preserved by the multiplication by $\mathbb{C}^*$; see [Z2, Lemma 1.1]. For each trivialization

$$\mathcal{N}_XV|_U\approx U\times\mathbb{C}$$

over an open subset U of V, there exists $\alpha \in \Gamma(U; T^*V \otimes_{\mathbb{R}} \mathbb{C})$ such that the image of $\pi_{X,V}^* TV$ corresponding to this splitting is given by

$$T^{\mathrm{hor}}_{(x,w)}(\mathcal{N}_X V) = \big\{(v, -\alpha_x(v)w) \colon v \in T_x V\big\} \qquad \forall\, (x,w) \in U \times \mathbb{C}.$$

The isomorphism $(x,w) \longrightarrow (x, w^{-1})$ of $U \times \mathbb{C}^*$ maps this vector space to

$$\begin{aligned} T^{\mathrm{hor}}_{(x,w^{-1})}\big((\mathcal{N}_X V)^*\big) &= \big\{(v, w^{-2}\alpha_x(v)w) \colon v \in T_x V\big\} \\ &= \big\{(v, \alpha_x(v)w^{-1}) \colon v \in T_x V\big\} \qquad \forall\, (x,w) \in U \times \mathbb{C}^*. \end{aligned}$$

Thus, the splitting of (4.15) induced by a connection in $(\mathcal{N}_X V, \mathfrak{i}_{X,V})$ extends to a splitting of the exact sequence

$$0 \longrightarrow T^{\mathrm{vrt}}(\mathbb{P}_X V) \longrightarrow T(\mathbb{P}_X V) \stackrel{\mathrm{d}\pi_{X,V}}{\longrightarrow} \pi_{X,V}^* TV \longrightarrow 0,$$

where $\mathbb{P}_X V$ is as in (1.13) and $\pi_{X,V} : \mathbb{P}_X V \longrightarrow V$ is the bundle projection map; this splitting restricts to the canonical splittings over

$$\mathbb{P}_{X,\infty} V \equiv \mathbb{P}(\mathcal{N}_X V \oplus 0) \qquad \text{and} \qquad \mathbb{P}_{X,0} V \equiv \mathbb{P}(0 \oplus \mathcal{O}_V)$$

and is preserved by the multiplication by $\mathbb{C}^*$. Via this splitting, the almost complex structure $J_V \equiv J_X|_V$ and the complex structure $\mathfrak{i}_{X,V}$ in the fibers of $\pi_{X,V}$ induce an almost complex structure $J_{X,V}$ on $\mathbb{P}_X V$ which restricts to almost complex structures on $\mathbb{P}_{X,\infty} V$ and $\mathbb{P}_{X,0} V$ and is preserved by the $\mathbb{C}^*$-action. Furthermore, the projection $\pi_{X,V} : \mathbb{P}_X V \longrightarrow V$ is $(J_V, J_{X,V})$-holomorphic. By [Z2, Lemma 2.2], $\xi \in \Gamma(V; \mathcal{N}_X V)$ is $(J_{X,V}, J_X|V)$-holomorphic if and only if ξ lies in the kernel of the $\bar{\partial}$-operator on $(\mathcal{N}_X V, \mathfrak{i}_{X,V})$ corresponding to the connection used above.

For each $m \in \mathbb{Z}^{\geq 0}$, let

$$\begin{gathered} X_m^V = \big(X \sqcup \{1\} \times \mathbb{P}_X V \sqcup \ldots \sqcup \{m\} \times \mathbb{P}_X V\big)/\!\sim, \qquad \text{where} \\ x \sim 1 \times \mathbb{P}_{X,\infty} V|_x, \quad r \times \mathbb{P}_{X,0} V|_x \sim (r+1) \times \mathbb{P}_{X,\infty} V|_x \quad \forall\, x \in V,\ r = 1, \ldots, m-1; \end{gathered} \tag{4.16}$$

see Figure 2. We denote by J_m the almost complex structure on X_m^V so that

$$J_m|_X = J_X \qquad \text{and} \qquad J_m|_{\{r\} \times \mathbb{P}_X V} = J_{X,V} \quad \forall\, r = 1, \ldots, m.$$

For each $(c_1, \ldots, c_m) \in \mathbb{C}^*$, define

$$\Theta_{c_1,\ldots,c_m} : X_m^V \longrightarrow X_m^V \qquad \text{by} \quad \Theta_{c_1,\ldots,c_m}(x) = \begin{cases} x, & \text{if } x \in X; \\ (r, [c_r v, w]), & \text{if } x = (r, [v,w]) \in r \times \mathbb{P}_X V. \end{cases} \tag{4.17}$$

This diffeomorphism is biholomorphic with respect to J_m and preserves the fibers of the projection $\mathbb{P}_X V \longrightarrow V$ and the sections $\mathbb{P}_{X,0} V$ and $\mathbb{P}_{X,\infty} V$.

The moduli space of relative stable maps into (X, V) is constructed in [IP4] under the additional assumption that

$$N_{J_X}(v, w) \in T_x V \qquad \forall\, v, w \in T_x X,\ x \in V. \tag{4.18}$$

In light of (4.13), this assumption insures that the operator $D_u^{\mathcal{N}_X V}$ is $\mathbb{C}$-linear for every $(J_X, \mathfrak{j})$-holomorphic map $u \colon \Sigma \longrightarrow V$ and thus the operator

$$\Gamma(V; TX|_V) \longrightarrow \Gamma\big(V; (T^*V)^{0,1} \otimes_{\mathbb{C}} TX|_V\big), \qquad \xi \longrightarrow \frac{1}{2}\big(\widetilde{\nabla}\xi + J_X \circ \widetilde{\nabla}\xi \circ J_X\big),$$

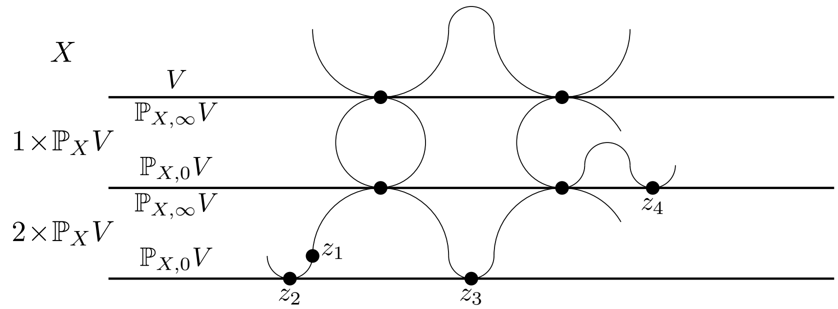

Figure 2: The image of a relative map with $k=1$ and $\mathbf{s}=(2,2,2)$ to the space X_2^V.

induces a $\bar{\partial}$-operator on $(\mathcal{N}_X V, \mathfrak{i}_{X,V})$ corresponding to some connection $\nabla^{\mathcal{N}_X V}$ in $(\mathcal{N}_X V, \mathfrak{i}_{X,V})$; see [Z2, Section 2.3]. Let $J_{X,V}$ be the complex structure on $\mathbb{P}_X V$ induced by J_X and $\nabla^{\mathcal{N}_X V}$ as in the paragraph above the previous one; it depends only on the above $\bar{\partial}$-operator and not on the connection $\nabla^{\mathcal{N}_X V}$ realizing it. Thus, for every $(J_X, \mathfrak{j})$-holomorphic map $u\colon \Sigma \longrightarrow V$ and $\xi \in \Gamma(\Sigma; u^*\mathcal{N}_X V)$, $\xi \in \ker D_u^{\mathcal{N}_X V}$ if and only if $\xi\colon \Sigma \longrightarrow \mathbb{P}_X V$ is a $(J_{X,V}, \mathfrak{j})$-holomorphic map.

If $A \in H_2(X;\mathbb{Z})$, $g,k,\ell \in \mathbb{Z}^{\ge 0}$, and $\mathbf{s}=(s_1,\ldots,s_\ell)\in(\mathbb{Z}^+)^\ell$ is a tuple satisfying (1.3), let

$$\mathcal{M}_{g,k;\mathbf{s}}^V(X,A) \subset \overline{\mathcal{M}}_{g,k+\ell}(X,A) \tag{4.19}$$

denote the subset of equivalence of stable J_X-holomorphic maps u from genus g marked nodal curves $(\Sigma, z_1, \ldots, z_{k+\ell})$ such that

$$u^{-1}(V) = \big\{z_{k+1}, \ldots, z_{k+\ell}\big\} \qquad \text{and} \qquad \mathrm{ord}_{z_{k+i}}^V u = s_i \quad \forall\, i=1,\ldots,\ell.$$

A "compactification" $\overline{\mathcal{M}}_{g,k;\mathbf{s}}^V(X,A)$ of $\mathcal{M}_{g,k;\mathbf{s}}^V(X,A)$ is described in [IP4, Section 7].[2] Each element of $\overline{\mathcal{M}}_{g,k;\mathbf{s}}^V(X,A)$ is an equivalence class of stable J_m-holomorphic maps from marked connected nodal domains $(\Sigma, z_1, \ldots, z_{k+\ell})$ into X_m^V for some $m \in \mathbb{Z}^{\ge 0}$. The composition of such a map u with the natural projection $X_m^V \longrightarrow X$ lies in the homology class A and the preimages of

$$\{1\}\times\big(\mathbb{P}_X V - \mathbb{P}_{X,0}V - \mathbb{P}_{X,\infty}V\big), \ldots, \{m\}\times\big(\mathbb{P}_X V - \mathbb{P}_{X,0}V - \mathbb{P}_{X,\infty}V\big)$$

under u are non-empty.[3] The restriction of u to each irreducible component Σ_j of Σ is either

- a map to X such that the set $u|_{\Sigma_j}^{-1}(V) - \{z_{k+1}, \ldots, z_{k+\ell}\}$ consists of the nodes joining Σ_j to irreducible components of Σ mapped to $\{1\}\times\mathbb{P}_X V$ and $\mathrm{ord}_{z_{k+i}}^V(u|_{\Sigma_j})=s_i$ for all $z_{k+i}\in\Sigma_j$, or
- a map to $\{r\}\times\mathbb{P}_X V$ for some $r=1,\ldots,m$ such that
  - the set $u|_{\Sigma_j}^{-1}(\{r\}\times\mathbb{P}_{X,\infty}V)$ consists of the nodes $z_{j,i}$ joining Σ_j to irreducible components of Σ mapped to $\{r-1\}\times\mathbb{P}_X V$ if $r>1$ and to X if $r=1$ and $\mathrm{ord}_{z_{j,i}}^{\mathbb{P}_{X,\infty}V}(u|_{\Sigma_j})$ equals to the order of contact of $u|_{\Sigma_{i,j}}$ at $z_{i,j}$ with $\{r-1\}\times\mathbb{P}_{X,0}V$ or $V\subset X$, respectively, where $z_{i,j}\in\Sigma_{i,j}$ is the point identified with $z_{j,i}$, while

[2] As in the case of the usual moduli space of stable maps, $\overline{\mathcal{M}}_{g,k+\ell}(X,A)$, $\overline{\mathcal{M}}_{g,k;\mathbf{s}}^V(X,A)$ is a natural compact space containing $\mathcal{M}_{g,k;\mathbf{s}}^V(X,A)$ as an open subspace, but which is generally not dense.

[3] The latter property is in fact implied by the stability requirement.

- ∘ the set $u|_{\Sigma_j}^{-1}(\{r\}\times\mathbb{P}_{X,0}V)-\{z_{k+1},\ldots,z_{k+\ell}\}$ consists of the nodes joining Σ_j to irreducible components of Σ mapped to $\{r+1\}\times\mathbb{P}_XV$ and $\mathrm{ord}_{z_{k+i}}^{\mathbb{P}_{X,0}V}(u|_{\Sigma_j})=s_i$ for all $z_{k+i}\in\Sigma_j$;

see Figure 2. Two such tuples $(\Sigma,z_1,\ldots,z_{k+\ell},u)$ and $(\Sigma',z_1',\ldots,z_{k+\ell}',u')$ are equivalent if there are $c_1,\ldots,c_m\in\mathbb{C}^*$ and a biholomorphic $\varphi\colon\Sigma'\longrightarrow\Sigma$ so that

$$\varphi(z_1')=z_1,\quad \ldots,\quad \varphi(z_{k+\ell}')=z_{k+\ell},\quad \text{and}\quad u'=\Theta_{c_1,\ldots,c_m}\circ u\circ\varphi.$$

A tuple as above is stable if it has finitely many automorphisms (self-equivalences). For each stable tuple $(\Sigma,z_1,\ldots,z_{k+\ell},u)$ as above and $r=1,\ldots,m$, either

- the degree of the composition of $u|_{u^{-1}(\{r\}\times\mathbb{P}_XV)}$ with the projection to V is not zero, or
- the arithmetic genus of some topological component of $u^{-1}(\{r\}\times\mathbb{P}_XV)$ is positive, or
- some topological component of $u^{-1}(\{r\}\times\mathbb{P}_XV)$ carries one of the marked points $z_1,\ldots,z_k$, or
- the restriction of u to some topological component of $u^{-1}(\{r\}\times\mathbb{P}_XV)$ has at least three special points: nodal, branch, or in the preimage of $\mathbb{P}_{X,0}V$ or $\mathbb{P}_{X,\infty}V$.

Suppose (Y,ω_Y) is another symplectic manifold containing V as a symplectic hypersurface so that (1.1) holds, J_Y is an ω_Y-compatible almost complex structure, such that $J_Y(TV)=TV$ and $J_Y|_{TV}=J_X|_{TV}\equiv J_V$, and we have chosen an isomorphism as in (1.2). Such an isomorphism identifies $\mathbb{P}_XV$ with $\mathbb{P}_YV$. For each $m\in\mathbb{Z}^{\ge0}$, let

$$\begin{gathered} X\cup_V^m Y=\big(X_m^V\sqcup Y_m^V\big)/\!\sim,\\ X_m^V\ni[r,x]\sim[m{+}1{-}r,x]\in Y_m^V\qquad\forall\ r=1,\ldots,m,\ x\in\mathbb{P}_XV=\mathbb{P}_YV\,; \end{gathered}\tag{4.20}$$

see Figure 3. We extend (4.17) to an isomorphism

$$\Theta_{c_1,\ldots,c_m}\colon X\cup_V^m Y\longrightarrow X\cup_V^m Y\tag{4.21}$$

by taking it to be the identity on Y. By the discussion following (4.15), the almost complex structures $J_{X,V}$ and $J_{Y,V}$ on $\mathbb{P}_XV=\mathbb{P}_YV$ agree if they are induced from J_V using dual connections in $\mathcal{N}_XV$ and $\mathcal{N}_YV$. The moduli spaces of stable maps into $X\cup_VY$ are defined under the additional assumptions that (4.18) holds,

$$N_{J_Y}(v,w)\in T_yV\qquad\forall\ v,w\in T_yY,\ y\in V,$$

and the linearized operator

$$D_u^{\mathcal{N}_YV}\colon\Gamma(\Sigma;u^*\mathcal{N}_YV)\longrightarrow\Gamma^{0,1}_{J_Y,\mathfrak{j}}(\Sigma;u^*\mathcal{N}_YV)$$

is dual to (4.14). These assumptions insure that the almost complex structures $J_{X,V}$ and $J_{Y,V}$ on $\mathbb{P}_XV=\mathbb{P}_YV$ agree and so induce a well-defined almost complex structure J_m on $X\cup_V^mY$, which is preserved by (4.21). They are satisfied by $J_X=J_{\mathcal{Z}}|_X$ and $J_Y=J_{\mathcal{Z}}|_Y$ if $J_{\mathcal{Z}}$ is as in Proposition 3.16.

Let $A\in H_2(X\cup_VY;\mathbb{Z})$ be an element in the image of $H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z})$ under the natural homomorphism

$$H_2(X;\mathbb{Z})\oplus H_2(Y;\mathbb{Z})\longrightarrow H_2(X\cup_VY;\mathbb{Z})$$

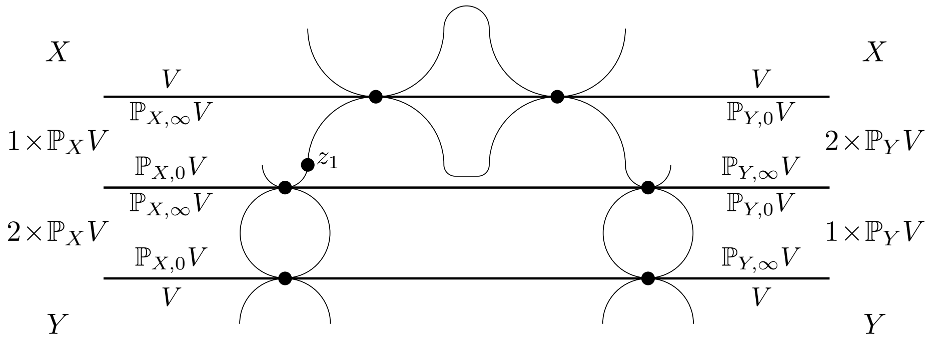

Figure 3: The image of a limit map with $k\!=\!1$ to the space $X\cup_V^2 Y$.

and $g,k\in\mathbb{Z}^{\ge0}$. The elements of the moduli space $\overline{\mathcal{M}}_{g,k}(X\cup_V Y,A)$ of stable maps into $X\cup_V Y$ are equivalence classes of J_m-holomorphic maps u from marked connected nodal domains $(\Sigma,z_1,\ldots,z_k)$ to $X\cup_V^m Y$ for some $m\in\mathbb{Z}^{\ge0}$ such that the preimages of

$$\{1\}\times\big(\mathbb{P}_XV-\mathbb{P}_{X,0}V-\mathbb{P}_{X,\infty}V\big),\ldots,\{m\}\times\big(\mathbb{P}_XV-\mathbb{P}_{X,0}V-\mathbb{P}_{X,\infty}V\big)$$

under u are non-empty. Furthermore, the restriction of u to each irreducible component Σ_j of Σ is either

- a map to X such that the set $u|_{\Sigma_j}^{-1}(V)$ consists of the nodes $z_{j,i}$ joining Σ_j to irreducible components of Σ mapped to $\{1\}\times\mathbb{P}_XV$ if $m\in\mathbb{Z}^+$ and to Y if $m\!=\!0$ and $\mathrm{ord}_{z_{j,i}}^V(u|_{\Sigma_j})$ equals to the order of contact of $u|_{\Sigma_{i,j}}$ at $z_{i,j}$ with $1\times\mathbb{P}_{X,\infty}V$ or $V\subset Y$, respectively, where $z_{i,j}\in\Sigma_{i,j}$ is the point identified with $z_{j,i}$, or
- a map to Y such that the set $u|_{\Sigma_j}^{-1}(V)$ consists of the nodes $z_{j,i}$ joining Σ_j to irreducible components of Σ mapped to $\{1\}\times\mathbb{P}_YV$ if $m\in\mathbb{Z}^+$ and to X if $m\!=\!0$ and $\mathrm{ord}_{z_{j,i}}^V(u|_{\Sigma_j})$ equals to the order of contact of $u|_{\Sigma_{i,j}}$ at $z_{i,j}$ with $1\times\mathbb{P}_{Y,\infty}V$ or $V\subset X$, respectively, where $z_{i,j}\in\Sigma_{i,j}$ is the point identified with $z_{j,i}$, or
- a map to $\{r\}\times\mathbb{P}_XV=\{m\!+\!1\!-\!r\}\times\mathbb{P}_YV$ for some $r=1,\ldots,m$ such that
  - the set $u|_{\Sigma_j}^{-1}(\{r\}\times\mathbb{P}_{X,\infty}V)$ consists of the nodes $z_{j,i}$ joining Σ_j to irreducible components of Σ mapped to $\{r\!-\!1\}\times\mathbb{P}_XV$ if $r\!>\!1$ and to X if $r\!=\!1$ and $\mathrm{ord}_{z_{j,i}}^{\mathbb{P}_{X,\infty}V}(u|_{\Sigma_j})$ equals to the order of contact of $u|_{\Sigma_{i,j}}$ at $z_{i,j}$ with $\{r\!-\!1\}\times\mathbb{P}_{X,0}V$ or $V\subset X$, respectively, where $z_{i,j}\in\Sigma_{i,j}$ is the point identified with $z_{j,i}$, while
  - the set $u|_{\Sigma_j}^{-1}(\{m\!+\!1\!-\!r\}\times\mathbb{P}_{Y,\infty}V)$ consists of the nodes $z_{j,i}$ joining Σ_j to irreducible components of Σ mapped to $\{m\!-\!r\}\times\mathbb{P}_YV$ if $r<m$ and to Y if $r=m$ and $\mathrm{ord}_{z_{j,i}}^{\mathbb{P}_{Y,\infty}V}(u|_{\Sigma_j})$ equals to the order of contact of $u|_{\Sigma_{i,j}}$ at $z_{i,j}$ with $\{m\!-\!r\}\times\mathbb{P}_{Y,0}V$ or $V\subset Y$, respectively, where $z_{i,j}\in\Sigma_{i,j}$ is the point identified with $z_{j,i}$;

see Figure 3. Two such tuples $(\Sigma,z_1,\ldots,z_k,u)$ and $(\Sigma',z_1',\ldots,z_k',u')$ are equivalent if there are $c_1,\ldots,c_m\in\mathbb{C}^*$ and a biholomorphic $\varphi\colon\Sigma'\longrightarrow\Sigma$ so that

$$\varphi(z_1')=z_1,\quad \ldots,\quad \varphi(z_k')=z_k,\quad\text{and}\quad u'=\Theta_{c_1,\ldots,c_m}\circ u\circ\varphi\,,$$

where $\Theta_{c_1,\ldots,c_m}$ is the extension of the automorphism (4.17) by the identity over Y. A tuple as above is stable if it has finitely many automorphisms (self-equivalences). There is no canonical splitting of an element of $\overline{\mathcal{M}}_{g,k}(X\cup_V Y,A)$ into elements of the relative moduli spaces $\widetilde{\mathcal{M}}^V_{\chi_X,k_X;\mathbf{s}}(X,A_X)$ and $\widetilde{\mathcal{M}}^V_{\chi_Y,k_Y;\mathbf{s}}(Y,A_Y)$, except along the subspace of maps with $m\!=\!0$. Nevertheless, the compact Hausdorff topologies on the relative moduli spaces described in [IP4, Sections 6,7] induce a compact Hausdorff topology on $\overline{\mathcal{M}}_{g,k}(X\cup_V Y;A)$. We denote by $\widetilde{\mathcal{M}}_{g,k}(X\cup_V Y,A)$ the analogue of the space $\overline{\mathcal{M}}_{g,k}(X\cup_V Y,A)$ with disconnected domains Σ.

Remark 4.4. The description of the convergence topology on $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A_X)$ in [IP4, Sections 6,7] has a minor omission, as it does not involve adding new components to the domain of a map to X. This would prevent the appearance of such maps as those from domains as in Figure 2 that restrict to a fiber map on the left component mapping into $\{1\}\!\times\!\mathbb{P}_X V$. The moduli space $\overline{\mathcal{M}}_{g,k}(X\cup_V Y,A)$ does not appear in [IP4] or [IP5]. The space similar to $\overline{\mathcal{M}}_{g,k}(X\cup_V Y,A)$ that appears at the top of page 1003 in [IP5] does not quotient the maps by the $(\mathbb{C}^*)^m$-action on $X\cup_V^m Y$ and thus cannot be Hausdorff by [IP4, Sections 6,7]. This space is also not relevant and leads to the mistaken appearance of the S-matrix in the main symplectic sum formulas in [IP5]; see Section 5.5 for more details.

The roles of the "components" X and $\mathbb{P}_X V$ of the target space for relative stable maps in the setting of [LR] are played by $\mathring{X}_V$ and $\mathbb{R}\!\times\!SV$, respectively, or alternatively by $\widehat{X}_V$ and $[-\infty,\infty]\!\times\!SV$; see (3.44) and (3.45). For each $m\!\in\!\mathbb{Z}^{\ge0}$, let

$$\mathring{X}^m_V = \mathring{X}_V \sqcup \bigsqcup_{r=1}^{m} \{r\}\!\times\!\mathbb{R}\!\times\!SV, \qquad \widehat{X}^m_V = \bigg(\widehat{X}_V \sqcup \bigsqcup_{r=1}^{m} \{r\}\!\times\![-\infty,\infty]\!\times\!SV\bigg)\Big/\!\sim, \tag{4.22}$$

where

$$(-\infty)\!\times\!x \sim 1\!\times\!\infty\!\times\!x\,, \quad r\!\times\!(-\infty)\!\times\!x \sim (r\!+\!1)\!\times\!\infty\!\times\!x \qquad \forall\, x\!\in\!SV,\ r=1,\ldots,m\!-\!1.$$

The homeomorphism (4.17) induces homeomorphisms

$$\mathring{\Theta}_{c_1,\ldots,c_m}\colon \mathring{X}^m_V \longrightarrow \mathring{X}^m_V \qquad\text{and}\qquad \widehat{\Theta}_{c_1,\ldots,c_m}\colon \widehat{X}^m_V \longrightarrow \widehat{X}^m_V; \tag{4.23}$$

the first is the restriction of the homeomorphism (4.17), while the second is the continuous extension of the first. As in the setting of [IP4] described above, an almost complex structure J_X on X such that $J_X(TV)\!=\!TV$ induces an almost complex structures $\mathring{J}_m$ on $\mathring{X}^m_V$ so that the first map in (4.23) is biholomorphic. In the approach of [LR], J_X is chosen so that it has an asymptotic behavior near V as at the end of Section 3.3. The almost complex structure $\mathring{J}_m$ then satisfies

$$\mathring{J}_m\frac{\partial}{\partial a_X} = \zeta_H \ \text{ on } (-\infty,-a_0)\!\times\!SV \subset X, \qquad \mathring{J}_m\frac{\partial}{\partial a_{\mathbb{R}}} = \zeta_H \ \text{ on } \{r\}\!\times\!\mathbb{R}\!\times\!SV,\ r\!=\!1,\ldots,m,$$

for some $a_0\!\in\!\mathbb{R}^+$ sufficiently large, where ζ_H is the characteristic vector of the S^1 action as before and $\frac{\partial}{\partial a_X}$ and $\frac{\partial}{\partial a_{\mathbb{R}}}$ are the coordinate vector fields in the $\mathbb{R}$-direction on $(-\infty,-a_0)\!\times\!SV$ and $\mathbb{R}\!\times\!SV$, respectively. It restricts to the pull-back of J_V on $\ker\alpha\!\subset\!T(SV)$, where α is a connection one-form on the S^1-bundle $SV\!\longrightarrow\!V$.

The roles of the components Σ_j of the domains of J_m-holomorphic maps u into X_V^m with contact with $V\subset X$ or $\mathbb{P}_{X,0}V,\mathbb{P}_{X,\infty}V\subset\mathbb{P}_XV$ of order $s_{j,i}$ at $z_{j,i}\in\Sigma_j$ are played by the punctured Riemann surfaces $\mathring{\Sigma}_j=\Sigma_j-\{z_{j,i}:\,i\}$. The relative maps in the sense of [LR, Definition 3.14] are $\mathring{J}_m$-holomorphic maps

$$\mathring{u}\colon\mathring{\Sigma}\equiv\bigsqcup_j\mathring{\Sigma}_j\longrightarrow\mathring{X}_V^m \tag{4.24}$$

for some $m\in\mathbb{Z}^{\ge0}$ satisfying certain limiting, stability, and degree conditions. Let

$$\pi_{\mathbb{R}},\pi_{SV}\colon\mathbb{R}^-\times SV\longrightarrow\mathbb{R}^-,SV,\qquad \pi_{\mathbb{R}},\pi_{SV}\colon\{r\}\times\mathbb{R}\times SV\longrightarrow\mathbb{R},SV$$

denote the projection maps. The punctures of each topological component $\mathring{\Sigma}_j$ are either positive or negative with respect to $\mathring{u}$. If $z=\mathrm{e}^{-t+\mathrm{i}\theta}$ is a local coordinate centered at a positive puncture $z_{j,i}$ of Σ_j, i.e. $t\longrightarrow\infty$ as $z\longrightarrow0$, then $\mathring{u}(\mathring{\Sigma}_j)\subset\{r\}\times\mathbb{R}\times SV$ for some $r=1,\ldots,m$ and

$$\lim_{t\longrightarrow\infty}\pi_{\mathbb{R}}\circ\mathring{u}\big(\mathrm{e}^{-t+\mathrm{i}\theta}\big)=\infty,\qquad \lim_{t\longrightarrow\infty}\pi_{SV}\circ\mathring{u}\big(\mathrm{e}^{-t+\mathrm{i}\theta}\big)=\gamma\big(\mathrm{e}^{\mathrm{i}k\theta}\big)\qquad\forall\,\theta\in S^1,$$

for some $k\in\mathbb{Z}^+$ and 1-periodic S^1-orbit $\gamma\colon S^1\longrightarrow SV$ over a point $x\in V$. In such a case, we will write

$$\mathcal{P}^+_{z_{j,i}}(\mathring{u})=(x,k),\qquad \mathrm{ord}^+_{z_{j,i}}(\mathring{u})=k.$$

If $z=\mathrm{e}^{t+\mathrm{i}\theta}$ is a local coordinate centered at a negative puncture of Σ_j, i.e. $t\longrightarrow-\infty$ as $z\longrightarrow0$, then either

$$\mathring{u}(\mathring{\Sigma}_j)\subset\{r\}\times\mathbb{R}\times SV\ \text{ for some }\ r=1,\ldots,m\qquad\text{or}\qquad \mathring{u}(\mathrm{e}^{t+\mathrm{i}\theta})\in\mathbb{R}^-\times SV\subset X$$

and

$$\lim_{t\longrightarrow-\infty}\pi_{\mathbb{R}}\circ\mathring{u}\big(\mathrm{e}^{t+\mathrm{i}\theta}\big)=-\infty,\qquad \lim_{t\longrightarrow-\infty}\pi_{SV}\circ\mathring{u}\big(\mathrm{e}^{t+\mathrm{i}\theta}\big)=\gamma\big(\mathrm{e}^{\mathrm{i}k\theta}\big)\qquad\forall\,\theta\in S^1,$$

for some $k\in\mathbb{Z}^+$ and 1-periodic S^1-orbit $\gamma\colon S^1\longrightarrow SV$ over a point x in V. In either of the two cases, we will write

$$\mathcal{P}^-_{z_{j,i}}(\mathring{u})=(x,k),\qquad \mathrm{ord}^-_{z_{j,i}}(\mathring{u})=k.$$

Any map (4.24) satisfying these conditions has a well-defined degree $A\in H_2(X;\mathbb{Z})$ obtained by composing $\mathring{u}$ with the projection to X_V^m (which sends each limiting orbit $\gamma\subset SV$ to a single point $x\in V$) and then with the projection $X_V^m\longrightarrow X$.

For any nodal Riemann surface, we denote by $\Sigma^*\subset\Sigma$ the subspace of smooth points. Let $A\in H_2(X;\mathbb{Z})$, $g,k,\ell\in\mathbb{Z}^{\ge0}$, and $\mathbf{s}=(s_1,\ldots,s_\ell)\in(\mathbb{Z}^+)^\ell$ be a tuple satisfying (1.3). The relative moduli space $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A)$ of [LR] consists of stable tuples $(\Sigma,z_1,\ldots,z_{k+\ell},\mathring{u})$, where $(\Sigma,z_1,\ldots,z_{k+\ell})$ is a genus g marked nodal connected compact Riemann surface,

$$\mathring{u}\colon\mathring{\Sigma}\equiv\bigsqcup_j\mathring{\Sigma}_j\longrightarrow\mathring{X}_V^m,\qquad \Sigma^*-\{z_{k+1},\ldots,z_\ell\}\subset\mathring{\Sigma}\subset\Sigma-\{z_{k+1},\ldots,z_\ell\},$$

such that $\mathring{u}$ is a $\mathring{J}_m$-holomorphic map of degree A,

$$\mathring{u}^{-1}\big(\{r\}\times\mathbb{R}\times SV\big)\neq\emptyset\quad\forall\,r=1,\ldots,m,\qquad \mathrm{ord}^-_{z_{k+i}}(\mathring{u})=s_i\quad\forall\,i=1,\ldots,\ell,$$

each node in $\Sigma-\mathring{\Sigma}$ gives rise to one positive and one negative puncture of $(\mathring{\Sigma},\mathring{u})$, and the positive punctures $z_{j,i}$ of any component Σ_j mapped into $\{r\}\times\mathbb{R}\times SV$ for some $r=1,\ldots,m$ correspond to the nodes of Σ joining Σ_j to the components mapped into $\{r-1\}\times\mathbb{R}\times SV$ if $r>1$ and to X if $r=1$ outside of the punctures and

$$\mathcal{P}^+_{z_{j,i}}(\mathring{u})=\mathcal{P}^-_{z_{i,j}}(\mathring{u}), \tag{4.25}$$

where $z_{i,j}\in\Sigma_{i,j}$ is the point identified with $z_{j,i}$. Two such relative maps $(\Sigma,z_1,\ldots,z_{k+\ell},\mathring{u})$ and $(\Sigma',z_1',\ldots,z_{k+\ell}',\mathring{u}')$ are equivalent if there are $c_1,\ldots,c_m\in\mathbb{C}^*$ and a biholomorphic $\varphi:\Sigma'\longrightarrow\Sigma$ so that

$$\varphi(z_1')=z_1,\quad \ldots,\quad \varphi(z_{k+\ell}')=z_{k+\ell},\quad\text{and}\quad \mathring{u}'=\Theta_{c_1,\ldots,c_m}\circ\mathring{u}\circ\varphi.$$

A tuple as above is stable if it has finitely many automorphisms (self-equivalences).

By the same construction as in (3.45), the punctured Riemann surfaces $\mathring{\Sigma}_j$ above can be compactified to bordered surfaces $\widehat{\Sigma}_j$. The matching condition (4.25) insures that the surfaces $\widehat{\Sigma}_j$ can be glued together along pairs of boundary components corresponding to the same node of Σ into a surface $\widehat{\Sigma}$ with ℓ boundary components in such a way that $\mathring{u}$ extends to a continuous map $\widehat{u}\colon\widehat{\Sigma}\longrightarrow\widehat{X}^m_V$. Composing $\widehat{u}$ with the projection $\widehat{X}^m_V\longrightarrow X^m_V$, we obtain a relative map in the sense of [IP4, Definitions 7.1,7.2]. Removing the preimages of $V\subset X$ and $\mathbb{P}_{X,0}V,\mathbb{P}_{X,\infty}V\subset\mathbb{P}_XV$ under a relative map $u:\Sigma\longrightarrow X^m_V$ in the sense of [IP4], we obtain a relative map $\mathring{u}:\mathring{\Sigma}\longrightarrow\mathring{X}^m_V$ in the sense of [LR]. Thus, the moduli spaces of relative maps $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A)$ in the sense of [IP4] and [LR] are canonically identified when the same almost complex structure J_X on X is used. While the space of admissible J_X is smaller in [LR], it is still non-empty and path-connected, possesses the same transversality properties as the larger space of J_X in [IP4], and so is just as good for defining relative invariants. On the other hand, the stronger restriction on J_X in [LR] simplifies the required gluing constructions; see Section 5.

Suppose (Y,ω_Y) is another symplectic manifold containing V as a symplectic hypersurface so that (1.1) holds, J_Y is an ω_Y-compatible almost complex structure, such that $J_Y(TV)=TV$ and $J_Y|_{TV}=J_X|_{TV}\equiv J_V$, and we have chosen an isomorphism as in (1.2). Such an isomorphism identifies S_YV with $SV=S_XV$ as in (3.43). Continuing with the setup at the end of Section 3.3, let

$$\begin{gathered} X\mathring{\cup}^m_VY=\big(\mathring{X}^m_V\sqcup\mathring{Y}^m_V\big)/\!\sim,\qquad X\widehat{\cup}^m_VY=\big(\widehat{X}^m_V\sqcup\widehat{Y}^m_V\big)/\!\sim,\\ \widehat{X}^m_V\ni[r,x]\sim[m+1-r,x]\in\widehat{Y}^m_V\qquad\forall\ r=1,\ldots,m,\ x\in[-\infty,\infty]\times SV. \end{gathered} \tag{4.26}$$

The homeomorphisms (4.23) extend to these spaces by taking them to be the identity on $\mathring{Y}_V$.

Let $A\in H_2(X\cup_VY;\mathbb{Z})$ be an element in the image of $H_2(X;\mathbb{Z})\times_VH_2(Y;\mathbb{Z})$ under the natural homomorphism

$$H_2(X;\mathbb{Z})\oplus H_2(Y;\mathbb{Z})\longrightarrow H_2(X\cup_VY;\mathbb{Z})$$

and $g,k\in\mathbb{Z}^{\ge0}$. For almost complex structures J_X on X and J_Y on Y satisfying $J_X|_V=J_Y|_V$ and the asymptotic condition at the end of Section 3.3, the notions of k-marked genus g degree A stable maps to $X\cup_VY$ in [LR] and in [IP5] differ in essentially the same way as the notions of relative maps to (X,V) described above. Such a map is a tuple $(\Sigma,z_1,\ldots,z_k,\mathring{u})$, where $(\Sigma,z_1,\ldots,z_k)$ is a genus g marked nodal connected compact Riemann surface,

$$\mathring{u}\colon \mathring{\Sigma} \equiv \bigsqcup_j \mathring{\Sigma}_j \longrightarrow X\mathring{\cup}_V^m Y, \qquad \Sigma^* \subset \mathring{\Sigma} \subset \Sigma,$$

such that

- $\mathring{u}$ is a $\mathring{J}_m$-holomorphic map of degree A,
- $\mathring{u}^{-1}\big(\{r\}\times\mathbb{R}\times SV\big) \neq \emptyset$ for every $r=1,\ldots,m$,
- each node in $\Sigma-\mathring{\Sigma}$ gives rise to one positive and one negative puncture of $(\mathring{\Sigma},\mathring{u})$,
- the positive punctures $z_{j,i}$ of any component Σ_j mapped into $\{r\}\times\mathbb{R}\times SV$ for some $r=1,\ldots,m$ correspond to the nodes of Σ joining Σ_j to the components mapped into $\{r-1\}\times\mathbb{R}\times SV$ if $r>1$ and to X if $r=1$ outside of the punctures,
- the negative punctures $z_{j,i}$ of any component Σ_j mapped into $\{m+1-r\}\times\mathbb{R}\times SV$ for some $r=1,\ldots,m$ correspond to the nodes of Σ joining Σ_j to the components mapped into $\{m-r\}\times\mathbb{R}\times SV$ if $r>1$ and to Y if $r=1$ outside of the punctures, and
- $\mathcal{P}^+_{z_{j,i}}(\mathring{u})=\mathcal{P}^-_{z_{i,j}}(\mathring{u})$, where $z_{i,j}\in\Sigma_{i,j}$ is the point identified with $z_{j,i}$.

The notion of equivalences is defined as before. The moduli spaces $\overline{\mathcal{M}}_{g,k}(X\cup_V Y,A)$ of stable maps in the sense of [IP4] and [LR] are again canonically identified, by the same procedure as in the relative case above.

Remark 4.5. The key definition of relative stable maps, [LR, Definition 3.14], is not remotely precise. It involves three different Riemann surfaces, without a clear connection between them, a continuous map into a vaguely described space, and a vaguely specified equivalence relation. The signs of the limiting periods are not properly defined. The definition of a stable map into $X\cup_V Y$, [LR, Definition 3.18], implies that the rubber components can be separated into X and Y-parts, which is incorrect. The analysis related to the compactness of the moduli spaces $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A)$ and $\overline{\mathcal{M}}_{g,k}(X\cup_V Y,A)$ is contained in [LR, Sections 3.1,3.2]. Nearly all arguments in [LR, Section 3.1], which is primarily concerned with rates of convergence for maps to $\mathbb{R}\times SV$, are either incorrect or incomplete, but the only desired claim is easy to establish; see Section 5.1 below. [LR, Section 3.2] applies this claim to study convergence for sequences of J-holomorphic maps from Riemann surfaces with punctures into $\mathring{X}_V$, $\mathbb{R}\times SV$, and $X\#_V Y$, though the targets are never specified. The assumptions $u'(\Sigma')\subset D_p(\epsilon)$ and $u'(\partial\Sigma')\subset\partial D_p(\epsilon)$ in [LR, Lemma 3.8], which is missing a citation, should be weakened to $u'(\partial\Sigma')\cap D_p(\epsilon)=\emptyset$. The bound on the energy of J-holomorphic maps to $\mathbb{R}\times\widetilde{V}$ claimed below [LR, (3.44)] needs a justification; it follows from the correspondence with maps to $\mathbb{P}_X V$. There is no specification of the target of the sequence of maps u_i central to the discussion of [LR, Section 3.2]. The sentence containing [LR, (3.48)] and the next one do not make sense as stated. There is no mention of what happens to nodal points of the domain or if $\widetilde{m}_0=\widetilde{m}(q)$ is zero (which can happen, since $\widetilde{m}_0$ measures only the horizontal energy). The main argument applies [LR, Theorem 3.7] to maps from disk, even though it is stated only for maps from $\mathbb{C}$ (as done in [H, HWZ1]). In (3) of the proof of [LR, Lemma 3.11], the horizontal distance bound [LR, (3.55)] is used (incorrectly) to draw a conclusion about the vertical distance in the last equation; it would have implied the last claim of (3) without [LR, (3.51),(3.52)]. Because of the arbitrary

choice of t_0 in [LR, (3.53)], the claim of [LR, Lemma 3.11(3)] in fact cannot be possibly true. The statement of [LR, Lemma 3.12] even explicitly excludes stable ghost bubbles with one puncture going into the rubber, which is incorrect. As the rescaling argument in [LR, Section 3.2] concerns one node at a time, it has the same kind of issue as described in the first part of Remark 4.4. On the other hand, the approach of [LR] is better suited to deal with this issue because it can be readily interpreted as a rescaling on the target. The proof of [LR, Lemma 3.15] has basically no content. Other, fairly minor misstatements in [LR, Sections 3.2,3.3] include
p178, lines 5-7: u has finite energy;
p178, (3.44): P is not used until Section 3.3;
p178, below (3.44): $E_\phi(u)$ is fixed, according to (3.43);
p180, (3.49),(3.50): follow from [MS2, Lemma 4.7.3];
p180, line -1: $\log\epsilon \le s \le \log\delta_i'$;
p181, line 2: $\delta_i < \delta_i'$, $\log\epsilon \le s \le \log\delta_i$;
p181, (3.52): $log \longrightarrow \log$;
p181, Lemma 3.11: N already denotes a space;
p181, line -1: Lemma (3.5) $\longrightarrow$ Lemma 3.5;
p182, (3.57): $lim \longrightarrow \lim$; $A(\epsilon, R\delta_i) \longrightarrow A(R\delta_i, \epsilon)$;
p182, (3.57): repeat of first sentence of (3);
p183, Rmk 3.13: the collapsed compact manifold is $\mathbb{P}(\widetilde{V}\times_{S^1}\mathbb{C}\oplus V\times\mathbb{C})$;
p183, Section 3, lines 1,2: $\Sigma_1\vee\Sigma_2\longrightarrow\mathbb{R}\times\widetilde{M}$ be a map;
p185, line 14: $\bigoplus \longrightarrow \bigsqcup$.

4.3 Relative invariants: [IP4], [IP5, Section 1], [LR, Section 4]

Let X, V, A, g, k, and $\mathbf{s}$ be as in Section 4.2. The moduli space $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A)$ carries a virtual fundamental class (VFC), which gives rise to relative GW-invariants of (X,ω,V) and is used in the proof of the symplectic sum formula in [Lj2, LR]. The proof in [IP5] is restricted to the cases when the relevant relative and absolute invariants can be realized more geometrically, but the principles of [IP5] apply in the general VFC setting as well, once the VFC is shown to exist. In the restricted setting of [IP4, IP5], it is not even necessary to consider the elaborate rubber structure (maps to $\mathbb{P}_X V$) described in Section 4.2. Below we review the geometric construction of the absolute GW-invariants, due to [RT1, RT2], and its adaption to relative invariants, due to [IP4]. We then comment on the general case considered in [LR].

We begin with two definitions which are later used to describe the cases when the absolute and relative invariants can be realized geometrically.

Definition 4.6. A $2n$-dimensional symplectic manifold (X,ω) is

(1) semi-positive if $\langle c_1(X),A\rangle\ge 0$ for all $A\in\pi_2(M)$ such that

$$\langle\omega,A\rangle > 0 \qquad \text{and} \quad c_1(A)\ge 3-n, \tag{4.27}$$

(2) strongly semi-positive if $\langle c_1(X),A\rangle > 0$ for all $A\in\pi_2(M)$ such that (4.27) holds.

Definition 4.7. Let (X,ω) be a $2n$-dimensional symplectic manifold and $V\subset X$ be a symplectic divisor. The triple (X,ω,V) is

(1) semi-positive if $\langle c_1(X),A\rangle \ge A\cdot_X V$ for all $A\in\pi_2(M)$ such that

$$A\cdot_X V \ge 0, \qquad \langle\omega,A\rangle>0, \quad \text{and} \quad \langle c_1(X),A\rangle \ge \max(3, A\cdot_X V+2)-n, \tag{4.28}$$

(2) strongly semi-positive if $\langle c_1(X),A\rangle > A\cdot_X V$ for all $A\in\pi_2(M)$ such that (4.28) holds.

A $2n$-dimensional symplectic manifold (X,ω) is semi-positive if $n\le 3$ and is strictly semi-positive if $n\le 2$. Similarly, if $V\subset X$ is a symplectic hypersurface, (X,ω,V) is semi-positive if $n\le 2$ and is strictly semi-positive if $n=1$.

Let $g,k\in\mathbb{Z}^{\ge 0}$ be such that $2g+k\ge 3$,

$$\widetilde{\mathcal{M}}_{g,k} \longrightarrow \overline{\mathcal{M}}_{g,k} \tag{4.29}$$

be the branched cover of the Deligne-Mumford space of stable genus g k-marked curves by the associated moduli space of Prym structures constructed in [Lo], and

$$\pi_{g,k}: \check{\mathcal{U}}_{g,k} \longrightarrow \widetilde{\mathcal{M}}_{g,k}$$

be the corresponding universal curve. A genus g k-marked nodal curve with a Prym structure is a connected compact nodal k-marked Riemann surface $(\Sigma,z_1,\ldots,z_k)$ of arithmetic genus g together with a holomorphic map $\mathrm{st}_\Sigma:\Sigma\longrightarrow\check{\mathcal{U}}_{g,k}$ which surjects on a fiber of $\pi_{g,k}$ and takes the marked points of Σ to the corresponding marked points of the fiber. If $A\in H_2(X;\mathbb{Z})$, J is an almost complex structure on X, and

$$\nu\in\Gamma_{g,k}(X,J)\equiv\Gamma\big(\check{\mathcal{U}}_{g,k}\times X, \pi_1^*(T^*\check{\mathcal{U}}_{g,k})^{0,1}\otimes\pi_2^*(TX,J)\big), \tag{4.30}$$

a degree A genus g k-marked (J,ν)-map is a tuple $(\Sigma,z_1,\ldots,z_k,\mathrm{st}_\Sigma,u)$ such that $(\Sigma,z_1,\ldots,z_k,u)$ is a genus g k-marked nodal curve with a Prym structure and $u:\Sigma\longrightarrow X$ is a smooth (or L^p_1, with $p>2$) map such that

$$u_*[\Sigma]=A \qquad \text{and} \qquad \bar\partial_{J,\mathfrak{j}}u\big|_z=\nu|_{(\mathrm{st}_\Sigma(z),u(z))} \quad \forall\, z\in\Sigma,$$

where $\mathfrak{j}$ is the complex structure on Σ. Two such tuples are equivalent if they differ by a reparametrization of the domain commuting with the maps to $\check{\mathcal{U}}_{g,k}$.

By [RT2, Corollary 3.9], the space $\overline{\mathcal{M}}_{g,k}(X,A;J,\nu)$ of equivalence classes of degree A genus g k-marked (J,ν)-maps is Hausdorff and compact (if X is compact) in Gromov's convergence topology. By [RT2, Theorem 3.16], for a generic J each stratum of $\overline{\mathcal{M}}_{g,k}(X,A;J,\nu)$ consisting of simple (not multiply covered) maps of a fixed combinatorial type is a smooth manifold of the expected even dimension, which is less than the expected dimension of the subspace of simple maps with smooth domains (except for this subspace itself). By [RT2, Theorem 3.11], the last stratum has a canonical orientation. By [RT2, Proposition 3.21], the images of the strata of $\overline{\mathcal{M}}_{g,k}(X,A;J,\nu)$ consisting of multiply covered maps under the morphism

$$\mathrm{ev}\times\mathrm{st}: \overline{\mathcal{M}}_{g,k}(X,A;J,\nu)\longrightarrow X^k\times\overline{\mathcal{M}}_{g,k} \tag{4.31}$$

are contained in images of maps from smooth even-dimensional manifolds of dimension less than this stratum if (J,ν) is generic and (X,ω) is semi-positive. Thus, (4.31) is a pseudocycle. Intersecting it with classes in the target and dividing by the order of the covering (4.29), we obtain (absolute) GW-invariants of a semi-positive symplectic manifold (X,ω) in the stable range, i.e. with (g,k) such that $2g+k\geq 0$. In the unstable range (which must be considered for the disconnected GT-invariants), the same reasoning applies with $\nu=0$ and yields the same conclusion if (X,ω) is strictly semi-positive.

Suppose in addition that $V\subset X$ is a symplectic divisor preserved by the almost complex structure J and $\mathbf{s}\in(\mathbb{Z}^+)^\ell$ is a tuple satisfying (1.3). For

$$\nu\in\Gamma_{g,k}(X,J)\qquad \text{s.t.}\quad \nu|_{\widetilde{\mathcal{U}}_{g,k}\times V}\in\Gamma_{g,k}(V,J|_V)\,, \tag{4.32}$$

we define the moduli space

$$\mathcal{M}^V_{g,k;\mathbf{s}}(X,A;J,\nu)\subset\overline{\mathcal{M}}_{g,k+\ell}(X,A;J,\nu)$$

analogously to (4.19). If $u\colon\Sigma\longrightarrow X$ is a (J,ν)-holomorphic map such that $u(\Sigma)\subset V$, the linearization of $\bar\partial_{J,\mathfrak{j}}+\nu$ at u again descends to a first-order differential operator

$$D_u^{\mathcal{N}_XV}\colon\Gamma(\Sigma;u^*\mathcal{N}_XV)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(\Sigma;u^*\mathcal{N}_XV).$$

If J satisfies (4.18) and

$$\big\{\nabla_w\nu+J\nabla_{Jw}\nu\big\}(v)\in T_xV\qquad\forall\ v\in T_xV,\ w\in T_xX,\ x\in V, \tag{4.33}$$

then this linearization is $\mathbb{C}$-linear and in fact is the same as the corresponding operator with $\nu=0$. A "compactification" $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A;J,\nu)$ of $\mathcal{M}^V_{g,k;\mathbf{s}}(X,A;J,\nu)$ similar to the $\nu=0$ case above is described in [IP4, Section 7] under the assumptions (4.18) and (4.33) on (J,ν).

Remark 4.8. There are a number of misstatements in the related part of [IP5]. In [IP5, (1.11)], $+(J\nabla_\xi J)$ should be $-(J\nabla_\xi J)$ to agree with [MS2, Proposition 3.1.1] in the $\nu=0$ case. This is also necessary to obtain [IP5, (1.14)] with 1/4 instead of 1/8 and

$$N_J(\xi,\zeta)=-[\xi,\zeta]-J[J\xi,\zeta]-J[\xi,J\zeta]+[J\xi,J\zeta]\qquad\forall\,\xi,\zeta\in\Gamma(X,TX),$$

as in (4.13) above and in [MS2]. Furthermore, $\Phi_f=0$ if f is $(J,\mathfrak{j},\nu)$-holomorphic; otherwise, there are lots of linearizations of $\bar\partial_{J,\mathfrak{j}}+\nu$. The three-term expression in parenthesis in [IP5, (1.11)] reduces to $\{\partial f-\nu\}(w)$, but should be just $\partial f(w)$ to be consistent with [MS2, Proposition 3.1.1]; otherwise, this term is not even $(J,\mathfrak{j})$-antilinear. In this equation, ∇ denotes the pull-back connection of the Levi-Civita connection ∇ for the metric [IP5, (1.1)] to a connection in u^*TX in the first two times it appears, but ∇ itself the last two times it appears (contrary to p945, line -4); the term $\nabla_\xi\nu$ should be replaced by $\widetilde\nabla_\xi\nu$. Via the first equation in [MS2, (C.7.5)], the correct version of [IP5, (1.11)] gives

$$\frac14N_J(\xi,\partial f)-\frac12T_\nu(\xi,w),\qquad\text{where}\quad T_\nu(\xi,w)=\{\widetilde\nabla_\xi\nu\}(w)+J\{\widetilde\nabla_{J\xi}\nu\}(w),$$

instead of [IP5, (1.14)]; the correct version is consistent with [MS2, (3.1.5)]. The reason [IP5, (1.15b)], with the above correction for T_ν, is equivalent to [IP4, (3.3bc)] is the restriction in (4.30) and that the right-hand side of [IP4, (3.3c)] is zero by [MS2, (C.7.1)]. Other related typos include

p943, (1.2): RHS should end with $\circ d\phi$;

p943, line 11: $\mathrm{Hom}(\pi_1^*T\mathbb{P}^N,\pi_2^*TX)\ \longrightarrow\mathrm{Hom}(\pi_2^*T\mathbb{P}^N,\pi_1^*TX)$;

p943, line 13: $J_{\mathbb{P}^n} \longrightarrow J_{\mathbb{P}^n}v$;
p943, line 17: $|(u,v)|^2$ presumably means $|u|^2+|v|^2$, in contrast to $|dF|^2$ in [IP5, (1.5)];
p943, (1.5): second half should read $\int_B F^*\hat{\omega} = \omega([f]) + \omega_{\mathbb{P}^n}([\phi])$;
p943, line -3: *smooth* is questionable across the boundary;
p945, bottom: since $h \in H^{0,1}(TC(-\sum p_i))$, which is a quotient, f_*h is not defined;
p947, lines 14,15: $\mathrm{coker} D_{\mathbf{s}} = 0$ after restricting the range of D.

By [IP4, Proposition 7.3], the space $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A;J,\nu)$ is (Hausdorff and) compact (if X is compact) in Gromov's convergence topology. By [IP4, Lemma 7.5], if J is generic each stratum of $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A;J,\nu)$ consisting of simple maps of a fixed combinatorial type is a smooth manifold of the expected even dimension, which is less than the expected dimension of the subspace of simple maps with smooth domains (except for this subspace itself). By [IP4, Theorem 7.4], the last stratum has a canonical orientation. The multiply covered maps in $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A;J,\nu)$ fall into two (overlapping) subsets: those with a multiply covered component mapped into V and those with a multiply covered component not contained in V. By [RT2, Proposition 3.21], the images of first type of multiply covered strata under the morphism

$$\mathrm{ev}\times\mathrm{ev}^V\times\mathrm{st}\colon \overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A;J,\nu) \longrightarrow X^k \times V^\ell \times \overline{\mathcal{M}}_{g,k+\ell} \tag{4.34}$$

are contained in images of maps from smooth even-dimensional manifolds of dimension less than the main stratum if $(J|_V,\nu|_V)$ is generic and $(V,\omega|_V)$ is semi-positive. By a similar dimension counting, the images of the second type of multiply covered strata under (4.34) are contained in images of maps from smooth even-dimensional manifolds of dimension less than the main stratum if (J,ν) is generic, subject to the conditions (4.18) and (4.33), and (X,ω,V) is semi-positive.[4] Thus, (4.34) is a pseudocycle and gives rise to relative GW-invariants of a semi-positive triple (X,ω,V) with a semi-positive $(V,\omega|_V)$. In the unstable range, similar reasoning applies with $\nu=0$ and yields the same conclusion if (X,ω,V) is strictly semi-positive and $(V,\omega|_V)$ is semi-positive. One key difference in this case is that the space of multiply covered relative degree A J-holomorphic maps from smooth domains with two relative marked points can be of the same dimension as the space of simple degree A J-holomorphic maps from smooth domains, but is then smooth.

While it is not stated in the assumptions for [IP5, (0.2),(10.14),(12.17)], the proof of these formulas in [IP5] is nominally restricted to the cases when $(X\#_V Y,\omega_\#)$, (X,ω_X,V), and (Y,ω_Y,V) satisfy suitable positivity conditions. By the above, these conditions are

(0) $(X\#_V Y,\omega_\#)$ is strongly semi-positive;

(1) $(V,\omega_X|_V)=(V,\omega_Y|_V)$ is semi-positive;

(2) (X,ω_X,V) and (Y,ω_Y,V) are strongly semi-positive.

Condition (0) is not implied by the other two conditions in general. However, it can still be ignored, since it holds when restricted to the classes $A\in\pi_2(X\#_V Y)$ which can be represented

[4] If (4.27) fails, the space of simple degree A (J,ν)-maps is empty for a generic J. If (4.28) fails, the space of simple relative degree A (J,ν)-maps with one relative marked point is empty. Irreducible components of the domain of a map in $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A;J,\nu)$ which carry at least two marked points are stable because they also carry at least one node; (J,ν)-maps from stable components are not multiply covered for a generic ν.

by $J_\#$-holomorphic curves for an almost complex structure $J_\#$ induced by generic almost complex structures J_X on (X,V) and J_Y on (Y,V) via the symplectic sum construction of Section 3.3, i.e. an almost complex structure $J_\#$ of the kind considered in [IP5]; see the second identity in (3.42). In light of (1.1), Condition (2) implies Condition (1). Thus, the setting in [IP5] is directly applicable whenever Condition (2) is satisfied.

Remark 4.9. In the semi-positive case, the relative moduli space described above can be replaced by a subspace of $\overline{\mathcal{M}}_{g,k+\ell}(X,A)$; see [IP4, Section 7]. There is some confusion in [IP4, IP5] regarding the proper semi-positivity requirements in the relative case. The only requirement stated in [IP4, Theorem 1.8] is that (X,ω) is semi-positive; [IP4, Theorem 8.1] also requires $(V,\omega|_V)$ to be semi-positive. The only condition stated in the bottom half of page 947 in [IP5], in the context of disconnected GT-invariants appearing on the following page, is that $\langle c_1(X),A\rangle \ge A\cdot_X V$ whenever

$$\langle\omega,A\rangle>0 \qquad \text{and} \qquad \langle c_1(X),A\rangle \ge \max(3, A\cdot_X V+1)-n\,.$$

The domain and the target of the linearized $\bar\partial$-operator D_s^N are described incorrectly below [IP4, (6.2)]; the index of the described operator is generally too small (because $s_i(s_i+1)/2$ contact conditions on the vector fields are imposed at each contact, but no conditions on the one-forms). The resulting bundle section in [IP4, (6.7)] cannot be transverse unless $s_i=1$. However, this issue can be resolved by using the twisting down construction of [1, Lemma 2.4.1]. The observation in the sentence before the preceding paragraph is not made in [IP4, IP5], but it is necessary to make sense of the invariants giving rise to the S-matrix in [IP5, Section 11]; see Section 5.5.

In order to define relative invariants without a semi-positivity assumption on (X,ω,V), it is necessary to describe neighborhoods of elements of the relative moduli space inside of a configuration space and to construct finite-rank vector bundles over them with certain properties. Unlike the situation with absolute GW-invariants in [FO] and [LT], describing such a neighborhood requires gluing maps with rubber components which are defined only up to a $\mathbb{C}^*$-action on the target. The aim of [LR, Section 4] is to justify the existence of such invariants. However, the gluing construction in [LR, Section 4] is limited to maps with a single node. Even in this very special case, the $\mathbb{C}^*$-action on the maps to the rubber ($\mathbb{R}\times SV$ in the approach of [LR]) is not considered, and the target space for the resulting glued maps, described by [LR, (4.12),(4.13)], is not the original space $\mathring{X}_V$, but a manifold diffeomorphism to $\mathring{X}_V$ (and not canonically or biholomorphically). Neither the injectivity nor surjectivity of the neighborhood description is even considered in [LR]. Thus, there is not even an attempted construction of a virtual fundamental class for $\overline{\mathcal{M}}^V_{g,k;\mathbf{s}}(X,A)$ in [LR]. Nevertheless, the suggested idea of stretching the necks on both the domain and the target of the maps fits naturally with the analytic problems involved in such a construction; we return to this point in Section 5.

Remark 4.10. The formula [LR, (4.2)] for the linearized $\bar\partial$-operator is incorrect, since J is not even tamed by the metric; see [MS2, Section 3.1]. The statement above [LR, Remark 4.1] requires a citation. The norms on the line bundle $u^*L\otimes\lambda$ on page 190 in [LR] are not specified; because of the poles at the nodes, it does not even seem possible to define norms on this line bundle in a way that extend over nearby smooth domains. Furthermore, the 3-4 pages spent on this line bundle are not necessary; it is used only to construct local finite-rank subbundles of the cokernel bundle $\mathcal{F}$. On the other hand, the deformations constructed from this line bundle need to respect the $\mathbb{C}^*$-action on $\mathbb{R}\times SV$ and thus need to be pulled back from V as in [IP4], of which no mention is made. The required bound on the radial component a in [LR, Lemma 4.6] and other statements is not part of any previous statement, such as [LR, Theorem 3.7]. In [LR, Section 4.2], the Implicit Function Theorem in an infinite-dimensional setting is involved twice (middle of page 200 and bottom of page 201) without any care. While the relevant bounds for the 0-th and 1-st order terms are at least discussed in [LR, Section 4.1], not a word is said about the quadratic term. The variable r is used to denote the norm of the gluing parameter $(r)=(r,\theta_0)$ in an ambiguous way. The issue is further confused by the notation i_r at the bottom of page 193 in [LR], I_r at the bottom of page 201, (ξ_r, h_r) in (4.51); in all cases, the subscript r should be replaced by the gluing parameter (r). The most technical part of the paper, roughly 4 pages, concerns the variation of various operators with respect to the norm r of (r), which is done without explicitly identifying the domains and targets of these spaces. This part is used only for showing that the integrals [LR, (4.50)] defining relative invariants converge. However, this is not necessary, since the relevant evaluation had supposedly been shown to be a rational pseudocycle before then. At the end of the first part of the proof of [LR, Proposition 4.1], it is claimed that the overlaps of the gluing maps are smooth; no one has shown this to be the case along the lower strata. The wording of [LR, Lemma 4.12] suggests the existence of a diffeomorphism between an odd-dimensional manifold and an even-dimensional manifold. The constant C_3 in [LR, (4.44)] depends on α; thus, it is unclear that $C_3|\alpha|$ can be made arbitrary small. The inequality [LR, (4.57)] is not justified. The paper does not even touch on the independence claims of [LR, Theorem 4.14]. Other, fairly minor misstatements in [LR, Section 4] include

p188, below Rmk 4.2: the implication goes the other way;
p189, lines 10,13: $\Sigma_1\wedge\Sigma_2 \longrightarrow \Sigma_1\vee\Sigma_2$; $h_{10}=h_{20} \longrightarrow \hat{h}_{10}=\hat{h}_{20}$;
p190, lines -7,-6: unjustified and irrelevant statement;
p192, line -2: x has not been defined;
p193, (4.16): δ as in (4.3);
p193, (4.17): $s_2+4r \longrightarrow s_2$;
p194, (4.20): P has very different meaning in (3.44);
p203, (4.60) would be more relevant without Q and $D\mathcal{S}$;
p204, (4.60): the middle term on RHS should be dropped;
p204, (4.65): the "other gluing parameter v" is denoted by θ_0 on p192;
p205, Thm 4.14 repeats Thm C on p158 (7 lines).

5 On the proof of Theorem 4.1

The analytic steps needed to establish Theorem 4.1 can be roughly split into four parts: á priori estimates on convergence and on stable maps to $X\cup_V Y$, a pregluing construction, uniform elliptic estimates, and a gluing construction; we review them below. While some statements in [IP5] implicitly assume suitable positivity conditions on $(X\#_V Y,\omega_\#)$, (X,V), and (Y,V), the approach described in [IP5] to comparing numerical GW-invariants would fit with all natural VFC constructions, such as in [FO, LT], once they are shown to apply to relative invariants. However, the analytic issues required for constructing and comparing the relevant VFCs appear to be much harder to deal with in the approach of [IP5] than of [LR].

5.1 Á priori estimates: [IP5, Sections 3-5], [LR, Section 3.1]

Let $V\subset X$ be a submanifold of real codimension two and J be an almost complex structure on X such that $J(TV)=TV$. Suppose $(\Sigma,\mathfrak{j})$ is a smooth Riemann surface,

$$\nu\in\Gamma\big(\Sigma\times X, T^*\Sigma^{0,1}\otimes_{\mathbb{C}}TX\big) \qquad \text{s.t.} \quad \nu|_{\Sigma\times V}\in\Gamma\big(\Sigma\times V, T^*\Sigma^{0,1}\otimes_{\mathbb{C}}TV\big),$$

and z is a complex coordinate on a neighborhood $\Sigma_{u;z_0}$ of z_0 with $z(z_0)=0$. If $u:\Sigma\longrightarrow X$ is a smooth map such that $u^{-1}(V)=\{z_0\}$ and

$$\bar{\partial}_{J,\mathfrak{j}}u|_z \equiv \frac{1}{2}\big(\mathrm{d}_z u + J(u(z))\circ \mathrm{d}_z u\circ \mathfrak{j}_z\big) = \nu\big(z,u(z)\big) \qquad \forall\, z\in\Sigma$$

and $\mathcal{N}_X V|_{W_{u(z_0)}}\approx W_{u(z_0)}\times\mathcal{N}_X V|_{u(z_0)}$ is a trivialization of $\mathcal{N}_X V$ over a neighborhood $W_{u(z_0)}$ of $u(z_0)$ in V, then there exist

- a neighborhood $\Sigma'_{u;z_0}$ of z_0 in $u^{-1}(\mathcal{N}_X V|_{W_{u(z_0)}})\cap\Sigma_{u;z_0}$ and
- $\Phi\in L^p_1(\Sigma'_{u;z_0};\mathcal{N}_X V|_{u(z_0)}-0)$, for any $p>2$, such that

$$\pi_2(u(z)) = \Phi(z)z^{\mathrm{ord}^V_{z_0}(u)} \quad \forall\, z\in\Sigma'_{u;z_0}\,; \tag{5.1}$$

see [FHS, Theorem 2.2].

Let $\pi:\mathcal{Z}\longrightarrow\Delta$ be a symplectic fibration associated with the symplectic sum $(X\#_V Y,\omega_\#)$ as in Proposition 3.16 and $J_{\mathcal{Z}}$ be an $\omega_{\mathcal{Z}}$-compatible almost complex structure on $\mathcal{Z}$ as before. By Gromov's Compactness Theorem [RT1, Proposition 3.1], a sequence of $(J_{\mathcal{Z}},\mathfrak{j}_k)$-holomorphic maps $u_k:\Sigma\longrightarrow\mathcal{Z}_{\lambda_k}$, with $\lambda_k\in\Delta^*$ and $\lambda_k\longrightarrow 0$, has a subsequence converging to a $(J_{\mathcal{Z}},\mathfrak{j})$-holomorphic map $u:\Sigma'\longrightarrow\mathcal{Z}_0$. By the previous paragraph,

$$\Sigma' = \Sigma'_X\cup\Sigma'_V\cup\Sigma'_Y\,,$$

where Σ'_V is the union of irreducible components of Σ' mapped into V, Σ'_X is the union of irreducible components mapped into $X-V$ outside of finitely many points $x_1,\ldots,x_\ell$, and Σ'_Y is the union of irreducible components mapped into $Y-V$ outside of finitely many points $x'_1,\ldots,x'_{\ell'}$. By [IP5, Lemma 3.3], which is the main statement of [IP5, Section 3], if $\Sigma'_V=\emptyset$, then $\ell=\ell'$ and

$$\big(\mathrm{ord}^V_{x'_1}u,u(x'_1)\big) = \big(\mathrm{ord}^V_{x_{\tau(1)}}u,u(x_{\tau(1)})\big), \quad \ldots \quad \big(\mathrm{ord}^V_{x'_\ell}u,u(x'_\ell)\big) = \big(\mathrm{ord}^V_{x_{\tau(\ell)}}u,u(x_{\tau(\ell)})\big) \tag{5.2}$$

for some automorphism $\tau\in S_\ell$ of $\{1,\ldots,\ell\}$. This conclusion also holds for sequences of $(J_{\mathcal{Z}},\nu)$-holomorphic maps with

$$\nu|_V\in\Gamma_{g,k}(V,J_{\mathcal{Z}}|_V),$$

similarly to (4.32).

Remark 5.1. The expansion [IP5, (5.5)], based on [IP4, Lemma 3.4], corresponds to Φ above being differentiable at $z=0$. As can be seen from (5.1), this is indeed the case if u is smooth. The proof of [IP5, Lemma 3.3] is purely topological and applies to convergent sequences of continuous maps. An explicit condition, called δ-flatness, insuring that $\Sigma'_V=\emptyset$ above is described in [IP5, Section 3]. Contrary to the suggestion after [IP5, Definition 3.1], the δ-flatness condition does not prevent the marked points from being sent into V and thus a δ-flat J-holomorphic map into $\mathcal{Z}_0$ need not be V-regular in the sense of [IP4, Definition 4.1]. Other, fairly minor misstatements in [IP5, Section 3] include
p954, (3.5): the limit is over $\lambda\neq 0$;
p954, line -12: there is no [IP4, Lemma 3.2]; [IP4, Lemma 3.4] alone suffices;
p955, Lemma 3.3, proof: f_k is an element of a sequence, but f_1,f_2 are parts of a limiting map;

p956, lines 7-8: stabilization does not fit with this map and there is no need for it, since $\widetilde{\mathcal{M}}_{\chi,n}$ consists of curves with finitely many components, not necessarily stable ones, according to p946, line -6;
p957, (3.11); p957, (3.12): the fiber products should be quotiented by $S_{\ell(s)}$.

A node of the limiting map u as in [IP5, Lemma 3.3] corresponds to special points $z_0\in\Sigma'_X$ and $w_0\in\Sigma'_Y$ with

$$u(z_0)=u(w_0)=q \qquad\text{and}\qquad \mathrm{ord}^V_{z_0}u=\mathrm{ord}^V_{w_0}u=s$$

for some $q\in V$ and $s\in\mathbb{Z}^+$. A neighborhood of this node in the total space of a versal family of deformations of Σ' is given by

$$U=\big\{(\mu',\mu,z,w)\in\mathbb{C}^{\ell-1}\times\mathbb{C}^3\colon\, zw=\mu\big\}, \tag{5.3}$$

with Σ' corresponding to $(\mu,\mu')=0$ and the node at $(z,w)=0$. Let

$$U_{\mu',\mu;\epsilon}=\big\{(\mu',\mu,z,w)\in U\colon\, |z|,|w|<\epsilon^{1/2}\big\}, \qquad \varrho(\mu',\mu,z,w)=\sqrt{|z|^2+|w|^2}\,.$$

Denote by

$$x\colon\mathcal{N}_XV|_{W_q}\longrightarrow\mathcal{N}_XV|_q \qquad\text{and}\qquad y\colon\mathcal{N}_YV|_{W_q}\longrightarrow\mathcal{N}_YV|_q$$

the projections induced by dual trivializations of $\mathcal{N}_X$ and $\mathcal{N}_Y$ over a neighborhood W_q of q in V, similarly to Section 3.3. Below we will assume that W_q is identified with $\mathbb{R}^{2n}$ using geodesics from q.

Let $(\mu_k,\mu'_k)\in\mathbb{C}\times\mathbb{C}^{\ell-1}$ be the parameters corresponding to Σ_k, the domain of u_k. For each $\epsilon<\epsilon_k$ such that $u_k(U_{\mu'_k,\mu_k;\epsilon})\subset\mathcal{Z}_{\mathrm{neck}}|_{W_q}$, let $\bar{u}^V_{k;\epsilon}\in W_q$ denote the average value of $\pi_V\circ u_k|_{U_{\mu'_k,\mu_k;\epsilon}}$ with respect to the cylindrical metric on $U_{\mu',\mu;\epsilon}$ and

$$\tilde{u}^V_{k;\epsilon}(z)=\pi\circ u_k(z)-\bar{u}^V_{k;\epsilon}\in W_q \qquad \forall\, z\in U_{\mu'_k,\mu_k;\epsilon}.$$

Under the assumptions of the paragraph above Remark 5.1,

$$x(u_k(z))\cdot y(u_k(z))=\lambda_k \qquad \forall\, z\in\Sigma_k\,.$$

By [FHS, Theorem 2.2],

$$\lim_{z\longrightarrow0}\frac{x(u(z))}{az^s}=1 \qquad\text{and}\qquad \lim_{z\longrightarrow0}\frac{y(u(w))}{bw^s}=1 \tag{5.4}$$

for some $a\in\mathcal{N}_XV|_q-0$ and $b\in\mathcal{N}_YV|_q-0$. By [IP5, Lemma 5.3],

$$\lim_{k\longrightarrow\infty}\frac{\lambda_k}{ab\mu_k^s}=1. \tag{5.5}$$

The factor of $\langle\mathbf{s}\rangle$ in (1.9) is a reflection of this statement and takes into account the number of solutions μ_k of the equation $ab\mu_k^s=\lambda_k$ for a fixed λ_k. By [IP5, Lemma 5.4],

$$\begin{split}\int_{U_{\mu'_k,\mu_k;\epsilon}}\Big(|\tilde{u}_{k;\epsilon}|^p+|\mathrm{d}\tilde{u}_{k;\epsilon}|^p+|\varrho^{1-s}x\circ u_k|^p+|\varrho^{1-s}\mathrm{d}(x\circ u_k)|^p\\ +|\varrho^{1-s}y\circ u_k|^p+|\varrho^{1-s}\mathrm{d}(y\circ u_k)|^p\Big)\varrho^{-p\delta'}\le C_p\epsilon^{p/3}\end{split} \tag{5.6}$$

for all $p \geq 2$, $\delta', \epsilon \in \mathbb{R}^+$ sufficiently small, $k \in \mathbb{Z}^+$ sufficiently large, and for some $C_p \in \mathbb{R}^+$ (dependent of the sequence $\{u_k\}$ only); the norms in (5.6) are defined using the cylindrical metric on $U_{\mu'_k, \mu_k; \epsilon}$ and the metric $g_{\mathcal{Z}}$ on $\mathcal{Z}$. Both statements, (5.5) and (5.6), make use of [IP5, Lemma 5.1], which is a version of the standard exponential decay of the energy of a J-holomorphic map in the "middle" of a long cylinder; see [MS2, Lemma 4.7.3].

Remark 5.2. The proof of [IP5, Lemma 5.3] uses a complete metric on the universal curve $\mathcal{U}_{g,n}$ over the moduli space $\mathcal{M}_{g,n}$ of smooth n-marked genus g curves (with Prym structures) constructed in [IP5, Section 4] by re-scaling a Kahler metric $g_{\mathcal{U}}$ on $\overline{\mathcal{U}}_{g,n}$ along the nodal strata $\mathcal{N}$. The apparent, implicit intention is to take the metric $g_{\mathcal{U}}$ in [IP5, (4.1)] so that it satisfies [IP5, (4.4)]. As the various local metrics are patched together, the resulting global metric is not of the form [IP5, (4.10)] everywhere near $\mathcal{N}$. This section also does not yield a compactification of $\mathcal{M}_{g,n}$ as described in the last paragraph, because it is unclear how the different tori fit together and because [IP5, (4.3)] describes the normal bundle to a certain immersion, not to a submanifold of $\overline{\mathcal{M}}_{g,n}$. Even outside of the singular locus of the immersion, this normal bundle may not be biholomorphic to a neighborhood; in particular, the construction described above [IP5, Remark 4.1] need not extend outside of the open strata $\mathcal{N}_\ell$ of curves with precisely ℓ nodes. For a related reason, the construction in this section does not lead to uniform estimates in the following sections, only fiber-uniform ones, contrary to a claim at the top of [IP5, p960]. The second sentence of [IP5, Remark 4.1] has no connection with the first. However, none of these additional statements is necessary for the purposes of [IP5]. Other, fairly minor misstatements in [IP5, Section 4] include

p958, line -2: separated by a minimum distance *in each fiber*;
p959, line 1: distance to the nodal set; this is not a smooth function;
p959, above Remark 4.1: all curves have Prym structures by assumption;
p959, line -1: $T_k \longrightarrow T_k(\mu)$;
p960, line 1: $T_k = \log(2/2\sqrt{|\mu_k|}) \longrightarrow T_k(\mu) = \frac{1}{2}\cosh^{-1}(1/|\mu_k|)$;
p960, line 10: the restriction of (4.1) to a fiber agrees with (4.5);
p960, line 14: no change of constants needed given conformal invariance;
p960, last 2 paragraphs: issues similar to p960, line 1;
p960, line -9: the two fractions should be μ'_k/μ_k; the map is defined only near each μ;
p960, line -3: this is for $\tilde{v}_k$ on each $B_k(2)$ and there should be no sum;
p961, line 2: j restricted to each fiber;
p961, line -1: $\mathrm{Re}\,(d\mu_k)^2$ should not be here;
p962, line 2: distance between μ and μ' is the sum of the logs, but $\mu_k = e^{t+i\theta}$ and $\mu'_k = e^{t'+i\theta'}$.

Remark 5.3. The statement of [IP5, Lemma 5.4] is not carefully formulated. In particular, $\bar{v}_n$ ($\bar{u}_{k;\epsilon}$ in our notation above) and ∇ are not defined. Based on the proof, the latter denotes a connection on $\mathcal{Z}$, not on $\mathcal{Z}_\lambda$. Since ρ (ϱ in our notation) is bounded above, the intended statement of [IP5, Lemma 5.4] is equivalent to the $\delta = 1/3$ case; this δ has no connection to the δ used to construct the symplectic sum $(\mathcal{Z}, \omega_{\mathcal{Z}})$ in [IP5, Section 2]. Other, fairly minor misstatements in [IP5, Section 5] include

p962, Section 5, line 6: C^∞-convergence on compact sets implies $L^{1,2}$ and C^0-convergence on the same sets; $L^{1,2}$ and C^0-convergence on entire domain does not make sense;
p962, line -4: in (3.11), $\mathcal{K}_\delta$ is contained in a different space;
p963, line 1: (4.4) $\longrightarrow$ (4.2);
p963, lines 2-7: n here is k in Section 4 and different from the subscript in f_n on line 9 and the superscripts on line 12;
p963, lines 7: $|\log(2/2\sqrt{|\mu_n|})| \longrightarrow \frac{1}{2}\cosh^{-1}(1/|\mu_n|)$;

p963, line -12: graph of $\longrightarrow$ locus;
pp963-964, Lemma 5.1: $c_1 = 1$; $Z \longrightarrow Z\times\overline{\mathcal{U}}$; $A_0 = [-T,T]\times S^1$; $\rho(t)^2 = 2|\mu_n|\cosh(2t)$, $C = \rho(T)^{-\frac{2}{3}}$;
p964, line 10: $\bar\partial F \longrightarrow 2\bar\partial F$;
p964, lines 11,12: $J \longrightarrow \hat J$;
p964, line 15: $\bar\partial F \equiv F_t + iF_\theta$ in the rest of the proof;
p964, line -2: $E(t) \longrightarrow E(F,t)$;
p965, p965, line 1: $[-T,T] \longrightarrow [-T/2,T/2]$;
p965, Definition 5.2: inconsistency in the definition of $\hat f_n$; $|\mu_n| \longrightarrow 2|\mu_n|$;
p966, line 6: near, not along, V; $J - J_0$ is $O(R)$;
p966, line 7: this long undisplayed expression has 3 typos, and the first inequality need not hold;
p966, (5.13): $\mathrm{d}x_n \longrightarrow \mathrm{d}x$, twice;
p966, line -8: $\sqrt{\mu} \longrightarrow \sqrt{2|\mu|}$;
p966, line -1: $|\bar G_n(\sqrt{|\mu_n|})| \longrightarrow 2|\bar G_n(\sqrt{|\mu_n|})|$;
p967, line 10: $[-T,T] \longrightarrow [-T_n,T_n]$;
p967, Lemma 5.4, proof: G is G_n of the proof of Lemma 5.3;
p967, displayed equation after (5.15) is not any of the CZ inequalities in the 190-page [IS].

The analytic approach of [LR] is motivated by the SFT type constructions of [H, HWZ1] involving J-holomorphic curves on infinite "cylinders". Let SV, α, and $\mathring{J}$ be as at the ends of Sections 3.3 and 4.3. For $\ell_1, \ell_2 \in \mathbb{R}^+$ with $\ell_1 < \ell_2$, denote by Φ_{ℓ_1,ℓ_2} the set of orientation-preserving diffeomorphisms $\phi\colon \mathbb{R} \longrightarrow (\ell_1,\ell_2)$. For each $\phi \in \Phi_{\ell_1,\ell_2}$,

$$\tilde\omega_\phi \equiv \pi^*\omega_V + \mathrm{d}(\phi\alpha)$$

is a closed two-form on $\mathbb{R}\times SV$; it is symplectic and tames $\mathring{J}$ if $|\ell_1|, |\ell_2|$ are sufficiently small. With such ℓ_1,ℓ_2 fixed, for any $(\mathring{J},\mathfrak{j})$-holomorphic map $u\colon \Sigma \longrightarrow \mathbb{R}\times SV$ from a (not necessarily compact) Riemann surface $(\Sigma,\mathfrak{j})$, let

$$E_{\ell_1,\ell_2}(u) = \sup_{\phi\in\Phi_{\ell_1,\ell_2}} \int_\Sigma u^*\tilde\omega_\phi\,, \qquad E_V(u) = \int_\Sigma u^*\pi^*\omega_V; \tag{5.7}$$

these numbers may not be finite. Let $\mathbb{D} \subset \mathbb{C}$ denote the closed unit ball and $\mathbb{D}^* = \mathbb{D} - \{0\}$.

Lemma 5.4 ([LR, Lemma 3.5]). *(1) Let $u : \mathbb{C} \longrightarrow \mathbb{R}\times SV$ be a $\mathring{J}$-holomorphic map such that $E_{\ell_1,\ell_2}(u)$ is finite. If $E_V(u) = 0$, then u is constant.*

(2) Let $u\colon \mathbb{R}\times S^1 \longrightarrow \mathbb{R}\times SV$ be a $\mathring{J}$-holomorphic map such that $E_{\ell_1,\ell_2}(u)$ is finite. If $E_V(u) = 0$, then there exist $s \in \mathbb{Z}$, $r_0 \in \mathbb{R}$, and a 1-periodic orbit $\gamma\colon S^1 \longrightarrow SV$ of the Hamiltonian H such that

$$u\big(r, \mathrm{e}^{\mathrm{i}\theta}\big) = \big(sr + r_0, \gamma(\mathrm{e}^{\mathrm{i}k\theta})\big) \qquad \forall\ \big(r, \mathrm{e}^{\mathrm{i}\theta}\big) \in \mathbb{R}\times S^1.$$

Corollary 5.5. *If $u\colon \mathbb{D}^* \longrightarrow \mathbb{R}\times SV$ is a $\mathring{J}$-holomorphic map such that $E_{\ell_1,\ell_2}(u)$ is finite, then*

$$\big|\partial_t u(\mathrm{e}^{t+\mathrm{i}\theta})\big|, \big|\partial_t u(\mathrm{e}^{t+\mathrm{i}\theta})\big| \le C_u$$

for some $C_u \in \mathbb{R}$.

The justification provided for [LR, Lemma 3.5] is that it can be obtained *using the same method as in [H]*, which treats the case when (SV,α) is contact, but the flow of the Reeb vector field ζ_H does not necessarily generate an S^1-action. If fact, the assumption $E_V(u)=0$ in this case implies that the image of u lies in $\mathbb{R}\times S_xV$ for some $x\in V$ and so the situation in [H] is directly applicable. The two statements of Lemma 5.4 are thus immediately implied by the statement of [H, Lemma 28] and by the proof of [H, Theorem 31] in the bottom half of page 538, respectively.

Let u be as in the statement of Corollary 5.5. By Gromov's Removable Singularity Theorem [MS2, Theorem 4.1.2], the J_V-holomorphic map $\pi\circ u\colon \mathbb{D}^*\longrightarrow V$ extends to a J_V-holomorphic map $u_V:\mathbb{D}\longrightarrow V$. The proof of [H, Proposition 27], which uses the standard rescaling argument to construct a $\mathring{J}$-holomorphic map $f\colon\mathbb{C}\longrightarrow\mathbb{R}\times SV$ out of a sequence with increasing derivatives, and Lemma 5.4(1) then yield Corollary 5.5.

Lemma 5.6. *For every $\mathring{J}$-holomorphic map $u=(u_{\mathbb{R}},u_{SV}):\mathbb{D}^*\longrightarrow\mathbb{R}\times SV$ such that $E_{\ell_1,\ell_2}(u)$ is finite, there exist $s\in\mathbb{Z}$ and a 1-periodic orbit $\gamma\colon S^1\longrightarrow SV$ of the Hamiltonian H with the following properties. If $r_i\in\mathbb{R}^+$ is a sequence with $r_i\longrightarrow 0$, there exist a subsequence, still denoted by r_i, and $\theta_0\in\mathbb{R}$ such that*

$$\lim_{i\longrightarrow\infty} u_{SV}\big(r_i\mathrm{e}^{\mathrm{i}t}\big)=\gamma\big(\mathrm{e}^{\mathrm{i}s\theta+\theta_0}\big)$$

in $C^\infty(S^1,SV)$. Furthermore, the function $u_{\mathbb{R}}$ is bounded if and only if $s=0$, and $u_{\mathbb{R}}(r\mathrm{e}^{\mathrm{i}\theta})\longrightarrow\mp\infty$ as $r\longrightarrow 0$ if and only if $s\in\mathbb{Z}^{\pm}$.

This lemma corrects, refines, and generalizes the statement of [LR, Lemma 3.6]; the wording and the usage of the latter suggest that $s\in\mathbb{Z}^+$. By Gromov's Removable Singularity Theorem [MS2, Theorem 4.1.2], the J_V-holomorphic map $\pi\circ u_{SV}:\mathbb{D}^*\longrightarrow V$ extends to a J_V-holomorphic map $u_V:\mathbb{D}\longrightarrow V$. Thus, the image of

$$S^1\longrightarrow SV,\qquad \mathrm{e}^{\mathrm{i}\theta}\longrightarrow u_{SV}(r\mathrm{e}^{\mathrm{i}s\theta}), \tag{5.8}$$

approaches $S_{u_V(0)}V$ as $r\longrightarrow 0$. Let $\gamma: S^1\longrightarrow SV$ be a 1-periodic orbit parameterizing $S_{u_V(0)}V$. The claims concerning the sequence, with some choice of s and θ_0, and the relation between the sign of s and the behavior of $u_{\mathbb{R}}$ follow from the proof of [H, Theorem 31], where the functions v and w are used interchangeably and $f+\mathrm{i}b$ should be replaced by $f-\mathrm{i}b$. However, in the present situation, α (denoted by λ in [LR]) has no relation to $\pi^*\omega_V$. Thus, the first equation in the second row of [H, (54)], the third equation on the first line of [H, (55)], and [H, (56)] no longer apply, and the long equation at the end of the proof can no longer be used to relate the period s (denoted by k in [LR] and by c in [H]) to the energy of u_V. The independence of s of the subsequence r_i follows from the fact that $u_{SV}(r\mathrm{e}^{\mathrm{i}\theta})$ is contained in a tubular neighborhood of $S_{u_V(0)}V\approx S^1$ for all r sufficiently small and thus the homology class of (5.8) is independent of s.

Remark 5.7. A completely different approach to the independence of γ and s of the subsequence in the statement of Lemma 5.6 appears in the proof of [LR, Theorem 3.7]. However, the argument in [LR] is incorrect (or at least far from complete). In particular, it *presupposes* that there exist $r_0\in\mathbb{R}^+$ and a periodic orbit $\gamma: S^1\longrightarrow SV$ such that the images of the maps (5.8) are contained in a small neighborhood $\mathcal{O}_{\gamma,\epsilon}$ of γ for all $r<r_0$; see the top of page 175 in [LR]. Without this assumption, the key action functional $\mathcal{A}=\mathcal{A}_\gamma$ is not even defined in [LR]. Most of the remainder of this argument is dedicated to using this $\mathcal{A}$ to show that such $\mathcal{O}_{\gamma,\epsilon}$ can be chosen arbitrarily small, but it was arbitrarily small to begin with. It is actually possible to define $\mathcal{A}$ on a

neighborhood of the entire space $\mathcal{O}_s$ of periodic orbits of period $s\in\mathbb{Z}$, but this cannot be used to show that s in Lemma 5.6 is independent of the subsequence (as attempted in [LR]). The proof of [LR, Theorem 3.7] also makes use of [LR, Proposition 3.4]; the proof of the latter is based on an infinite-dimensional version of the Morse lemma, for which no justification or citation is provided. The desired conclusion of this Morse lemma involves the inner-product [LR, (3.14)] with respect to which the domain $W_r^1(S^1,SV)$ is not even complete. The second equality in [LR, (3.25] does not appear obvious either.

Proposition 5.8. *Let $u=(u_{\mathbb{R}},u_{SV})\colon \mathbb{D}^*\longrightarrow\mathbb{R}\times SV$ be a $\mathring{J}$-holomorphic map. If $E_{\ell_1,\ell_2}(u)$ is finite, then there exist $s\in\mathbb{Z}$, a 1-periodic orbit $\gamma\colon S^1\longrightarrow SV$ of the Hamiltonian H, $r_0\in\mathbb{R}$, and $C_u\in\mathbb{R}^+$ such that*

$$\big|u_{\mathbb{R}}\big(\mathrm{e}^{t+\mathrm{i}\theta}\big)-(st+r_0)\big|,d_{SV}\big(u_{SV}(\mathrm{e}^{t+\mathrm{i}\theta}),\gamma(\mathrm{e}^{\mathrm{i}s\theta})\big)\le C_u\mathrm{e}^t\quad\forall\,(t,\theta)\in(-\infty,-1)\times S^1\,,\tag{5.9}$$

$$\big|\mathrm{d}u_{\mathbb{R}}\big(\mathrm{e}^{t+\mathrm{i}\theta}\big)-s\,\mathrm{d}t\big|,d_{SV}\big(\mathrm{d}u_{SV}(\mathrm{e}^{t+\mathrm{i}\theta}),\mathrm{d}\gamma(\mathrm{e}^{\mathrm{i}s\theta})\big)\le C_u\mathrm{e}^t\quad\forall\,(t,\theta)\in(-\infty,-1)\times S^1\,.\tag{5.10}$$

Furthermore, the function $u_{\mathbb{R}}$ is bounded if and only if $s=0$, and $u_{\mathbb{R}}(r\mathrm{e}^{\mathrm{i}\theta})\longrightarrow\mp\infty$ as $r\longrightarrow0$ if and only if $s\in\mathbb{Z}^{\pm}$.

This proposition corrects, refines, and generalizes the statement of [LR, Theorem 3.7], the main conclusion of [LR, Section 3.1]. The contrast of the second bound in (5.9) with the first statement of Lemma 5.6 is that θ_0 is now independent of the choice of the sequence. The convergence property for $\pi\circ u_{SV}$ is standard; see [MS2, Lemmas 4.3.1,4.7.3]. Along with [LR, (3.33),(3.34)] and the ellipticity of the $\bar\partial$-operator, this implies the convergence statements for the vertical direction; see [HWZ1, Lemma 4.1]. The convergence estimates (5.9) and (5.10), formulated in the cylindrical metric on the target, are analogous to the estimates in [IP5, Lemma 5.1] and on $\hat{x}_n,\hat{y}_n$ in the proof of [IP5, Lemma 5.3].

Proposition 5.8 is needed for the convergence arguments of [LR, Section 3.2]; [LR, Theorem 3.7], which is a similar statement with $\mathbb{D}^*$ replaced by $\mathbb{C}$, does not suffice for these purposes. The topological reasoning in the paragraph preceding above Remark 5.7 also implies that the ends of the components of broken limits of J-holomorphic maps have matching orders, as described by (4.25) and the last bullet above Remark 4.5. The proof of this statement in [LR, Lemma 3.11(3)] is incorrect, as explained in Remark 4.5.

Remark 5.9. For [LR, (3.18),(3.20),(3.22)] to hold, the sign in the definition of the operator S above [LR, (3.18)] should be reversed. The symmetry of [LR, (3.18)] in ζ and η is not obvious. It follows from

$$\big\langle\nabla_vX_H,w\big\rangle=\frac12\varpi(v,w),\quad\big\langle(\nabla_{X_H}J)v,w\big\rangle=\frac12\big(\varpi(v,Jw)+\varpi(Jv,w)\big)\qquad\forall\,v,w\in\ker\lambda\,,$$

where $\mathrm{d}\lambda=\pi^*\varpi$. For the statement of [LR, Proposition 3.4] to make sense, it needs to be shown that $\mathcal{A}$ is well-defined on $\mathcal{O}$. Equation (3.22) should read

$$\|\mathrm{d}_\gamma\mathcal{A}\|_{L^2(S^1)}\ge C|\mathcal{A}(\gamma)|^{\frac12}\qquad\forall\,\gamma\in\mathcal{O}\,,$$

and $\|\mathrm{d}_\gamma\mathcal{A}\|_{L^2(S^1)}$ needs to be defined. The second displayed equation in the proof of this proposition should read

$$\|\mathrm{d}_\gamma\mathcal{A}\|_{L^2(S^1)}\ge\big\|\mathrm{d}_y\mathcal{A}(P(x)y/(P(x)y,P(x)y)^{1/2})\big\|_{L^2(S^1)}\,.$$

The statement after the proof of [LR, Theorem 3.7] does not make sense, because the constants there depend on the map $\mathbb{C}\longrightarrow\mathbb{R}\times SV$. Other, fairly minor misstatements in [LR, Section 3.1] include

p172, lines 6,-2: dfn of $T_\gamma^\perp$, $T_{x_k}^\perp$ should involve pointwise inner-products;
p172, (3.18): Π not necessary by the previous equation;
p172, above Rmk 3.1: accumulate *only* at;
p172, Rmk 3.1 is meaningless, since (3.18) is derived for any γ in a fiber of π;
p173, Rmk 3.3 is irrelevant and debatable;
p173, Prop 3.4: $x\in S_k$;
p173, line -10: no need to introduce P';
p175, (3.29): $\tilde{d}(\tilde{u}(s,t),\tilde{u}(s_i,t)) \longrightarrow \tilde{d}(\pi(\tilde{u}(s,t)),\pi(\tilde{u}(s_i,t)))$;
p175, line -6: Lemma (3.6) $\longrightarrow$ Lemma 3.6;
p176, line 1: defined just above;
p177, (3.39)-(3.41) do not make sense, given the definition of r.

The use of the sup-energy (5.7) introduced in [H] is not necessary in the setting of [LR]. It can be replaced by the energy with respect to the restriction to $\mathbb{R}\times SV$ of the symplectic form on

$$\mathbb{P}\big((SV\times_{S^1}\mathbb{C})\times\mathbb{C}\big)\approx\mathbb{P}_X V$$

given by

$$\widehat{\omega}_\epsilon=\pi_V^*\omega-\epsilon\,\mathrm{d}\bigg(\frac{\alpha}{1+\rho^2}\bigg), \qquad \text{where} \qquad \rho\big(\big[[x,c_1],c_2\big]\big)=|c_1|^2\,,$$

with $\epsilon>0$ small (if ϵ is not sufficiently small, $\widehat{\omega}_\epsilon$ may be degenerate). If the target is $\mathring{X}_V$ or $\mathring{Y}_V$, instead of $\mathbb{R}\times SV$, the restrictions of the symplectic forms ω_X and ω_Y can be used. This is also related to the reason why the sup energy of the maps appearing in [LR] is finite (for which no explanation is provided).

The convergence topology arising from [LR, Section 3.2] involves pulling the domains of the stable maps apart via long cylinders on which an $\widehat{\omega}_\epsilon$-type energy disappears. Along with (5.9) and (5.10), this leads to analogues of (5.5) and (5.6). The gluing construction on the domains in [LR] is the same as on the target in (3.46) and is parametrized by pairs $(r,\theta)\in\mathbb{R}^+\times S^1$ at each node with $r\longrightarrow\infty$ with $\mu=\mathrm{e}^{-r-\mathrm{i}\theta}$. In the notation around Remark 5.1, if

$$x\big(u(\mathrm{e}^{t+\mathrm{i}\theta'})\big)\approx\mathrm{e}^{t-r_X+\mathrm{i}(\theta'-\theta_X)} \quad \text{as} \;\; t\longrightarrow-\infty, \quad y\big(u(\mathrm{e}^{-t+\mathrm{i}\theta'})\big)\approx\mathrm{e}^{-(t-r_Y)+\mathrm{i}(\theta'-\theta_Y)} \quad \text{as} \;\; t\longrightarrow\infty,$$

then the relation between the gluing parameters for the target (a_k,ϑ_k) in (3.46) and the domains of the converging maps is described by

$$\lim_{k\longrightarrow\infty}\big((a_k+\mathrm{i}\vartheta_k)-s(r_k+\mathrm{i}\theta_k)\big)=r_X+r_Y+\mathrm{i}(\theta_X+\theta_Y)\in\mathbb{C}/2\pi\mathrm{i}\mathbb{Z}. \tag{5.11}$$

This is the analogue of (5.5) in the setup of [LR].

In both approaches, it is necessary to consider sequences $u_k:\Sigma\longrightarrow\mathcal{Z}_{\lambda_k}$ that limit to maps $u:\Sigma'\longrightarrow\mathcal{Z}_0$ with $\Sigma'_V\neq\emptyset$; see the notation above Remark 5.1. Such limits are considered briefly at the top of page 1003 in [IP5], with an incorrect conclusion; see Section 5.5 for more detail. On the other hand, the approach of [LR, Section 3.2] can be corrected to show that any such limit lies in a moduli space $\overline{\mathcal{M}}_{g,k}(X\cup_V Y, A)$ defined in Section 4.2, whenever the almost complex structures J_λ satisfy the more restrictive conditions of [LR]. The condition (5.11) then extends as a relation between smoothing parameters for the target and the domain at each transition between different levels of the target space; see Section 5.2.

5.2 Pregluing: [IP5, Section 6], [LR, Section 4.2]

The pregluing steps of gluing constructions typically involve constructing approximately J-holomorphic maps and defining Sobolev spaces suitable for studying their deformations. The former is done in essentially the same way in [IP5] and [LR]; the latter is done very differently.

For $A\in H_2(X;\mathbb{Z})$, $\chi\in\mathbb{Z}$, $k,\ell\in\mathbb{Z}^{\geq 0}$, and a tuple $\mathbf{s}=(s_1,\ldots,s_\ell)\in(\mathbb{Z}^+)^\ell$ satisfying (1.3), let

$$\mathcal{M}^{V*}_{\chi,k;\mathbf{s}}(X,A)\subset\widetilde{\mathcal{M}}^{V}_{\chi,k;\mathbf{s}}(X,A)$$

denote the subspace of morphisms from smooth, but not necessarily connected, domains. For each $i=1,\ldots,\ell$, let

$$L_i\longrightarrow\widetilde{\mathcal{M}}^{V}_{\chi,k;\mathbf{s}}(X,A)$$

be the universal tangent line bundle at the i-th relative marked point (i.e. $(k+i)$-th marked point overall). By (5.1), every marked map representing an element of $\mathcal{M}^{V*}_{\chi,k;\mathbf{s}}(X,A)$ has a well-defined s_i-th derivative in the normal direction to V at the i-th relative marked point. By (5.4), these derivatives induce a nowhere zero section of the line bundle

$$L_i^{*\otimes s_i}\otimes\mathrm{ev}_i^{V*}\mathcal{N}_XV\longrightarrow\mathcal{M}^{V*}_{\chi,k;\mathbf{s}}(X,A)\,,$$

which we denote by $\mathfrak{D}^{(s_i)}_X$.

If $u\colon\Sigma'\longrightarrow\mathcal{Z}_0$ is the limit of a sequence of $(J_{\mathcal{Z}},\mathfrak{j})$-holomorphic maps $u_k\colon\Sigma\longrightarrow\mathcal{Z}_{\lambda_k}$, with $\lambda_k\in\Delta^*$, and has no component mapped into V, u determines an element of

$$\mathcal{M}^{V*}_{\chi,k;\mathbf{s}}(\mathcal{Z}_0,C)\equiv\bigsqcup_{\substack{\chi_X+\chi_Y=\chi\\ k_X+k_Y=k}}\;\bigsqcup_{A_X\#A_Y=C}\big\{(u_X,u_Y)\in\mathcal{M}^{V*}_{\chi_X,k_X;\mathbf{s}}(X,A_X)\times\mathcal{M}^{V*}_{\chi_Y,k_Y;\mathbf{s}}(Y,A_Y)\colon\;\mathrm{ev}^V(u_X)=\mathrm{ev}^V(u_Y)\big\}$$

for some $\mathbf{s}=(s_1,\ldots,s_\ell)$ and $\ell\in\mathbb{Z}^{\geq 0}$. Denote by

$$\pi_X,\pi_Y\colon\mathcal{M}^{V*}_{\chi,k;\mathbf{s}}(\mathcal{Z}_0,C)\longrightarrow\bigsqcup_{\chi_X,k_X,A_X}\mathcal{M}^{V*}_{\chi_X,k_X;\mathbf{s}}(X,A_X)\,,\;\bigsqcup_{\chi_Y,k_Y,A_Y}\mathcal{M}^{V*}_{\chi_Y,k_Y;\mathbf{s}}(X,A_Y)$$

the projection maps. In [IP5, Sections 6-9], a gluing construction is carried out on the $\langle\mathbf{s}\rangle$-fold cover

$$\widetilde{\mathcal{M}}^{V*}_{\chi,k;\mathbf{s}}(\mathcal{Z}_0,C)_\lambda\equiv\big\{(\mu_{X;i}\otimes\mu_{Y;i})_{i=1,\ldots,\ell}\in\bigoplus_{i=1}^{\ell}\pi_X^*L_i\otimes\pi_Y^*L_i\colon\;\mathfrak{D}^{(s_i)}_X\mu_{X;i}^{\otimes s_i}\otimes\mathfrak{D}^{(s_i)}_Y\mu_{Y;i}^{\otimes s_i}=\lambda\;\forall\,i\big\}\qquad(5.12)$$

of $\mathcal{M}^{V*}_{\chi,k;\mathbf{s}}(\mathcal{Z}_0,C)$, with the last equality viewed via the identification (1.2). This cover accounts for the convergence property (5.5).

Fix a smooth map $\beta\colon \mathbb{R}\longrightarrow[0,1]$ so that

$$\beta(r)=\begin{cases}1, & \text{if } r\le 1;\\ 0, & \text{if } r\ge 2.\end{cases}$$

For each $\epsilon>0$, let $\beta_\epsilon(r)=\beta(\epsilon^{-1}r)$. Denote by ∇ the Levi-Civita connection of the metric $g_{\mathcal{Z}}=\omega_{\mathcal{Z}}(\cdot,J_{\mathcal{Z}}\cdot)$ and by $\nabla^{\mathbb{C}}$ the associated $J_{\mathcal{Z}}$-linear connection. Using the ∇-geodesics from q, we identify the ball of injectivity radius of $g_{\mathcal{Z}}|_V$ in T_qV with a neighborhood W_q of q in V. Using the parallel transport with respect to $\nabla^{\mathbb{C}}$ along the ∇-geodesics from q, we identify $\mathcal{N}_XV$ and $\mathcal{N}_YV$ with $W_q\times\mathcal{N}_XV|_q$ and $W_q\times\mathcal{N}_YV|_q$, respectively. The proof of [FHS, Theorem 2.2] then ensures that the map Φ in (5.1) can be chosen to depend smoothly on u.

For $\mu\in\widetilde{\mathcal{M}}^{V*}_{\mathcal{X},k;\mathbf{s}}(\mathcal{Z}_0,C)_\lambda$, an approximately $(J_{\mathcal{Z}},\nu)$-holomorphic map $u_\mu:\Sigma_\mu\longrightarrow\mathcal{Z}_\lambda$ can be constructed as follows. Given an element $([u_X,u_Y])$ of $\mathcal{M}^{V*}_{\mathcal{X},k;\mathbf{s}}(\mathcal{Z}_0,C)$, with $u_X:\Sigma_X\longrightarrow X$ and $u_Y\colon\Sigma_Y\longrightarrow Y$, denote by Σ_0 the Riemann surface obtained by identifying the i-th relative marked point $z_i\in\Sigma_X$ with the i-th relative marked point $w_i\in\Sigma_Y$ for all $i=1,\ldots,\ell$ and by $\Sigma_0^*\subset\Sigma_0$ the complement of the nodes. Define

$$u_0\colon\Sigma_0\longrightarrow\mathcal{Z}_0 \qquad\text{by}\quad u_0(z)=\begin{cases}u_X(z), & \text{if } z\in\Sigma_X;\\ u_Y(z), & \text{if } z\in\Sigma_Y\,.\end{cases}$$

Given $i=1,\ldots,\ell$, let z and w be coordinates on $\Sigma_{X;i}\subset\Sigma_X$ and $\Sigma_{Y;i}\subset\Sigma_Y$ centered at z_i and w_i, respectively. For each sufficiently small $\mu\equiv(\mu_i)_{i=1,\ldots,\ell}$ in $\mathbb{C}^\ell$, we define

$$\Sigma_{\mu;i}\equiv\big\{(z,w)\in\mathbb{C}^2\colon\, zw=\mu_i\big\}\quad\forall\, i=1,\ldots,\ell,$$
$$\Sigma_0^*(\mu)=\Sigma_0-\bigcup_{i=1}^{\ell}\big(\{z_i\in\Sigma_X\colon\,|z_i|\le|\mu_i|^{\frac12}\}\cup\{w_i\in\Sigma_Y\colon\,|w_i|\le|\mu_i|^{\frac12}\}\big),$$
$$\Sigma_\mu=\Bigg(\Sigma_0^*(\mu)\sqcup\bigsqcup_{i=1}^{\ell}\Sigma_{\mu;i}\Bigg)\Big/\!\sim\qquad (z,w)\sim\begin{cases}z\in\Sigma_X, & \text{if } |z|>|w|;\quad \forall\,(z,w)\in\Sigma_{\mu;i},\\ w\in\Sigma_Y, & \text{if } |z|<|w|;\qquad i=1,\ldots,\ell.\end{cases}$$

For each $i=1,\ldots,\ell$ and $\epsilon>0$, we also define

$$\varrho_{\mu;i},\beta_{\mu;i}\colon\Sigma_{\mu;i}\longrightarrow\mathbb{R}\quad\text{by}\quad \varrho_{\mu;i}(z,w)=\sqrt{|z|^2+|w|^2},\quad \beta_{\mu;i}(z,w)=\beta_{|\mu_i|^{\frac14}}\big(\varrho_{\mu;i}(z,w)\big)\,;$$
$$\Sigma_{\mu;i}(\epsilon)=\big\{(z,w)\in\Sigma_{\mu;i}\colon\,\varrho_{\mu;i}(z,w)<\epsilon\big\}.$$

Let $\epsilon>0$ be such that the restrictions of u_X and u_Y to

$$\Sigma_{X;i}(\epsilon)\equiv\big\{z\in\Sigma_{X;i}\colon\,|z|<\epsilon\big\}\qquad\text{and}\qquad\Sigma_{Y;i}(\epsilon)\equiv\big\{w\in\Sigma_{Y;i}\colon\,|w|<\epsilon\big\}$$

respectively, satisfy (5.1) for some $\Phi=\Phi_{X;i},\Phi_{Y;i}$. In particular, $u_X(\Sigma_{X;i}(\epsilon))$ and $u_Y(\Sigma_{Y;i}(\epsilon))$ are contained in the open subset $\mathcal{Z}_{\text{neck}}$ of $\mathcal{Z}$ defined in (3.32) and in the total spaces of $\mathcal{N}_XV$ and $\mathcal{N}_YV$ over the geodesics ball W_{q_i}, where $q_i=u_X(z_i)=u_Y(w_i)$. Thus, there exist smooth functions

$$u_{X;i}\colon\Sigma_{X;i}(\epsilon)\longrightarrow T_{q_i}V\quad\text{and}\quad u_{Y;i}\colon\Sigma_{Y;i}(\epsilon)\longrightarrow T_{q_i}V\qquad\text{s.t.}$$
$$u_X(z)=\big(u_{X;i}(x),\Phi_{X;i}(z)z^{s_i}\big)\quad\forall\,z\in\Sigma_{X;i}(\epsilon)\,,\quad u_Y(z)=\big(u_{Y;i}(w),\Phi_{Y;i}(w)w^{s_i}\big)\quad\forall\,w\in\Sigma_{Y;i}(\epsilon),$$

under the identifications of the previous paragraph.

For any $\mu\in\mathbb{C}^\ell$ sufficiently small, let

$$\Phi_{\mu;X;i}\colon \Sigma_{\mu;i}(\epsilon) \longrightarrow \mathcal{N}_X V|_{q_i}\,, \quad \Phi_{\mu;X;i}(z) = \Phi_{X;i}(0)\bigg(\beta_{\mu;i}(z,w) + (1-\beta_{\mu;i}(z,w))\frac{\Phi_{X;i}(z)}{\Phi_{X;i}(0)}\bigg)z^{s_i}\,,$$
$$\Phi_{\mu;Y;i}\colon \Sigma_{\mu;i}(\epsilon) \longrightarrow \mathcal{N}_Y V|_{q_i}\,, \quad \Phi_{\mu;Y;i}(z) = \Phi_{Y;i}(0)\bigg(\beta_{\mu;i}(z,w) + (1-\beta_{\mu;i}(z,w))\frac{\Phi_{Y;i}(w)}{\Phi_{Y;i}(0)}\bigg)w^{s_i}\,.$$

With $\lambda=\mu^{s_i}\Phi_{X;i}(0)\Phi_{Y;i}(0)$, we define $u_\mu\colon \Sigma_\mu \longrightarrow \mathcal{Z}_\lambda$ by requiring that

$$u_\mu(z,w) = \begin{cases} \big((1-\beta_{\mu;i}(z,w))u_{X;i}(z), \Phi_{\mu;X;i}(z), \frac{\lambda}{\Phi_{\mu;X;i}(z)}\big), & \text{if } |z|\ge|w|; \\ \big((1-\beta_{\mu;i}(z,w))u_{Y;i}(w), \frac{\lambda}{\Phi_{\mu;Y;i}(w)}, \Phi_{\mu;Y;i}(w)\big), & \text{if } |z|\le|w|; \end{cases} \tag{5.13}$$

for all $(z,w)\in\Sigma_{\mu;i}(\epsilon)$ and $i=1,\ldots,\ell$ and extending as u over the complement of $\Sigma_0(\epsilon/2)$ in Σ_0^*.

The relevant Sobolev norms for sections of $u_\mu^*T\mathcal{Z}_\lambda$ and for $(0,1)$-forms with values in $u_\mu^*T\mathcal{Z}_\lambda$ are defined by the $m=1$ case of [IP5, (6.10)] and the $m=0$ case of [IP5, (6.11)], respectively, with $p>2$ in [IP5, (6.9)]. The failure of the map $u_\mu\colon \Sigma_\mu \longrightarrow \mathcal{Z}_\lambda$ to be $(J_\mathcal{Z},\nu)$-holomorphic is described by

$$\big\|\{\bar\partial_{J_\mathcal{Z}}-\nu\}(u_\mu)\big\|_{\mu,0} \le C|\mu|^{\frac{1}{6}} \le C|\lambda|^{\frac{1}{6|\mathbf{s}|}}\,, \tag{5.14}$$

with C independent of μ, but depending continuously on the projection of μ to $\mathcal{M}^{V*}_{X,k;\mathbf{s}}(\mathcal{Z}_0,C)$; this can be deduced from the proof of [IP5, Lemma 6.9].

Remark 5.10. The pregluing construction done in the first half of [IP5, Section 6] is not needed for the purposes of [IP5, Lemma 6.8(a)], which is about properties of moduli spaces of maps into the singular fiber $\mathcal{Z}_0$. Based on the proof, the wording of [IP5, Lemma 6.8(a)] is incorrect: *for every* $(f,C)\in\mathcal{K}_\delta$ should appear after $\le\epsilon$ and again after $\ge c$ (so that ρ_0 in the first part and c in the second part are independent of (f,C)); there is a similar problem with the wording of [IP5, Lemma 6.8(d)]. [IP5, Lemma 6.8(a)] also has nothing to do with c_i, c_i'. The proof of the first part of [IP5, Lemma 6.8(b)] is not complete because [IP5, Lemma 5.1] is about finite cylinders, not wedges of disks. The pregluing setup in [IP5, Section 6] implicitly assumes that the domains of the nodal maps are stable, since it is based on [IP5, Section 4]. The stability assumption need not hold in general; it is not necessary though. The domains can be stabilized as in [IP5, Remark 1.1], but not across an entire stratum of maps; in particular, [IP5, Observation 6.7] may not always apply. The definition of the norms in [IP5, Section 6] makes no mention that $p>2$, which is necessary for the control of the C^0-norm. The statement about uniform C^0-bound in [IP5, Remark 6.6] needs a justification, since the domains C_μ change (which is standard) and the metric on the targets $\mathcal{Z}_\lambda$ collapses (which is not standard). Without a local trivialization of the normal bundle, the formula [IP5, (6.4)] does not make sense. The crucial bound of [IP5, Lemma 6.9] is incorrect. Its proof neglects to consider the first two components of $F-f$ with respect to the decomposition in [IP5, (6.14)]; contrary to the statement immediately after [IP5, (6.14)], these two components are not zero, as f does not involve β. However, the weaker bound of (5.14) suffices. Other, fairly minor misstatements in [IP5, Section 6] include

p968, above (6.1): if f_0 is in the limit of a sequence, then (6.1) holds;

p968, line -4: $C \longrightarrow C_1$, with the notation as before;

p969, line 2: $\mathfrak{L}_k$ is used for L_k in (4.3);
p969, par. above Dfn 6.2: not *from* (4.2) and (5.4); *as in* (5.11);
p970, 1st par.: there are no (a) and (b) in (2.6) or (6.4);
p970, after (6.5): were $\longrightarrow$ where;
p971, line 11: $\sqrt{|\mu_k|} \longrightarrow \sqrt{2|\mu_k|}$;
p971, above (6.8): geodesics and parallel transport with respect to what connection?
p971, below (6.8): average value zero only for the horizontal part ξ^V; *as in* (5.11);
p971, (6.9): $k\!=\!m$ below; only $k\!=\!1$ is used; 2 can be absorbed into δ;
p971, below (6.9): there are no *coordinates* in (5.3); (4.5) is closer;
p971, Dfn 6.5: there is no triple in (6.8);
p972, top: $k \longrightarrow h$; not just *Finsler* metric;
p972, Lemma 6.8(a), line 1: $\mathrm{dist}(f(A(\rho_0)),V) \longrightarrow \max_{z\in A(\rho_0)}\mathrm{dist}(f(z),V)$;
p973, lines 3,8: $p_n\!\in\!C_n$; $p_n\!\in\!C_n\setminus A(\rho_0)$;
p974, (6.12): $|\nu_F\!-\!\nu_f|$ should not be multiplied by $|dF|$, similarly to (6.15);
p974, (6.14): $\beta \longrightarrow \beta_\mu$;
p974, below (6.14): $(J_F\!-\!J_f)\circ dF \longrightarrow (J_F\!-\!J_f)\circ df$.

The approximately J-holomorphic map u_μ in (5.13) is constructed in the same way at the bottom of page 192 in [LR]. Because of the regular nature of the almost complex structures J_X and J_Y used in [LR] on neighborhoods of V in X and Y, the gluing approach of [LR] extends to maps into $X\overset{\circ}{\cup}{}_V^m Y$ with $m\!\ge\!1$. As formally explained at the end of this section, the gluing of the target spaces in (3.46) extends to $X\overset{\circ}{\cup}{}_V^m Y$. This extension is parametrized by the tuples

$$(\underline{\mathfrak{a}},\underline{\vartheta}) \equiv \big(\mathfrak{a}_0,\ldots,\mathfrak{a}_m,\vartheta_0,\ldots,\vartheta_m\big) \in (\mathbb{R}^+)^{m+1}\times(\mathbb{R}/2\pi\mathbb{Z})^{m+1} \tag{5.15}$$

so that $\mathcal{Z}_{\underline{\mathfrak{a}},\underline{\vartheta}}\!=\!\mathcal{Z}_{|\underline{\mathfrak{a}}|,|\underline{\vartheta}|}$ as far as the almost complex structures are concerned, where

$$|\underline{\mathfrak{a}}| = \mathfrak{a}_0+\ldots+\mathfrak{a}_m\,, \qquad |\underline{\vartheta}| = \vartheta_0,\ldots,\vartheta_m\,.$$

In the next paragraph, we formally define the space of gluing parameters, generalizing (5.12) from the $m\!=\!0$ case.

Given $m\!\in\!\mathbb{Z}^{\ge0}$, let $\mathbb{C}_{m+1}$ denote the quotient of $\mathbb{C}^{m+1}$ by the $(\mathbb{C}^*)^m$-action

$$(c_1,\ldots,c_m)\cdot(\lambda_0,\ldots,\lambda_m) = \big(c_1^{-1}\lambda_0, c_1c_2^{-1}\lambda_1,\ldots,c_{m-1}c_m^{-1}\lambda_{m-1},c_m\lambda_m\big).$$

The map $(\lambda_0,\ldots,\lambda_m)\!\longrightarrow\!\lambda_0\ldots\lambda_m$ then descends to $\mathbb{C}_{m+1}$. For each $\lambda\!\in\!\mathbb{C}$, let $\mathbb{C}_{m+1;\lambda}\!\subset\!\mathbb{C}_{m+1}$ be preimage of λ. Let $u:\Sigma\longrightarrow X\cup_V^m Y$ be a representative of an element of $\overline{\mathcal{M}}_{g,k}(X\cup_V Y,A)$ and $i\!=\!1,\ldots,\ell$ be an index set for its nodes on the divisors

$$V\subset X,Y \qquad \text{and} \qquad \{r\}\!\times\!\mathbb{P}_{X,0}V, \{r\}\!\times\!\mathbb{P}_{X,\infty}V \subset \{r\}\times\mathbb{P}_XV\,.$$

For each such i, let $|r|\!=\!0$ if the node lies on $V\!\subset\!X$ and $|i|\!=\!r$ if it lies on $\{r\}\!\times\!\mathbb{P}_{X,0}V$. Denote by $s_i\!\in\!\mathbb{Z}^+$ the order of contact with the divisor of either of the two branches at the i-th node, by $L_{u;i}$ the line of smoothings of this node (denoted by $\pi_X^*L_i\!\otimes\!\pi_Y^*L_i$ in (5.12)), and by $\mathfrak{D}_i^{(s)}\!\in\!L_{u;i}^*$ the s_i-th derivative (denoted by $\mathfrak{D}_X^{(s_i)}\!\otimes\!\mathfrak{D}_Y^{(s_i)}$ in (5.12)). The admissible relative smoothing parameters at u for maps to $\mathcal{Z}_\lambda$ are the elements of the space

$$L_{u;\lambda} = \Big\{(\mu_i)_{i=1,\ldots,\ell}\!\in\!\bigoplus_{i=1}^{\ell}L_{u;i}\!:\ \exists[\lambda_0,\ldots,\lambda_m]\!\in\!\mathbb{C}_{m+1;\lambda} \text{ s.t. } \mathfrak{D}_i^{(s_i)}(\mu_i)=\lambda_{|i|}\ \forall\,i\Big\}.$$

While $\mathfrak{D}_i^{(s_i)}$ depends on the choice of representative u for $[u]\in\overline{\mathcal{M}}_{g,k}(X\cup_V Y, A)$, $L_{u;\lambda}$ is determined by $[u]$ and the choice of ordering of the relative nodes of u, since the action of $(\mathbb{C}^*)^m$ on $\mathbb{C}^{m+1}$ defined above corresponds to the action of $(\mathbb{C}^*)^m$ on $X\cup_V^m Y$.

We now define the spaces $\mathcal{Z}_{\underline{\mathfrak{a}},\underline{\vartheta}}$, with $(\underline{\mathfrak{a}},\underline{\vartheta})$ as in (5.15) and $\mathfrak{a}_0$ and $\mathfrak{a}_m$ sufficiently large, and identify them with $\mathcal{Z}_{|\underline{\mathfrak{a}}|,|\underline{\vartheta}|}$; see (3.46). For each $r=1,\ldots,m$, let

$$|\mathfrak{a}|_r^- = \mathfrak{a}_0+\ldots+\mathfrak{a}_{r-1}, \qquad |\mathfrak{a}|_r^+ = \mathfrak{a}_r+\ldots+\mathfrak{a}_m\,,$$
$$|\vartheta|_r^- = \vartheta_0+\ldots+\vartheta_{r-1}, \qquad |\vartheta|_r^+ = \vartheta_r+\ldots+\vartheta_m\,.$$

We assume that $m\in\mathbb{Z}^+$. Let

$$\mathcal{Z}_{\underline{\mathfrak{a}},\underline{\vartheta}} = \Big(X_{\mathfrak{a}_0}\sqcup\bigsqcup_{r=1}^m\{r\}\times[-\frac{3}{4}\mathfrak{a}_r,\frac{3}{4}\mathfrak{a}_{r-1}]\times SV\sqcup Y_{\mathfrak{a}_m}\Big)\Big/\sim,$$

with the equivalence relation defined by

$$\begin{aligned}
(1,a,x) &\sim \big(a-\mathfrak{a}_0,\mathrm{e}^{-\mathrm{i}\vartheta_0}x\big)\subset X_{\mathfrak{a}_0} && \forall\, 4a\in(\mathfrak{a}_0,3\mathfrak{a}_0),\\
(r,a,x) &\sim \big(r+1,a+\mathfrak{a}_r,\mathrm{e}^{\mathrm{i}\vartheta_r}x\big) && \forall\, 4a\in(-\mathfrak{a}_r,-3\mathfrak{a}_r),\ r=1,\ldots,m-1,\\
(m,a,x) &\sim \big(a+\mathfrak{a}_m,\mathrm{e}^{\mathrm{i}\vartheta_m}x\big)\subset Y_{\mathfrak{a}_m} && \forall\, 4a\in(-\mathfrak{a}_m,-3\mathfrak{a}_m)\,.
\end{aligned}$$

These identifications respect the almost complex structure $\mathring{J}$ and thus induce an almost complex structure on $\mathcal{Z}_{\underline{\mathfrak{a}},\underline{\vartheta}}$. The bijection $\mathcal{Z}_{\underline{\mathfrak{a}},\underline{\vartheta}}\longrightarrow\mathcal{Z}_{|\underline{\mathfrak{a}}|,|\underline{\vartheta}|}$ given by

$$x\longrightarrow\begin{cases}x\in X_{|\mathfrak{a}|}, & \text{if } x\in X_{\mathfrak{a}_0};\\ x\in Y_{|\mathfrak{a}|}, & \text{if } x\in Y_{\mathfrak{a}_m};\end{cases}\qquad (r,a,x)\longrightarrow\begin{cases}(a-|\mathfrak{a}|_r^-,\mathrm{e}^{-\mathrm{i}|\vartheta|_r^-}x)\in X_{|\mathfrak{a}|}, & \text{if } 4a\geq|\mathfrak{a}|_r^--3|\mathfrak{a}|_r^+;\\ (a+|\mathfrak{a}|_r^+,\mathrm{e}^{\mathrm{i}|\vartheta|_r^+}x)\in Y_{|\mathfrak{a}|}, & \text{if } 4a\leq 3|\mathfrak{a}|_r^--|\mathfrak{a}|_r^+;\end{cases}$$

is well-defined on the overlaps and identifies the two spaces with their almost complex structures, as needed for the general gluing construction. However, the just described construction and identification do not fit with the more general almost complex structures of [IP5], as they are not regularized on neighborhoods of V in X and Y.

Remark 5.11. The only gluing constructions described in [LR] involve smoothing a single node. In particular, there is no mention of the above identification $\mathcal{Z}_{\underline{\mathfrak{a}},\underline{\vartheta}}=\mathcal{Z}_{|\underline{\mathfrak{a}}|,|\underline{\vartheta}|}$, which is needed to make sense of the target of the smoothed out maps, or of the space $L_{u;\lambda}$ of admissible smoothings.

5.3 Uniform estimates: [IP5, Sections 7,8], [LR, Section 4.2]

Gluing constructions in GW-theory typically require defining linearizations $\mathbf{D}_{u_\mu}$ of the $\bar\partial$-operator at the approximately J-holomorphic maps u_μ (these are not unique away from J-holomorphic maps) and establishing uniform bounds on these linearizations and their right inverses. Establishing the former is typically fairly straightforward, with appropriate choices of the linearizations and the Sobolev norms on their domains and targets. Uniform bounds on the right inverses can be obtained either by bounding the eigenvalues of the Laplacians $\mathbf{D}_{u_\mu}\mathbf{D}_{u_\mu}^*$ from below, by a direct computation for explicit right inverses, or by establishing a uniform elliptic estimate on $\mathbf{D}_{u_\mu}$ with suitable Sobolev norms. As stated at the beginning of [IP5, Section 8], such uniform Fredholm bounds are the key analytic step in the proof. As we explain below, the argument in [IP5] has several material, consecutive errors, i.e. with each sufficient to break it.

The approach taken in [IP5, Sections 7,8] is to bound the eigenvalues of the Laplacians $\mathbf{D}_{u_\mu}\mathbf{D}^*_{u_\mu}$ from below. With the definitions at the beginning of [IP5, Section 7], the index of $\mathbf{D}_u$ (denoted by $\mathbf{D}_f$ in [IP5]) is generally larger than the index of $\mathbf{D}_{u_\mu}$ (denoted by $\mathbf{D}_F$), as the former does not see the order of contact. In particular, $\mathbf{D}_u$ does not fit into any kind of continuous Fredholm setup, though by itself this issue need not be material as far as the estimates on $\mathbf{D}_{u_\mu}$ are concerned.

In the displayed expression above [IP5, (7.5)], $\langle\zeta_1,\zeta_2\rangle$ has two different meanings in the same equation. This equation defines an inner-product only on the first part of the domain of $\mathbf{D}_{u_\mu}=\mathbf{D}_F$ and so does not define $\mathbf{D}^*_{u_\mu}$. The explicit formula for the first component of $\mathbf{D}^*_F$ in [IP5, (7.5)] cannot be correct because it does not satisfy the average value condition on the elements of $L_{1;\mathbf{s};0}$ for $F=u_\mu$ and even more conditions for $f=u$ (the average value condition is described above [IP5, (7.1)]). This formula has to be corrected by an element of the L^2-orthogonal complement of $L_{1;\mathbf{s},0}(u_\mu^*T\mathcal{Z}_\lambda)$ in $L_{1;\mathbf{s}}(u_\mu^*T\mathcal{Z}_\lambda)$; unfortunately, the orthogonal complement does not lie in $L_{1;\mathbf{s}}(u_\mu^*T\mathcal{Z}_\lambda)$. Thus, [IP5, Proposition 7.3] says *nothing* about the uniform boundness of $\mathbf{D}^*_F=\mathbf{D}^*_{u_\mu}$. Without taking out the average, the norms of [IP5, Definition 6.5] would not be finite over f, as used in [IP5] to obtain uniform bounds over F.

Remark 5.12. The crucial Sections 7 and 8 in [IP5] are written in a confusing way with the same notation used for different objects, including in the same equation at times. With the definition as in [IP5, (1.11),(7.2)], the image of the operator in [IP5, (7.4)] would not be in the $(0,1)$-forms because of the F_*h term (which is not a $(0,1)$-form if F is not J-holomorphic; F_*h needs to be replaced by $\partial F\circ h$). Since F is defined on a smooth domain, the operators in [IP5, (7.4),(7.6)] are Fredholm because they differ from real Cauchy-Riemann operators by finite-dimensional pieces; uniform boundness in μ as in [IP5, Proposition 7.3] is a separate issue. With a reasonable interpretation of the inner-product above [IP5, (7.5)], the last component of $\mathbf{D}^*_F$ in [IP5, (7.5)] is missing $\frac{1}{2}$. The expression for $A\eta$ in [IP5, (7.5)] cannot be correct either, since it should produce a tuple indexed by the relative marked points, not a sum. Furthermore, this expression should have more terms, as the proof of [IP5, Proposition 7.3] suggests, and should depend on the vertical part of η as well. However, the exact forms of the second and last components of $\mathbf{D}^*_F$ do not matter as long as they are uniformly bounded; this is the case because the restrictions of $\mathbf{D}_F$ to the second and last components in [IP5, (7.4)] are uniformly bounded. The bound on $\nabla\nu$ at the beginning of the proof of [IP5, Proposition 7.3] is not obvious, because ∇ there denotes the Levi-Civita connection with respect to the metric on $\mathcal{Z}_\lambda$, which degenerates as $\lambda\longrightarrow 0$; this bound depends on the requirement on the second fundamental form in [IP5, Definition 2.2]. Other, fairly minor misstatements in [IP5, Section 7] include

- p975, Section 7, line 2: there are no Sobolev spaces in Definition 6.5;
- p975, line -5: Lemma 7.3 $\longrightarrow$ Proposition 7.3; same on p976, line 5;
- p976, 2nd paragraph: there is nothing about generic δ or Fredholm in Proposition 7.3; there seems to be no connection with Lemma 3.4 at all;
- p976, line -3: no such verification in Lemma 3.4;
- p977, lines 1,2: there is no stabilization in Observation 6.7;
- p977, lines 4,6: ev $\longrightarrow$ ev;
- p977, Lemma 7.2: ζ should be a vector field along F, not on a chart;
- p980, line 7: X already denotes a symplectic manifold;
- p978, line -10: L $\longrightarrow$ L_F;

p978, line -9: no use of Lemma 7.2 in addition to (7.7);
p978, line -7: with this description, $\tilde{\nabla}$ and ∇ are connections in different spaces;
p979, line 5: there is no h^v and $\tilde{x}$ in Definition 6.4.

There is a *crucial* sign error in the proof of [IP5, Proposition 8.2]: the two terms on the second line of [IP5, (8.7)], a Gauss curvature equation written in a rather unusual way, should have the opposite signs; see [L, Theorem 13.38], which uses the same (more standard) sign convention for the curvature tensor R (defined at the beginning of [L, Section 13.2]). Thus, the minus sign in [IP5, (8.8)] should be a plus, which destroys the argument. Conceptually, it seems implausible to have a negative sign in [IP5, (8.8)], because it should allow to make the right-hand side of [IP5, (8.6)] negative by taking a local solution of L_F^* and sending μ and λ to 0.

The proof of [IP5, Lemma 8.4] is also incomplete. At the very bottom of page 984 in [IP5], it is stated that $\mathbf{D}_0^*\eta = \mathbf{D}_u^*\eta$ lies in the image of the map $\mathbf{D}_0^*$ in [IP5, (7.6)]. However, it had not been shown that the limiting $(0,1)$-form η lies in the domain of $\mathbf{D}_0^*$, which involves bounding the first derivative over the entire domain. The preceding argument shows that the L_1^2-norm of η outside of the nodes of the domain is bounded, but that does not imply that the L_1^2-norm of η is bounded everywhere. Furthermore, since the metrics on the targets $\mathcal{Z}_\lambda$ degenerate, a proof is needed to show that the elliptic estimate used in the proof of [IP5, Lemma 8.5] is uniform; it is not so clear that it is.

Remark 5.13. The bound on ∇J on line 10 on page 981 of [IP5] is not obvious, because ∇ there denotes the Levi-Civita connection with respect to the metric on $\mathcal{Z}_\lambda$, which degenerates as $\lambda \longrightarrow 0$; this bound depends on the requirement on the second fundamental form in [IP5, Definition 2.2]. Since the metric on the horizontal tangent space in N_Z^V varies in the normal direction (according to the bottom half on p951), the formula for g_λ on line -5 on page 982 cannot be precisely correct; this has an effect on the formulas for Christoffel symbols on the last line on this page (though this gets absorbed into the error term in the next sentence, which should include s_k in front of tanh). There is a similar issue with the statement concerning the independence of F^*g_λ. Other, fairly minor misstatements in [IP5, Section 8] include
p980, line 9: (1.4) $\longrightarrow$ (1.5);
p981, line 15: ω already denotes a symplectic form;
p981, (8.6): $-d(\rho^\delta)\wedge\omega$ is part of the first integrand on RHS;
p981, line -6: this has nothing to do with the connection on the domain (which is also not flat);
p981, line -5: V already denotes the symplectic divisor;
p982, line 5: A_k as defined in the proof of Lemma 6.9 is a subset of C_μ, not of $\mathcal{Z}_\lambda$;
p982, line 7: ν already denotes the key $(0,1)$-form; missing ν at the end;
p982, line 17: first inequality does not hold because of z^s in (6.14);
p982, line 18: there is no bound on $|\nu^N|$ in the sentence preceding (6.17);
p982, line 20: $U - JV \longrightarrow V - JU$, twice;
p982, line 21: no connection to the preceding statement;
p982, line -6: $\theta \longrightarrow \Theta$;
p982, line -3: $F_*\partial_\theta$ also involves a V-component;
p983, lines 15,16: *multiply* and *adding* do not help here;
p984, top: δ generic does not appear in this section again;
p984, lines 13,14: by definition of $\{F_n\}$, not *Bubble Tree Convergence Theorem*;
p984, line 21: $N = \{\rho \leq \delta\}$, and this δ is different from the δ in the norm;
p984, bottom third: X already denotes a symplectic manifold;

p984, line -9: βh is not in $T_{C_0}\mathcal{M}$.

In the approach of [LR], the metrics on the targets do not collapse. A family of uniformly bounded right inverses for the linearized operators $\mathbf{D}_{u_\mu}$ is constructed in the proof of [LR, Lemma 4.8] directly via the approach of [MS2, Section 10.5]. Conceptually, the existence of such inverses follows from uniform elliptic estimates in the metrics of [LR] on the target; see the proofs of [LT, Lemmas 3.9,3.10].

5.4 Gluing: [IP5, Sections 9,10], [LR, Sections 4.2,5]

The final step in gluing constructions involves showing that every approximately J-holomorphic map u_μ can be perturbed to an actual J-holomorphic map, in a unique way subject to suitable restrictions, and that every nearby J-holomorphic map can be obtained in such a way. The last part is often established by showing that all nearby maps, J-holomorphic or not, are of the form $\exp_{u_\mu}\xi$ with ξ small. The uniqueness part can be established by showing that each nearby map can be written uniquely in the form $\exp_{u_\mu}\xi$, subject to suitable conditions on ξ. The nearby solutions of the $\bar\partial$-equations are then determined by locally trivializing the bundle of $(0,1)$-forms and expanding the $\bar\partial$-equation as

$$\bar\partial \exp_{u_\mu}\xi = \bar\partial u_\mu + \mathbf{D}_{u_\mu}\xi + Q_{u_\mu}(\xi)\,, \tag{5.16}$$

where $\mathbf{D}_{u_\mu}$ is the linearization of the $\bar\partial$-operator determined by the given trivialization and $Q_{u_\mu}(\xi)$ is the error term, quadratic in ξ. The equation (5.16) can be solved for all μ sufficiently small if the norm of $\bar\partial u_\mu$ tends to 0 as $\mu\longrightarrow 0$, $\mathbf{D}_{u_\mu}$ admits a right inverse which is uniformly bounded in μ, and the error term Q_{u_μ} is also uniformly bounded in μ.

The bijectivity of the gluing map is the subject of [IP5, Proposition 9.1], though its wording is not quite correct. Based on the proof and the usage, the intended wording is that there exist $\varepsilon_0, c>0$ such that the map Φ_λ is a diffeomorphism as described whenever $\varepsilon, |\lambda|<\varepsilon_0$. The proof of [IP5, Proposition 9.1] is incorrect at the end of the injectivity argument, even ignoring the problems with the prerequisite statements: even if $(f_n, C_{0,n}, \mu_n)=(f'_n, C'_{0,n}, \mu'_n)$, η_n and η'_n need not lie in the injectivity radius of Φ_{λ_n} for n large, as this radius likely collapses as $n\longrightarrow\infty$, because the injectivity radius of the metric g_λ collapses as $\lambda\longrightarrow 0$ and the norms are not scaled to address this. In order to show that the injectivity radius of Φ_λ does not collapse, one needs to show that the vertical part of $P_F\eta$ on suitable necks is bounded by something like $|\lambda|^{\frac{1}{2}}\|P_F\eta\|$. In light of (5.6), this appears plausible for the nearby J-holomorphic maps, but less so for arbitrary nearby maps. It thus seems quite possible that the injectivity part of the intended statement of [IP5, Proposition 9.1] is not correct with the norms of [IP5, Definition 6.5], which impose a rather mild weight in the collapsing direction.

The proof of [IP5, Proposition 9.4] is incomplete, as a justification is required for why the constant C in the bound [IP5, (9.11)] on the quadratic error term in (5.16) is uniform in μ. This is not obvious in this case, since the metrics on $\mathcal{Z}_\lambda$ degenerate and the constant C depends on the curvature of the metric; see [Z2, Section 3]. Thus, this is also a significant issue in the approach of [IP5].

Remark 5.14. The proof of [IP5, Lemma 9.2] ignores the regions $|\mu_k|^{\frac{1}{4}}\le\rho\le 2|\mu_k|^{\frac{1}{4}}$. The statement of [IP5, Proposition 9.3] is essentially correct, but the last part of its proof does not make sense.

For example, since f_0 is a map from a wedge of two disks and f_n is a map from a cylinder, f_0-f_n is not defined. Furthermore, the equations $F_n-f_n=(\hat{\zeta}_n,\bar{\xi}_n)$, $\hat{\zeta}_n=\zeta_n+(F_n-f_0)$, and $\zeta_n=f_0-f_n$ are inconsistent. Other, fairly minor misstatements in [IP5, Section 9] and in the first part of [IP5, Section 10] include

p986, above (9.2): determined by $\longrightarrow$ related to;
p986, below (9.2): Φ_λ is defined everywhere and is the identity along the zero section;
p986, (9.3): it is only an isomorphism, since the first summand on RHS is not a subspace of LHS;
p986, below (9.3): Lemma 5.3 is not needed here;
p986, line -3: the image of F_0 in TZ_λ $\longrightarrow$ F_0;
p986, line -1: RHS describes only the vector field component of LHS and only for $\eta_0=0$;
p987, line 13: B is the two-dimensional manifold underlying C_0 and C_0';
p987, line -4: there is no such extension in Section 4;
p987, bottom: h_1 is a variation of μ, which is basically fixed;
p988, lines 4,5: not extended over Z_λ;
p988, line 6: ξ_0 has not been defined;
p988, (9.5): second line is missing $\frac{1}{2}$;
p988, line 13: $\rho^{-|s|}$, not $\rho^{1-|s|}$, according to (6.15), which is still good enough;
p988, after (9.6): the estimates in the proof of Proposition 7.3;
p988, after (9.7): there is no equation (6.4a);
p989, line 9: (9.8) $\longrightarrow$ (9.6);
p989, line -3: Lemma 5.4 does not say this;
p990, (9.10) holds only after some identifications;
p991, below (10.2): this sentence does not make sense;
p991, (10.3): since s is fixed, there should be no $\bigsqcup$;
p992, lines 3,4: Φ_λ^1 maps into $\mathcal{M}_s^{V,\delta}(Z_\lambda)$ according (10.3);
p992, lines 16,17: this sentence makes no sense.

The correspondence between approximately J-holomorphic maps and actual J-holomorphic maps in [LR] is the subject of Proposition 4.10. The expansion (5.16) does not even appear in its proof, with the Implicit Function Theorem applied in an infinite-dimensional setting without any justification. On the other hand, the above issues with the collapsing metric do not arise in the setting of [LR], and so the required uniform estimates are fairly straightforward to obtain.

Remark 5.15. The approach of [LR, Section 5] to the symplectic sum formula involves the existence of a virtual fundamental class for $\overline{\mathcal{M}}_{g,k}(X\cup_V Y, A)$. The justification for its existence consists of a few lines after [LR, Lemma 5.4], which is far from even mentioning all the required issues. The comparison of GW-invariants for $X\cup_V Y$ and $X\#_V Y$ in [LR, Section 5] again involves integration instead of pseudocycles (top of p208 and p209), and does not explain the key multiplicity factor k in [LR, Theorem 5.7]. The top of page 209 again suggests an isomorphism between an even and odd-dimensional manifolds. The index formula [LR, (5.1)] cannot possibly follow from the proof of [LR, Lemma 4.9], as the latter has no numerical expressions for the index. Since this index also depends α (according to [LR, Remark 4.1]), how can there be a natural correspondence between the domains and targets of the operators D_u and $D_{\bar{u}}$ in [LR, Remark 5.2]? With the definitions in [LR, Section 4], the dimension of $\ker L_\infty$ is $2n$, not $2n+2$, as stated after [LR, (5.1)]. Mayer-Vietoris has nothing to do with a pseudoholomorphic map defining a homology class at the bottom of page 206 in [LR]. [LR, Remark 5.9] is irrelevant, since there had been no assumption that the divisor is connected. Remark 4.10 contains additional related comments.

In general, one has to consider smoothings of nodes that do not map to the junctions between the smooth pieces of $X\cup_V^m Y$. However, such nodes can be handled in a standard way, such as in [LT, Section 3], as mentioned in [IP5, Remark 6.3].

5.5 The S-matrix [IP5, Sections 11,12]

The symplectic sum formula of [IP5] contains two features not present in the formulas of [Lj2] and [LR]: a rim tori refinement of relative invariants and the so-called S-matrix. Section 3.2 explains why the first additional feature should not appear; this section explains why the second feature should not appear. We also show that in fact the S-matrix does not matter because it *acts as* the identity in all cases and not just in the cases considered in [IP5, Sections 14,15], when the S-matrix *is* the identity. The fundamental reason for the latter is the same as for the former: a group action is forgotten in [IP5].

By Gromov's Compactness Theorem [RT1, Proposition 3.1], a sequence of $(J_{\mathcal{Z}},\mathfrak{j}_k)$-holomorphic maps $u_k:\Sigma\longrightarrow\mathcal{Z}_{\lambda_k}$, with $\lambda_k\in\Delta^*$ and $\lambda_k\longrightarrow 0$, has a subsequence converging to a $(J_{\mathcal{Z}},\mathfrak{j})$-holomorphic map $u\colon\Sigma'\longrightarrow\mathcal{Z}_0$. As explained in Section 5.1,

$$\Sigma'=\Sigma'_X\cup\Sigma'_V\cup\Sigma'_Y\,,$$

where Σ'_V is the union of irreducible components of Σ' mapped into V, Σ'_X is the union of irreducible components mapped into $X-V$ outside of finitely many points $x_1,\ldots,x_\ell$, and Σ'_Y is the union of irreducible components mapped into $Y-V$ outside of finitely many points $x'_1,\ldots,x'_{\ell'}$. The symplectic sum formulas of [Lj2] and [LR] arise only from the limits with $\Sigma'_V=\emptyset$; these are also the limits considered in [IP5, Sections 6-10].

The S-matrix arises at the top of page 1003 in [IP5] from the consideration of limits with $\Sigma'_V\neq\emptyset$. Such maps are interpreted as maps to the singular spaces $X\cup_V^m Y$, with $m\in\mathbb{Z}^+$, defined in (4.20). This interpretation is obtained by viewing the sequences of maps which give rise to such limits as having their images inside the total space $\mathcal{Z}_m$ of an $(m+1)$-dimensional family of smoothings of $X\cup_V^m Y$, instead of the total space $\mathcal{Z}$ of a one-dimensional family of smoothing of $X\cup_V Y$. However, it is not possible to associate a sequence of maps to $\mathcal{Z}_m$ to a sequence of maps to $\mathcal{Z}$ in a systematic way which is consistent with the aims of [IP5, Section 12]. Contrary to the implicit view in [IP5, Section 12], the resulting limiting map to $X\cup_V^m Y$ is well-defined by the original sequence of maps to $\mathcal{Z}$ not up to a finite number of ambiguities, but up to an action of finitely m copies of $\mathbb{C}^*$ on the target. Furthermore, the entire setup at the top of page 1003 in [IP5] is incorrect because the almost complex structure on $\mathcal{Z}_\lambda$ viewed as a fiber of $\mathcal{Z}$ is different from what it would have been as a fiber in $\mathcal{Z}_m$ (the latter would be effected by $m+1$ copies of V). However, these almost complex structure would be the same in the case of the more restricted almost complex structures of [LR].

The situation is nearly identical to [IP4, Sections 6,7], where limits of sequences of relative maps into (X,V) are shown to correspond to maps to X_m^V *up to* a natural $(\mathbb{C}^*)^m$-action; see Section 4.3 above. The same reasoning as in [IP4, Sections 6,7] shows that limits of sequences of maps into $\mathcal{Z}$ correspond to maps to $X\cup_V^m Y$ *up to* a natural $(\mathbb{C}^*)^m$-action.

As in the situation in Section 4.3, which reviews [IP4, Sections 6,7], the virtual dimension of the spaces of morphisms into $X\cup_V^m Y$ is $2m$ less than the expected dimension of the corresponding spaces of morphisms into $X\cup_V Y$ (with the matching conditions imposed) and into $\mathcal{Z}_\lambda$. Thus, the spaces of morphisms into $X\cup_V^m Y$ with $m\geq 1$ have no effect on the symplectic sum formula. The S-matrix, which takes such spaces into account, enters at the top of page 1003 in [IP5] because the spaces of such morphisms are mistakenly not quotiented out by $(\mathbb{C}^*)^m$; this is done in [Lj2] and in [LR].

While the S-matrix is generally not the identity, it acts as the identity in the symplectic sum formulas of [IP5], i.e. in equations (0.2) and (12.7) in [IP5], for the following reason. For all

$$\chi\in\mathbb{Z}, \qquad (A,B)\in H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z}),$$

and a generic collection of constraints of appropriate total codimension, the symplectic sum formula presents the corresponding GT-invariant of $X\#_V Y$ as the sum of weighted cardinalities of finitely many finite sets enumerating morphisms into $X\cup_V^m Y$, with $m\geq 0$, meeting the constraints. The group $(\mathbb{C}^*)^m$ acts on the set of such morphisms with at most finite stabilizers (the constraints inside each $\{r\}\times\mathbb{P}_V X$ are pull-backs from V). Thus, the sets with $m\geq 1$ are empty, i.e. there is no contribution to the symplectic sum formula from morphisms to $X\cup_V^m Y$ with $m\geq 1$. Since these are the morphisms that make up the difference between the S-matrix and the identity, the S-matrix acts as the identity in the symplectic sum formulas of [IP5].

The next observation illustrates one of the problems with the normalizations of generating functions in [IP5, Section 1] and thus another problem with the symplectic sum formulas of [IP5]. The last statement of [IP5, Lemma 11.2(a)] is key to even making sense of the action of the S-matrix. However, it does not hold with the definitions in the paper. By [IP5, (1.24)], the $\mathcal{M}_{\mathbb{I}}$-part of $\mathrm{GW}_{\mathbb{P}_V}^{V_\infty,V_0}(1)$ is given by

$$\begin{aligned}\mathrm{GW}_{\mathbb{P}_V}^{V_\infty,V_0}(1)_{\mathcal{M}_{\mathbb{I}}}&=\sum_{d=1}^{\infty}\big[\mathrm{ev}_1\times\mathrm{ev}_2\colon\mathcal{M}_{0,2,(d,d)}^{V_\infty\sqcup V_0}(\mathbb{P}_V,d)\longrightarrow\mathcal{H}_{\mathbb{P}_V,dF,(d,d)}^{V_\infty\sqcup V_0}\big]t_{dF}\lambda^{-2}\\&=\sum_{d=1}^{\infty}\frac{1}{d}\Delta_{dF,(d,d)}t_{dF}\lambda^{-2},\end{aligned}$$

where $\Delta_{dF,(d,d)}\subset\mathcal{H}_{\mathbb{P}_V,dF,(d,d)}^{V_\infty\sqcup V_0}$ is the preimage of the diagonal in $V_\infty\times V_0=V\times V$. The exponential of an element of $H_*(\widetilde{\mathcal{M}}\times\mathcal{H}_X^V)$ and the product of two elements of $H_*(\mathcal{H}_X^V)$ are never defined, but under reasonable definitions

$$\begin{aligned}\mathrm{GT}_{\mathbb{P}_V}^{V_\infty,V_0}(1)_{\mathcal{M}_{\mathbb{I}}}\equiv e^{\mathrm{GW}_{\mathbb{P}_V}^{V_\infty,V_0}(1)_{\mathcal{M}_{\mathbb{I}}}}&=\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\big(\mathrm{GW}_{\mathbb{P}_V}^{V_\infty,V_0}(1)_{\mathcal{M}_{\mathbb{I}}}\big)^{\ell}\\&=1+\sum_{\ell=1}^{\infty}\sum_{d_1,\ldots,d_\ell>0}\frac{1}{\ell!}\,\frac{1}{d_1\ldots d_\ell}\Delta_{d_1F,(d_1,d_1)}\times\ldots\times\Delta_{d_\ell F,(d_\ell,d_\ell)}\,t_{(d_1+\ldots+d_\ell)F}\lambda^{-2\ell};\end{aligned}$$

this definition seems to be consistent with [IP5, (A.3)] and the description of the coefficients in the following paragraph. Let $\eta\in\mathcal{H}_{\mathbb{P}_V,B,(s_1,\ldots,s_m)}^{V_0}t_B$. By [IP5, (10.6)], the only nonzero term in

$\eta * \mathrm{GT}^{V_\infty,V_0}_{\mathbb{P}_V}(1)_{\mathcal{M}_{\mathbb{I}}}$ arises from the summand $\ell = m$ and $(d_1,\ldots,d_\ell)=(s_1,\ldots,s_m)$ and equals

$$\frac{s_1\ldots s_m}{m!}\lambda^{2m}\eta\cdot\frac{1}{\ell!}\,\frac{1}{d_1\ldots d_\ell}\lambda^{-2\ell}=\frac{1}{m!m!}\eta\neq\eta\quad\text{if}\quad m>1.$$

The proof of the symplectic sum formula, [IP5, (12.7)], makes use of (11.3); otherwise, there would be dependence on N.

Remark 5.16. Other, fairly minor misstatements in [IP5, Sections 11,12] include
p998, (11.1): no need for square brackets; the superscripts on $\mathcal{H}$ should be the same;
p998, line -6: before (1.4) $\longrightarrow$ after (1.5);
p999, line 1: the *irreducible* $\mathbb{P}_V$-trivial;
p999, (11.3),(11.4): LHS missing $*$; $R^{V_\infty,V_0}\longrightarrow R$ in the notation below;
p999, line -3: (J,ν) $\longrightarrow$ (A,n,χ);
p1000, Dfn 11.3: there is no dependence on (J,ν);
p1000, bottom: this sentence does not make sense;
p1001, lines -13,-9: $2N$ $\longrightarrow$ $2N-1$;
p1001, line -4: both identities are incorrect;
p1002, line 6: there is no t in (2.6);
p1002, below (12.2): μ is on the domain, λ is on the target;
p1002, line 18: $\varepsilon=\alpha_V$;
p1002, line 22: *nonempty* subset;
p1003, Thm 12.3, line 5: (11.3) $\longrightarrow$ of Definition 11.3.

6 Applications: [IP5, Sections 14,15]

The purpose of [IP5, Sections 14,15] is to give three powerful applications of the symplectic sum formula. The authors make clear what geometric reasoning should lead to the three main formulas; fully implementing the arguments they sketch leads to quick proofs of these formulas, which had been previously established through significantly more complicated arguments. Unfortunately, the arguments in the paper are not completely precise and contain multiple, sometimes self-canceling, errors (as well as typos), and none of the three formulas is stated fully correctly. In order to illustrate the beauty of the intended arguments in this part of [IP5], we reproduce them completely here, but with all the details and without the errors, and then list the errors and typos made in [IP5]; the substance and organization of these arguments come entirely from [IP5].

6.1 Invariants of $\mathbb{P}^1$ and $\mathbb{T}^2$: [IP5, Section 14.1]

This section computes some relative GW-invariants of $\mathbb{P}^1$ and $\mathbb{T}^2$. If $V_1,V_2\subset X$ are two disjoint symplectic divisors, we will denote by $\mathrm{GW}^{V_1,V_2}_{X,A,g;\mathbf{s}_1,\mathbf{s}_2}$ the relative GW-invariants of $(X,V_1\cup V_2)$ with the contacts with V_1 and V_2 described by $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively. We will use similar notation for the disconnected GT-invariants and for the moduli spaces.

Lemma 6.1 ([IP5, Lemma 14.1])**.** *Let $0,\infty$ denote two distinct points in $\mathbb{P}^1$, $V=\{0,\infty\}$, and $d\in\mathbb{Z}^+$. The relative degree d GW-invariants of $(\mathbb{P}^1,V)$ with no constraints from $\mathbb{P}^1$ or $\overline{\mathcal{M}}$ are given by*

$$\mathrm{GW}^{0,\infty}_{\mathbb{P}^1,d,g;\mathbf{s}_0,\mathbf{s}_\infty}()=\begin{cases}1/d, & \text{if } g=0,\ \mathbf{s}_0,\mathbf{s}_\infty=(d);\\ 0, & \text{otherwise.}\end{cases}$$

Proof. By [IP5, (1.21)],

$$\begin{aligned}\dim_{\mathbb{C}} \overline{\mathcal{M}}^{0,\infty}_{g,0;\mathbf{s}_0,\mathbf{s}_\infty}(\mathbb{P}^1,d) &= 2d + (1-3)(1-g) + \ell(\mathbf{s}_0) + \ell(\mathbf{s}_\infty) - \deg \mathbf{s}_0 - \deg \mathbf{s}_\infty \\ &= 2g-2+\ell(\mathbf{s}_0)+\ell(\mathbf{s}_\infty) \ge 2g \ge 0.\end{aligned} \tag{6.1}$$

This dimension is 0 only if $g=0$ and $\ell(\mathbf{s}_0),\ell(\mathbf{s}_\infty)=1$. If $g=0$ and $\mathbf{s}_0,\mathbf{s}_\infty=(d)$, $\overline{\mathcal{M}}^{0,\infty}_{g,0;\mathbf{s}_0,\mathbf{s}_\infty}(\mathbb{P}^1,d)$ consists of a single element, the map $z\longrightarrow z^d$. Since the order of the group of automorphisms of this map is d, it contributes $1/d$ to the GW-invariant. □

Lemma 6.2 ([IP5, Lemma 14.2])**.** *Let* $0,\infty,1$ *denote three distinct points in* $\mathbb{P}^1$, $V=\{0,\infty,1\}$, $d\in\mathbb{Z}^+$ *with* $d\ge 2$, *and*

$$\mathbf{s}_1=(2,\underbrace{1,\ldots,1}_{d-2}). \tag{6.2}$$

The relative degree d *GW-invariants of* $(\mathbb{P}^1,V)$ *enumerating maps with simple branching over 1 and no constraints from* $\mathbb{P}^1$ *or* $\overline{\mathcal{M}}$ *are given by*

$$\begin{aligned}\mathrm{GW}^{0,\infty}_{\mathbb{P}^1,d,g;\mathbf{s}_0,\mathbf{s}_\infty}(b) &\equiv \frac{1}{(d-2)!}\mathrm{GW}^{0,\infty,1}_{\mathbb{P}^1,d,g;\mathbf{s}_0,\mathbf{s}_\infty,\mathbf{s}_1}() \\ &= \begin{cases} 1, & \text{if } g=0,\ \{\ell(\mathbf{s}_0),\ell(\mathbf{s}_\infty)\}=\{1,2\},\ \deg \mathbf{s}_0,\deg \mathbf{s}_\infty = d; \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Proof. Similarly to (6.1),

$$\dim_{\mathbb{C}} \overline{\mathcal{M}}^{0,\infty,1}_{g,0;\mathbf{s}_0,\mathbf{s}_\infty,\mathbf{s}_1}(\mathbb{P}^1,d) = 2g-3+\ell(\mathbf{s}_0)+\ell(\mathbf{s}_\infty) \ge 2g-1 \ge -1.$$

This dimension is 0 only if $g=0$ and $\ell(\mathbf{s}_0)+\ell(\mathbf{s}_\infty)=3$. Every holomorphic function on $\mathbb{C}$ with a pole of order d at ∞ and zeros at 0 and 1 of orders a and b, respectively, with $a+b=d$, is of the form $z\longrightarrow Cz^a(z-1)^b$. There is a unique value of C so that this function sends the remaining critical point, $z=a/d$, to 1. Thus, $\overline{\mathcal{M}}^{0,\infty,1}_{g,0;\mathbf{s}_0,\mathbf{s}_\infty,\mathbf{s}_1}(\mathbb{P}^1,d)$ consists of $(d-2)!$ automorphism-free elements (corresponding to the orderings of the simple preimages of 1). □

Remark 6.3. [IP5, Lemma 14.3] is not used in the rest of the paper. Furthermore, its statement is wrong, as the authors forget to divide by the order of the automorphism group of covers of the torus. The notation for GW-invariants in [IP5, Sections 14.1-14.5] is inconsistent with earlier parts of the paper, as the first subscript is supposed to indicate the target space. The notation for the simple branch point invariant of [IP5, Lemma 14.2], which is never formally defined, is even more confusing, since an insertion in parenthesis is supposed to indicate a class on a product of $\overline{\mathcal{M}}$ and copies of X. The conclusion in the proof of Lemma 14.1 about the S-matrix does not follow from the rest of the argument, since it may have contributions from higher genus and classes coming from $\overline{\mathcal{M}}$.

6.2 Genus 1 invariants of $\mathbb{P}^1\times\mathbb{T}^2$: [IP5, Section 14.2]

This section computes some genus 1 GW-invariants of $\mathbb{P}^1\times\mathbb{T}^2$ and $(\mathbb{P}^1\times\mathbb{T}^2,F)$, where $F=p\times\mathbb{T}^2$ is a fiber of the projection to the first component. We denote by s and f the homology classes of $\mathbb{P}^1\times q$ and F, respectively. Let

$$G(q)=\sum_{d=1}^{\infty}\sigma(d)q^d, \qquad \text{where} \quad \sigma(d)=\sum_{k|d}k. \tag{6.3}$$

Lemma 6.4 ([IP5, Lemma 14.4]). *The genus 1 genus GW-invariants of* $\mathbb{P}^1\times\mathbb{T}^2$ *satisfy*

$$\sum_{d=1}^{\infty} d\mathrm{GW}_{\mathbb{P}^1\times\mathbb{T}^2,df,1}()q^d = 2G(q).$$

Proof. Let $L=\mathcal{O}_{\mathbb{T}^2}(p-q)\longrightarrow\mathbb{T}^2$ be a non-torsion line bundle ($L^{\otimes k}\not\approx\mathcal{O}_{\mathbb{T}^2}$ for all $k\in\mathbb{Z}^+$). The only holomorphic maps in $\mathbb{P}(L\oplus\mathcal{O}_{\mathbb{T}^2})\approx\mathbb{P}^1\times\mathbb{T}^2$ in the homology class df are then covers of

$$F_0\equiv\mathbb{P}\big(0\oplus\mathcal{O}_{\mathbb{T}^2}\big) \qquad\text{and}\qquad F_\infty\equiv\mathbb{P}\big(L\oplus 0\big),$$

and these maps are regular. Since the number of degree d covers $\mathbb{T}^2\longrightarrow\mathbb{T}^2$ (or equivalently of subgroups of $\mathbb{Z}^2$ of index d) is $\sigma(d)$, $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^1\times\mathbb{T}^2,df)$ consists of $2\sigma(d)$ elements. Since the order of the automorphism group of each of these elements is d, we conclude that

$$\mathrm{GW}_{\mathbb{P}^1\times\mathbb{T}^2,df,1}() = 2\sigma(d)/d,$$

as claimed. □

Lemma 6.5 ([IP5, Lemma 14.5]). *The genus 1 genus GW-invariants of* $(\mathbb{P}^1\times\mathbb{T}^2,F)$ *with two point constraints satisfy*

$$\sum_{d=0}^{\infty}\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,s+df,1}\big(p;C_1(p)\big)q^d = qG'(q).$$

Proof. Suppose Σ is a connected nodal genus 1 curve and $u\colon\Sigma\longrightarrow\mathbb{P}^1\times\mathbb{T}^2$ is a degree $s+df$ stable map. Since $\pi_1\circ u:\Sigma\longrightarrow\mathbb{P}^1$ has degree 1 and every holomorphic map $\mathbb{P}^1\longrightarrow\mathbb{T}^2$ is constant, Σ contains a unique irreducible component $\Sigma_0\approx\mathbb{P}^1$ such that $u\colon\Sigma_0\longrightarrow\mathbb{P}^1\times q_2$ is an isomorphism for some $q_2\in\mathbb{T}^2$. If Σ_i is another irreducible rational component of Σ, then $u|_{\Sigma_i}$ is constant. Since Σ is of genus 1, Σ contains at most one (precisely one if $d>0$) irreducible genus 1 component Σ_1; furthermore, $f|_{\Sigma_1}$ is a degree d (unbranched) cover of $q_1\times\mathbb{T}^2$ for some $q_1\in\mathbb{P}^1$. Every such stable map is regular.

Thus, the subspace

$$\big\{[u,x_1,y_1]\in\overline{\mathcal{M}}^F_{1,1;(1)}(\mathbb{P}^1\times\mathbb{T}^2,s+df)\colon u(x_1)=p_1,\ u(y_1)=p_2\big\}$$

consists of maps $u:\Sigma_0\cup\Sigma_1\longrightarrow\mathbb{P}^1\times\mathbb{T}^2$ such that $u:\Sigma_0\longrightarrow\mathbb{P}^1\times\pi_2(p_2)$ is an isomorphism and $u:\Sigma_1\longrightarrow\pi_1(p_1)\times\mathbb{T}^2$ is a degree d cover.[5] There are $\sigma(d)$ such maps u, each of which has an automorphism of order d. For each choice of the map u, there are d choices for the preimage of p_1 and d choices for the nodes on Σ_1. Thus,

$$\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,df,1}\big(p;C_1(p)\big) = \sigma(d)/d\cdot d\cdot d = d\sigma(d);$$

this establishes the claim. □

[5]The elements $[u,x_1,x_2]$ of $\overline{\mathcal{M}}_{1,2}(\mathbb{P}^1\times\mathbb{T}^2,s+df)$ such that $u:\Sigma_0\longrightarrow\mathbb{P}^1\times\pi_2(p_1)$ is an isomorphism and $u:\Sigma_1\longrightarrow\pi_1(p_2)\times\mathbb{T}^2$ is a degree d cover correspond to the elements $[u,x_1,y_1]$ of $\overline{\mathcal{M}}^F_{1,1;(1)}(\mathbb{P}^1\times\mathbb{T}^2,s+df)$ such that $u\colon\Sigma_0\longrightarrow\mathbb{P}^1\times\pi_2(p_1)$ is an isomorphism, $u\colon\Sigma_{1;0}\cup\Sigma_{1;1}\longrightarrow\mathbb{P}^1\times\mathbb{T}^2$ is a map into the rubber, $u\colon\Sigma_{1;0}\longrightarrow\mathbb{P}^1\times\pi_2(p_1)$ is an isomorphism, and $u\colon\Sigma_1\longrightarrow q'\times\mathbb{T}^2$ is a degree d cover for some $q'\in\mathbb{P}^1-\{0,\infty\}$; this map does not pass through the relative constraint p_2, which now lies on $0\times\mathbb{T}^2$ in the rubber.

Remark 6.6. Of the three statements in [IP5, Lemmas 14.4], only the first is used in the rest of the paper; of the three statements in [IP5, Lemmas 14.5], only the second is used. In particular, there is no reason for the rim tori discussion in this section at all. The first two statements of [IP5, Lemma 14.4] are wrong, as the authors forget to divide by the order of the automorphism group of covers of the torus. The proof of [IP5, Lemma 14.5] has two mutually canceling errors, as the authors forget to divide by the order of the automorphism group as well as forget to account for the number of choices of the node on the genus 1 component. The statement at the end of the first paragraph of the proof is true only generically or after imposing the constraints; otherwise, there could be maps with more components of the domain. At the end of the second paragraph, *the domain of* F should be *the preimage of* F. In the third statement of [IP5, Lemma 14.5], t_{s+df+R} should be t^d, as t is the variable used on the right-hand side of this expression.

6.3 Invariants of $\mathbb{F}_n$: [IP5, Section 14.3]

This section computes some relative GW-invariants of $(\mathbb{F}_n, S_0 \cup S_\infty)$, where

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}), \qquad S_0 = \mathbb{P}(0 \oplus \mathcal{O}_{\mathbb{P}^1}) \subset \mathbb{F}_n, \qquad S_\infty = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus 0) \subset \mathbb{F}_n.$$

We denote by s_0 and f the homology classes of S_0 and of a fiber of $\mathbb{F}_n \longrightarrow \mathbb{P}^1$. For $A \in H_2(\mathbb{F}_n)$, ordered partitions $\mathbf{s}_0$ and $\mathbf{s}_\infty$ of $A \cdot S_0$ and $A \cdot S_\infty$, respectively, and $\alpha \in \mathbb{T}^*(\mathbb{F}_n)$,

$$\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,A,g;\mathbf{s}_0,\mathbf{s}_\infty}(\alpha) \in H_*(S_0^{\ell(\mathbf{s}_0)}) \otimes H_*(S_\infty^{\ell(\mathbf{s}_\infty)}).$$

If $A = as_0 + bf$, then

$$A \cdot S_\infty = b, \qquad A \cdot S_0 = na + b, \qquad \langle c_1(T\mathbb{F}_n), A \rangle = (2+n)a + 2b.$$

Thus, by [IP5, (1.21)],

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,A,g;\mathbf{s}_0,\mathbf{s}_\infty}(\alpha) &= (2+n)a + 2b + (2-3)(1-g) + \ell(\alpha) + \ell(\mathbf{s}_0) + \ell(\mathbf{s}_\infty) \\ &\quad - \deg \alpha - \deg \mathbf{s}_0 - \deg \mathbf{s}_\infty \\ &= g - 1 + 2a + \ell(\alpha) + \ell(\mathbf{s}_0) + \ell(\mathbf{s}_\infty) - \deg \alpha, \end{aligned} \tag{6.4}$$

if $\alpha \in H^*(\mathbb{F}_n^{\ell(\alpha)})$. In particular, $\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,A,g;\mathbf{s}_0,\mathbf{s}_\infty}(\alpha) = 0$ unless

$$g + 2a \le 1 + \deg \alpha - \ell(\alpha). \tag{6.5}$$

Lemma 6.7 ([IP5, Lemma 14.6])**.** *The relative degree A GW-invariants of $(\mathbb{F}_n, S_0 \cup S_\infty)$ with no constraints from $\mathbb{F}_n$ or $\overline{\mathcal{M}}$ are given by*

$$\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,A,g;\mathbf{s}_0,\mathbf{s}_\infty}() = \begin{cases} \frac{1}{b}(S_0 \otimes 1 + 1 \otimes S_\infty), & \text{if } g=0,\ A=bf,\ b \in \mathbb{Z}^+,\ \mathbf{s}_0, \mathbf{s}_\infty = (b); \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By (6.5), $\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,A,g;\mathbf{s}_0,\mathbf{s}_\infty}() = 0$ unless $a = 0$ and $g = 0, 1$. Since all elements of $\overline{\mathcal{M}}^{S_0,S_\infty}_{g,0;\mathbf{s}_0,\mathbf{s}_\infty}(\mathbb{F}_n, bf)$ are maps to a fiber, $\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,A,g;\mathbf{s}_0,\mathbf{s}_\infty}()$ lies in the image of the homomorphism

$$H_*(\Delta) \longrightarrow H_*(S_0^{\ell(\mathbf{s}_0)}) \otimes H_*(S_\infty^{\ell(\mathbf{s}_\infty)}), \qquad \text{where} \quad \Delta = \big\{ (p, \ldots, p) \in S_0^{\ell(\mathbf{s}_0)} \times S_\infty^{\ell(\mathbf{s}_\infty)} \big\},$$

induced by the inclusion. Since $\dim_{\mathbb{C}}\Delta=1$, (6.4) then implies that $\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,bf,g;\mathbf{s}_0,\mathbf{s}_\infty}()=0$ unless $g=0$ and $\ell(\mathbf{s}_0),\ell(\mathbf{s}_\infty)=1$. In the case $\mathbf{s}_0,\mathbf{s}_\infty=(b)$, for every element $(p,p)\in\Delta\subset S_0\times S_\infty$, there is a unique element $[u,y_1,y_2]$ of $\overline{\mathcal{M}}^{S_0,S_\infty}_{0,0;\mathbf{s}_0,\mathbf{s}_\infty}(\mathbb{F}_n,bf)$ such that $u(y_1)=p\in S_0$ and $u(y_2)=p\in S_\infty$; this is the map $z\longrightarrow z^b$ onto the fiber of $\mathbb{F}_n\longrightarrow\mathbb{P}^1$ over p. Since the order of the automorphism group of this map is b, we conclude that

$$\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,bf,g;\mathbf{s}_0,\mathbf{s}_\infty}()=\frac{1}{b}\Delta=\frac{1}{b}(S_0\otimes 1+1\otimes S_\infty)\in H_2(S_0\times S_\infty),$$

by the Kunneth decomposition of the diagonal. □

Lemma 6.8 ([IP5, Lemma 14.7])**.** *The relative degree A GW-invariants of $(\mathbb{F}_n,S_0\cup S_\infty)$ with one point insertion from $\mathbb{F}_n$ and no other constraints from $\mathbb{F}_n$ or $\overline{\mathcal{M}}$ are given by*

$$\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,A,g;\mathbf{s}_0,\mathbf{s}_\infty}(p)=\begin{cases}1, & \text{if } g=0,\ A=bf,\ b\in\mathbb{Z}^+,\ \mathbf{s}_0,\mathbf{s}_\infty=(b);\\ S_0^{\ell(\mathbf{s}_0)}\times S_\infty^{\ell(\mathbf{s}_\infty)}, & \text{if } g=0,\ A=s_0+bf,\ b\in\mathbb{Z}^{\ge 0},\ \deg\mathbf{s}_0=n+b,\ \deg\mathbf{s}_\infty=b;\\ 0, & \text{otherwise.}\end{cases}$$

Proof. By (6.5), $\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,A,g;\mathbf{s}_0,\mathbf{s}_\infty}(p)=0$ unless either $a=0$ and $g=0,1,2$ or $a=1$ and $g=0$.

In the first case, all elements of $\overline{\mathcal{M}}^{S_0,S_\infty}_{g,1;\mathbf{s}_0,\mathbf{s}_\infty}(\mathbb{F}_n,bf)$ are maps to a fiber and $\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,A,g;\mathbf{s}_0,\mathbf{s}_\infty}(p)$ lies in the image of the homomorphism

$$H_*\big(q^{\ell(\mathbf{s}_0)}\times q^{\ell(\mathbf{s}_\infty)}\big)\longrightarrow H_*(S_0^{\ell(\mathbf{s}_0)})\otimes H_*(S_\infty^{\ell(\mathbf{s}_\infty)}),$$

where $q=\pi(p)\in\mathbb{P}^1$. Thus, by (6.4), $\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,bf,g;\mathbf{s}_0,\mathbf{s}_\infty}(p)=0$ unless either $b=0$ and $g=2$ or $g=0$ and $\mathbf{s}_0,\mathbf{s}_\infty=(b)$; otherwise, this class would not be zero-dimensional. In the $g=2,\ b,\ell(\mathbf{s}_0),\ell(\mathbf{s}_\infty)=0$ subcase,

$$\big\{[u,x_1]\in\overline{\mathcal{M}}^{S_0,S_\infty}_{g,1;\mathbf{s}_0,\mathbf{s}_\infty}(\mathbb{F}_n,bf):\ u(x_1)=p\big\}\approx\overline{\mathcal{M}}_{2,1},$$

while the restriction of the obstruction bundle to this subspace is isomorphic to $\mathbb{E}_2^*\otimes T_p\mathbb{F}_n$, where $\mathbb{E}_2\longrightarrow\overline{\mathcal{M}}_{2,1}$ is the Hodge bundle. Since $\mathbb{E}_2$ is the pull-back of the Hodge bundle over $\overline{\mathcal{M}}_{2,0}$ by the forgetful map,

$$\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,0,2;,}(p)=\langle c_2(\mathbb{E}_2^*\otimes T_p\mathbb{F}_n),\overline{\mathcal{M}}_{2,1}\rangle=\langle c_2(\mathbb{E}_2)^2,\overline{\mathcal{M}}_{2,1}\rangle=0.$$

In the $g=0$, $\mathbf{s}_0,\mathbf{s}_\infty=(b)$ subcase, there is a unique element $[u,x_1,y_1,y_2]$ of $\overline{\mathcal{M}}^{S_0,S_\infty}_{0,1;\mathbf{s}_0,\mathbf{s}_\infty}(\mathbb{F}_n,bf)$ such that $u(x_1)=p$; this is the map $z\longrightarrow z^b$ onto the fiber of $\mathbb{F}_n\longrightarrow\mathbb{P}^1$ containing p. Unlike the case considered in the proof of the previous lemma, this element is automorphism free, due to the three marked points on its domain; so the corresponding GW-invariant is 1.

In the case $g=0$, $A=s+bf$ with $b\ge 0$, $\deg\mathbf{s}_0=b+n$, and $\deg\mathbf{s}_\infty=b$,

$$\dim_{\mathbb{C}}\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,A,g;\mathbf{s}_0,\mathbf{s}_\infty}(p)=\ell(\mathbf{s}_0)+\ell(\mathbf{s}_\infty)$$

by (6.4) and thus $\mathrm{GW}^{S_0,S_\infty}_{\mathbb{F}_n,A,g;\mathbf{s}_0,\mathbf{s}_\infty}(p)$ is a multiple of the fundamental class of $S_0^{\ell(\mathbf{s}_0)}\times S_\infty^{\ell(\mathbf{s}_\infty)}$. This multiple is 1 because b points on S_∞ determine poles of a section of $\mathcal{O}(n)\longrightarrow\mathbb{P}^1$ and $b+n$ points on S_0 determine the unique section of $\mathcal{O}(n)$ with these poles that passes through p. □

Remark 6.9. We denote the divisors $S,E\subset\mathbb{F}_n$ of [IP5, Sections 14.3,15.1] by S_0,S_∞ in order to avoid confusion with the rational elliptic surface of [IP5, Sections 14.4,15.3], which is also denoted by E. The conclusion in the proof of [IP5, Lemma 14.6] about the S-matrix does not follow from the rest of the argument, since it may have contributions from higher genus and classes coming from $\overline{\mathcal{M}}$. The proof of [IP5, Lemma 14.7] ignores the possibility of $b=0$ considered above. In the second case considered in this proof, the dimension of the moduli space is $\ell(s)+\ell(s')$ after cutting down by the point constraint. An irreducible curve representing $S+bF$ is genus 0 and embedded, because its projection to S is of degree 1. The above argument gives a simpler reason why the multiple is 1. In the statement of [IP5, Lemma 14.7], the degree conditions on $\mathbf{s}$ and $\mathbf{s}'$ are reversed (and are implied by the notation). Other, minor typos in [LR, Sections 2,3.0] include

p1010, lines -2,-1: $X \longrightarrow \mathbb{F}_n$;

statement and proof of Lemma 14.7: $SV_{\mathbf{s}} \longrightarrow V_{\mathbf{s}}$; $SV_{\mathbf{s}'} \longrightarrow V_{\mathbf{s}'}$.

6.4 Invariants of rational elliptic surface: [IP5, Section 14.4]

This section makes some observations concerning absolute GW-invariants of E and relative GW-invariants of (E,F), where E is the blowup of $\mathbb{P}^2$ at the 9 points of the intersection of two general cubic curves and $F\approx\mathbb{T}^2$ is a fiber of the projection $E\longrightarrow\mathbb{P}^1$ corresponding to the filtration of E by the proper transforms of the cubic curves passing through the 9 points (this is the pencil of cubics spanned by the first two cubics). The fibration $E\longrightarrow\mathbb{P}^1$ has 9 sections $S_1,\ldots,S_9$ corresponding to the 9 exceptional divisors. We denote by $s_1,\ldots,s_9$ and f the homology classes of $S_1,\ldots,S_9$ and F; these classes form a basis for $H_2(E;\mathbb{Z})$. Since

$$(s_i+df)\cdot f=1,\qquad \big\langle c_1(TE),f\big\rangle=0,\qquad\text{and}\quad \big\langle c_1(TE),s_i\big\rangle=1, \tag{6.6}$$

[IP5, (1.21)] gives

$$\begin{aligned}\dim_{\mathbb{C}}\overline{\mathcal{M}}_{1,0}(E,df)&=0+(2-3)(1-1)=0,\\ \dim_{\mathbb{C}}\overline{\mathcal{M}}_{g,0}(E,s_i+df)&=\dim_{\mathbb{C}}\overline{\mathcal{M}}^F_{g,0;(1)}(E,s_i+df)=1+(2-3)(1-g)=g.\end{aligned}$$

Thus, $\mathrm{GW}_{E,df,1}()$, $\mathrm{GW}_{E,s+df,g}(p^g)$, and $\mathrm{GW}^F_{E,s+df,g;(1)}(p^g)$, where p^g denotes g absolute point constraints, are rational numbers. These invariants are independent of the choice of the complex structure on E and (E,F).

Lemma 6.10. *If* $d\in\mathbb{Z}^+$,

$$\mathrm{GW}_{E,df,1}()=\sigma(d)/d,\qquad \textit{where}\quad \sigma(d)=\sum_{k|d}k.$$

Proof. If E is obtained by blowing up $\mathbb{P}^2$ at 9 general points, there is only one degree f holomorphic curve; this is the proper transform of the unique cubic passing through the 9 points. In this case, $\overline{\mathcal{M}}_{1,0}(E,df)$ consists of the $\sigma(d)$ unbranched covers of this cubic, all of which are regular and have an automorphism of order d. This establishes the claim. □

Lemma 6.11 ([IP5, Lemma 14.7])**.** *Let* $d,g\in\mathbb{Z}^{\ge0}$*. The absolute and relative degree* s_i+df *genus* g *GW-invariants of* E *and* (E,F) *with* g *point insertions satisfy*

$$\mathrm{GW}^F_{E,s_i+df,g;(1)}(p^g)=\mathrm{GW}_{E,s_i+df,g}(p^g).$$

Proof. Let J be a generic almost complex structure on (E,F). Suppose Σ is a connected nodal genus g curve and $u\colon \Sigma\longrightarrow E$ is a degree s_i+df J-holomorphic stable map. If Σ_i is an irreducible component of Σ such that $u\colon \Sigma\longrightarrow F$ is not constant, then the genus of Σ_i is at least one and the sum of the genera of the remaining components of Σ is at most $g-1$. Therefore, if the g points are in general position, $u(\Sigma)$ does not contain all of them. It follows that all of the maps contributing to the absolute invariant with g point insertions are F-regular and thus contribute in the same way to the relative invariant. □

Remark 6.12. There is no standard notion of the term rational elliptic surface in algebraic geometry, and it is never defined in [IP5]. Based on [IP5, Sections 14.4,15,3], including the authors' references to [BL, Theorem 1.2], the understood meaning is apparently the one described above. Lemma 6.10 above is equivalent to the content at the bottom of page 1019 in [IP5]. The second equality in [IP5, Lemma 14.8] and the description of its meaning do not make sense. By the definition of relative invariants, e.g. [IP5, (1.23)] and the description of $\mathcal{H}^V_{X,A;\mathbf{s}}$ at the end of [IP4, Section 5],

$$\mathrm{GW}^F_{E,s_i+df,g,(1)}(p^g;\cdot)\in H_0(\mathcal{H}^F_{E,s_i+df;(1)};\mathbb{Q})=H_0(\mathbb{R}^2;\mathbb{Q})\approx\mathbb{Q}.$$

Thus, the point classes in $\mathcal{H}^F_{E,s_i+df;(1)}$ are all the same and are not indexed by the rim tori $R\in\mathcal{R}^F_E$; only the elements of the fibers of $\mathcal{H}^F_{E,s_i+df;(1)}\longrightarrow F$ can be indexed in this way, but not in a continuous way.

Remark 6.13. Neither of the two propositions in [IP5, Section 14.5] is used in the rest of the paper. In the proof of [IP5, Proposition 14.9], presumably $A_0\in H_2(V)$ is a preimage of $A\in H_2(X)$ under the inclusion homomorphism; so $c_1(N_XV)\cdot A$ should be $c_1(N_XV)\cdot A_0$. For the equality on the second-to-last line to hold, $\mathcal{M}_{V,A_0,g}$ should be $\mathcal{M}_{g,0}(V,A_0)$. However, for the equality on the last line to hold, with $\mathcal{M}^V_{X,A,g}$ denoting either $\mathcal{M}_{g,0}(X,A)$ or $\mathcal{M}^V_{g,0;\mathbf{s}}(X,A)$, on the previous line $\mathcal{M}_{V,A_0,g}$ should be $\mathcal{M}_{g,\ell(\mathbf{s})}(V,A_0)$ and RHS of that line should be increased by $\ell(\mathbf{s})$; then citing [IP4,(6.3)] at the top of page 1014 would make sense. In the proof of [IP5, Proposition 14.10], there is a reference to Lemma 11.3, which does not exist. The first γ in the displayed expression should be $\pi_*\gamma$ to collect the constraints from V_∞. While constraints from $\overline{\mathcal{M}}_{0,*}$ are not explicitly considered in the proof, the argument applies to them as well.

6.5 Enumeration of plane curves: [IP5, Section 15.1]

This section deduces the Caporaso-Harris formula enumerating curves in $\mathbb{P}^2$, [CH, Theorem 1.1], from the symplectic sum formula. Fix a line $L\subset\mathbb{P}^2$. For tuples

$$\alpha\equiv(\alpha_1,\alpha_2,\ldots),\beta\equiv(\beta_1,\beta_2,\ldots)\in(\mathbb{Z}^{\ge0})^{\mathbb{Z}^+}$$

with finitely many nonzero entries, let

$$|\alpha|=\alpha_1+\alpha_2+\ldots,\quad \alpha!=\alpha_1!\cdot\alpha_2!\cdot\ldots,\quad I\alpha=\alpha_1+2\alpha_2+\ldots,\quad I^\alpha=1^{\alpha_1}2^{\alpha_2}\ldots,$$
$$\binom{\alpha}{\beta}=\binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\ldots,\quad \mathbf{C}_{\alpha;\beta}=C_1(p)^{\alpha_1}C_2(p)^{\alpha_2}\ldots C_1(L)^{\beta_1}C_2(L)^{\beta_2}\ldots\in\mathrm{Sym}^*\big(\mathbb{N}\times H^*(L)\big),$$

where $p,L\in H^*(L)$ denote the Poincare duals of a point in L and of the fundamental class of L. For each $k\in\mathbb{Z}^+$, let $\varepsilon_k\in(\mathbb{Z}^{\ge0})^{\mathbb{Z}^+}$ be the tuple with the k-th coordinate equal to 1 and the remaining coordinates equal to 0.

Given $d\in\mathbb{Z}^+$, $\delta\in\mathbb{Z}^{\ge0}$, and $\alpha,\beta\in(\mathbb{Z}^{\ge0})^{\mathbb{Z}^+}$ such that $I\alpha+I\beta=d$, let $N^{d,\delta}(\alpha,\beta)$ denote the number of degree d curves in $\mathbb{P}^2$ that have δ nodes, have contact of order k with L at α_k fixed points and β_k arbitrary points for each $k=1,2,\ldots$, and pass through

$$r=\frac{d(d+1)}{2}-\delta+|\beta|$$

general points in $\mathbb{P}^2$. Thus,

$$\begin{aligned}\beta!N^{d,\delta}(\alpha,\beta)&=\mathrm{GT}^L_{\mathbb{P}^2,dL,\chi_\delta(d)}(p^r;\mathbf{C}_{\alpha;\beta}),\\ &\equiv \mathrm{GT}^L_{\mathbb{P}^2,dL,\chi_\delta(d);\mathbf{s}_\alpha,\mathbf{s}_\beta}\big(p^r;\underbrace{p,\ldots,p}_{|\alpha|},\underbrace{L,\ldots,L}_{|\beta|}\big)\end{aligned}\tag{6.7}$$

where $\chi_\delta(d)=2\delta-d(d–3)$ is the geometric euler characteristic of the curves (the euler characteristic of the normalization). Since a degree d curve in $\mathbb{P}^2$ can have at most $d(d–1)/2$ nodes, the number r above is positive whenever $N^{d,\delta}(\alpha,\beta)\neq0$. This numbers r is at least 2 if $N^{d,\delta}(\alpha,\beta)\neq0$ and $(d,\alpha)\neq(1,\mathbf{0})$.

Corollary 6.14 ([CH, Theorem 1.1])**.** *Let $d\in\mathbb{Z}^+$, $\delta\in\mathbb{Z}^{\ge0}$, and $\alpha,\beta\in(\mathbb{Z}^{\ge0})^{\mathbb{Z}^+}$ with $(d,\alpha)\neq(1,\mathbf{0})$. If $I\alpha+I\beta=d$,*

$$\begin{aligned}N^{d,\delta}(\alpha,\beta)=&\sum_{\substack{k\in\mathbb{Z}^+\\ \beta_k>0}}kN^{d,\delta}(\alpha+\varepsilon_k,\beta-\varepsilon_k)\\ &+\sum_{\substack{\delta'\in\mathbb{Z}^{\ge0},\,\alpha',\beta'\in(\mathbb{Z}^{\ge0})^{\mathbb{Z}^+}\\ I\alpha'+I\beta'=d-1\\ \delta-\delta'+|\beta'-\beta|=d-1}}\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}I^{\beta'-\beta}N^{d-1,\delta'}(\alpha',\beta').\end{aligned}$$

As sketched in [IP5, Section 15.1], this formula can be proved by applying the natural extension of the symplectic sum formula (1.12) to the decomposition

$$(\mathbb{P}^2,L)=(\mathbb{P}^2,L)\underset{L=S_\infty}{\#}(\mathbb{F}_1,S_\infty,S_0),$$

with $(\mathbb{F}_1,S_\infty,S_0)$ as in Section 6.3, and moving one of the r absolute point insertions to the $\mathbb{F}_1$ side. Since the divisor $L=S_\infty$ is simply connected, the connect sum

$$\#\colon H_2(\mathbb{P}^2;\mathbb{Z})\underset{L=S_\infty}{\times}H_2(\mathbb{F}_1;\mathbb{Z})\longrightarrow H_2(\mathbb{P}^2;\mathbb{Z})$$

is well-defined in this case. Since $dL\cdot L=(aS_0+bF)\cdot S_\infty$ if and only if $d=b$ and

$$dL\#(aS_0+dF)=(d+a)L,$$

the symplectic sum formula (1.12) and (6.7) give

$$\begin{aligned}&\beta!N^{d,\delta}(\alpha,\beta)\\ &=\sum_{\substack{d'\in\mathbb{Z}^+,d''\in\mathbb{Z}^{\ge0}\\ d'+d''=d}}\ \sum_{\substack{\delta'\in\mathbb{Z}^{\ge0},\chi''\in\mathbb{Z},\alpha',\beta'\in(\mathbb{Z}^{\ge0})^{\mathbb{Z}^+}\\ I\alpha'+I\beta'=d'\\ \chi_{\delta'}(d')+\chi''=\chi_\delta(d)+2|\alpha'|+2|\beta'|\\ \frac{d'(d'+1)}{2}-\delta'+|\beta'|=r-1}}\frac{I^{\alpha'}I^{\beta'}}{\alpha'!}N^{d',\delta'}(\alpha',\beta')\cdot\mathrm{GT}^{S_\infty,S_0}_{\mathbb{F}_1,d''S_0+d'F,\chi''}(p;\mathbf{C}_{\beta';\alpha'},\mathbf{C}_{\alpha;\beta}),\end{aligned}\tag{6.8}$$

with the GT-invariant defined analogously to (6.7) for each component of the relative divisor. By Lemmas 6.7 and 6.8, there are two types of configurations that contribute to the GT-invariant in (6.8):

(1) genus 0 multiple covers of fibers, each with a single point of contact with S_∞ and a single point of contact with S_0, with one of these fiber maps passing through the constraint point in $\mathbb{F}_1$;

(2) genus 0 multiple covers of fibers, each with a single point of contact with S_∞ and a single point of contact with S_0, and one genus 0 degree $S_0+d'L$ map passing through the constraint point in $\mathbb{F}_1$.

By Lemma 6.7, a genus 0 multiple cover of a fiber not passing through the constraint point passes through either a fixed point on S_0 (i.e. a point with contact specified by α) and an arbitrary point on S_∞ (i.e. a point with contact encoded by α') or an arbitrary point on S_0 (i.e. a point with contact encoded by β) and a fixed point on S_∞ (i.e. a point with contact specified by β'). The orders of contact on the two ends are the same number k, which is the degree of the cover. Such a cover contributes a factor of $1/k$ to the GT-invariant in (6.8).

In the first case above, $d'=d$, $\delta'=\delta$, and $\alpha'=\alpha+\varepsilon_k$ and $\beta'=\beta-\varepsilon_k$ for some $k\in\mathbb{Z}^+$, as both relative conditions on the distinguished fiber map into $\mathbb{F}_1$ must be single arbitrary points by the first statement in Lemma 6.8. For each $k\in\mathbb{Z}^+$ with $\beta_k>0$, there are

(a) β_k choices for the relative marked point on the S_0 end of the distinguished fiber map into $\mathbb{F}_1$ and α'_k choices on the S_∞ end of this map,

(b) $\beta'!=\beta!/\beta_k$ choices of ordering the "arbitrary" points on the S_0 end of the other fiber maps (the ordering on the S_∞ end of these maps can be fixed by the fixed points; along with (a), this contributes a factor of $\beta!$ to the right-hand side of (6.8)),

(c) $\alpha!=\alpha'!/\alpha'_k$ choices of ordering the "arbitrary" points on the S_∞ end of the other fiber maps (the ordering on the S_0 end of these maps can be fixed by the fixed points; along with (a), this eliminates $1/\alpha'!$ from the right-hand side of (6.8)).

Furthermore, the non-distinguished fiber maps in a single configuration contribute

$$\frac{1}{I^\alpha I^{\beta'}}=\frac{k}{I^{\alpha'}I^{\beta'}}$$

to the invariant. Thus, Case 1 contributes $\beta!\cdot kN^{d,\delta}(\alpha+\varepsilon_k,\beta-\varepsilon_k)$ to the right-hand side of (6.8).

In the second case above, $d'=d-1$, $\alpha'=\alpha-\alpha_0$ for some $\alpha_0\in(\mathbb{Z}^{\geq 0})^{\mathbb{Z}^+}$, and $\beta=\beta'-\beta'_0$ for some $\beta'_0\in(\mathbb{Z}^{\geq 0})^{\mathbb{Z}^+}$, as both relative conditions on the distinguished map into $\mathbb{F}_1$ must be fixed points by the second statement in Lemma 6.8. For each pair (α_0,β'_0), there are

(a) $\binom{\alpha}{\alpha_0}$ choices of fixed points on the S_0 end of $\mathbb{F}_1$ and $\binom{\beta'}{\beta'_0}$ choices of fixed points on the S_∞ end of $\mathbb{F}_1$ (these go on the non-fiber curve),

(b) $\beta!=(\beta'-\beta'_0)!$ choices of ordering the "arbitrary" points on the S_0 end of the fiber maps (the ordering on the S_∞ end of these maps can be fixed by the fixed points; this contributes a factor of $\beta!$ to the right-hand side of (6.8)),

(c) $\alpha'! = (\alpha - \alpha_0)!$ choices of ordering the "arbitrary" points on the S_∞ end of the fiber maps (the ordering on the S_0 end of these maps can be fixed by the fixed points; this eliminates $1/\alpha'!$ from the formula).

Furthermore, the fiber maps in a single configuration contribute

$$\frac{1}{I^{\alpha'} I^{\beta}} = \frac{I^{\beta'-\beta}}{I^{\alpha'} I^{\beta'}}$$

to the invariant. Thus, Case 2 contributes $\beta! \cdot \binom{\alpha}{\alpha'}\binom{\beta'}{\beta} I^{\beta'-\beta} N^{d-1,\delta'}(\alpha', \beta')$ to the right-hand side of (6.8). This establishes Corollary 6.14.

Remark 6.15. Throughout [IP5, Section 15.1], $\mathbb{P}$ denotes the surface $\mathbb{F}_1$ of [IP5, Section 14.3]. The first identity on page 1015 cannot possibly be true, since $\mathrm{GT}^L_{\mathbb{P}^2,dL,\chi}$, however its definition is interpreted, groups the relative constraints of the same type together and treats the resulting sets in the same way, while $N^{d,\delta}(\alpha, \beta)$ treats the α and β constraints differently (the α-contacts are fixed and so the corresponding contact points of the domain can be ordered). The definition in the second displayed expression and the symplectic sum formula in the third displayed expression have the same issue. The former is unnecessary, since the symplectic sum formula involves GW/GT-invariants and these are also the numbers computed in [IP5, Lemma 14.7]. Finally:

p1014, -3: $\mathbb{P} \longrightarrow \mathbb{P}^2$;
p1015, lines 3,18,19,21,29; p1016, line 38: $\mathbb{P} \longrightarrow \mathbb{P}_1$;
p1015, line 10: $\gamma^1 \longrightarrow \gamma_1$;
line 12: $m!$ and $|m|$ correspond to $\alpha!\beta!$ and $|\alpha| \cdot |\beta|$; $\prod_i \alpha_i \longrightarrow \prod_i \alpha_i!$;
line 14: $\mathrm{GT}^L_{\chi,dL,\mathbb{P}^2} \longrightarrow \mathrm{GT}^L_{\mathbb{P}^2,dL,\chi}$; $C_m \longrightarrow \mathbf{C}_m$;
line 15: χ is geometric euler characteristic;
line 18: $\mathrm{GT}^{E,L}_{\chi,aL+bF,\mathbb{P}} \longrightarrow \mathrm{GT}^{E,L}_{\mathbb{P},aL+bF,\chi}$; $(C_m; p; C_{m'}) \longrightarrow (\mathbf{C}_{m'}; p; \mathbf{C}_m)$
line 23: the S-matrix is the identity;
bottom: $\alpha' = \alpha + \varepsilon_k$, $\beta' = \beta - \varepsilon_k$, $\chi' = \chi$;
p1016, lines 7-9: it is unclear what this sentence is saying;
line 11: $N^{d,\delta'}(\alpha - \varepsilon_k, \beta + \varepsilon_k) \longrightarrow N^{d,\delta}(\alpha + \varepsilon_k, \beta - \varepsilon_k)$.

6.6 Hurwitz numbers: [IP5, Section 15.2]

This section deduces a cut and paste formula for branched covers of $\mathbb{P}^1$, [GJV, Lemma 3.1], from the natural extension of the symplectic sum formula (1.12) to relative invariants. Fix a point $p \in \mathbb{P}^1$. For tuples

$$\alpha \equiv (\alpha_1, \alpha_2, \ldots), \alpha' \equiv (\alpha'_1, \alpha'_2, \ldots) \in (\mathbb{Z}^{\ge 0})^{\mathbb{Z}^+}$$

with finitely many nonzero entries, let

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots, \quad \alpha! = \alpha_1! \cdot \alpha_2! \cdot \ldots, \quad I\alpha = \alpha_1 + 2\alpha_2 + \ldots, \quad I^\alpha = 1^{\alpha_1} 2^{\alpha_2} \ldots,$$
$$\binom{\alpha}{\alpha'} = \binom{\alpha_1}{\alpha'_1}\binom{\alpha_2}{\alpha'_2} \ldots, \quad \mathbf{s}_\alpha = (\underbrace{1, \ldots, 1}_{\alpha_1}, \underbrace{2, \ldots, 2}_{\alpha_2}, \ldots) \in (\mathbb{Z}^+)^{|\alpha|}.$$

For each $k \in \mathbb{Z}^+$, let $\varepsilon_k \in (\mathbb{Z}^{\ge 0})^{\mathbb{Z}^+}$ be the tuple with the k-th coordinate equal to 1 and the remaining coordinates equal to 0.

Given $d\in\mathbb{Z}^+$, $g\in\mathbb{Z}^{\ge0}$, and $\alpha\in(\mathbb{Z}^{\ge0})^{\mathbb{Z}^+}$ such that $I\alpha=d$, let $N_{d,g}(\alpha)$ denote the number of genus g degree d branched covers of $\mathbb{P}^1$ with α_k branch points of order k over p for each $k\in\mathbb{Z}^+$ and simple branching over

$$r = d+|\alpha|+2g-2$$

other fixed points $p_1,\ldots,p_r$ in $\mathbb{P}^1$. Thus,

$$N_{d,g}(\alpha)=\frac{1}{\alpha!\,(d-2)!^r}\,\deg \mathrm{GW}^{V_r}_{\mathbb{P}^1,d,g;\mathbf{s}_1^r,\mathbf{s}_\alpha}()\,, \tag{6.9}$$

where $V_r=\{p_1,...,p_r,p\}$ and $\mathbf{s}_1^r$ denotes r copies of the tuple $\mathbf{s}_1$ defined in (6.2). Since $d,|\alpha|\ge1$ and $g\ge0$, the number r above is positive unless $(d,g,\alpha)=(1,0,(1))$; in this exceptional case, $N_{d,g}(\alpha)=1$.

Corollary 6.16 ([GJV, Lemma 3.1])**.** *The generating function*

$$F(\lambda,u,z_1,z_2,\ldots)=\sum_{g=0}^{\infty}\sum_{d=1}^{\infty}\sum_{\substack{\alpha\in(\mathbb{Z}^{\ge0})^\infty\\ I\alpha=d}}N_{d,g}(\alpha)\bigg(\prod_{k=1}^{\infty}z_k^{\alpha_k}\bigg)\frac{u^{d+|\alpha|+2g-2}}{(d+|\alpha|+2g-2)!}\lambda^{2g-2} \tag{6.10}$$

satisfies the PDE

$$\partial_u F=\frac{1}{2}\sum_{i,j\ge1}\big(ijz_{i+j}\lambda^2[\partial_{z_i}\partial_{z_j}F+\partial_{z_i}F\cdot\partial_{z_j}F]+(i+j)z_iz_j\partial_{z_{i+j}}F\big). \tag{6.11}$$

As sketched in [IP5, Section 15.2], this statement can be proved by applying the symplectic sum formula to the decomposition

$$(\mathbb{P}^1,p_1,\ldots,p_r,p)=(\mathbb{P}^1,p_1,\ldots,p_{r-1},x)\underset{x=y}{\#}(\mathbb{P}^1,y,p_r,p),$$

i.e. by separating off the distinguished branch point and one of the simple branch points onto a second copy of $\mathbb{P}^1$. In this case, the connect sum is well-defined on H_2 and is given by

$$\#\colon H_2(\mathbb{P}^1;\mathbb{Z})\underset{x=y}{\times}H_2(\mathbb{P}^1;\mathbb{Z})=\big\{(d\mathbb{P}^1,d\mathbb{P}^1)\colon\ d\in\mathbb{Z}\big\}\longrightarrow H_2(\mathbb{P}^1;\mathbb{Z}),$$
$$(d\mathbb{P}^1,d\mathbb{P}^1)\longrightarrow d\mathbb{P}^1\,.$$

Thus, the symplectic sum formula and (6.9) give

$$\begin{aligned}&\alpha!\,(d-2)!^rN_{d,g}(\alpha)\\ &=\sum_{\substack{\Gamma=(\Gamma_1,\Gamma_2)\\ g(\Gamma)=g+|\mathrm{V}_\Gamma|-|\mathrm{E}_\Gamma|-1}}\ \sum_{\substack{\alpha'\in(\mathbb{Z}^{\ge0})^{\mathbb{Z}^+}\\ |\alpha'|=|\mathrm{E}_\Gamma|,\ I\alpha'=d}}\frac{I^{\alpha'}}{\alpha'!}\Big(\deg \mathrm{GT}^{V_{r-1}}_{\mathbb{P}^1,\Gamma';\mathbf{s}_1^{r-1},\mathbf{s}_{\alpha'}}\Big)\Big(\deg \mathrm{GT}^{y,p_r,p}_{\mathbb{P}^1,\Gamma'';\mathbf{s}_{\alpha'},\mathbf{s}_1,\mathbf{s}_\alpha}\Big),\end{aligned} \tag{6.12}$$

with the outer sum taken over all bipartite connected graphs $\Gamma=(\Gamma',\Gamma'')$ with vertices V_Γ decorated by nonnegative integers, as in Figure 4; we denote the sum of these numbers by $g(\Gamma)$. Each vertex of Γ' (resp. Γ'') corresponds to a map from a connected curve of the genus given by the vertex label into the first (resp. second) $\mathbb{P}^1$. Each edge in Γ represents paired relative marked points of the domains mapped into the two copies of $\mathbb{P}^1$. By Lemmas 6.1 and 6.2, there are two types of configurations that contribute to the right-hand side of (6.12):

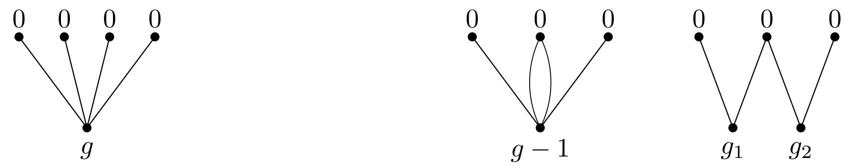

Figure 4: Graph types Γ contributing to the right-hand side of (6.12).

(1) genus 0 branched covers of the second $\mathbb{P}^1$, each with a single preimage of y and a single preimage of p, and a genus 0 branched cover of the second $\mathbb{P}^1$ with a single preimage of y, two preimages of p, and a simple branching over p_r;

(2) genus 0 branched covers of the second $\mathbb{P}^1$, each with a single preimage of y and a single preimage of p, and a genus 0 branched cover of the second $\mathbb{P}^1$ with a single preimage of p, two preimages of y, and a simple branching over p_r.

By Lemma 6.1, a degree k branched cover of the second $\mathbb{P}^1$ without the branching condition over p_r contributes a factor of $1/k$ to the last GT-invariant in (6.12). Such a cover has contact of order k with p (i.e. a point with contact specified by α) and y (i.e. a point with contact encoded by α'). For all curve types Γ, there are $(d-2)!$ choices of ordering the non-branched preimages of each of the r simple branch points, which together contribute a factor of $(d–1)!^r$ to the right-hand side of (6.12).

In the first case above, Γ' consists of a single vertex with label g and

$$\alpha' = \alpha - \varepsilon_i - \varepsilon_j + \varepsilon_{i+j}$$

for some $i, j \in \mathbb{Z}^+$, as there are two contact conditions on the p end of a branched cover of the second $\mathbb{P}^1$ (corresponding to α) and only one on the y end (corresponding to α'). Whenever $i \neq j$ and $\alpha_i, \alpha_j > 0$, there are

(a) $\alpha_i\alpha_j$ choices for the relative marked points on the p end of the distinguished map into the second $\mathbb{P}^1$,

(b) $\alpha'!$ choices for ordering the points on the y end of the second $\mathbb{P}^1$ for a given ordering of the points on the p end (this eliminates $1/\alpha'!$ from the formula).

Furthermore, the non-distinguished fiber maps in a single configuration contribute

$$\frac{1}{I^\alpha/(ij)} = \frac{1}{I^{\alpha'}/(i+j)}.$$

Thus, the contribution from this case is

$$\alpha_i\alpha_j \cdot (i+j) \cdot \alpha'! N_{g,d}(\alpha') = \alpha! \cdot (i+j)(\alpha_{i+j}+1) N_{g,d}(\alpha').$$

In the $i=j$ case, there are $\alpha_i(\alpha_i-1)/2$ choices in (a) above and the same number of choices in (b). So, the contribution now is

$$\frac{\alpha_i(\alpha_i-1)}{2} \cdot (i+i) \cdot \alpha'! N_{g,d}(\alpha') = \alpha! \cdot \frac{1}{2}(i+j)(\alpha_{i+j}+1) N_{g,d}(\alpha').$$

Both cases correspond to the last term in (6.11), since α' is obtained from α by reducing α_i and α_j and increasing α_{i+j} by 1 (thus, $N_{g,d}(\alpha')$ is the coefficient of the product of $z_1,\ldots$ with one smaller power of z_i and z_j and one larger power of z_{i+j}; the factor of $\alpha_{i+j}{+}1$ above corresponds to differentiating $z_{i+j}^{\alpha_{i+j}+1}$).

In the second case above, Γ' consists either of a single vertex with label $g{-}1$ or two vertices with labels adding up to g and

$$\alpha' = \alpha + \varepsilon_i + \varepsilon_j - \varepsilon_{i+j}$$

for some $i,j\in\mathbb{Z}^+$, as there is one contact condition on the p end of a branched cover of the second $\mathbb{P}^1$ (corresponding to α) and two on the y end (corresponding to α'). Whenever $i\neq j$ and $\alpha'_i,\alpha'_j>0$, there are

(a) $\alpha'_i\alpha'_j$ choices for the relative marked points on the y end of the distinguished map into the second $\mathbb{P}^1$,

(b) $\alpha!$ choices for ordering the points on the p end of the second $\mathbb{P}^1$ for a given ordering of the points on the y end (this contributes a factor of $\alpha!$ to the right-hand side of (6.12)).

Furthermore, the non-distinguished fiber maps in a single configuration contribute

$$\frac{1}{I^\alpha/(i+j)} = \frac{1}{I^{\alpha'}/(ij)}.$$

Thus, the contribution from this case is

$$\alpha!\cdot\alpha'_i\alpha'_j\cdot\frac{ij}{\alpha'!}\cdot\alpha'!N'_{g,d}(\alpha') = \alpha!\cdot ij(\alpha_i{+}1)(\alpha_j{+}1)N'_{g,d}(\alpha'), \tag{6.13}$$

where $N'_{g,d}(\alpha')$ denotes the sum of the numbers from the two possible configurations into the first $\mathbb{P}^1$, divided by $(d{-}2)!^{r-1}$ and $\alpha'!$. In the $i{=}j$ case, there are $\alpha'_i(\alpha'_i{-}1)/2$ choices in (a) above and the same number of choices in (b). So, the contribution now is

$$\alpha!\cdot\frac{1}{2}\alpha'_i(\alpha'_i{-}1)\frac{ii}{\alpha'!}\cdot\alpha'!N'_{g,d}(\alpha') = \alpha!\cdot\frac{1}{2}ij(\alpha_i{+}2)(\alpha_i{+}1)N'_{g,d}(\alpha'). \tag{6.14}$$

The connected configuration into the first $\mathbb{P}^1$ contributes $N_{g-1,d}(\alpha')$ to the number $N'_{g,d}(\alpha')$. Combined with the factors (6.13) and (6.14), this corresponds to the first term on the right-hand side of (6.11), since α' is obtained from α by increasing α_i and α_j and reducing α_{i+j} by 1 (thus, $N_{g-1,d}(\alpha')$ is the coefficient of the product of $z_1,\ldots$ with one larger power of z_i and z_j and one smaller power of z_{i+j} and λ^2).

Finally, the contribution of the two-component configuration to $N'_{g,d}(\alpha')$ is

$$\begin{aligned}
&\frac{1}{\alpha'!}\sum_{r_1+r_2=r-1}\sum_{g_1+g_2=g}\sum_{\substack{\alpha'_1+\alpha'_2=\alpha'\\ \alpha'_{1;i},\alpha'_{2;j}>0}}\binom{r-1}{r_1}\binom{\alpha'-\varepsilon_i-\varepsilon_j}{\alpha'_1-\varepsilon_i}\alpha'_1!N_{d_1,g_1}(\alpha'_1)\alpha'_2!N_{d_2,g_2}(\alpha'_2)\\
&\quad = (r{-}1)!\frac{(\alpha'-\varepsilon_i-\varepsilon_j)!}{\alpha'!}\sum_{r_1+r_2=r-1}\sum_{g_1+g_2=g}\sum_{\alpha'_1+\alpha'_2=\alpha'}\frac{\alpha'_{1;i}N_{d_1,g_1}(\alpha'_1)}{r_1!}\frac{\alpha'_{2;j}N_{d_2,g_2}(\alpha'_2)}{r_2!},
\end{aligned}$$

where d_i is determined by g_i, α'_i, and r_i. Combined with the factors (6.13) and (6.14) and summed over ordered pairs (i,j), this contributes

$$\alpha!\cdot(r-1)!\cdot\frac{1}{2}\sum_{i,j}ij\sum_{r_1+r_2=r-1}\sum_{g_1+g_2=g}\sum_{\alpha'_1+\alpha'_2=\alpha'}\frac{\alpha'_{1;i}N_{d_1,g_1}(\alpha'_1)}{r_1!}\frac{\alpha'_{2;j}N_{d_2,g_2}(\alpha'_2)}{r_2!}$$

to the right-hand side of (6.12). This corresponds to the middle term on the right-hand side of (6.11) (the factorials in the above expression precisely correspond to $u^r/r!$ in the definition of F).

Remark 6.17. Our notation in this section differs from that of [IP5, Section 15.2] and [GJV]. The k-th component of our tuple α is the number of entries in the tuple α of [IP5, Section 15.2] and [GJV] that equal k. Thus, our usage of α is consistent with Section 6.5, which is essentially [IP5, Section 15.1], while the tuples α of [IP5, Section 15.2] and [GJV] are denoted by $\mathbf{s}$ in the rest of [IP5]. Similarly to the situation with [IP5, Lemma 14.2], $\mathrm{GW}^{p}_{\mathbb{P}^1,g,d}(b^r;\mathbf{C}_m)$ is not defined in [IP5, Section 15.2]; its intended meaning is inconsistent with the notation used in the rest of the paper. The generating function on line 3 on page 1017 in [IP5] is not [IP5, (A.6)]. More significantly, it is also not the generating function of [GJV, (3.1)]. Dropping t^d from this definition is not material, since t does not appear in the PDE for F and d is encoded by m, but dropping $m_a!$ is material; otherwise, F would not satisfy the PDE. It is not immediately clear whether the sum in [GJV, (3.1)] is over ordered or unordered partitions α of $n=d$ (neither of which would correspond to the generating function in [IP5]), but summing over the unordered partitions α (which is the same as summing over our tuples α) gives a solution of the desired PDE. As stated, [IP5, (15.3)] is incorrect, since the sum is only over certain configurations of curves (as explained after this formula). The last term in [IP5, (15.3)] is not even defined in the paper (though its meaning could be guessed); it is also unnecessary, since the symplectic sum formula involves GW/GT-invariants and these are also the numbers computed in [IP5, Lemmas 14.1,14.2]. Other, fairly minor misstatements in [LR, Section 6] include

p1016, Section 15.2, line 17,20; p1017, line 3,7: $C_m \longrightarrow \mathbf{C}_m$;
p1016, line -1: there should be only one -2 in this formula;
p1017, line 6: the S-matrix is the identity;
line 10: it is not clear what $\mathrm{GT}=\exp\mathrm{GW}$ means here or why it is relevant;
2.: $-\chi_1=2g-4$, $g_1+g_2=g$, $d_1+d_2=d$.

6.7 Curves on the rational elliptic surface: [IP5, Section 15.3]

This section deduces a formula enumerating curves on the rational elliptic surface E of Section 6.4, [BL, Theorem 1.2], from a natural extension of the symplectic sum formula (1.12). Continuing with the notation of Section 6.4, for each $g\in\mathbb{Z}^{\geq 0}$, let

$$\mathcal{F}_g(q)=\sum_{d=0}^{\infty}\mathrm{GW}_{E,s_i+df,g}(p^g)q^d. \tag{6.15}$$

Since $s_i^2=-1$, there is only one holomorphic curve in the homology s_i and thus

$$\mathcal{F}_0(q)\in 1+q\mathbb{Q}[[q]],\qquad \mathcal{F}_g(q)\in q\mathbb{Q}[[q]]\ \ \forall\, g\in\mathbb{Z}^{+}. \tag{6.16}$$

Corollary 6.18 ([BL, Theorem 1.1])**.** *For every $g\in\mathbb{Z}^{\geq 0}$,*

$$\mathcal{F}_g(q)=\bigg(\prod_{d=1}^{\infty}(1-q^d)\bigg)^{-12}\big(qG'(q)\big)^g\,, \tag{6.17}$$

with $G(q)$ given by (6.3).

For any symplectic manifold X, we denote by

$$\psi_1\in H^2\big(\overline{\mathcal{M}}_{g,k}(X,A)\big)$$

the first chern class of the universal cotangent line bundle for the first marked point. For each $d\in\mathbb{Z}^{\geq 0}$, let

$$\mathrm{GW}_{E,s_i+df,1}\big(\tau_1[f]\big)=\deg\big([\overline{\mathcal{M}}_{1,1}(E,s_i+df)]^{\mathrm{vir}}\cap\psi_1\cap\mathrm{ev}_1^*f\big)\equiv\int_{[\overline{\mathcal{M}}_{1,1}(E,s_i+df)]^{\mathrm{vir}}}\psi_1\mathrm{ev}_1^*f,$$

where $f\in H^2(E)$ is the Poincare dual of the fiber class. The $g=0$ case of (6.17) is proved by obtaining two different expressions for

$$H(q)=\sum_{d=0}^{\infty}\mathrm{GW}_{E,s_i+df,1}\big(\tau_1[f]\big)q^d \tag{6.18}$$

and setting them equal.

Lemma 6.19 ([IP5, Lemma 15.1])**.** *Let X be a symplectic 4-manifold with canonical class K_X.*

(a) For every $f\in H^2(X)$,

$$\mathrm{GW}_{X,0,1}(f)=\frac{1}{24}K_X\cdot f\,.$$

(b) If $A\in H_2(X)$ with $A\cdot K_X<0$ and $f\in H^2(X)$, then

$$\begin{aligned}\mathrm{GW}_{X,A,1}\big(\tau_1(f),p^{-K_X\cdot A-1}\big)&=\frac{f\cdot A}{24}\big(A\cdot A+K_X\cdot A\big)\mathrm{GW}_{X,A,0}\big(p^{-K_X\cdot A-1}\big)\\&+\sum_{\substack{A_0,A_1\in H_2(X)-0\\A_0+A_1=A}}\binom{-K_X\cdot A-1}{-K_X\cdot A_0-1}(f\cdot A_0)(A_0\cdot A_1)\mathrm{GW}_{X,A_0,0}\big(p^{-K_X\cdot A_0-1}\big)\mathrm{GW}_{X,A_1,1}\big(p^{-K_X\cdot A_1}\big),\end{aligned}$$

where $p\in H^4(X)$ denotes the Poincare dual of a point.

Proof. (a) If $h\colon Y\longrightarrow X$ represents the Poincare dual of f (after passing to a multiple if necessary),

$$\big\{(y,[u,x_1])\in Y\times\overline{\mathcal{M}}_{1,1}(X,0):\ h(y)=u(x_1)\big\}\approx Y\times\overline{\mathcal{M}}_{1,1}$$

and the obstruction bundle is isomorphic to $\pi_1^*h^*TX\otimes\pi_2^*\mathbb{E}^*$, where $\mathbb{E}\longrightarrow\overline{\mathcal{M}}_{1,1}$ is the Hodge line bundle. Thus,

$$\begin{aligned}\mathrm{GW}_{X,0,1}(f)&=\big\langle e\big(\pi_1^*h^*TX\otimes\pi_2^*\mathbb{E}^*\big),Y\times\overline{\mathcal{M}}_{1,1}\big\rangle\\&=-\langle h^*c_1(TX),Y\rangle\langle c_1(\mathbb{E}),\overline{\mathcal{M}}_{1,1}\rangle=\frac{1}{24}K_X\cdot f.\end{aligned}$$

(b) Let $k=-K_X\cdot A$ and $\{H_i\},\{\check{H}_i\}\subset H^2(X)$ be dual bases. Choose a representative $F\subset X$ for f and $k-1$ general points $p_2,\ldots,p_k\in X$. By the genus 1 topological recursion relation, illustrated in Figure 5 and explained in [Liu],

$$\psi_1 = \frac{1}{12}\Delta_0 + \Delta_{;1}\,,$$

where $\Delta_0, \Delta_{;1} \subset \overline{\mathcal{M}}_{1,k}(X,A)$ are the virtual divisors whose virtually generic elements are morphisms from the genus 1 irreducible nodal curve and from a smooth genus 1 curve with a rational tail which carries the first marked point.

By the Kunneth decomposition of the diagonal in X^2 and the divisor relation, the degree of the intersection of

$$\overline{\mathcal{M}}'_{1,k}(X,A) \equiv \big\{[u,x_1,\ldots,x_k]\in\overline{\mathcal{M}}_{1,k}(X,A):\ u(x_1)\in f,\ u(x_2)=p_2,\ldots,u(x_k)=p_k\big\}$$

with Δ_0 is

$$\begin{aligned}\frac{1}{2}\sum_i \mathrm{GW}_{X,A,0}(H_i,\check{H}^i,f,p^{k-1}) &= \frac{1}{2}\sum_i (H_i\cdot A)(\check{H}_i\cdot A)(f\cdot A)\mathrm{GW}_{X,A,0}(p^{k-1})\\ &= \frac{1}{2}(A\cdot A)(f\cdot A)\mathrm{GW}_{X,A,0}(p^{k-1}).\end{aligned}$$

This gives the first term in our formula.

The intersection of $\overline{\mathcal{M}}'_{1,k}(X,A)$ with the components of $\Delta_{;1}$ whose generic element restricts to a morphism of degree $A_1=0$ on the genus 1 component is the same as with the subset of these components consisting of morphisms from domains with no marked points on the genus 1 component (since the virtual complex dimension of $\overline{\mathcal{M}}_{1,1}(X,0)$ is 1, it contains no elements passing through any of the points $p_2,\ldots,p_k$). Thus, similarly to the above, the degree of this intersection is

$$\begin{aligned}\sum_i \mathrm{GW}_{X,A,0}(H_i,f,p^{k-1})\mathrm{GW}_{X,0,1}(\check{H}^i) &= \sum_i (H_i\cdot A)(f\cdot A)\mathrm{GW}_{X,A,0}(p^{k-1})\frac{1}{24}K_X\cdot\check{H}_i\\ &= \frac{1}{24}(f\cdot A)(K_X\cdot A)\mathrm{GW}_{X,A,0}(p^{k-1});\end{aligned}$$

the first equality follows from part (a). This gives the second term in our formula. The intersection of $\overline{\mathcal{M}}'_{1,k}(X,A)$ with the components of $\Delta_{;1}$ whose generic element restricts to a morphism of degree $A_1=A$ on the genus 1 component is empty, since the domain of any morphism in the intersection would contain a union of irreducible components on which the morphism is of degree 0 and which carries at least one of the last $k-1$ points (for stability), but F does not contain any of the points $p_2,\ldots,p_r$.

For dimensional reasons, the intersection of $\overline{\mathcal{M}}'_{1,k}(X,A)$ with the components of $\Delta_{;1}$ whose generic element restricts to a morphism of degree $A_1\neq 0$ on the genus 1 component and $A_0\neq 0$ on the genus 0 tail consists of morphisms from the domains so that the rational tail carries $-K_X\cdot A_0-1$

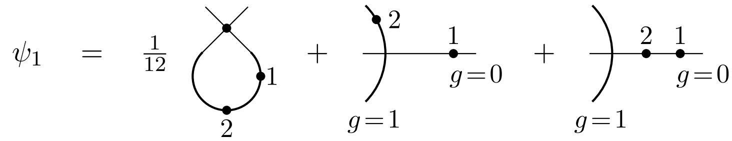

Figure 5: The genus 1 TRR on $\overline{\mathcal{M}}_{1,2}(X,A)$

of the last $k-1$ marked points. Thus, similarly to the above, the degree of this intersection is

$$\begin{aligned}
&\sum_i \binom{-K_X\cdot A-1}{-K_X\cdot A_0-1}\mathrm{GW}_{X,A_0,0}\big(H_i,f,p^{-K_X\cdot A_0-1}\big)\mathrm{GW}_{X,A_1,1}\big(\check{H}_i,p^{-K_X\cdot A_1}\big)\\
&\quad=\sum_i \binom{-K_X\cdot A-1}{-K_X\cdot A_0-1}(H_i\cdot A_0)(f\cdot A_0)\mathrm{GW}_{X,A_0,0}\big(p^{-K_X\cdot A_0-1}\big)(\check{H}_i\cdot A_1)\mathrm{GW}_{X,A_1,1}\big(p^{-K_X\cdot A_1}\big)\\
&\quad=\binom{-K_X\cdot A-1}{-K_X\cdot A_0-1}(f\cdot A_0)(A_0\cdot A_1)\mathrm{GW}_{X,A_0,0}\big(p^{-K_X\cdot A_0-1}\big)\mathrm{GW}_{X,A_1,1}\big(p^{-K_X\cdot A_1}\big).
\end{aligned}$$

This gives the last term in our formula. □

If $X=E$, $\overline{\mathcal{M}}_{0,k}(X,df)=\emptyset$ for all $d\in\mathbb{Z}^+$ and $-K_X\cdot(s+df)=-1$, where $f\in H_2(X;\mathbb{Z})$ is the fiber class. Applying Lemma 6.19(b) with $X=E$ and $A=s+df$, we thus obtain

$$\begin{aligned}
H(q)&=\sum_{d=0}^{\infty}\frac{d-1}{12}\mathrm{GW}_{E,s+df,0}()q^d+\sum_{\substack{d_0\in\mathbb{Z}^{\ge0},d_1\in\mathbb{Z}^+\\ d_0+d_1=d}}\mathrm{GW}_{E,s+d_0f,0}()q^{d_0}\cdot d_1\mathrm{GW}_{E,d_1f,1}()q^{d_1}\\
&=\frac{1}{12}\big(q\mathcal{F}_0'(q)-\mathcal{F}_0(q)\big)+\mathcal{F}_0(q)\cdot G(q)\,;
\end{aligned}\tag{6.19}$$

the second equality follows from (6.15), Lemma 6.10, and (6.3).

We next obtain a different expression for $H(q)$ by applying the symplectic sum formula to the decomposition

$$E=(E,F)\underset{F}{\#}(\mathbb{P}^1\times\mathbb{T}^2,F)\tag{6.20}$$

and moving the fiber constraint to the $\mathbb{P}^1\times\mathbb{T}^2$ side. Since $\mathcal{R}^F_{\mathbb{P}^1\times\mathbb{T}^2}=0$, the homomorphism

$$\#\colon H_2(E;\mathbb{Z})\times_F H_2(\mathbb{P}^1\times\mathbb{T}^2;\mathbb{Z})\longrightarrow H_2(\mathbb{P}^2;\mathbb{Z})$$

is well-defined; see Corollary 3.4(4). Since

$$(a_1s_1+\ldots+a_9s_9+d'f)\cdot F=(as+d''f)\cdot F$$

if and only if $a_1+\ldots+a_9=a$ and $s_i\#s=s_i$, the symplectic sum formula gives

$$\begin{aligned}
\mathrm{GW}_{E,s_i+df,1}\big(\tau_1[f]\big)=&\sum_{\substack{d',d''\in\mathbb{Z}^{\ge0}\\ d'+d''=d}}\mathrm{GW}^F_{E,s_i+d'f,0;(1)}(;F)\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,s+d''f,1;(1)}\big(\tau_1[f];p\big)\\
&+\sum_{\substack{d',d''\in\mathbb{Z}^{\ge0}\\ d'+d''=d}}\mathrm{GW}^F_{E,s_i+d'f,1;(1)}(;p)\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,s+d''f,0;(1)}\big(\tau_1[f];F\big),
\end{aligned}\tag{6.21}$$

where the relative constraints are listed after the semi-columns. Since the composition of a degree $s+d''f$ morphism to $\mathbb{P}^1\times\mathbb{T}^2$ with the projection to the second factor is a degree d'' morphism to $\mathbb{T}^2$ and there are no such morphisms from $\mathbb{P}^1$ if $d''\in\mathbb{Z}^+$, and

$$\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1\times\mathbb{T}^2,s+d''f),\overline{\mathcal{M}}^F_{0,1;(1)}(\mathbb{P}^1\times\mathbb{T}^2,s+d''f)=\emptyset \qquad \forall\, d''\in\mathbb{Z}^+. \tag{6.22}$$

On the other hand, the morphism

$$\big\{[u,x,y]\in\overline{\mathcal{M}}^F_{0,1;(1)}(\mathbb{P}^1\times\mathbb{T}^2,s):\ u(x)\in f\big\}\longrightarrow f, \qquad [u,x,y]\longrightarrow u(x),$$

is an isomorphism and the restriction of ψ_1 under this isomorphism is the first chern class of the conormal bundle to a fiber $\mathbb{T}^2$ in $\mathbb{P}^1\times\mathbb{T}^2$, i.e. 0. Thus, the second sum in (6.21) vanishes.

We next observe that

$$\big\{[u,x,y]\in\overline{\mathcal{M}}^F_{1,1;(1)}(\mathbb{P}^1\times\mathbb{T}^2,s+df):\ u(x)\in f,\ u(y)=p\big\}$$
$$\approx \big\{[u,x_1,x_2]\in\overline{\mathcal{M}}_{1,2}(\mathbb{P}^1\times\mathbb{T}^2,s+df):\ u(x_1)\in f,\ u(x_2)=p\big\},$$

where $p\in F$ is a fixed point. Both spaces contain three irreducible components, which we describe below and which have essentially the same deformation/obstruction theory; see Figure 6. This implies that

$$\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,s+d''f,1;(1)}\big(\tau_1[f];p\big)=\mathrm{GW}_{\mathbb{P}^1\times\mathbb{T}^2,s+d''f,1}\big(\tau_1[f],p\big). \tag{6.23}$$

One of the components common to both spaces is isomorphic to

$$\mathbb{P}^1\times\big\{[u,x_1']\in\overline{\mathcal{M}}_{1,1}(\mathbb{T}^2,d):\ u(x_1')=p_2\big\},$$

if $p\equiv(p_1,p_2)\in\mathbb{P}^1\times\mathbb{T}^2$. A generic element of this component is a morphism from a smooth genus 1 curve and a rational tail carrying the two marked points which restricts to a degree d morphism to a fiber of π_1 (specified by $\mathbb{P}^1$) on the genus 1 curve and an isomorphism from the tail to the section s_p through p. Another component is isomorphic to

$$\big\{[u,x_1,x_2']\in\overline{\mathcal{M}}_{1,2}(f,d):\ u(x_2')=p_2\big\}.$$

A generic element of this component is a morphism from a smooth genus 1 curve carrying the first marked point and a rational tail carrying the second marked point which restricts to a degree d morphism to the fiber f of π_1 on the genus 1 curve and an isomorphism from the tail to s_p. The last component of the absolute moduli space is isomorphic to

$$\big\{[u,x_1',x_2]\in\overline{\mathcal{M}}_{1,2}(F,d):\ u(x_2)=p\big\}.$$

A generic element of this component is a morphism from a smooth genus 1 curve carrying the second marked point and a rational tail carrying the first marked point which restricts to a degree d morphism to the fiber F of π_1 on the genus 1 curve and an isomorphism from the tail to a section of π_1 (through the image of the first marked point of an element of $\overline{\mathcal{M}}_{1,2}(F,d)$). The last component of the relative moduli space is described in the same way, except the morphism on the genus 1 component above is replaced by the $\mathbb{C}^*$-equivalence class of a morphism into the rubber $\mathbb{P}^1\times\mathbb{T}^2$ from a smooth genus 1 component with a rational tail carrying two marked points which restricts to a degree d morphism into a fiber of π_1, but not over $0,\infty\in\mathbb{P}^1$, and an isomorphism from the

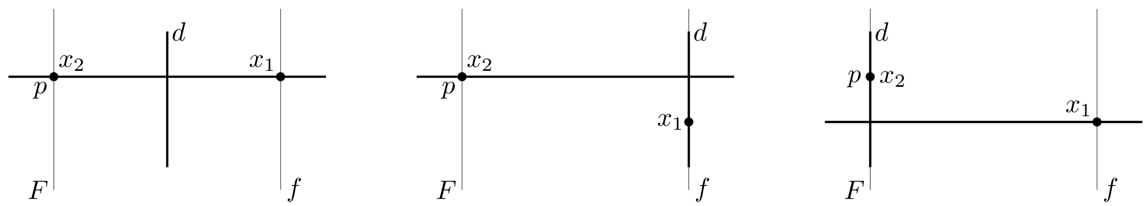

Figure 6: The three components of $\overline{\mathcal{M}}_{1,2}(\mathbb{P}^1\!\times\!\mathbb{T}^2, s\!+\!df)$

tail to a section of π_1. The restriction of ψ_1 to the third component, in either case, vanishes for the same reason as in the previous paragraph.

Finally, we apply Lemma 6.19(b) with $X=\mathbb{P}^1\!\times\!\mathbb{T}^2$ and $A=s\!+\!d''f$ to the right-hand side of (6.23). In this case,

$$-K_X\cdot A = -2, \qquad A^2 = 2d'', \qquad \text{and} \qquad f\cdot A_0=0 \quad \forall\ A_0=d''_1 f,\ d''_1\in\mathbb{Z}.$$

Thus, by (6.23) and Lemma 6.19(b),

$$\begin{aligned}\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,s+d''f,1;(1)}\big(\tau_1[f];p\big) = \;& \frac{d''-1}{12}\mathrm{GW}_{\mathbb{P}^1\times\mathbb{T}^2,s+d''f,0}(p)\\ &+\sum_{\substack{d''_0\in\mathbb{Z}^{\ge0},d''_1\in\mathbb{Z}^+\\ d''_0+d''_1=d''}} d''_1\mathrm{GW}_{\mathbb{P}^1\times\mathbb{T}^2,s+d''_0f,0}(p)\mathrm{GW}_{\mathbb{P}^1\times\mathbb{T}^2,d''_1f,1}().\end{aligned}$$

By (6.22), the first genus 0 term above vanishes unless $d''=0$ and the second unless $d''_0=0$; in the exceptional cases, they equal 1. Thus,

$$\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,s+d''f,1;(1)}\big(\tau_1[f];p\big) = \begin{cases}-\frac{1}{12}, & \text{if } d''=0;\\ d''\mathrm{GW}_{\mathbb{P}^1\times\mathbb{T}^2,d''f,1}(), & \text{if } d''>0.\end{cases} \tag{6.24}$$

Since there are no genus 0 curves mapping into the fiber F of E,

$$\mathrm{GW}^F_{E,s_i+d'f,0;(1)}(;F) = \mathrm{GW}_{E,s_i+d'f,0}(f) = (f\cdot s_i)\mathrm{GW}_{E,s_i+d'f,0}() = \mathrm{GW}_{E,s_i+d'f,0}().$$

Combining this with (6.18), (6.21), (6.24), (6.15), and Lemma 6.4, we find that

$$H(q) = -\frac{1}{12}\mathcal{F}_0(q) + \mathcal{F}_0(q)\cdot 2G(q). \tag{6.25}$$

By (6.16), (6.19), and (6.25),

$$\mathcal{F}_0(0) = 1, \qquad q\frac{\mathrm{d}}{\mathrm{d}q}\log\mathcal{F}_0(q) = 12G(q). \tag{6.26}$$

Since

$$\frac{1}{12}q\frac{\mathrm{d}}{\mathrm{d}q}\log\bigg(\prod_{d=1}^{\infty}(1-q^d)\bigg)^{-12} = \sum_{d=1}^{\infty}\frac{dq^d}{1-q^d} = \sum_{d=1}^{\infty}\sigma(d)q^d = G(q),$$

(6.26) implies the $g=0$ case of (6.17).

The $g>0$ cases of (6.17) are deduced from the $g=0$ case by relating $\mathcal{F}_g(q)$ to the power series

$$\tilde{\mathcal{F}}_g(q) \equiv \sum_{d=0}^{\infty} \mathrm{GW}_{E,s_i+df,g;(1)}\big(p^{g-1};p\big)q^d,$$

where $g\in\mathbb{Z}^+$.

Lemma 6.20 ([IP5, Lemma 15.2])**.** *For every* $g\in\mathbb{Z}^+$,

$$\mathcal{F}_g(q) = \tilde{\mathcal{F}}_g(q) + qG'(q)\cdot\mathcal{F}_{g-1}(q), \qquad \mathcal{F}_0(q)\tilde{\mathcal{F}}_g(q) + \tilde{\mathcal{F}}_1(q)\mathcal{F}_{g-1}(q) = 0. \tag{6.27}$$

Proof. The first statement is proved using the splitting (6.20) and moving one point to the $\mathbb{P}^1\times\mathbb{T}^2$ side. Since $g-1$ points stay on the E side, the genus on the E side in the symplectic sum formula (1.12) must be at least $g-1$ for the invariants not to vanish. Thus, similarly to (6.21), we obtain

$$\begin{aligned}\mathrm{GW}_{E,s_i+df,g}\big(p^g\big) = &\sum_{\substack{d',d''\in\mathbb{Z}^{\ge0}\\ d'+d''=d}} \mathrm{GW}^F_{E,s_i+d'f,g;(1)}\big(p^{g-1};p\big)\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,s+d''f,0;(1)}(p;F)\\ &+\sum_{\substack{d',d''\in\mathbb{Z}^{\ge0}\\ d'+d''=d}} \mathrm{GW}^F_{E,s_i+d'f,g-1;(1)}\big(p^{g-1};F\big)\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,s+d''f,1;(1)}(p;p).\end{aligned} \tag{6.28}$$

By (6.22), $\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,s+d''f,0;(1)}(p;f)=0$ unless $d''=0$, and

$$\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,s,0;(1)}(p;F) = \mathrm{GW}_{\mathbb{P}^1\times\mathbb{T}^2,s,0}(p,f) = (f\cdot s)\,\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,s,0}(p) = 1,$$

as the relative and absolute invariants are the same in this case (because there are no genus 0 curves contained in F) and there is a unique section passing through a point $p\in F$. Combining these observations with (6.28) and Lemmas 6.11 and 6.5, we find that

$$\begin{aligned}\sum_{d=0}^{\infty}\mathrm{GW}_{E,s_i+df,g}\big(p^g\big)q^d &= \sum_{d=0}^{\infty}\mathrm{GW}^F_{E,s_i+df,g;(1)}\big(p^{g-1};p\big)q^d\\ &\quad+\sum_{d',d''\in\mathbb{Z}^{\ge0}}\mathrm{GW}_{E,s_i+d'f,g-1}\big(p^{g-1}\big)q^{d'}\cdot\mathrm{GW}^F_{\mathbb{P}^1\times\mathbb{T}^2,s+d''f,1;(1)}(p;p)q^{d''}\\ &= \tilde{\mathcal{F}}_g(q) + \mathcal{F}_{g-1}(q)\cdot qG'(q).\end{aligned}$$

This establishes the first statement in (6.27).

The second equation in (6.27) is proved by applying the symplectic sum formula to the decomposition

$$\mathbb{K}_3 = (E,F)\underset{F}{\#}(E,F), \tag{6.29}$$

where $\mathbb{K}_3$ is a K3 surface; see [GfS, Section 3.1]. Since $H_2(F;\mathbb{Z})=\mathbb{Z}$, by Corollary 3.4(1) and the Mayer-Vietoris sequence for $E\cup_F E$ as in the proof of Lemma 3.3, the kernel of the homomorphism

$$\#\colon H_2(E;\mathbb{Z})\times_F H_2(E;\mathbb{Z}) \longrightarrow H_2(\mathbb{K}_3;\mathbb{Z})/\mathcal{R}^F_{E,E}, \quad (A,B)\longrightarrow A\#_F B,$$

is generated by $(f,-f)$. It follows that

$$\Big(\sum_{k=1}^{9} a_k s_k + d'f\Big)\#\Big(\sum_{k=1}^{9} b_k s_k + d''f\Big) = \langle s_i\#s_i{+}df\rangle$$
$$\Longrightarrow \qquad \sum_{k=1}^{9} a_k s_k, \sum_{k=1}^{9} b_k s_k = s_i,\ d'{+}d''=d.$$

By [Le, Theorem 2.4], $\mathbb{K}_3$ admits an almost complex structure J for which there are no J-holomorphic curves and thus all GW-invariants of nonzero degree vanish. Thus,

$$\sum_{C\in\langle s_i+df\rangle} \mathrm{GW}_{\mathbb{K}_3,C,g}(p^{g-1}) = 0 \qquad \forall\, d\in\mathbb{Z}. \tag{6.30}$$

We next compute the left-hand side of (6.30) by applying the symplectic sum formula (1.12) and moving all points to the first copy of E in (6.29). The genus going to this copy of E must then be at least $g{-}1$ for the invariants not to vanish. Thus, similarly to (6.28), we obtain

$$\begin{aligned} 0 = &\sum_{\substack{d',d''\in\mathbb{Z}^{\ge 0}\\ d'+d''=d}} \mathrm{GW}^F_{E,s_i+d'f,g;(1)}\big(p^{g-1};p\big)\mathrm{GW}^F_{E,s_i+d''f,0;(1)}(;F)\\ &+\sum_{\substack{d',d''\in\mathbb{Z}^{\ge 0}\\ d'+d''=d}} \mathrm{GW}^F_{E,s_i+d'f,g-1;(1)}\big(p^{g-1};F\big)\mathrm{GW}^F_{E,s+d''f,1;(1)}(;p). \end{aligned}$$

Combining this with (6.30) and Lemma 6.11, we obtain

$$\begin{aligned} 0 = &\sum_{\substack{d',d''\in\mathbb{Z}^{\ge 0}\\ d'+d''=d}} \mathrm{GW}^F_{E,s_i+d'f,g;(1)}\big(p^{g-1};p\big)q^{d'}\cdot \mathrm{GW}_{E,s_i+d''f,0}()q^{d''}\\ &+\sum_{\substack{d',d''\in\mathbb{Z}^{\ge 0}\\ d'+d''=d}} \mathrm{GW}_{E,s_i+d'f,g-1}\big(p^{g-1}\big)q^{d'}\cdot \mathrm{GW}^F_{E,s+d''f,1;(1)}(;p)q^{d''}\\ =\ &\tilde{\mathcal{F}}_g(q)\mathcal{F}_0(q)+\mathcal{F}_{g-1}(q)\tilde{\mathcal{F}}_1(q)\,. \end{aligned}$$

This establishes the second statement in (6.27). □

By the $g\!=\!1$ case of the second statement in (6.27) and the first statement in (6.16), $\tilde{\mathcal{F}}_1(q)\!=\!0$. Thus, the second statement in (6.27) becomes

$$\mathcal{F}_0(q)\tilde{\mathcal{F}}_g(q)=0.$$

Along with the first statement in (6.16), this gives $\tilde{\mathcal{F}}_g(q)\!=\!0$ for all $g\!\in\!\mathbb{Z}^+$. Combining this with the first statement in (6.27), we obtain

$$\mathcal{F}_g(q)=qG'(q)\cdot\mathcal{F}_0(q),$$

which inductively confirms (6.17) for $g\!>\!0$.

Remark 6.21. The statement of the symplectic sum formula in the middle of page 1020 in [IP5] is wrong: it should involve relative invariants. As stated, the last factor is not even zero-dimensional. The next displayed expression has the same problem. Following the stated symplectic formula and the text after it would lead to [IP5, (15.8)] without the power series G. Part (b) of [IP5, Lemma 15.1] treats only the $K_X \cdot A = -1$ case of part (b) of Lemma 6.19; the $K_X \cdot A = -2$ case, which is proved similarly, is needed to obtain [IP5, (15.8)]. The notation in [IP5, Section 15.1] is inconsistent with the rest of the paper and imprecise. In particular, the first subscript in GW should indicate the target space. Throughout most of this section, the understood target space is E, but in the displayed expression above (15.8) the target space is $\mathbb{P} = \mathbb{P}^1 \times \mathbb{T}^2$. In the preceding expression, the target space is indicated in parenthesis, in place of the constraints as done in the rest of the paper. The extra factor of t_s in the definition of F_g is unnecessary and only complicates other formulas: (15.5), the next displayed equation, and the first displayed expression on page 1021. Finally,
p1017, line -1; p1018, line 11: subscript $d \longrightarrow s + df$;
p1018, line 2; p1021, Lemma 15.2(a): $G'(t) \longrightarrow tG'(t)$;
(15.5); p1021, line 5: subscript $d \geq 1$;
p1019, line 14: the $(g_2, A_2) = (0, 0)$ invariant is not even defined;
p1020, lines 11-14: these refer to $\overline{\mathcal{M}}_{0,1}(\mathbb{P}, s) \cap \mathrm{ev}_1^* F$;
p1021, Lemma 15.2, Proof, 1.,2: $s + df \longrightarrow s + af$;
p1022, last paragraph of Section 15, line 2: Lemma 15.2a $\longrightarrow$ Lemma 15.2c.

7 A little more on [IP5]

[GJV]: Vainstein $\longrightarrow$ Vainshtein
[IP4]: 1993 $\longrightarrow$ 2003
[T]: curves $\longrightarrow$ submanifolds

[DK], [L], [V] do not appear to be cited anywhere in the text.

8 A little more on [LR]

Connections of this paper with birational geometry are described extensively on page 152, at the beginning of the introduction. There are many other instances of the discussion diverging in this direction which have little to do with the content of the paper. These include the entire page 153, the paragraph preceding Definition 1.1, the last three sentences on page 155, the sentence after (1.8), the short and long paragraphs on page 159, the sentence before Definition 2.4, and the three paragraphs of Remarks 2.15 and 2.16.

There are many statements that come with no citations or imprecise citations. These include

(1) the sentence before Corollary A.3;

(2) bottom of page of 160 (Gray's Theorem);

(3) the paragraph below (2.24);

(4) some statements in Remarks 2.14 and 2.15;

(5) citations of [H] above Lemma 3.5 on p174 and of [HWZ1] at the top of p177;

(6) statement above Remark 4.1 on p188;

(7) reference to Siebert's construction at the bottom of p190;

(8) reference to Siebert's book on at the top of p198.

The statements of Theorems A and B do not make the assumptions on the manifold M clear. Based on the proofs, M is a threefold in both cases. Symplectic sum formulas are not necessary to establish these formulas; a nearly complete geometric argument for them is already given in the paper.

The fourth sentence of the long paragraph on page 163 in [LR] makes it sound that every two smooth CY 3-folds are related by a sequence of flops, but it is apparently meant to apply to every pair of birational smooth CY 3-folds. The flop construction is never formally defined, but apparently the sentence after [LR, (2.15)] is part of the definition. Other, fairly minor misstatements in [LR, Section 1] include

p155, line -7: the above corollary $\longrightarrow$ Corollary A.2;
p156, Crl B: there exists such a φ;
p157, line 12: this equality does not hold, as LHS is degenerate along $\pi^{-1}(Z)$;
p157, lines 14-16: this sentence makes no sense;
p157, line 19: positive $\longrightarrow$ nonnegative;
p157, line 23: is tangent to $\longrightarrow$ has contact with;
p157, bottom: no connection to justification;
p158, line 1: Theorem 5.3 $\longrightarrow$ Theorem 4.14;
p158, Theorem C(iii): need an almost complex structure on M;
p158, line 1: Theorem 5.6,5.7 $\longrightarrow$ Theorems 5.6,5.8;

Remark 8.1. Theorems A and B are deduced from the symplectic sum formula in [LR, Section 6]. However, the latter is barely used in their proofs and the arguments indicate how to avoid it entirely. Let $\hat{X}$ be the symplectic blowup of a threefold X along an embedded curve C, $A\in H_2(X;\mathbb{Z})$ be such that $\langle c_1(X),A\rangle>0$, and $\alpha_i\in H^4(X)\cup H^6(X)$ be a collection of classes of total codimension corresponding to GW-invariants of degree A. These invariants are then counts of (J,ν)-holomorphic curves, for a generic (J,ν), passing through representatives of $\mathrm{PD}_X(\alpha_i)$. The latter can be chosen to be disjoint from C; J can be chosen to be Kahler along C and so that all curves of degree A passing through the constraints are disjoint from C. These representatives and curves then lift to $\hat{X}$, contributing to the corresponding GW-invariants of $\hat{X}$; any other curve in $\hat{X}$ contributing to this count would descend to X and thus be disjoint from C. Other, fairly minor misstatements in [LR, Section 6] include

p212, above (6.3): there is no $\mathrm{Ind}\, D_u$ in (5.2);
p212, below (6.4): $\bar{M}^+$ just defined on the previous page;
p212, above (6.6): Remark 3.24 $\longrightarrow$ Remark 5.2;
p213, above Pf of Crl A.2: Corollaries A.1 and A.3 are immediate consequences;
p215, line 6: $Y \longrightarrow M^+$;
p215, above Pf of Crl B.2: Corollary B.1 is an immediate consequence.

[Ba], [D], [DH], [Gr], [K1], [KM], [LT1], [M1], [Pan], [Wi1],[Wi2], [W1] do not appear to be cited anywhere in the text.

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