Normal Crossings Singularities for Symplectic Topology

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Abstract

We introduce topological notions of normal crossings symplectic divisor and variety and establish that they are equivalent, in a suitable sense, to the desired geometric notions. Our proposed concept of equivalence of associated topological and geometric notions fits ideally with important constructions in symplectic topology. This partially answers Gromov's question on the feasibility of defining singular symplectic (sub)varieties and lays foundation for rich developments in the future. In subsequent papers, we establish a smoothability criterion for symplectic normal crossings varieties, in the process providing the multifold symplectic sum envisioned by Gromov, and introduce symplectic analogues of logarithmic structures in the context of normal crossings symplectic divisors.

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1 Introduction

Algebraic and complex analytic (sub)varieties are the central objects of study in the fields of algebraic geometry and of complex geometry, respectively. Curves and divisors, i.e. subvarieties of dimension and codimension 1 over the ground field, have long been of particular importance in these fields; they can be viewed as dual to each other. Gromov's introduction [12] of pseudoholomorphic curve techniques into symplectic topology has led to numerous connections with algebraic geometry and to the appearance of symplectic divisors in different contexts, including relations with complex line bundles [6], symplectic sum constructions [10, 18], degeneration and decomposition formulas for Gromov-Witten invariants [28, 3, 16, 17, 25], affine symplectic geometry [20, 21], and homological mirror symmetry [26]. While most applications of divisors in symplectic topology have so far concerned *smooth* divisors in (*smooth*) symplectic manifolds, recent developments in symplectic topology and algebraic geometry suggest the need for notions of *singular* symplectic varieties and subvarieties (at least with certain kinds of singularities). Gromov [11, p343] in fact asked about the feasibility of introducing such notions by the mid 1980s. They should involve only some soft intrinsic symplectic data, but at the same time be compatible with rigid auxiliary almost Kähler data needed for making such notions useful.

In this paper, we propose a new perspective on the fundamental quandary conceived in [11] and demonstrate that it is suitable for introducing normal crossings singularities into symplectic topology. In symplectic topology, it is common to study a "topological" object (such as a symplectic manifold) by adding some auxiliary "geometric" data (such as a compatible complex structure) and then constructing an invariant which is independent of this auxiliary data. One then shows that such an invariant is also an invariant of the deformation equivalence class of the corresponding topological object. This approach works well when studying symplectic manifolds, but is much more difficult to carry out once singularities are introduced since there is no Darboux theorem in this case; see in particular Remark 4.5. We propose an alternative philosophy involving the entire deformation equivalence class of the topological object, as opposed to the topological object itself, and looking at the subspace of "nice" objects in this deformation equivalence class. These objects should have particularly well-behaved auxiliary geometric data which can be used to construct invariants in usual ways. The subspace of "nice" objects should topologically reflect the space of all objects. Our perspective is summarized by the principle below and a more detailed nexus on page 5.

Principle. *A symplectic variety/subvariety should be viewed as a deformation equivalence class of objects with the same topology, not as a single object.*

The viewpoint on the compatibility between the topological and geometric sides we propose in the nexus is symmetric in taking deformation equivalence classes on both sides, in contrast to the presently standard viewpoint of taking individual objects on the topological side and deformation equivalence classes on the geometric side. Our focus on the deformation equivalence classes to begin with fits ideally with the concern of symplectic topology with the deformation equivalence classes of symplectic manifolds, instead of individual symplectic manifolds, as illustrated below.

1.1 Topological vs. geometric symplectic data

Every symplectic manifold (X,ω) admits a tame (and compatible) almost complex structure J. Furthermore, the fibers of the projection

$$\mathrm{AK}(X)\longrightarrow \mathrm{Symp}(X), \qquad (\omega,J)\longrightarrow \omega, \tag{1.1}$$

from the space of pairs (ω,J) consisting of a symplectic form ω on X and an ω-tame almost complex structure J to the space of symplectic forms on X are contractible. This fibration is thus a weak homotopy equivalence, i.e. it induces isomorphisms

$$\pi_k\big(\mathrm{AK}(X)\big)\longrightarrow \pi_k\big(\mathrm{Symp}(X)\big) \tag{1.2}$$

between the homotopy groups π_k with $k\in\mathbb{Z}^+$ and the sets π_0 of deformation equivalence classes; see [5, Theorem 6.3]. The bijectivity of (1.2) for $k=0$ is key to the program initiated by Gromov in the 1980s to bring algebro-geometric techniques into symplectic topology.

For a symplectic submanifold V in a symplectic manifold (X,ω), the normal bundle

$$\mathcal{N}_XV\equiv\frac{TX|_V}{TV}\approx TV^{\omega}\equiv\big\{v\in T_xX:\ x\in V,\ \omega(v,w)=0\ \forall\, w\in T_xV\big\} \tag{1.3}$$

of V in X inherits a fiberwise symplectic form $\Omega\equiv\omega|_{\mathcal{N}_XV}$ from ω. A smooth divisor in a symplectic manifold (X,ω) is a closed symplectic submanifold V of real codimension 2. For every such triple (X,V,ω), there is an ω-tame almost complex structure J such that $J(TV)=TV$. It can be chosen to be very regular near V in the following sense. An Ω-compatible (fiberwise) complex structure $\mathfrak{i}$ on $\mathcal{N}_XV$ and a compatible connection ∇ on $\mathcal{N}_XV$ determine a closed 2-form $\widehat{\omega}$ on the total space of $\mathcal{N}_XV$, which is symplectic on a neighborhood of V in $\mathcal{N}_XV$. By the Symplectic Neighborhood Theorem [19, Theorem 3.30], there is an identification Ψ of small neighborhoods of V in $(\mathcal{N}_XV,\widehat{\omega})$ and in (X,ω). The tuple $(\mathfrak{i},\nabla,\Psi)$ is equivalent to an ω-regularization (ρ,∇,Ψ) for V in X in the terminology of Definition 2.12(1); we view it as an auxiliary structure for (X,V,ω). We call an ω-tame almost complex structure J compatible with (ρ,∇,Ψ) if Ψ^*J agrees with the almost complex structure $\widehat{J}$ determined by $J|_V$, $\mathfrak{i}$, and ∇; such a J is integrable in the normal direction to V (i.e. the image of its Nijenhuis tensor on $TX|_V$ is contained in TV). The fibers of the projection

$$\mathrm{AK}(X,V)\longrightarrow \mathrm{Aux}(X,V), \qquad (\omega,\mathcal{R},J)\longrightarrow(\omega,\mathcal{R}), \tag{1.4}$$

from the space of triples $(\omega,\mathcal{R},J)$ consisting of a symplectic form ω on (X,V), an ω-regularization $\mathcal{R}$ for V in X, and an $\mathcal{R}$-compatible almost complex structure J to the space of pairs consisting of a symplectic form ω on (X,V) and an ω-regularization for V in X are contractible. This fibration thus induces isomorphisms

$$\pi_k\big(\mathrm{AK}(X,V)\big)\longrightarrow \pi_k\big(\mathrm{Aux}(X,V)\big)$$

between the homotopy groups π_k with $k\in\mathbb{Z}^+$ and the sets π_0 of deformation equivalence classes.

For a closed codimension 2 submanifold V of X, the fibers of the projection

$$\mathrm{Aux}(X,V)\longrightarrow \mathrm{Symp}(X,V), \qquad (\omega,\mathcal{R})\longrightarrow \omega, \tag{1.5}$$

to the space of symplectic forms on X restricting to symplectic forms on V are also contractible. The composition of (1.4) and (1.5),

$$\mathrm{AK}(X,V) \longrightarrow \mathrm{Symp}(X,V), \qquad (\omega,\mathcal{R},J) \longrightarrow \omega, \tag{1.6}$$

thus has contractible fibers and induces isomorphisms

$$\pi_k\big(\mathrm{AK}(X,V)\big) \longrightarrow \pi_k\big(\mathrm{Symp}(X,V)\big) \tag{1.7}$$

between the homotopy groups π_k with $k\in\mathbb{Z}^+$ and the sets π_0 of deformation equivalence classes. The former in particular implies that the map (1.6) is surjective and ensures that paths in the base can be lifted to paths with specified endpoints. While these two properties of (1.6) feature prominently in the standard perspective on applications of symplectic divisors, only the bijectivity of the map (1.7) with $k=0$ is material for applications in symplectic topology. This is consistent with symplectic topology being fundamentally about deformation equivalence classes of symplectic manifolds, not about individual manifolds, as illustrated by the well-known applications recalled in the next two paragraphs.

The approach to relative Gromov-Witten invariants for (X,V,ω) in [16] involves choosing an ω-regularization $\mathcal{R}$ for V in X and an $\mathcal{R}$-compatible almost complex structure J. The resulting numbers do not change along a path $(\omega_t,\mathcal{R}_t,J_t)$ in $\mathrm{AK}(X,V)$. Since a path ω_t in $\mathrm{Symp}(X,V)$ can be lifted to a path $(\omega_t,\mathcal{R}_t,J_t)$ with specified endpoints $(\omega_0,\mathcal{R}_0,J_0)$ and $(\omega_1,\mathcal{R}_1,J_1)$, the relative Gromov-Witten invariants of (X,V,ω) depend only on the path-component of $\mathrm{Symp}(X,V)$ containing ω. It would thus have been sufficient to show that

- (X,V) admits ω-regularizations $\mathcal{R}$ for *at least some* symplectic forms ω on (X,V),
- every path ω_t in the subspace of such "good" symplectic forms lifts to a path $\mathcal{R}_t$ of ω_t-regularizations for V in X with given endpoints,
- the inclusion of the subspace of "good" symplectic forms into the space of all symplectic forms on (X,V) induces a bijection between the corresponding sets of path components.

This change in perspective turns out to be useful when dealing with NC symplectic divisors.

The symplectic sum construction of [10] smooths the union of two symplectic manifolds (X,ω_X) and (Y,ω_Y) glued along a common smooth symplectic divisor V such that

$$c_1\big(\mathcal{N}_XV,\omega_X\big)+c_1\big(\mathcal{N}_YV,\omega_Y\big)=0\in H^2(V;\mathbb{Z}) \tag{1.8}$$

into a new symplectic manifold $(X_\#,\omega_\#)$. In the terminology of Definition 2.5, the tuples

$$\big((X_1\!\equiv\! X,X_2\!\equiv\! Y,X_{12}\!\equiv\! V),(\omega_1\!\equiv\!\omega_X,\omega_2\!=\!\omega_Y)\big) \quad\text{and}\quad \big(X\cup_V Y,(\omega_X,\omega_Y)\big) \tag{1.9}$$

are a 2-fold simple crossings symplectic configuration and the associated simple crossings symplectic variety. The topological type of $X_\#$ depends only on the choice of the homotopy class of isomorphisms

$$\big(\mathcal{N}_XV,\omega_X\big)\otimes_{\mathbb{C}}\big(\mathcal{N}_YV,\omega_Y\big)\approx V\times\mathbb{C}$$

as complex line bundles. With such a choice fixed, the construction of [10] involves choosing an ω_X-regularization for V in X, an ω_Y-regularization for V in Y, and a representative for the above homotopy class. Because of these choices, the resulting symplectic manifold $(X_\#,\omega_\#)$ is determined by (X,ω_X), (Y,ω_Y), and the choice of the homotopy class only up to symplectic deformation equivalence. The symplectic sum construction of [10] can thus be viewed as a map

$$\begin{split}\Big\{\big([\omega_X],[\omega_Y]\big)\in\pi_0\big(\mathrm{Symp}(X,V)\big)\times\pi_0\big(\mathrm{Symp}(Y,V)\big):\\ \big[\omega_X|_V\big]=\big[\omega_Y|_V\big]\in\pi_0\big(\mathrm{Symp}(V)\big), c_1\big(\mathcal{N}_XV,\omega_X\big)+c_1\big(\mathcal{N}_YV,\omega_Y\big)=0\Big\}\longrightarrow\bigsqcup_{X_\#}\pi_0\big(\mathrm{Symp}(X_\#)\big).\end{split}$$

It would have been sufficient to carry it out only on a path-connected set of representatives for each deformation equivalence class of the tuples (1.9). This change in perspective turns out to be useful for smoothing out more elaborate simple and normal crossings symplectic varieties.

The above observations concerning (1.6) and (1.7) motivate the principle introduced in the present paper for adapting algebro-geometric notions of singularities to symplectic topology. It can be summarized as follows.

Nexus. *(1) A symplectic variety should be a stratified space X with some additional smooth-type structure and symplectic-type structure ω so that the restriction of ω to each smooth stratum X_i of X is a symplectic form in the usual sense. The set $\mathrm{Symp}(X)$ of the "symplectic structures" on X compatible with a given "smooth structure" should have a natural topology.*

(2) There should be a notion of a regularization $\mathfrak{R}$ for a symplectic structure ω on X which models neighborhoods of the strata X_i on subspaces of complex vector bundles $\mathcal{N}_i$ over X_i consisting of fiberwise subvarieties in a compatible fashion. The set $\mathrm{Aux}(X)$ of such pairs $(\omega,\mathfrak{R})$ should have a natural topology so that the projection

$$\mathrm{Aux}(X)\longrightarrow\mathrm{Symp}(X),\qquad(\omega,\mathfrak{R})\longrightarrow\omega,\tag{1.10}$$

induces a bijection between the connected components of the two spaces (or better yet, is a weak homotopy equivalence). However, this projection need not be surjective.

(3) There should be a notion of an almost complex structure J on X compatible with an ω-regularization $\mathfrak{R}$ which restricts on each X_i to an almost complex structure in the usual sense and is integrable in the normal directions to X_i. The set $\mathrm{AK}(X)$ of such triples $(\omega,\mathfrak{R},J)$ should have a natural topology so that the fibers of the projection

$$\mathrm{AK}(X)\longrightarrow\mathrm{Aux}(X),\qquad(\omega,\mathfrak{R},J)\longrightarrow(\omega,\mathfrak{R}),\tag{1.11}$$

are contractible.

A symplectic subvariety V in a symplectic variety X should be a topological subspace of X with associated spaces $\mathrm{Symp}(X,V)$, $\mathrm{Aux}(X,V)$, and $\mathrm{AK}(X,V)$ which are related as above.

1.2 Normal crossings singularities

A normal crossings (or NC) complex analytic variety X of (complex) dimension n is a Hausdorff topological space covered by charts

$$\varphi_x\colon U_x\longrightarrow\mathbb{C}^{n-k}\times\big\{(z_1,\ldots,z_{k+1})\in\mathbb{C}^{k+1}:\ z_1\ldots z_{k+1}=0\big\}\quad\text{with}\quad k=k(x)\in\{0,1,\ldots,n\},\ x\in X,$$

that overlap analytically. An NC divisor V in a Kähler manifold X of complex dimension n is a subspace of X locally of the form

$$\mathbb{C}^{n-k}\times\big\{(z_1,\ldots,z_k)\in\mathbb{C}^k:\, z_1\ldots z_k=0\big\}\quad\text{with}\quad k=k(x)\in\{0,1,\ldots,n\},\, x\in X,$$

in holomorphic coordinates on X. Such a divisor is the image of a generically injective proper Kähler immersion $\iota: \widetilde{V}\longrightarrow X$ from a Kähler manifold $\widetilde{V}$ of complex dimension $n-1$ such that all self-intersections of ι are transverse. A basic example, which we call a simple crossings (or SC) divisor, is provided by the union of transversely intersecting closed Kähler hypersurfaces V_i in X. NC singularities are the simplest, non-trivial singularities and are also of the most direct relevance to symplectic topology.

It has long been a mystery what an NC or even SC divisor in the symplectic category should be. In this paper, we introduce soft topological notions of an SC symplectic divisor in a symplectic manifold and of an SC symplectic variety and show that they are compatible, in a suitable sense, with associated rigid geometric notions. As all of our arguments are essentially local, they readily apply to the arbitrary NC case as well. However, the latter involves a more elaborate setup, with the normal bundle of an immersion replacing the normal bundle of a submanifold. For this reason, we defer the arbitrary NC case to [8] in order to highlight the ideas involved.

For a subspace V of a symplectic manifold (X,ω) to be an SC symplectic divisor, it should at least be the union of transversely intersecting closed symplectic submanifolds $\{V_i\}_{i\in S}$ of (X,ω) of real codimension 2. However, as [15, Example 1.9] illustrates, the intersection number of a pair of symplectic submanifolds V_1 and V_2 in a compact symplectic 4-manifold X can be negative; in such a case, there is no ω-tame almost complex structure J on X which restricts to almost complex structures on V_1 and V_2. If J is an ω-tame almost complex structure on X which restricts to an almost complex structure on each V_i, then the intersection V_I of the smooth divisors in any subcollection of $\{V_i\}_{i\in S}$ is a symplectic submanifold of (X,ω) and the ω-orientation of each V_I agrees with its intersection orientation induced by the orientations of X and $\{V_i\}_{i\in I}$; see Section 2.1. These two properties, appearing in Definition 2.1, are thus necessary for the existence of an ω-tame J which restricts to an almost complex structure on each V_i. It turns out that these two, essentially topological, properties suffice for a kind of virtual existence of such a J as well as of compatible collections of ω-regularizations $(\rho_i,\nabla^{(i)},\Psi_i)$ for V_i in X; see Definition 2.12(1) and Theorem 2.13.

The compatibility-of-orientations condition of Definition 2.1, which is equivalent to the positively intersecting notion of [20, Definition 5.1], is preserved under deformations of ω that keep every intersection V_I symplectic. Thus, it is a necessary condition for the existence of an ω'-tame almost complex structure J that restricts to an almost complex structure on each V_I for some deformation ω' of ω through symplectic structures ω_t on $\{V_I\}_{I\subset S}$ (i.e. symplectic forms ω_t on X such that $\omega_t|_{V_I}$ is symplectic for all $I\subset S$). By Theorem 2.13 with B being the point, this condition suffices not only for the existence of such an ω'-tame J, but also for the existence of compatible collections of ω-regularizations $(\rho_i,\nabla^{(i)},\Psi_i)$ for V_i in X. By Theorem 2.13 with $B=[0,1]$, for every path ω_t of symplectic structures on $\{V_I\}_{I\subset S}$ and all ω_0- and ω_1-regularizations $\mathcal{R}_0$ and $\mathcal{R}_1$ for V_i in X, there exists a path ω'_t homotopic to the path ω_t through paths of symplectic structures on $\{V_I\}_{I\subset S}$ and a path $\mathcal{R}_t$ of compatible ω'_t-regularizations for V_i in X. By the general case of Theorem 2.13, the projection

$$\mathrm{Aux}\big(X,(V_i)_{i\in S}\big)\longrightarrow \mathrm{Symp}^+\big(X,(V_i)_{i\in S}\big),\qquad (\omega,\mathcal{R})\longrightarrow\omega, \tag{1.12}$$

from the space of symplectic forms ω on X with compatible regularizations $\mathcal{R}$ for $(V_i)_{i\in S}$ in X to the space of symplectic structures ω on $\{V_I\}_{I\subset S}$ such that the ω-orientation of each V_I agrees with its intersection orientation is a weak homotopy equivalence. Theorem 2.17 is the analogue of Theorem 2.13 for SC symplectic varieties as in Definition 2.5. These are collections of symplectic manifolds identified along SC symplectic divisors. Some applications of these four theorems are described in the next two paragraphs.

Two versions of an NC divisor V in an almost Kähler manifold X are described in [15, Definition 1.3] and [24, Section 2]; see also [22, Definition 14.6]. The main objective of [15] and one of the two main objectives of [23, 25] are to define Gromov-Witten type invariants of X relative to such V. The constructions in [25, 15] automatically imply that the resulting invariants do not change under deformations of the *almost Kähler* data compatible with (X,V). In [24, Section 3], it is shown that the relevant almost Kähler data exists for a certain, fairly rigid, class of symplectic forms on X (for which the branches of V are symplectic and meet orthogonally) and that deformations of the symplectic form within this class extend to deformations of compatible almost Kähler data. However, it would be desirable to know that the resulting invariants depend only on some topological deformation equivalence class of *symplectic* structures on (X,V) and apply to all classes that satisfy a specific simple condition. An ω-regularization for an NC symplectic divisor V in (X,ω) can be used to construct an almost Kähler structure on X so that V becomes an NC almost Kähler divisor in the sense of [15, Definition 1.3] and [22, Definition 14.6]. By Theorem 2.13, every deformation equivalence class of SC symplectic divisors in the sense of Definition 2.1 contains a representative ω admitting a regularization and any two such representatives with compatible regularizations can be joined by a path. Thus, Theorem 2.13 implies that any invariants arising from [25, 15] depend only on the deformation equivalence class of symplectic structures on (X,V) and specifies to which classes the constructions of [25, 15] can be applied.

Theorem 2.17 is used in [7] to show an NC symplectic variety $(X_\emptyset,(\omega_i)_{i=1,\ldots,N})$ is the central fiber of a one-parameter family of degenerations with a smooth total space if and only if it satisfies a simple topological condition on the Chern class of a complex line bundle over the singular locus X_∂ of $X_\emptyset$. In the $N\!=\!2$ case, this condition reduces to (1.8) and every non-central fiber of the resulting family is a representative of the deformation equivalence class of the associated symplectic sum $(X_\#,\omega_\#)$ of [10]. In general, a non-central fiber of such a family is a representative of the deformation equivalence class of the multifold symplectic construction on $(X_\emptyset,(\omega_i)_{i=1,\ldots,N})$ envisioned in [11, p343]. It yields a multitude of new symplectic manifolds; some of them contain closed non-orientable hypersurfaces. Going in the opposite direction, the symplectic cut/degeneration construction of [9] produces one-parameter families as above out of symplectic manifolds with compatible Hamiltonian torus actions on open subsets. The second main objective of [23, 25] is to obtain decomposition formulas for Gromov-Witten invariants under certain almost Kähler splittings. An important consequence of Theorem 2.17 is that the decomposition formulas arising from [25] include splitting formulas for the Gromov-Witten invariants of the N-fold symplectic sums constructed in [7].

While the present paper connects directly with deep questions raised by Gromov [11] over 3 decades ago, the immediate inspiration for our overall project comes from the Gross-Siebert program [13] for a direct proof of mirror symmetry and from distinct recent developments in symplectic topology. Theorems 2.13 and 2.17, along with the deformation principle behind them, lay the foundation for symplectic topology versions of logarithmic structures of algebraic geometry and of stable logarithmic maps of [14, 4, 1] that are central to the Gross-Siebert program. The almost Kähler and exploded manifold versions of these objects proposed in [15, 23] are more rigid than desirable and have so far proved too unwieldy for practical applications. We expect Theorem 2.13 to be also useful for studying smooth affine varieties and isolated singularities from a symplectic perspective. For instance, an affine variety can be embedded into a smooth projective variety so that its complement is an NC divisor; see [20, Section 2.1]. Theorem 2.13 describes what a neighborhood of this divisor looks like and hence what the affine variety looks like at infinity; this is useful for analyzing the symplectic cohomology of such varieties. In contrast to [20, Theorem 5.14], Theorem 2.13 describes such neighborhoods for families of affine varieties. Links of isolated singularities or families of isolated singularities (viewed as contact manifolds) can also be described explicitly by looking at neighborhoods of the exceptional curves of some resolution, using Theorem 2.13 to put such neighborhoods in a standard form, and then applying techniques from [21].

1.3 Outline and acknowledgments

We formally define SC symplectic divisors and varieties in Section 2.1, regularizations for the former in Section 2.2, and regularizations for the latter in Section 2.3. While the precise definitions of regularizations are a bit technical, their substance is that a neighborhood of each point in the divisor or variety looks as expected. In particular, the branches of the divisor symplectically correspond to hyperplane subbundles of a split complex vector bundle; this implies that they are symplectically orthogonal. Sections 2.2 and 2.3 conclude with theorems stating that the spaces of symplectic forms with regularizations are weakly homotopy equivalent to the spaces of all admissible symplectic forms. The necessary deformation arguments on split vector bundles are carried out in Section 3, especially in the proof of Proposition 3.6. Section 4 contains stratified versions of the usual smooth Tubular Neighborhood Theorem. We prove Theorems 2.13 and 2.17 in Section 5 by applying Theorem 3.1 via Proposition 4.2; the crucial compatibility-of-orientations condition in Definition 2.1 allows us to apply Proposition 3.6.

We would like to thank E. Ionel and B. Parker for enlightening discussions related to normal crossings divisors in the symplectic category and E. Lerman for pointing out related literature.

2 Simple crossings divisors and varieties

We begin by introducing the most commonly used notation. If $N\in\mathbb{Z}^{\ge0}$ and $I\subset\{1,\ldots,N\}$, let

$$[N]=\{1,\ldots,N\},\qquad \mathbb{C}^N_I=\big\{(z_1,\ldots,z_N)\in\mathbb{C}^N\colon\, z_i=0\ \forall\, i\in I\big\}.$$

Denote by $\mathcal{P}(N)$ the collection of subsets of $[N]$ and by $\mathcal{P}^*(N)\subset\mathcal{P}(N)$ the collection of nonempty subsets. If in addition $i\in[N]$, let

$$\mathcal{P}_i(N)=\big\{I\in\mathcal{P}(N)\colon\, i\in I\big\}.$$

If $\mathcal{N}\longrightarrow V$ is a vector bundle, $\mathcal{N}'\subset\mathcal{N}$, and $V'\subset V$, we define

$$\mathcal{N}'|_{V'}=\mathcal{N}|_{V'}\cap\mathcal{N}'\,. \tag{2.1}$$

Let $\mathbb{I}=[0,1]$.

2.1 Definitions and examples

Let X be a (smooth) manifold. For any submanifold $V\subset X$, let

$$\mathcal{N}_X V \equiv \frac{TX|_V}{TV} \longrightarrow V$$

denote the normal bundle of V in X. For a collection $\{V_i\}_{i\in S}$ of submanifolds of X and $I\subset S$, let

$$V_I \equiv \bigcap_{i\in I} V_i \subset X\,.$$

Such a collection is called transverse if any subcollection $\{V_i\}_{i\in I}$ of these submanifolds intersects transversely, i.e. the homomorphism

$$T_xX \oplus \bigoplus_{i\in I} T_xV_i \longrightarrow \bigoplus_{i\in I} T_xX, \qquad \big(v,(v_i)_{i\in I}\big) \longrightarrow (v+v_i)_{i\in I}\,, \tag{2.2}$$

is surjective for all $x\in V_I$. By the Inverse Function Theorem [29, Theorem 1.30], each subspace $V_I\subset X$ is then a submanifold of X of codimension

$$\mathrm{codim}_X V_I = \sum_{i\in I} \mathrm{codim}_X V_i$$

and the homomorphisms

$$\begin{gathered} \mathcal{N}_X V_I \longrightarrow \bigoplus_{i\in I} \mathcal{N}_X V_i\big|_{V_I} \quad \forall\, I\subset S, \qquad \mathcal{N}_{V_{I-i}} V_I \longrightarrow \mathcal{N}_X V_i\big|_{V_I} \quad \forall\, i\in I\subset S, \\ \bigoplus_{i\in I-I'} \mathcal{N}_{V_{I-i}} V_I \longrightarrow \mathcal{N}_{V_{I'}} V_I \quad \forall\, I'\subset I\subset S \end{gathered} \tag{2.3}$$

induced by inclusions of the tangent bundles are isomorphisms.

Let X be an oriented manifold. If $V\subset X$ is an oriented submanifold of even codimension, the short exact sequence of vector bundles

$$0 \longrightarrow TV \longrightarrow TX|_V \longrightarrow \mathcal{N}_X V \longrightarrow 0 \tag{2.4}$$

over V induces an orientation on $\mathcal{N}_X V$ (if the codimension and dimension of V are odd, the induced orientation on $\mathcal{N}_X V$ depends also on a sign convention). If $\{V_i\}_{i\in S}$ is a transverse collection of oriented submanifolds of X of even codimensions, the orientations on $\mathcal{N}_X V_i$ induced by the orientations of X and V_i induce an orientation on $\mathcal{N}_X V_I$ via the first isomorphism in (2.3). The orientations of X and $\mathcal{N}_X V_I$ then induce an orientation on V_I via the short exact sequence (2.4). Thus, a transverse collection $\{V_i\}_{i\in S}$ of oriented submanifolds of X of even codimensions induces an orientation on each submanifold $V_I\subset X$ with $|I|\ge 2$, which we call the intersection orientation of V_I. If V_I is zero-dimensional, it is a discrete collection of points in X and the homomorphism (2.2) is an isomorphism at each point $x\in V_I$; the intersection orientation of V_I at $x\in V_I$ then corresponds to a plus or minus sign, depending on whether this isomorphism is orientation-preserving or orientation-reversing. For convenience, we call the original orientations of $X=V_\emptyset$ and $V_i=V_{\{i\}}$ the intersection orientations of these submanifolds V_I of X with $|I|<2$.

Suppose (X,ω) is a symplectic manifold and $\{V_i\}_{i\in S}$ is a transverse collection of submanifolds of X such that each V_I is a symplectic submanifold of (X,ω). Each V_I then carries an orientation induced by $\omega|_{V_I}$, which we call the ω-orientation. If V_I is zero-dimensional, it is automatically a symplectic submanifold of (X,ω); the ω-orientation of V_I at each point $x\in V_I$ corresponds to the plus sign by definition. By the previous paragraph, the ω-orientations of X and V_i with $i\in I$ also induce intersection orientations on all V_I.

Definition 2.1. Let (X,ω) be a symplectic manifold. A simple crossings (or SC) symplectic divisor in (X,ω) is a finite transverse collection $\{V_i\}_{i\in S}$ of closed submanifolds of X of codimension 2 such that V_I is a symplectic submanifold of (X,ω) for every $I\subset S$ and the intersection and ω-orientations of V_I agree.

The intersection and symplectic orientations of V_I agree if $|I|<2$. Thus, an SC symplectic divisor $\{V_i\}_{i\in S}$ with $|S|=1$ is a smooth symplectic divisor in the usual sense. If (X,ω) is a 4-dimensional symplectic manifold, a finite transverse collection $\{V_i\}_{i\in S}$ of closed symplectic submanifolds of X of codimension 2 is an SC symplectic divisor if and only if all points of the pairwise intersections $V_{i_1}\cap V_{i_2}$ with $i_1\neq i_2$ are positive. By [15, Example 1.9], the latter need not be the case in general. By Example 2.7 below, in higher dimensions it is not sufficient to consider either the pairwise intersections or the deepest (non-empty) intersections .

As with symplectic manifolds and smooth symplectic divisors, it is natural to consider the space of all structures compatible with an SC symplectic divisor.

Definition 2.2. Let X be a manifold and $\{V_i\}_{i\in S}$ be a finite transverse collection of closed submanifolds of X of codimension 2. A symplectic structure on $\{V_i\}_{i\in S}$ in X is a symplectic form ω on X such that V_I is a symplectic submanifold of (X,ω) for all $I\subset S$.

For X and $\{V_i\}_{i\in S}$ as in Definition 2.2, we denote by $\mathrm{Symp}(X,\{V_i\}_{i\in S})$ the space of all symplectic structures on $\{V_i\}_{i\in S}$ in X and by

$$\mathrm{Symp}^+\big(X,\{V_i\}_{i\in S}\big)\subset \mathrm{Symp}\big(X,\{V_i\}_{i\in S}\big)$$

the subspace of the symplectic forms ω such that $\{V_i\}_{i\in S}$ is an SC symplectic divisor in (X,ω). The latter is a union of topological components of the former.

We next introduce analogous notions for SC varieties. A 3-fold SC configuration and the associated SC variety are shown in Figure 1.

Definition 2.3. Let $N\in\mathbb{Z}^+$. An N-fold transverse configuration is a tuple $\{X_I\}_{I\in\mathcal{P}^*(N)}$ of manifolds such that $\{X_{ij}\}_{j\in[N]-i}$ is a transverse collection of submanifolds of X_i for each $i\in[N]$ and

$$X_{\{ij_1,\ldots,ij_k\}}\equiv\bigcap_{m=1}^{k}X_{ij_m}=X_{ij_1\ldots j_k}\qquad\forall\ j_1,\ldots,j_k\in[N]-i.$$

Definition 2.4. Let $N\in\mathbb{Z}^+$ and $\mathbf{X}\equiv\{X_I\}_{I\in\mathcal{P}^*(N)}$ be an N-fold transverse configuration such that X_{ij} is a closed submanifold of X_i of codimension 2 for all $i,j\in[N]$ distinct. A symplectic structure on $\mathbf{X}$ is a tuple

$$(\omega_i)_{i\in[N]}\in\prod_{i=1}^{N}\mathrm{Symp}\big(X_i,\{X_{ij}\}_{j\in[N]-i}\big)$$

such that $\omega_{i_1}|_{X_{i_1i_2}}=\omega_{i_2}|_{X_{i_1i_2}}$ for all $i_1,i_2\in[N]$.

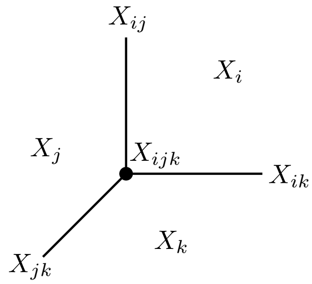

Figure 1: A 3-fold simple crossings configuration and variety.

For an N-fold transverse configuration as in Definition 2.3, let

$$X_\emptyset = \left(\bigsqcup_{i=1}^{N} X_i\right)\Big/\!\sim, \quad X_i \ni x \sim x \in X_j \quad \forall\, x \in X_{ij} \subset X_i, X_j,\ i \neq j\,, \tag{2.5}$$

$$X_\partial \equiv \bigcup_{I\in\mathcal{P}(N),|I|=2} X_I \subset X_\emptyset\,. \tag{2.6}$$

For $k\in\mathbb{Z}^{\geq 0}$, we call a tuple $(\omega_i)_{i\in[N]}$ a k-form on $X_\emptyset$ if ω_i is a k-form on X_i for each $i\in[N]$ and

$$\omega_i\big|_{X_{ij}} = \omega_j\big|_{X_{ij}} \qquad \forall\, i,j\in[N].$$

For $\mathbf{X}$ as in Definition 2.4, let $\mathrm{Symp}(\mathbf{X})$ denote the space of all symplectic structures on $\mathbf{X}$ and

$$\mathrm{Symp}^+\big(\mathbf{X}\big) = \mathrm{Symp}\big(\mathbf{X}\big) \cap \prod_{i=1}^{N} \mathrm{Symp}^+\big(X_i, \{X_{ij}\}_{j\in[N]-i}\big)\,. \tag{2.7}$$

Thus, if $(\omega_i)_{i\in[N]}$ is an element of $\mathrm{Symp}^+(\mathbf{X})$, then $\{X_{ij}\}_{j\in[N]-i}$ is an SC symplectic divisor in (X_i,ω_i) for each $i\in[N]$.

Definition 2.5. Let $N\in\mathbb{Z}^+$. An N-fold simple crossings (or SC) symplectic configuration is a tuple

$$\mathbf{X} = \big((X_I)_{I\in\mathcal{P}^*(N)}, (\omega_i)_{i\in[N]}\big) \tag{2.8}$$

such that $\{X_I\}_{I\in\mathcal{P}^*(N)}$ is an N-fold transverse configuration, X_{ij} is a closed submanifold of X_i of codimension 2 for all $i,j\in[N]$ distinct, and $(\omega_i)_{i\in[N]} \in \mathrm{Symp}^+(\mathbf{X})$. The SC symplectic variety associated to such a tuple $\mathbf{X}$ is the pair $(X_\emptyset, (\omega_i)_{i\in[N]})$.

Example 2.6. An SC symplectic divisor $\{V_i\}_{i\in S}$ in (X,ω) gives rise to an N-fold SC symplectic configuration with $N=|S|+1$. For each $i\in[N]$, let

$$\pi_1,\pi_2\colon V_i\times\mathbb{C} \longrightarrow V_i,\mathbb{C}$$

be the component projection maps. We identify S with $[N-1]$ and denote by $\omega_{\mathbb{C}}$ the standard symplectic form on $\mathbb{C}$. For $I\in\mathcal{P}^*(N)$ and $i\in[N]$, we define

$$X_I = \begin{cases} V_I\times\mathbb{C}, & \text{if } N\notin I;\\ V_{I-\{N\}}, & \text{if } N\in I; \end{cases} \qquad \omega_i = \begin{cases} \pi_1^*(\omega|_{V_i})+\pi_2^*\omega_{\mathbb{C}}, & \text{if } i\neq N;\\ \omega, & \text{if } i=N. \end{cases}$$

The resulting tuple $\mathbf{X}$ as in (2.8) is then an N-fold SC symplectic configuration.

Suppose ω is a symplectic structure on $\{V_i\}_{i\in S}$ in X in the sense of Definition 2.2. The symplectic part of the requirements on an SC almost Kähler divisor

$$V \equiv \bigcup_{i\in S} V_i \subset X$$

in [15, Definition 1.3] is equivalent to the existence for each $p \in V$ of an oriented chart ψ on X which restricts to oriented charts on the smooth divisors V_i after projecting to some coordinate hyperplanes. The existence of an ω-tame almost complex structure J on X which restricts to an almost complex structure on each V_i implies the existence of such charts. However, the symplectic part of the requirements on an SC almost Kähler divisor in [15, Definition 1.3] sees the orientations only of X, each V_i, and their zero-dimensional intersections, but not of the intermediate-dimensional intersections of the divisors V_i. By the $a>1$ case in the next example, this part does not by itself ensure the existence of a J compatible with every V_i. By the $-1<a<-\frac{1}{2}$ case in this example, the consideration of the orientations of the pairwise intersections only does not suffice either.

Example 2.7. Let $X=\mathbb{C}^3$ and

$$\begin{aligned}\omega =& \mathrm{d}x_1\wedge\mathrm{d}y_1 + \mathrm{d}x_2\wedge\mathrm{d}y_2 + \mathrm{d}x_3\wedge\mathrm{d}y_3\\ &+ a\big(\mathrm{d}x_1\wedge\mathrm{d}y_2 - \mathrm{d}y_1\wedge\mathrm{d}x_2\big) + a\big(\mathrm{d}x_1\wedge\mathrm{d}y_3 - \mathrm{d}y_1\wedge\mathrm{d}x_3\big) + a\big(\mathrm{d}x_2\wedge\mathrm{d}y_3 - \mathrm{d}y_2\wedge\mathrm{d}x_3\big)\end{aligned}$$

for some $a\in\mathbb{R}$. We note that

$$\begin{aligned}\omega^3 &= 6(1-a)^2(1+2a)\,\mathrm{d}x_1\wedge\mathrm{d}y_1\wedge\mathrm{d}x_2\wedge\mathrm{d}y_2\wedge\mathrm{d}x_3\wedge\mathrm{d}y_3\,,\\ \omega^2|_{\mathbb{C}^3_i} &= 2\big(1-a^2\big)\mathrm{d}x_j\wedge\mathrm{d}y_j\wedge\mathrm{d}x_k\wedge\mathrm{d}y_k \quad \text{if } \{i,j,k\}=\{1,2,3\}.\end{aligned}$$

Thus, ω is a symplectic structure on $\{\mathbb{C}^3_i\}_{i\in[3]}$ in $\mathbb{C}^3$ if $a \neq \pm1, -\frac{1}{2}$. The ω-orientations on the coordinate lines $\mathbb{C}^3_{ij}$ with $i\neq j$ and on the point $\mathbb{C}^3_{123}=\{0\}$ are the canonical complex orientations. If $a>1$,

- the ω-orientation on $\mathbb{C}^3$ is the canonical complex orientation,
- the ω-orientations on the hyperplanes $\mathbb{C}^3_i$ are the opposite of the canonical complex orientations,
- the intersection and ω-orientations on the coordinate lines $\mathbb{C}^3_{ij}$ with $i\neq j$ are the same,
- the intersection and ω-orientations on the point $\mathbb{C}^3_{123}$ are opposite.

If $-1<a<-\frac{1}{2}$,

- the ω-orientation on $\mathbb{C}^3$ is the opposite of the canonical complex orientation,
- the ω-orientations on the hyperplanes $\mathbb{C}^3_i$ are the canonical complex orientations,
- the intersection and ω-orientations on the coordinate lines $\mathbb{C}^3_{ij}$ with $i\neq j$ are opposite,
- the intersection and ω-orientations on the point $\mathbb{C}^3_{123}$ are the same.

2.2 Regularizations for SC symplectic divisors

In this section, we formally define the notions of regularizations for a submanifold $V \subset X$, for a symplectic submanifold with a split normal bundle, and for a transverse collection $\{V_i\}_{i\in S}$ of submanifolds with a symplectic structure ω; see Definitions 2.8, 2.9, and 2.12(1), respectively. A regularization in the sense of Definition 2.12(1) symplectically models a neighborhood of $x \in V_I$ in X on a neighborhood of the zero section V_I in the normal bundle $\mathcal{N}_X V_I$ split as in (2.3) with a standardized symplectic form. The existence of such a regularization requires the smooth symplectic divisors V_i to meet ω-orthogonally at V_I, which is rarely the case. However, by Theorem 2.13 at the end of this section, a virtual kind of existence, which suffices for many important applications in symplectic topology, is always the case if $\{V_i\}_{i\in S}$ is an SC symplectic divisor in the sense of Definition 2.1. This implies that our notion of an SC symplectic divisor is natural from the point of view of symplectic topology and its connections with algebraic geometry simultaneously.

If B is a manifold, possibly with boundary, and $k \in \mathbb{Z}^{\ge 0}$, we call a family $(\omega_t)_{t\in B}$ of k-forms on X smooth if the k-form $\widetilde{\omega}$ on $B\times X$ given by

$$\widetilde{\omega}_{(t,x)}(v_1,\ldots,v_k) = \begin{cases} \omega_t|_x(v_1,\ldots,v_k), & \text{if } v_1,\ldots v_k \in T_xX; \\ 0, & \text{if } v_1 \in T_tB; \end{cases}$$

is smooth. Smoothness for families of other objects is defined similarly.

For a vector bundle $\pi\colon \mathcal{N} \longrightarrow V$, we denote by $\zeta_{\mathcal{N}}$ the radial vector field on the total space of $\mathcal{N}$; it is given by

$$\zeta_{\mathcal{N}}(v) = (v,v) \in \pi^*\mathcal{N} = T\mathcal{N}^{\mathrm{ver}} \hookrightarrow T\mathcal{N}\,.$$

Let Ω be a fiberwise 2-form on $\mathcal{N} \longrightarrow V$. A connection ∇ on $\mathcal{N}$ induces a projection $T\mathcal{N} \longrightarrow \pi^*\mathcal{N}$ and thus determines an extension Ω_∇ of Ω to a 2-form on the total space of $\mathcal{N}$. If ω is a closed 2-form on V, the 2-form

$$\widehat{\omega} \equiv \pi^*\omega + \frac{1}{2}\mathrm{d}\iota_{\zeta_{\mathcal{N}}}\Omega_\nabla \equiv \pi^*\omega + \frac{1}{2}\mathrm{d}\big(\Omega_\nabla(\zeta_{\mathcal{N}},\cdot)\big) \tag{2.9}$$

on the total space of $\mathcal{N}$ is then closed and restricts to Ω on $\pi^*\mathcal{N} = T\mathcal{N}^{\mathrm{ver}}$. If ω is a symplectic form on V and Ω is a fiberwise symplectic form on $\mathcal{N}$, then $\widehat{\omega}$ is a symplectic form on a neighborhood of V in $\mathcal{N}$.

We call $\pi\colon (L,\rho,\nabla) \longrightarrow V$ a Hermitian line bundle if V is a manifold, $L \longrightarrow V$ is a smooth complex line bundle, ρ is a Hermitian metric on L, and ∇ is a ρ-compatible connection on L. We use the same notation ρ to denote the square of the norm function on L and the Hermitian form on L which is $\mathbb{C}$-antilinear in the second input. Thus,

$$\rho(v) \equiv \rho(v,v), \quad \rho(\mathfrak{i}v,w) = \mathfrak{i}\rho(v,w) = -\rho(v,\mathfrak{i}w) \qquad \forall\ (v,w) \in L\times_V L.$$

Let $\rho^{\mathbb{R}}$ denote the real part of ρ. A smooth map $h\colon V' \longrightarrow V$ pulls back a Hermitian line bundle (L,ρ,∇) over V to a Hermitian line bundle

$$h^*(L,\rho,\nabla) \equiv (h^*L, h^*\rho, h^*\nabla) \longrightarrow V'.$$

A Riemannian metric on an oriented real vector bundle $L \longrightarrow V$ of rank 2 determines a complex structure on the fibers of L. A Hermitian structure on an oriented real vector bundle $L \longrightarrow V$ of rank 2 is a pair (ρ,∇) such that (L,ρ,∇) is a Hermitian line bundle with the complex structure $\mathfrak{i}_\rho$ determined by the Riemannian metric $\rho^{\mathbb{R}}$. If Ω is a fiberwise symplectic form on an oriented vector bundle $L\longrightarrow V$ of rank 2, an Ω-compatible Hermitian structure on L is a Hermitian structure (ρ,∇) on L such that $\Omega(\cdot,\mathfrak{i}_\rho\cdot)=\rho^{\mathbb{R}}(\cdot,\cdot)$.

Let $(L_i,\rho_i,\nabla^{(i)})_{i\in I}$ be a finite collection of Hermitian line bundles over V. If each $(\rho_i,\nabla^{(i)})$ is compatible with a fiberwise symplectic form Ω_i on L_i and

$$(\mathcal{N},\Omega,\nabla)\equiv\bigoplus_{i\in I}\big(L_i,\Omega_i,\nabla^{(i)}\big),$$

then the 2-form (2.9) is given by

$$\widehat{\omega}=\widehat{\omega}^{\bullet}_{(\rho_i,\nabla^{(i)})_{i\in I}}\equiv\pi^*\omega+\frac{1}{2}\bigoplus_{i\in I}\pi^*_{I;i}\mathrm{d}\big((\Omega_i)_{\nabla^{(i)}}(\zeta_{L_i},\cdot)\big),\tag{2.10}$$

where $\pi_{I;i}\colon\mathcal{N}\longrightarrow L_i$ is the component projection map.

If in addition $\Psi: V'\longrightarrow V$ is an embedding, $I'\subset I$, and $(L'_i,\rho'_i,\nabla'^{(i)})_{i\in I'}$ is a finite collection of Hermitian line bundles over V', a vector bundle homomorphism

$$\widetilde{\Psi}\colon\bigoplus_{i\in I'}L'_i\longrightarrow\bigoplus_{i\in I}L_i$$

covering Ψ is a product Hermitian inclusion if

$$\widetilde{\Psi}\colon(L'_i,\rho'_i,\nabla'^{(i)})\longrightarrow\Psi^*(L_i,\rho_i,\nabla^{(i)})$$

is an isomorphism of Hermitian line bundles over V' for every $i\in I'$. We call such a morphism a product Hermitian isomorphism covering Ψ if $|I'|=|I|$.

Definition 2.8. Let X be a manifold and $V\subset X$ be a submanifold with normal bundle $\mathcal{N}_XV\longrightarrow V$. A regularization for V in X is a diffeomorphism $\Psi:\mathcal{N}'\longrightarrow X$ from a neighborhood of V in $\mathcal{N}_XV$ onto a neighborhood of V in X such that $\Psi(x)=x$ and the isomorphism

$$\mathcal{N}_XV|_x=T_x^{\mathrm{ver}}\mathcal{N}_XV\hookrightarrow T_x\mathcal{N}_XV\stackrel{\mathrm{d}_x\Psi}{\longrightarrow}T_xX\longrightarrow\frac{T_xX}{T_xV}\equiv\mathcal{N}_XV|_x$$

is the identity for every $x\in V$.

By this definition, a regularization for $V=X$ in X is the identity map on $X=\mathcal{N}_XX$.

If (X,ω) is a symplectic manifold and V is a symplectic submanifold in (X,ω), then ω induces a fiberwise symplectic form $\omega|_{\mathcal{N}_XV}$ on the normal bundle $\mathcal{N}_XV$ of V in X via the isomorphism (1.3). We denote the restriction of $\omega|_{\mathcal{N}_XV}$ to a subbundle $L\subset\mathcal{N}_XV$ by $\omega|_L$.

Definition 2.9. Let X be a manifold, $V\subset X$ be a submanifold, and

$$\mathcal{N}_XV=\bigoplus_{i\in I}L_i$$

be a fixed splitting into oriented rank 2 subbundles.

(1) If ω is a symplectic form on X such that V is a symplectic submanifold and $\omega|_{L_i}$ is nondegenerate for every $i\in I$, then an ω-regularization for V in X is a tuple $((\rho_i,\nabla^{(i)})_{i\in I},\Psi)$, where $(\rho_i,\nabla^{(i)})$ is an $\omega|_{L_i}$-compatible Hermitian structure on L_i for each $i\in I$ and Ψ is a regularization for V in X, such that

$$\Psi^*\omega=\widehat{\omega}^{\bullet}_{(\rho_i,\nabla^{(i)})_{i\in I}}\big|_{\mathrm{Dom}(\Psi)}. \tag{2.11}$$

(2) If B is a manifold, possibly with boundary, and $(\omega_t)_{t\in B}$ is a smooth family of symplectic forms on X which restrict to symplectic forms on V, then an $(\omega_t)_{t\in B}$-family of regularizations for V in X is a smooth family of tuples

$$(\mathcal{R}_t)_{t\in B}\equiv\big((\rho_{t;i},\nabla^{(t;i)})_{i\in I},\Psi_t\big)_{t\in B} \tag{2.12}$$

such that $\mathcal{R}_t$ is an ω_t-regularization for V in X for each $t\in B$ and

$$\big\{(t,v)\in B\times\mathcal{N}_XV:\ v\in\mathrm{Dom}(\Psi_t)\big\}\longrightarrow X,\qquad (t,v)\longrightarrow\Psi_t(v),$$

is a smooth map from a neighborhood of $B\times V$ in $B\times\mathcal{N}_XV$.

We next extend these definitions to SC divisors. Suppose $\{V_i\}_{i\in S}$ is a transverse collection of codimension 2 submanifolds of X. For each $I\subset S$, the last isomorphism in (2.3) with $I'=\emptyset$ provides a natural decomposition

$$\pi_I\colon\mathcal{N}_XV_I=\bigoplus_{i\in I}\mathcal{N}_{V_{I-i}}V_I\longrightarrow V_I$$

of the normal bundle of V_I in X into oriented rank 2 subbundles. We take this decomposition as given for the purposes of applying Definition 2.9. If in addition $I'\subset I$, let

$$\pi_{I;I'}\colon\mathcal{N}_{I;I'}\equiv\bigoplus_{i\in I-I'}\mathcal{N}_{V_{I-i}}V_I=\mathcal{N}_{V_{I'}}V_I\longrightarrow V_I\,.$$

There are canonical identifications

$$\mathcal{N}_{I;I-I'}=\mathcal{N}_XV_{I'}|_{V_I},\quad \mathcal{N}_XV_I=\pi_{I;I'}^*\mathcal{N}_{I;I-I'}=\pi_{I;I'}^*\mathcal{N}_XV_{I'}\qquad\forall\ I'\subset I\subset[N]. \tag{2.13}$$

The first equality in the second statement above is used in particular in (2.17).

Definition 2.10. Let X be a manifold and $\{V_i\}_{i\in S}$ be a transverse collection of submanifolds of X. A system of regularizations for $\{V_i\}_{i\in S}$ in X is a tuple $(\Psi_I)_{I\subset S}$, where Ψ_I is a regularization for V_I in X in the sense of Definition 2.8, such that

$$\Psi_I\big(\mathcal{N}_{I;I'}\cap\mathrm{Dom}(\Psi_I)\big)=V_{I'}\cap\mathrm{Im}(\Psi_I) \tag{2.14}$$

for all $I'\subset I\subset S$.

Given a system of regularizations as in Definition 2.10 and $I'\subset I\subset S$, let

$$\mathcal{N}'_{I;I'}=\mathcal{N}_{I;I'}\cap\mathrm{Dom}(\Psi_I),\qquad \Psi_{I;I'}\equiv\Psi_I\big|_{\mathcal{N}'_{I;I'}}\colon\mathcal{N}'_{I;I'}\longrightarrow V_{I'}\,.$$

The map $\Psi_{I;I'}$ is a regularization for V_I in $V_{I'}$. Let

$$\iota\colon\pi_{I;I'}^*\mathcal{N}_{I;I-I'}\hookrightarrow\pi_{I;I'}^*\mathcal{N}_XV_I=\pi_I^*\mathcal{N}_XV_I\big|_{\mathcal{N}_{I;I'}}\hookrightarrow T\mathcal{N}_XV_I\big|_{\mathcal{N}_{I;I'}}$$

denote the canonical inclusion as a subspace of the vertical tangent bundle. By (2.14),

$$\mathrm{d}\Psi_I\colon T\mathcal{N}_X V_I|_{\mathcal{N}'_{I;I'}} \longrightarrow TX|_{V_{I'}\cap \mathrm{Im}(\Psi_I)} \quad \text{and} \quad \mathrm{d}\Psi_I\colon T\mathcal{N}_{I;I'}|_{\mathcal{N}'_{I;I'}} \longrightarrow TV_{I'}\big|_{V_{I'}\cap \mathrm{Im}(\Psi_I)}$$

are isomorphisms of vector bundles for all $I'\subset I\subset S$. This implies that the composition

$$\begin{aligned}\mathfrak{D}\Psi_{I;I'}\colon \pi^*_{I;I'}\mathcal{N}_{I;I-I'}|_{\mathcal{N}'_{I;I'}} \xhookrightarrow{\iota} T\mathcal{N}_X V_I|_{\mathcal{N}'_{I;I'}} \xrightarrow{\mathrm{d}\Psi_I} TX|_{V_{I'}\cap\mathrm{Im}(\Psi_I)} \\ \longrightarrow \frac{TX|_{V_{I'}}}{TV_{I'}}\Big|_{V_{I'}\cap\mathrm{Im}(\Psi_I)} = \mathcal{N}_X V_{I'}|_{V_{I'}\cap\mathrm{Im}(\Psi_I)}\end{aligned} \tag{2.15}$$

is an isomorphism respecting the natural decompositions of $\mathcal{N}_{I;I-I'} = \mathcal{N}_X V_{I'}|_{V_I}$ and $\mathcal{N}_X V_{I'}$. For example,

$$\mathfrak{D}\Psi_{I;\emptyset} = \Psi_I, \qquad \mathfrak{D}\Psi_{I;I} = \mathrm{id}_{\mathcal{N}_X V_I}\,.$$

By the last assumption in Definition 2.8,

$$\mathfrak{D}\Psi_{I;I'}\big|_{\pi^*_{I;I'}\mathcal{N}_{I;I-I'}|_{V_I}} = \mathrm{id}\colon \mathcal{N}_{I;I-I'} \longrightarrow \mathcal{N}_X V_{I'}|_{V_I} \tag{2.16}$$

under the canonical identification of $\mathcal{N}_{I;I-I'}$ with $\mathcal{N}_X V_{I'}|_{V_I}$.

Definition 2.11. Let X be a manifold and $\{V_i\}_{i\in S}$ be a transverse collection of submanifolds of X. A regularization for $\{V_i\}_{i\in S}$ in X is a system of regularizations $(\Psi_I)_{I\subset S}$ for $\{V_i\}_{i\in S}$ in X such that

$$\mathrm{Dom}(\Psi_I) = \mathfrak{D}\Psi^{-1}_{I;I'}\big(\mathrm{Dom}(\Psi_{I'})\big), \quad \Psi_I = \Psi_{I'} \circ \mathfrak{D}\Psi_{I;I'}|_{\mathrm{Dom}(\Psi_I)} \tag{2.17}$$

for all $I'\subset I\subset S$.

Definition 2.12. Let X be a manifold and $\{V_i\}_{i\in S}$ be a finite transverse collection of closed submanifolds of X of codimension 2.

(1) If $\omega \in \mathrm{Symp}(X, \{V_i\}_{i\in S})$, then an ω-regularization for $\{V_i\}_{i\in S}$ in X is a tuple

$$(\mathcal{R}_I)_{I\subset S} \equiv \big((\rho_{I;i}, \nabla^{(I;i)})_{i\in I}, \Psi_I\big)_{I\subset S} \tag{2.18}$$

such that $\mathcal{R}_I$ is an ω-regularization for V_I in X for each $I\subset S$, $(\Psi_I)_{I\subset S}$ is a regularization for $\{V_i\}_{i\in S}$ in X, and the induced vector bundle isomorphisms

$$\mathfrak{D}\Psi_{I;I'}\colon \pi^*_{I;I'}\mathcal{N}_{I;I-I'}\big|_{\mathcal{N}'_{I;I'}} \longrightarrow \mathcal{N}_X V_{I'}\big|_{V_{I'}\cap\mathrm{Im}(\Psi_I)}$$

in (2.15) are product Hermitian isomorphisms for all $I'\subset I\subset S$.

(2) If B is a manifold, possibly with boundary, and $(\omega_t)_{t\in B}$ is a smooth family of symplectic forms in $\mathrm{Symp}(X, \{V_i\}_{i\in S})$, then an $(\omega_t)_{t\in B}$-family of regularizations for $\{V_i\}_{i\in S}$ in X is a smooth family of tuples

$$(\mathcal{R}_{t;I})_{t\in B, I\subset S} \equiv \big((\rho_{t;I;i}, \nabla^{(t;I;i)})_{i\in I}, \Psi_{t;I}\big)_{t\in B, I\subset S} \tag{2.19}$$

such that $(\mathcal{R}_{t;I})_{I\subset S}$ is an ω_t-regularization for $\{V_i\}_{i\in S}$ in X for each $t\in B$ and $(\mathcal{R}_{t;I})_{t\in B}$ is an $(\omega_t)_{t\in B}$-family of regularizations for V_I in X for each $I\subset S$.

Let X, $\{V_i\}_{i\in S}$, and $(\omega_t)_{t\in B}$ be as in Definition 2.12 and

$$\begin{aligned}(\mathcal{R}^{(1)}_{t;I})_{t\in B,I\subset S} &\equiv \big((\rho_{t;I;i},\nabla^{(t;I;i)})_{i\in I},\Psi^{(1)}_{t;I}\big)_{t\in B,I\subset S},\\ (\mathcal{R}^{(2)}_{t;I})_{t\in B,I\subset S} &\equiv \big((\rho_{t;I;i},\nabla^{(t;I;i)})_{i\in I},\Psi^{(2)}_{t;I}\big)_{t\in B,I\subset S}\end{aligned}$$

be two $(\omega_t)_{t\in B}$-families of regularizations for $(V_i)_{i\in S}$ in X. We define

$$(\mathcal{R}^{(1)}_{t;I})_{t\in B,I\subset S} \cong (\mathcal{R}^{(2)}_{t;I})_{t\in B,I\subset S} \tag{2.20}$$

if the two families of regularizations agree on the level of germs, i.e. there exists an $(\omega_t)_{t\in B}$-family of regularizations as in (2.19) such that

$$\mathrm{Dom}(\Psi_{t;I})\subset \mathrm{Dom}(\Psi^{(1)}_{t;I}),\mathrm{Dom}(\Psi^{(2)}_{t;I}) \qquad\text{and}\qquad \Psi_{t;I}=\Psi^{(1)}_{t;I}\big|_{\mathrm{Dom}(\Psi_{t;I})},\Psi^{(2)}_{t;I}\big|_{\mathrm{Dom}(\Psi_{t;I})}$$

for all $t\in B$ and $I\subset S$.

Definition 2.12(2) topologizes the set $\mathrm{Aux}(X,\{V_i\}_{i\in S})$ of pairs $(\omega,(\mathcal{R}_I)_{I\subset S})$ consisting of a symplectic structure ω on $\{V_i\}_{i\in S}$ in X and an ω-regularization $(\mathcal{R}_I)_{I\subset S}$ for $\{V_i\}_{i\in S}$ in X. Families of regularizations satisfying (2.20) are homotopic.

By Theorem 2.13 below, a family $(\omega_t)_{t\in B}$ of symplectic forms on X so that $\{V_i\}_{i\in S}$ is an SC symplectic divisor in (X,ω_t) can be deformed through such symplectic forms to a family $(\omega_{t,1})_{t\in B}$ which admits a family $(\widetilde{\mathcal{R}}_{t;I})_{t\in B,I\subset S}$ of regularizations for $(V_i)_{i\in S}$ in X. If $\partial B\neq\emptyset$ and the family $(\omega_t)_{t\in\partial B}$ admits a family $(\mathcal{R}_{t;I})_{t\in\partial B,I\subset S}$ of regularizations for $(V_i)_{i\in S}$ in X, then $(\omega_t)_{t\in B}$ can be deformed keeping it fixed for $t\in\partial B$ and $(\widetilde{\mathcal{R}}_{t;I})_{t\in B,I\subset S}$ can be chosen to extend $(\mathcal{R}_{t;I})_{t\in\partial B,I\subset S}$. This implies that the projection (1.12) is a weak homotopy equivalence. Furthermore, the family $(\omega_t)_{t\in B}$ can be deformed without changing the cohomology class of each ω_t or the restriction of ω_t to the complement X^* of an arbitrarily small neighborhood of the singular locus of the divisor $\{V_i\}_{i\in S}$. Since this locus is empty if $|S|=1$, the case $|S|=1$ and $X^*=X$ of Theorem 2.13 is a parametrized version of the standard Symplectic Neighborhood Theorem [19, Theorem 3.30].

Theorem 2.13. *Let X be a manifold, $\{V_i\}_{i\in S}$ be a finite transverse collection of closed submanifolds of X of codimension 2, and $X^*\subset X$ be an open subset, possibly empty, such that $\overline{X^*}\cap V_I=\emptyset$ for all $I\subset S$ with $|I|=2$. Suppose*

- *B is a compact manifold, possibly with boundary,*
- *$N(\partial B), N'(\partial B)$ are neighborhoods of ∂B in B such that $\overline{N'(\partial B)}\subset N(\partial B)$,*
- *$(\omega_t)_{t\in B}$ is a smooth family of symplectic forms in $\mathrm{Symp}^+(X,\{V_i\}_{i\in S})$, and*
- *$(\mathcal{R}_{t;I})_{t\in N(\partial B),I\subset S}$ is an $(\omega_t)_{t\in N(\partial B)}$-family of regularizations for $(V_i)_{i\in S}$ in X.*

Then there exist a smooth family $(\mu_{t,\tau})_{t\in B,\tau\in\mathbb{I}}$ of 1-forms on X such that

$$\begin{gathered}\omega_{t,\tau}\equiv\omega_t+\mathrm{d}\mu_{t,\tau}\in\mathrm{Symp}^+\big(X,\{V_i\}_{i\in S}\big)\quad\forall\,(t,\tau)\in B\times\mathbb{I},\\ \mu_{t,0}=0\quad\forall\,t\in B,\qquad \mathrm{supp}\big(\mu_{\cdot,\tau}\big)\subset\big(B-N'(\partial B)\big)\times(X-X^*)\quad\forall\,\tau\in\mathbb{I},\end{gathered}$$

and an $(\omega_{t,1})_{t\in B}$-family $(\widetilde{\mathcal{R}}_{t;I})_{t\in B,I\subset S}$ of regularizations for $(V_i)_{i\in S}$ in X such that

$$\big(\widetilde{\mathcal{R}}_{t;I}\big)_{t\in N'(\partial B),I\subset S}\cong\big(\mathcal{R}_{t;I}\big)_{t\in N'(\partial B),I\subset S}.$$

This theorem is an immediate consequence of Theorem 2.17 applied to

- the N-fold transverse configuration $\{X_I\}_{I\in\mathcal{P}^*(N)}$ and the family $(\omega_{t;i})_{t\in B,i\in[N]}$ of elements of $\mathrm{Symp}^+(\{X_I\}_{I\in\mathcal{P}^*(N)})$ induced by $(X,\{V_i\}_{i\in S})$ and $(\omega_t)_{t\in B}$ as in Example 2.6,
- the family $(\mathfrak{R}_t)_{t\in N(\partial B)}$ of regularizations for $\{X_I\}_{I\in\mathcal{P}^*(N)}$ induced by $(\mathcal{R}_{t;I})_{t\in N(\partial B),I\subset S}$ as in Example 2.16.

The family of tuples $(\widetilde{\mathcal{R}}_{t;I})_{t\in B}$ with $I\in\mathcal{P}_N(N)$ provided by Theorem 2.17 then satisfies the requirements of Theorem 2.13.

Theorem 2.13 can also be obtained without going through Theorem 2.17. The argument would be fundamentally the same, but Corollary 3.3 would no longer be needed and Lemma 4.4 would suffice in place of Proposition 4.2.

2.3 Regularizations for SC symplectic varieties

In this section, we define a regularization for a transverse configuration $\mathbf{X}$ of manifolds with a symplectic structure $(\omega_i)_{i\in[N]}$ as a tuple of ω_i-regularizations for $\{X_{ij}\}_{j\in[N]-i}$ in X_i that agree on the overlaps; see (2.22) and Definition 2.15(1). We conclude with Theorem 2.17: the space of SC symplectic varieties in the sense of Definition 2.5 is weakly homotopy equivalent to the space of those with regularizations.

Suppose $\{X_I\}_{I\in\mathcal{P}^*(N)}$ is a transverse configuration in the sense of Definition 2.3. For each $I\in\mathcal{P}^*(N)$ with $|I|\geq 2$, let

$$\pi_I\colon \mathcal{N}X_I\equiv\bigoplus_{i\in I}\mathcal{N}_{X_{I-i}}X_I\longrightarrow X_I\,.$$

If in addition $I'\subset I$, let

$$\pi_{I;I'}\colon \mathcal{N}_{I;I'}\equiv\bigoplus_{i\in I-I'}\mathcal{N}_{X_{I-i}}X_I\longrightarrow X_I\,.$$

By the last isomorphism in (2.3) with $X=X_i$ for any $i\in I'$ and $\{V_j\}_{j\in S}=\{X_{ij}\}_{j\in[N]-i}$,

$$\mathcal{N}_{I;I'}=\mathcal{N}_{X_{I'}}X_I\qquad\forall\ I'\subset I\subset[N],\ I'\neq\emptyset.$$

Similarly to (2.13), there are canonical identifications

$$\mathcal{N}_{I;I-I'}=\mathcal{N}X_{I'}|_{X_I},\quad \mathcal{N}X_I=\pi_{I;I'}^*\mathcal{N}_{I;I-I'}=\pi_{I;I'}^*\mathcal{N}X_{I'}\qquad\forall\ I'\subset I\subset[N];\tag{2.21}$$

the first and last identities above hold if $|I'|\geq 2$.

Definition 2.14. Let $N\in\mathbb{Z}^+$ and $\mathbf{X}=\{X_I\}_{I\in\mathcal{P}^*(N)}$ be a transverse configuration. A regularization for $\mathbf{X}$ is a tuple $(\Psi_{I;i})_{i\in I\subset[N]}$, where for each $i\in I$ the tuple $(\Psi_{I;i})_{I\in\mathcal{P}_i(N)}$ is a regularization for $\{X_{ij}\}_{j\in[N]-i}$ in X_i in the sense of Definition 2.11, such that

$$\Psi_{I;i_1}\big|_{\mathcal{N}_{I;i_1i_2}\cap\mathrm{Dom}(\Psi_{I;i_1})}=\Psi_{I;i_2}\big|_{\mathcal{N}_{I;i_1i_2}\cap\mathrm{Dom}(\Psi_{I;i_2})}\tag{2.22}$$

for all $i_1,i_2\in I\subset[N]$.

Given a regularization as in Definition 2.14 and $I'\subset I\subset[N]$ with $|I|\ge 2$ and $I'\neq\emptyset$, let

$$\mathcal{N}'_{I;I'}=\mathcal{N}_{I;I'}\cap\mathrm{Dom}(\Psi_{I;i}),\quad \Psi_{I;I'}=\Psi_{I;i}|_{\mathcal{N}'_{I;I'}}\colon \mathcal{N}'_{I;I'}\longrightarrow X_{I'} \qquad \text{if } i\in I'; \tag{2.23}$$

by (2.22), $\Psi_{I;I'}(v)$ does not depend on the choice of $i\in I'$. Let

$$\mathfrak{D}\Psi_{I;i;I'}\colon \pi^*_{I;I'}\mathcal{N}_{I;i\cup(I-I')}\big|_{\mathcal{N}'_{I;I'}}\longrightarrow \mathcal{N}_{I';i}\big|_{\mathrm{Im}(\Psi_{I;I'})} \tag{2.24}$$

be the associated vector bundle isomorphism as in (2.15). If $|I'|\ge 2$, we define an isomorphism of split vector bundles

$$\begin{aligned}&\mathfrak{D}\Psi_{I;I'}\colon \pi^*_{I;I'}\mathcal{N}_{I;I-I'}\big|_{\mathcal{N}'_{I;I'}}\longrightarrow \mathcal{N}X_{I'}\big|_{\mathrm{Im}(\Psi_{I;I'})},\\ &\mathfrak{D}\Psi_{I;I'}\big|_{\pi^*_{I;I'}\mathcal{N}_{I;i\cup(I-I')}|_{\mathcal{N}'_{I;I'}}}=\mathfrak{D}\Psi_{I;i;I'}\quad\forall\, i\in I';\end{aligned} \tag{2.25}$$

by (2.22), the last maps agree on the overlaps.

Definition 2.15. Let $N\in\mathbb{Z}^+$ and $\mathbf{X}\equiv\{X_I\}_{I\in\mathcal{P}^*(N)}$ be a transverse configuration.

(1) If $(\omega_i)_{i\in[N]}$ is a symplectic structure on $\mathbf{X}$ in the sense of Definition 2.4, an $(\omega_i)_{i\in[N]}$-regularization for $\mathbf{X}$ is a tuple

$$\mathfrak{R}\equiv(\mathcal{R}_I)_{I\in\mathcal{P}^*(N)}\equiv\big(\rho_{I;i},\nabla^{(I;i)},\Psi_{I;i}\big)_{i\in I\subset[N]} \tag{2.26}$$

such that $(\Psi_{I;i})_{i\in I\subset[N]}$ is a regularization for $\mathbf{X}$ in the sense of Definition 2.14 and for each $i\in[N]$ the tuple

$$\big((\rho_{I;j},\nabla^{(I;j)})_{j\in I-i},\Psi_{I;i}\big)_{I\in\mathcal{P}_i(N)}$$

is an ω_i-regularization for $\{X_{ij}\}_{j\in[N]-i}$ in X_i in the sense of Definition 2.12(1).

(2) If B is a smooth manifold, possibly with boundary, and $(\omega_{t;i})_{t\in B,i\in[N]}$ is a smooth family of symplectic structures on $\mathbf{X}$, then an $(\omega_{t;i})_{t\in B,i\in[N]}$-family of regularizations for $\mathbf{X}$ is a family of tuples

$$(\mathfrak{R}_t)_{t\in B}\equiv(\mathcal{R}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}\equiv\big(\rho_{t;I;i},\nabla^{(t;I;i)},\Psi_{t;I;i}\big)_{t\in B,i\in I\subset[N]} \tag{2.27}$$

such that $(\mathcal{R}_{t;I})_{I\in\mathcal{P}^*(N)}$ is an $(\omega_{t;i})_{i\in[N]}$-regularization for $\mathbf{X}$ for each $t\in B$ and for each $i\in[N]$ the tuple

$$\big((\rho_{t;I;j},\nabla^{(t;I;j)})_{j\in I-i},\Psi_{t;I;i}\big)_{t\in B,I\in\mathcal{P}_i(N)}$$

is an $(\omega_{t;i})_{t\in B}$-family of regularizations for $\{X_{ij}\}_{j\in[N]-i}$ in X_i in the sense of Definition 2.12(2).

The assumptions in Definition 2.15(1) imply that the corresponding isomorphisms (2.25) are product Hermitian isomorphisms covering the maps (2.23).

Example 2.16. Suppose X is a manifold, $\{V_i\}_{i\in S}$ is a transverse collection of closed submanifolds of X of codimension 2, $(\omega_t)_{t\in B}$ is a smooth family of symplectic structures on $\{V_i\}_{i\in S}$ in X, and $(\mathcal{R}_{t;I})_{t\in B,I\subset S}$ is an $(\omega_t)_{t\in B}$-family of regularizations for $\{V_i\}_{i\in S}$ in X as in (2.19). Let $\mathbf{X}$ and $(\omega_{t;i})_{t\in B,i\in[N]}$ be the associated transverse configuration and family of symplectic structures on it constructed as in Example 2.6. Denote by $(\rho_{\mathbb{C}},\nabla^{(\mathbb{C})})$ the standard Hermitian structure on $\mathbb{C}$. With notation as in Example 2.6, for $i\in I\subset[N]$ define

$$\widetilde{\Psi}_{t;I;i}=\begin{cases}(\Psi_{t;I;i},\mathrm{id}_{\mathbb{C}}) & \text{if } i\neq N;\\ \Psi_{t;I}, & \text{if } i=N;\end{cases}\qquad (\widetilde{\rho}_{t;I;i},\widetilde{\nabla}^{(t;I;i)})=\begin{cases}\pi_1^*(\rho_{t;I;i},\nabla^{(t;I;i)}) & \text{if } i\neq N;\\ \pi_2^*(\rho_{\mathbb{C}},\nabla^{(\mathbb{C})}), & \text{if } i=N.\end{cases}$$

The tuple

$$(\mathfrak{R}_t)_{t\in B} \equiv (\widetilde{\mathcal{R}}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)} \equiv \big(\widetilde{\rho}_{t;I;i}, \widetilde{\nabla}^{(t;I;i)}, \widetilde{\Psi}_{t;I;i}\big)_{t\in B,i\in I\subset[N]}$$

is then an $(\omega_{t;i})_{t\in B,i\in[N]}$-family of regularizations for $\mathbf{X}$.

Let $\mathbf{X}\equiv\{X_I\}_{I\in\mathcal{P}^*(N)}$ be an N-fold transverse configuration such that X_{ij} is a closed submanifold of X_i of codimension 2 for all $i,j\in[N]$ distinct and $(\omega_{t;i})_{t\in B,i\in[N]}$ be a family of symplectic structures on $\mathbf{X}$. Suppose the tuples

$$\begin{aligned}\big(\mathfrak{R}_t^{(1)}\big)_{t\in B} &\equiv \big(\rho_{t;I;i}, \nabla^{(t;I;i)}, \Psi^{(1)}_{t;I;i}\big)_{t\in B,i\in I\subset[N]},\\ \big(\mathfrak{R}_t^{(2)}\big)_{t\in B} &\equiv \big(\rho_{t;I;i}, \nabla^{(t;I;i)}, \Psi^{(2)}_{t;I;i}\big)_{t\in B,i\in I\subset[N]}\end{aligned}$$

are $(\omega_{t;i})_{t\in B,i\in[N]}$-families of regularizations for $\mathbf{X}$. We define

$$\big(\mathfrak{R}_t^{(1)}\big)_{t\in B} \cong \big(\mathfrak{R}_t^{(2)}\big)_{t\in B} \tag{2.28}$$

if the two families of regularizations agree on the level of germs, i.e. there exists an $(\omega_{t;i})_{t\in B,i\in[N]}$-family of regularizations as in (2.27) such that

$$\mathrm{Dom}(\Psi_{t;I;i}) \subset \mathrm{Dom}(\Psi^{(1)}_{t;I;i}), \mathrm{Dom}(\Psi^{(2)}_{t;I;i}), \quad \Psi_{t;I;i} = \Psi^{(1)}_{t;I;i}\big|_{\mathrm{Dom}(\Psi_{t;I;i})}, \Psi^{(2)}_{t;I;i}\big|_{\mathrm{Dom}(\Psi_{t;I;i})}$$

for all $t\in B$ and $i\in I\subset[N]$.

Definition 2.15(2) topologizes the set $\mathrm{Aux}(\mathbf{X})$ of pairs $((\omega_i)_{i\in[N]},\mathfrak{R})$ consisting of a symplectic structure $(\omega_i)_{i\in[N]}$ on $\mathbf{X}$ and an $(\omega_i)_{i\in[N]}$-regularization $\mathfrak{R}$ for $\mathbf{X}$. Families of regularizations satisfying (2.28) are homotopic. By the following theorem, the projection map

$$\mathrm{Aux}(\mathbf{X}) \longrightarrow \mathrm{Symp}^+(\mathbf{X}), \qquad \big((\omega_i)_{i\in[N]},\mathfrak{R}\big) \longrightarrow (\omega_i)_{i\in[N]},$$

is a weak homotopy equivalence.

Theorem 2.17. *Let $N\in\mathbb{Z}^+$, $\mathbf{X}\equiv\{X_I\}_{I\in\mathcal{P}^*(N)}$ be a transverse configuration such that X_{ij} is a closed submanifold of X_i of codimension 2 for all $i,j\in[N]$ distinct, and $X_i^*\subset X_i$ for each $i\in[N]$ be an open subset, possibly empty, such that $\overline{X_i^*}\cap X_I=\emptyset$ for all $i\in I\subset[N]$ with $|I|=3$. Suppose*

- *B, $N(\partial B)$, and $N'(\partial B)$ are as in Theorem 2.13,*
- *$(\omega_{t;i})_{t\in B,i\in[N]}$ is a smooth family of elements of $\mathrm{Symp}^+(\mathbf{X})$, and*
- *$(\mathfrak{R}_t)_{t\in N(\partial B)}$ is an $(\omega_{t;i})_{t\in N(\partial B),i\in[N]}$-family of regularizations for $\mathbf{X}$.*

Then there exist a smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ of 1-forms on $X_\emptyset$ such that

$$\begin{gathered}\big(\omega_{t,\tau;i}\equiv\omega_{t;i}+\mathrm{d}\mu_{t,\tau;i}\big)_{i\in[N]} \in \mathrm{Symp}^+(\mathbf{X}) \quad \forall\,(t,\tau)\in B\times\mathbb{I},\\ \mu_{t,0;i}=0 \quad \forall\, t\in B,\, i\in[N], \quad \mathrm{supp}\big(\mu_{\cdot,\tau;i}\big) \subset \big(B-N'(\partial B)\big)\times(X_i-X_i^*) \quad \forall\,\tau\in\mathbb{I},\, i\in[N],\end{gathered} \tag{2.29}$$

and an $(\omega_{t,1;i})_{t\in B,i\in[N]}$-family $(\widetilde{\mathfrak{R}}_t)_{t\in B}$ of regularizations for $\mathbf{X}$ such that

$$\big(\widetilde{\mathfrak{R}}_t\big)_{t\in N'(\partial B)} \cong \big(\mathfrak{R}_t\big)_{t\in N'(\partial B)}. \tag{2.30}$$

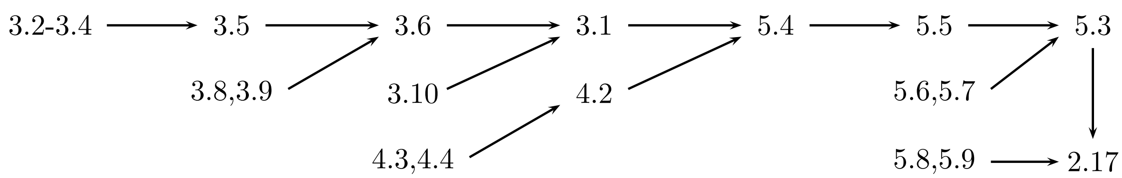

Figure 2: The statements used in the proof of Theorem 2.17.

This theorem is proved in Section 5 by induction on the strata of $\mathbf{X}$ using the essentially local notion of a weak regularization of Definition 5.2. By Proposition 5.3 and Corollary 5.5, a family of elements of $\mathrm{Symp}^+(\mathbf{X})$ with compatible *weak* regularizations over an open subset W of X which contains all X_I with $I \supsetneq I^*$ extends over a neighborhood of all of X_{I^*}. Lemma 5.4 implements the deformations for symplectic forms on split vector bundles obtained in Theorem 3.1 via Proposition 4.2; the latter is a stratified version of the Tubular Neighborhood Theorem that respects symplectic forms along the base. Proposition 3.6, the main step in the proof of Theorem 3.1, makes use of the compatibility-of-orientations assumptions in Definitions 2.1 and 2.5. By Lemma 5.8 and Corollary 5.9, weak regularizations and equivalences between them can be cut down to regularizations and equivalences between regularizations. The connections between the different parts of the proof of Theorem 2.17 are indicated in Figure 2.

3 Deformations of structures on vector bundles

Let V be an oriented manifold, I be a finite set, $L_i \longrightarrow V$ be an oriented rank 2 real vector bundle for each $i \in I$, and

$$\pi\colon \mathcal{N} \equiv \bigoplus_{i\in I} L_i \longrightarrow V\,. \tag{3.1}$$

We show that a symplectic structure $\widetilde{\omega}$ on a neighborhood $\mathcal{N}'$ of V in $\mathcal{N}$ can be deformed, keeping it fixed outside of a smaller neighborhood $\mathcal{N}''$ and keeping all natural submanifolds $\mathcal{N}_{I'}$ symplectic, to a very standard symplectic structure $\widehat{\omega}^\bullet$ near V as long as $\widetilde{\omega}$ satisfies a simple topological condition. By Proposition 3.6, this can be done for a symplectic structure $\widehat{\omega}$ on $\mathcal{N}'$ induced in a standard way from a symplectic form ω on V and a fiberwise symplectic structure Ω on $\mathcal{N}$. By Lemma 3.10, any symplectic structure $\widetilde{\omega}$ on a neighborhood $\mathcal{N}'$ of V in $\mathcal{N}$ can be deformed, keeping it fixed outside of a smaller neighborhood $\mathcal{N}''$ and keeping the submanifolds $\mathcal{N}_{I'}$ symplectic, so that it restricts to a standard symplectic structure $\widehat{\omega}$ on a smaller neighborhood $\widetilde{\mathcal{N}}$. The main statement of this section is Theorem 3.1; the remaining statements are used in its proof, but not in the remainder of the paper.

By Theorem 3.1, a finite collection $\{V_i\}_{i\in I}$ of smooth symplectic divisors in (X,ω) intersecting positively at V_I can be deformed inside an arbitrarily small neighborhood W_I of V_I so that the pairwise intersections $V_i\cap V_j$ are symplectically orthogonal inside W_I. For $|I|=2$ and V_I compact, this is [10, Lemma 2.3]. The compactness assumption is technical and is not fundamental to the three-page proof in [10], but the condition $|I|=2$ is. The latter is clearly illustrated by the one-page proof of [27, Lemma 3.2.3] treating the $V_I=\{\mathrm{pt}\}$ case of [10, Lemma 2.3] (and thus $\dim_{\mathbb{R}}X=4$). Our proof of Proposition 3.6, the main ingredient in the proof Theorem 3.1, follows a completely different approach. It starts with the linear algebra observation of Lemma 3.2 and deforms the symplectic forms in three stages as described below (3.16) and indicated in Figure 4.

3.1 Notation and key statement

For a finite set I, denote by $\mathcal{P}^*(I)$ the collection of non-empty subsets of I. With $\mathcal{N}$ as in (3.1), let

$$\mathcal{N}_{I'}=\bigoplus_{i\in I-I'}L_i \quad \forall\, I'\subset I, \qquad \mathcal{N}_\partial=\bigcup_{i\in I}\mathcal{N}_i\,. \tag{3.2}$$

For any $\mathcal{N}'\subset\mathcal{N}$, we define

$$\mathcal{N}'_{I'}=\mathcal{N}_{I'}\cap\mathcal{N}' \quad \forall\, I'\subset I, \qquad \mathcal{N}'_\partial=\mathcal{N}_\partial\cap\mathcal{N}'\,.$$

For any neighborhood $\mathcal{N}'$ of V in $\mathcal{N}$, $\{\mathcal{N}'_{I'}\}_{I'\in\mathcal{P}^*(I)}$ is a transverse configuration in the sense of Definition 2.3 such that $\mathcal{N}'_{ij}$ is a closed submanifold of $\mathcal{N}'_i$ of codimension 2 for all $i,j\in I$ distinct.

For $k\in\mathbb{Z}^{\ge 0}$, denote by

$$\pi\colon \Lambda^k_{\mathbb{C}}\mathcal{N}^*\longrightarrow V \qquad \text{and} \qquad \pi\colon \Lambda^k_{\mathbb{C}}\mathcal{N}_i^*\longrightarrow V, \quad i\in I,$$

the bundles of alternating k-tensors on $\mathcal{N}$ and $\mathcal{N}_i$, respectively. For a tensor α on $\mathcal{N}$ and $j\in I$, we view $\alpha|_{L_j}$ as a tensor on $\mathcal{N}$ via the projection to L_j. For a tensor α_i on $\mathcal{N}_i$ and $j\in I-\{i\}$, we view $\alpha_i|_{L_j}$ as a tensor on $\mathcal{N}_i$ via the projection to L_j. For such α and α_i, let

$$\alpha^\bullet=\sum_{j\in I}\alpha|_{L_j}\in\Lambda^k_{\mathbb{C}}\mathcal{N}^* \qquad \text{and} \qquad \alpha_i^\bullet=\sum_{j\in I-\{i\}}\alpha_i|_{L_j}\in\Lambda^k_{\mathbb{C}}\mathcal{N}_i^* \tag{3.3}$$

be the **diagonal parts** of α and α_i. Define

$$\Lambda^k_{\mathbb{C}}\mathcal{N}^*_\partial\equiv\Big\{(\alpha_i)_{i\in I}\in\bigoplus_{i\in I}\Lambda^k_{\mathbb{C}}\mathcal{N}_i^*\colon \alpha_{i_1}\big|_{\mathcal{N}_{i_1i_2}|_{\pi(\alpha_{i_1})}}=\alpha_{i_2}\big|_{\mathcal{N}_{i_1i_2}|_{\pi(\alpha_{i_2})}}\ \forall\, i_1,i_2\in I\Big\}\longrightarrow V\,.$$

This subspace is preserved under taking the diagonal part. Denote by

$$r_{\mathcal{N};\partial}\colon \Lambda^k_{\mathbb{C}}\mathcal{N}^*\longrightarrow\Lambda^k_{\mathbb{C}}\mathcal{N}^*_\partial$$

the natural restriction homomorphism. It commutes with taking the diagonal part.

We call a section $(\Omega_i)_{i\in I}$ of $\Lambda^k_{\mathbb{C}}\mathcal{N}^*_\partial$ a **fiberwise k-form on $\mathcal{N}_\partial$**. Each Ω_i is then a fiberwise linear k-form on $\mathcal{N}_i$ and

$$\Omega_{i_1}\big|_{\mathcal{N}_{i_1i_2}}=\Omega_{i_2}\big|_{\mathcal{N}_{i_1i_2}} \qquad \forall\, i_1,i_2\in I\,.$$

By Lemma 3.8, any such form is the restriction of a fiberwise k-form on $\mathcal{N}$. We call a fiberwise 2-form $(\Omega_i)_{i\in I}$ on $\mathcal{N}_\partial$ a **fiberwise symplectic form** if each Ω_i is a symplectic form on each fiber of $\mathcal{N}_i$. Let

$$\mathrm{Symp}^+_V(\mathcal{N}_\partial)\equiv\mathrm{Symp}^+_V\big(\{\mathcal{N}_{I'}\}_{I'\in\mathcal{P}^*(I)}\big)$$

be the subspace of fiberwise symplectic forms $(\Omega_i)_{i\in I}$ on $\mathcal{N}_\partial$ such that for all $i\in I'\subset I$ the fiberwise 2-form $\Omega_i|_{\mathcal{N}_{I'}}$ is symplectic and the Ω_i-orientation of each fiber of $\mathcal{N}_{I'}$ agrees with its canonical orientation, i.e. the one induced by the orientations of L_i.

Let $\mathcal{N}'$ be a neighborhood of V in $\mathcal{N}$. We call a tuple $(\widetilde{\omega}_i)_{i\in I}$ a (closed) k-form on $\mathcal{N}'_\partial$ if each $\widetilde{\omega}_i$ is a (closed) k-form on $\mathcal{N}'_i$ and

$$\widetilde{\omega}_{i_1}\big|_{T\mathcal{N}'_{i_1i_2}}=\widetilde{\omega}_{i_2}\big|_{T\mathcal{N}'_{i_1i_2}} \qquad \forall\, i_1,i_2\in I\,.$$

By a symplectic structure on $\mathcal{N}'_\partial$, we mean an element $(\widetilde{\omega}_i)_{i\in I}$ of $\mathrm{Symp}(\{\mathcal{N}'_{I'}\}_{I'\in\mathcal{P}^*(I)})$, i.e. a closed 2-form on $\mathcal{N}'_\partial$ which restricts to a symplectic form on $\mathcal{N}'_{I'}$ for each $I'\in\mathcal{P}^*(I)$. Let

$$\mathrm{Symp}^+\big(\mathcal{N}'_\partial\big)\equiv\mathrm{Symp}^+\big(\{\mathcal{N}'_{I'}\}_{I'\in\mathcal{P}^*(I)}\big)$$

be the subspace of symplectic structures $(\widetilde{\omega}_i)_{i\in I}$ on $\mathcal{N}'_\partial$ such that for all $i\in I'\subset I$ the $\widetilde{\omega}_i$-orientation of $\mathcal{N}_{I'}$ agrees with its canonical orientation, i.e. the one induced by the orientations of V and L_i.

A symplectic structure $(\widetilde{\omega}_i)_{i\in I}$ on $\mathcal{N}'_\partial$ restricts to a symplectic form ω on V and determines fiberwise symplectic structures $(\Omega_i)_{i\in I}$ and $(\Omega_i^\bullet)_{i\in I}$ on $\mathcal{N}_\partial$ via (1.3) and (3.3). If $(\widetilde{\omega}_i)_{i\in I}$ lies in $\mathrm{Symp}^+(\mathcal{N}'_\partial)$, then

$$(\Omega_i)_{i\in I},(\Omega_i^\bullet)_{i\in I}\in\mathrm{Symp}_V^+\big(\mathcal{N}_\partial\big)\,.$$

We call $(\Omega_i^\bullet)_{i\in I}$ the diagonalized fiberwise 2-form on $\mathcal{N}_\partial$ determined by $(\widetilde{\omega}_i)_{i\in I}$. A tuple $(\nabla^{(i)})_{i\in I}$ of connections on L_i determines a connection ∇ on each $\mathcal{N}_i$. We call the 2-form $(\widehat{\omega}_i^\bullet)_{i\in I}$ on $\mathcal{N}_\partial$ determined by ω, $(\Omega_i^\bullet)_{i\in I}$, and these connections ∇ as in (2.9) the diagonalized 2-form on $\mathcal{N}_\partial$ determined by $(\widetilde{\omega}_i)_{i\in I}$ and $(\nabla^{(i)})_{i\in I}$.

Theorem 3.1. *Let V be a manifold, I be a finite set with $|I|\ge 2$, $L_i\longrightarrow V$ be an oriented rank 2 real vector bundle for each $i\in I$, and $U\subset V$ be an open subset, possibly empty. Suppose*

- *B is a compact manifold, possibly with boundary, $N(\partial B)$ is a neighborhood of ∂B in B, and $\mathcal{N}'$ is a neighborhood of V in $\mathcal{N}$,*
- *$(\nabla^{(t;i)})_{t\in B}$ is a smooth family of connections on L_i for each $i\in I$,*
- *$(\widetilde{\omega}_{t;i})_{t\in B,i\in I}$ is a smooth family in $\mathrm{Symp}^+(\mathcal{N}'_\partial)$ such that*

$$\big(\widetilde{\omega}_{t;i}\big)_{i\in I}=\big(\widehat{\omega}_{t;i}^\bullet|_{\mathcal{N}'_i}\big)_{i\in I}\quad\forall\,t\in N(\partial B),\quad \big(\widetilde{\omega}_{t;i}|_{\mathcal{N}'_i|_U}\big)_{i\in I}=\big(\widehat{\omega}_{t;i}^\bullet|_{\mathcal{N}'_i|_U}\big)_{i\in I}\quad\forall\,t\in B, \tag{3.4}$$

where $(\widehat{\omega}_{t;i}^\bullet)_{t\in B,i\in I}$ is the smooth family of diagonalized 2-forms on $\mathcal{N}_\partial$ determined by $(\widetilde{\omega}_{t;i})_{t\in B,i\in I}$ and $(\nabla^{(t;i)})_{t\in B,i\in I}$.

Then there exist neighborhoods $\widehat{\mathcal{N}}\subset\mathcal{N}''$ of V in $\mathcal{N}'$ such that $\overline{\mathcal{N}''}\subset\mathcal{N}'$ and a smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in I}$ of 1-forms on $\mathcal{N}'_\partial$ such that

$$\big(\widetilde{\omega}_{t,\tau;i}\big)_{i\in I}\equiv\big(\widetilde{\omega}_{t;i}+\mathrm{d}\mu_{t,\tau;i}\big)_{i\in I} \tag{3.5}$$

is a symplectic structure on $\mathcal{N}'_\partial$ for all $(t,\tau)\in B\times\mathbb{I}$ and

$$\mu_{t,0;i}=0,\quad \widetilde{\omega}_{t,\tau;i}|_V=\widetilde{\omega}_{t;i}|_V,\quad \widetilde{\omega}_{t,1;i}|_{\widehat{\mathcal{N}}_i}=\widehat{\omega}_{t;i}^\bullet|_{\widehat{\mathcal{N}}_i},\quad \mathrm{supp}\big(\mu_{\cdot,\tau;i}\big)\subset\big(B-N(\partial B)\big)\times\mathcal{N}''|_{V-U} \tag{3.6}$$

for all $t\in B$, $\tau\in\mathbb{I}$, and $i\in I$.

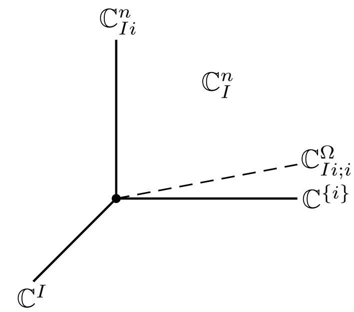

Figure 3: An illustration for the proof of Lemma 3.2.

3.2 Some linear algebra

This section collects some basic, but crucial, observations. Lemmas 3.2 and 3.4 can be seen as versions of [20, Lemmas 5.5,5.8]. According to these lemmas and Corollary 3.3, every linear 2-form form Ω on $\mathbb{C}^n$ such that

$$\big(\Omega|_{\mathbb{C}^n_i}\big)_{i\in[n]} \in \mathrm{Symp}^+_{\{0\}}\big(\mathbb{C}^n_\partial\big) \subset \mathrm{Symp}^+\big(\mathbb{C}^n_\partial\big),$$

i.e. $\Omega|_{\mathbb{C}^n_I}$ is symplectic and induces the complex orientation of $\mathbb{C}^n_I$ for every $I\in\mathcal{P}^*(n)$, can be homotoped in a canonical way to the standard symplectic form

$$\Omega_{\mathrm{std}} \equiv \mathrm{d}x_1\wedge\mathrm{d}y_1 + \ldots + \mathrm{d}x_n\wedge\mathrm{d}y_n$$

while keeping each coordinate subspace $\mathbb{C}^n_I$ symplectic. For a 2-form form Ω on $\mathbb{C}^n$ and $s\in\mathbb{R}$, let

$$\Omega_{i;s} = \Omega + s\,\mathrm{d}x_i\wedge\mathrm{d}y_i \quad \forall\, i\in[n], \qquad \Omega_s = \Omega + s\,\Omega_{\mathrm{std}}.$$

Lemma 3.2. *Let Ω be a linear symplectic form on $\mathbb{C}^n$ such that $\Omega|_{\mathbb{C}^n_I}$ is symplectic for every $I\in\mathcal{P}(n)$. If the Ω-orientation of $\mathbb{C}^n_I$ agrees with its complex orientation for every $I\in\mathcal{P}(n)$, then $\Omega_{i;s}|_{\mathbb{C}^n_I}$ is symplectic for all $I\in\mathcal{P}(n)$, $s\in\mathbb{R}^{\ge 0}$, and $i\in[n]$.*

Proof. If $i\in I$, then $\Omega_{i;s}|_{\mathbb{C}^n_I}=\Omega|_{\mathbb{C}^n_I}$ and there is nothing to prove. Suppose $i\notin I$, as in Figure 3. Let $\mathbb{C}^\Omega_{Ii;i}\subset\mathbb{C}^n_I$ be the Ω-orthogonal complement of $\mathbb{C}^n_{Ii}$. Since the Ω-orientations of $\mathbb{C}^n_{Ii}$ and $\mathbb{C}^n_{Ii}\oplus\mathbb{C}^\Omega_{Ii;i}$ agree with the complex orientations of $\mathbb{C}^n_{Ii}$ and $\mathbb{C}^n_I$, respectively, the Ω-orientation of $\mathbb{C}^\Omega_{Ii;i}$ is the same as the orientation induced by the restriction of $\mathrm{d}x_i\wedge\mathrm{d}y_i$. It follows that the restrictions of $\Omega_{i;s}$ to $\mathbb{C}^\Omega_{Ii;i}$ and $\mathbb{C}^n_I$ are symplectic. □

Corollary 3.3. *Let Ω be a linear 2-form on $\mathbb{C}^n$ such that $\Omega|_{\mathbb{C}^n_I}$ is symplectic for every $I\in\mathcal{P}^*(n)$. If the Ω-orientation of $\mathbb{C}^n_I$ agrees with its complex orientation for every $I\in\mathcal{P}^*(n)$, then $\Omega_{i;s}|_{\mathbb{C}^n_I}$ is symplectic for all $I\in\mathcal{P}^*(n)$, $s\in\mathbb{R}^{\ge 0}$, and $i\in[n]$.*

Proof. If $i\in I$, $\Omega_{i;s}|_{\mathbb{C}^n_I}=\Omega|_{\mathbb{C}^n_I}$ and there is nothing to prove. If $j\in I-i$, the claim follows from Lemma 3.2 with n replaced by $n-1$ (drop j from I and $[n]$). □

Lemma 3.4. *If Ω and Ω° are 2-forms on $\mathbb{C}^n$, then there exists $s_0\in\mathbb{R}^{\ge 0}$ such that $(\Omega_s+\tau\Omega^\circ)|_{\mathbb{C}^n_I}$ is symplectic for all $I\in\mathcal{P}(n)$, $\tau\in\mathbb{I}$, and $s\ge s_0$.*

Proof. This statement is equivalent to the restriction of the 2-form $\Omega_{\mathrm{std}}+\frac{1}{s}\Omega+\frac{\tau}{s}\Omega^\circ$ to each $\mathbb{C}^n_I$ being symplectic for all s sufficiently large. This is clear, since being symplectic is an open condition. □

Corollary 3.5. *Let V, I, $\mathcal{N}$, and B be as in Theorem 3.1 and $(\Omega_t)_{t\in B,i\in I}$ be a smooth family of fiberwise 2-forms on $\mathcal{N}$ such that $(\Omega_t|_{\mathcal{N}_i})_{i\in I}\in\mathrm{Symp}^+_V(\mathcal{N}_\partial)$ for every $t\in B$.*

(1) For all $t\in B$ and $s\ge 0$, $((\Omega_t+s\Omega^\bullet_t)|_{\mathcal{N}_i})_{i\in I}\in\mathrm{Symp}^+_V(\mathcal{N}_\partial)$.

(2) For every smooth family $(\Omega^\circ_t)_{t\in B}$ of fiberwise 2-forms on $\mathcal{N}$ and every compact subset $K\subset V$, there exist $s_0\in\mathbb{R}^{\ge 0}$ such that

$$\big((\Omega_t+s\Omega^\bullet_t+\tau\Omega^\circ_t)|_{\mathcal{N}_i}\big)_{i\in I}\in\mathrm{Symp}^+_K\big(\mathcal{N}_\partial|_K\big)\qquad\forall\,t\in B,\ \tau\in\mathbb{I},\ s\in\mathbb{R}^{\ge 0},\ s\ge s_0.$$

Proof. This follows immediately from Corollary 3.3 and Lemma 3.4. □

3.3 Deformations of standard structures

Let ω be a symplectic form on a manifold V and $\mathcal{N}_{I'}\subset\mathcal{N}$ be as in (3.2). For a fiberwise 2-form $(\Omega_i)_{i\in I}$ on $\mathcal{N}_\partial$, (2.9) induces a closed 2-form $(\widehat{\omega}_i)_{i\in I}$ on $\mathcal{N}_\partial$. If $(\Omega_i)_{i\in I}$ is an element of $\mathrm{Symp}^+_V(\mathcal{N}_\partial)$, then

- $(\Omega_i)_{i\in I}$ induces a fiberwise symplectic form on the subbundle $\mathcal{N}_{I'}$ compatible with its canonical orientation for every $I'\in\mathcal{P}^*(I)$,
- $(\Omega^\bullet_i)_{i\in I}$ is a fiberwise symplectic form on $\mathcal{N}_\partial$, and
- $\widehat{\omega}_i|_{\mathcal{N}'_{I'}}$ is a symplectic form for all $i\in I'\subset I$ and for some neighborhood $\mathcal{N}'$ of $V\subset\mathcal{N}$.

By Proposition 3.6 below, the tuple $(\widehat{\omega}_i)_{i\in I}$ can then be deformed, while keeping it fixed outside of some neighborhood $\mathcal{N}''\subsetneq\mathcal{N}'$ and keeping all submanifolds $\mathcal{N}'_{I'}$ with $I'\in\mathcal{P}^*(I)$ symplectic, to a symplectic form $(\widehat{\omega}_{1;i})_{i\in I}$ on $\mathcal{N}'_\partial$ so that $(\widehat{\omega}_{1;i})_{i\in I}$ agrees with the 2-form $(\widehat{\omega}^\bullet_i)_{i\in I}$ induced by $(\Omega^\bullet_i)_{i\in I}$ on a smaller neighborhood $\widehat{\mathcal{N}}_\partial\subset\mathcal{N}''_\partial$ of V.

Proposition 3.6. *Let $U\subset V$, I, $\mathcal{N}$, $N(\partial B)\subset B$, and $(\nabla^{(t;i)})_{t\in B,i\in I}$ be as in Theorem 3.1. Suppose*

- *$(\omega_t)_{t\in B}$ is a smooth family of symplectic forms on V,*
- *$(\Omega_{t;i})_{t\in B,i\in I}$ and $(\Omega'_{t;i})_{t\in B,i\in I}$ are smooth families in $\mathrm{Symp}^+_V(\mathcal{N}_\partial)$ such that*

$$\begin{gathered}\big(\Omega^\bullet_{t;i}\big)_{i\in I}=\big(\Omega'^\bullet_{t;i}\big)_{i\in I}\quad\forall\,t\in B,\\ \big(\Omega_{t;i}\big)_{i\in I}=\big(\Omega'_{t;i}\big)_{i\in I}\quad\forall\,t\in N(\partial B),\qquad\big(\Omega_{t;i}|_U\big)_{i\in I}=\big(\Omega'_{t;i}|_U\big)_{i\in I}\quad\forall\,t\in B,\end{gathered}\tag{3.7}$$

- *$(\widehat{\omega}_{t;i})_{t\in B,i\in I}$ (resp. $(\widehat{\omega}'_{t;i})_{t\in B,i\in I}$) is the family of closed 2-forms on $\mathcal{N}_\partial$ induced as in (2.9) by the families $(\omega_t)_{t\in B}$ of symplectic forms on V, $(\Omega_{t;i})_{t\in B,i\in I}$ (resp. $(\Omega'_{t;i})_{t\in B,i\in I}$) of fiberwise symplectic forms on $\mathcal{N}_\partial$, and $(\nabla^{(t;i)})_{t\in B,i\in I}$ of connections on L_i.*

If $V-U$ is compact, then there exist neighborhoods $\widehat{\mathcal{N}}\subset\mathcal{N}''\subset\mathcal{N}'$ of $V\subset\mathcal{N}$ such that $\overline{\mathcal{N}''}\subset\mathcal{N}'$ and a smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in I}$ of 1-forms on $\mathcal{N}_\partial$ such that

$$\big(\widehat{\omega}_{t,\tau;i}\big)_{i\in I}\equiv\big((\widehat{\omega}_{t;i}+\mathrm{d}\mu_{t,\tau;i})|_{\mathcal{N}'_i}\big)_{i\in I}\tag{3.8}$$

is a symplectic structure on $\mathcal{N}'_\partial$ for all $(t,\tau)\in B\times\mathbb{I}$ and

$$\mu_{t,0;i}=0,\quad \widehat{\omega}_{t,\tau;i}|_V=\omega_t,\quad \widehat{\omega}_{t,1;i}|_{\widehat{\mathcal{N}}_i}=\widehat{\omega}'_{t;i}|_{\widehat{\mathcal{N}}_i},\quad \operatorname{supp}\big(\mu_{\cdot,\tau;i}\big)\subset\big(B-N(\partial B)\big)\times\mathcal{N}''|_{V-U} \tag{3.9}$$

for all $t\in B$, $\tau\in\mathbb{I}$, and $i\in I$.

Remark 3.7. If $(\Omega_i)_{i\in I}$ and $(\Omega'_i)_{i\in I}$ are diagonal elements of $\mathrm{Symp}^+_V(\mathcal{N}_\partial)$, then the associated closed 2-forms $(\widehat{\omega}_i)_{i\in I}$ and $(\widehat{\omega}'_i)_{i\in I}$ can be directly deformed into each other using the bump function of Lemma 3.9 so that the restrictions of these deformations stay in $\mathrm{Symp}^+(\mathcal{N}'_\partial)$ for some neighborhood $\mathcal{N}'_\partial$ of V in $\mathcal{N}_\partial$. Thus, the first assumption in (3.7) is needed for the very last conclusion only.

Lemma 3.8. *Let V, I, and $\mathcal{N}$ be as in Theorem 3.1 and $k\in\mathbb{Z}^{\geq 0}$. There exists a smooth bundle map*

$$\Phi_{\mathcal{N};\partial}\colon \Lambda^k_{\mathbb{C}}\mathcal{N}^*_\partial\longrightarrow\Lambda^k_{\mathbb{C}}\mathcal{N}^*$$

such that $r_{\mathcal{N};\partial}\circ\Phi_{\mathcal{N};\partial}=\mathrm{id}$.

Proof. Let $(\alpha_i)_{i\in I}$ be an element of $\Lambda^k_{\mathbb{C}}\mathcal{N}^*_\partial$ in the fiber over a point $x\in V$. Thus,

$$\alpha_{i_1}|_{\mathcal{N}_{i_1i_2}|_x}=\alpha_{i_2}|_{\mathcal{N}_{i_1i_2}|_x}\quad\forall\, i_1,i_2\in I. \tag{3.10}$$

Assume that $I=[\ell^*]$ for some $\ell^*\geq 2$. For $i,\ell\in I$, let

$$\pi_i\colon\mathcal{N}\longrightarrow\mathcal{N}_i\qquad\text{and}\qquad\pi_{i;\ell}\colon\mathcal{N}_i\longrightarrow\mathcal{N}_{i\ell}$$

denote the projection maps. Define

$$\alpha'_1=\pi_1^*\alpha_1,\quad \alpha'_{\ell+1}=\alpha'_\ell+\pi^*_{\ell+1}\big(\alpha_{\ell+1}-\alpha'_\ell|_{\mathcal{N}_{\ell+1}}\big)\quad\forall\,\ell\in[\ell^*-1],\quad \alpha=\alpha'_{\ell^*}.$$

Since $\pi_i|_{\mathcal{N}_i}=\mathrm{id}_{\mathcal{N}_i}$ and $\pi_\ell|_{\mathcal{N}_i}=\pi_{i;\ell}$, it follows that

$$\alpha'_i|_{\mathcal{N}_i}=\alpha_i,\quad \alpha'_\ell|_{\mathcal{N}_i}=\alpha'_{\ell-1}|_{\mathcal{N}_i}+\pi^*_{i;\ell}\big(\alpha_\ell|_{\mathcal{N}_{i\ell}}-\alpha'_{\ell-1}|_{\mathcal{N}_{i\ell}}\big)\quad\forall\,\ell\in[\ell^*]-[i].$$

By (3.10) and induction, these identities imply that

$$\alpha'_\ell|_{\mathcal{N}_i}=\alpha_i\quad\forall\,\ell\in[\ell^*]-[i-1],\ i\in[\ell^*].$$

Thus, the constructed smooth bundle homomorphism

$$\Phi_{\mathcal{N};\partial}\colon\Lambda^k_{\mathbb{C}}\mathcal{N}^*_\partial\longrightarrow\Lambda^k_{\mathbb{C}}\mathcal{N}^*,\qquad \Phi_{\mathcal{N};\partial}\big((\alpha_i)_{i\in I}\big)=\alpha,$$

is a right inverse for $r_{\mathcal{N};\partial}$. □

Lemma 3.9. *There exists a smooth function $\chi\colon(0,1)\times(1,\infty)\times\mathbb{R}\longrightarrow\mathbb{R}^{\geq 0}$ such that*

$$\chi(\delta,s,r)=\begin{cases}s, & \text{if } r\leq\delta;\\ 0, & \text{if } r\geq\delta\mathrm{e}^{4s/\delta};\end{cases}\qquad \chi(\delta,s,r)\leq s,\qquad \left|\frac{\partial}{\partial r}\chi(\delta,s,r)\right|r\leq\delta. \tag{3.11}$$

Proof. If $\delta\in(0,1)$ and $s\in(1,\infty)$, then $2\delta\le\delta e^{4s/\delta}-1$. Let $\eta\colon\mathbb{R}\longrightarrow\mathbb{I}$ be a smooth function such that

$$\eta(r)=\begin{cases}0, & \text{if } r\le 0;\\ 1, & \text{if } r\ge 1;\end{cases}\qquad \big|\eta'(r)\big|\le 2.$$

The smooth function

$$\chi(\delta,s,r)=\eta\big(\delta e^{4s/\delta}-r\big)\Big(s-\eta(r/\delta-1)\frac{\delta}{4}\ln\big(r/\delta\big)\Big)$$

then satisfies (3.11). □

Proof of Proposition 3.6. Let $\Phi_{\mathcal{N};\partial}$ be as in Lemma 3.8. For each $t\in B$, define

$$\Omega_t=\Phi_{\mathcal{N};\partial}\big((\Omega_{t;i})_{i\in I}\big),\quad \Omega'_t=\Phi_{\mathcal{N};\partial}\big((\Omega'_{t;i})_{i\in I}\big),\quad \Omega^\circ_t=\Omega'_t-\Omega_t\,, \tag{3.12}$$

$$\widehat\omega_t=\pi^*\omega_t+\frac{1}{2}d\iota_{\zeta_{\mathcal{N}}}\{\Omega_t\}_{\nabla^{(t)}}\,,\quad \widehat\omega'_t=\pi^*\omega_t+\frac{1}{2}d\iota_{\zeta_{\mathcal{N}}}\{\Omega'_t\}_{\nabla^{(t)}}=\widehat\omega_t+\frac{1}{2}d\iota_{\zeta_{\mathcal{N}}}\{\Omega^\circ_t\}_{\nabla^{(t)}}\,. \tag{3.13}$$

By (3.7),

$$\Omega^\bullet_t=\widehat\Omega'^\bullet_t\quad\forall\,t\in B,\qquad \operatorname{supp}\big(\Omega^\circ_\cdot\big)\subset\big(B-N(\partial B)\big)\times\big(V-U\big)\,. \tag{3.14}$$

Since $r_{\mathcal{N};\partial}\circ\Phi_{\mathcal{N};\partial}=\mathrm{id}$,

$$\Omega_t|_{\mathcal{N}_i}=\Omega_{t;i},\quad \Omega'_t|_{\mathcal{N}_i}=\Omega'_{t;i},\quad \widehat\omega_t|_{\mathcal{N}_i}=\widehat\omega_{t;i},\quad \widehat\omega'_t|_{\mathcal{N}_i}=\widehat\omega'_{t;i}\quad\forall\,t\in B,\ i\in I. \tag{3.15}$$

We construct the desired families of 1-forms by pasting together three families of 1-forms via smooth functions $\eta_{\mathbb{I};1},\eta_{\mathbb{I};2},\eta_{\mathbb{I};3}\colon\mathbb{R}\longrightarrow\mathbb{I}$ such that

$$\eta_{\mathbb{I};1}(\tau)=\begin{cases}0, & \text{if } \tau\le 0;\\ 1, & \text{if } \tau\ge\frac{1}{3};\end{cases}\quad \eta_{\mathbb{I};2}(\tau)=\begin{cases}0, & \text{if } \tau\le\frac{1}{3};\\ 1, & \text{if } \tau\ge\frac{2}{3};\end{cases}\quad \eta_{\mathbb{I};3}(\tau)=\begin{cases}0, & \text{if } \tau\le\frac{2}{3};\\ 1, & \text{if } \tau\ge 1.\end{cases} \tag{3.16}$$

We first increase the diagonal part $\Omega^\bullet_t$ of Ω_t and Ω'_t as in Corollary 3.5(1). We then add the difference Ω°_t with Ω'_t as in Corollary 3.5(2). Finally, we reduce the diagonal part back to where it started. We cut off all three deformations by bump functions supported near V so that the forms do not change too far away from V, i.e. on $\mathcal{N}-\mathcal{N}''$. This construction is illustrated in Figure 4.

Fix a metric on V and a norm $\rho(\cdot)=|\cdot|^2$ on $\mathcal{N}$. For any $\varrho\in\mathbb{R}^+$, let

$$\mathcal{N}(\varrho)=\big\{v\in\mathcal{N}:\ |v|<\varrho\big\}.$$

Since B is compact, we can choose the norm on $\mathcal{N}$ so that the 2-form $\widehat\omega_t$ is nondegenerate on $\mathcal{N}'\equiv\mathcal{N}(1)$ for every $t\in B$. Since B and $V-U$ are compact, for every smooth family $\Xi\equiv(\Xi_t)_{t\in B}$ of fiberwise 2-forms on $\mathcal{N}$ there exists $C_\Xi\in\mathbb{R}^+$ such that

$$\begin{gathered}\Big|\iota_{\zeta_{\mathcal{N}}}\{\Xi_t\}_{\nabla^{(t)}}\Big|_v,\ \Big|\frac{1}{2}d\,\iota_{\zeta_{\mathcal{N}}}\{\Xi_t\}_{\nabla^{(t)}}-\{\Xi_t\}_{\nabla^{(t)}}\Big|_v\le C_\Xi|v|,\\ \Big|\frac{d\rho}{\rho}\wedge\iota_{\zeta_{\mathcal{N}}}\{\Xi_t\}_{\nabla^{(t)}}\Big|_v\le C_\Xi\end{gathered}\qquad\forall\ v\in\mathcal{N}'|_{V-U}\,. \tag{3.17}$$

For $\mathfrak{s}\in C^\infty(B\times V;\mathbb{R})$ and $\tau\in\mathbb{R}$, let $\Omega_{t;\mathfrak{s}}$ and $\Omega_{t;\mathfrak{s},\tau}$ be the fiberwise 2-forms on $\mathcal{N}$ given by

$$\Omega_{t;\mathfrak{s}}\big|_x=\Omega_t\big|_x+\mathfrak{s}(t,x)\,\Omega_t^\bullet\big|_x, \quad \Omega_{t;\mathfrak{s},\tau}\big|_x=\Omega_{t;\mathfrak{s}}\big|_x+\tau\,\Omega_t^\circ\big|_x=\Omega'_{t;\mathfrak{s}}\big|_x-(1-\tau)\Omega_t^\circ\big|_x \tag{3.18}$$

for all $x\in V$; the second equality in the second statement above holds by the first property in (3.14). By the first two equalities in (3.15) and Corollary 3.5(1), the restrictions of $\Omega_{t;\mathfrak{s},0}=\Omega_{t;\mathfrak{s}}$ and of $\Omega_{t;\mathfrak{s},1}=\Omega'_{t;\mathfrak{s}}$ to $\mathcal{N}_{I'}$ are nondegenerate for all $I'\in\mathcal{P}^*(I)$, $t\in B$, and $\mathfrak{s}\in C^\infty(B\times V;\mathbb{R}^{\ge0})$. By Corollary 3.5(2), there exists $s_0\in\mathbb{R}^+$ such that $\Omega_{t;\mathfrak{s},\tau}|_{\mathcal{N}_{I'}}$ is nondegenerate over $x\in V-U$ whenever

$$I'\in\mathcal{P}^*(I), \quad t\in B, \quad \tau\in\mathbb{I}, \quad \mathfrak{s}\in C^\infty(B\times V;\mathbb{R}^{\ge0}), \quad \text{and} \quad \mathfrak{s}(t,x)\ge s_0.$$

We assume that $s_0\ge2$.

By the second property in (3.14) and the first equality in (3.15),

$$\Omega_{t;0,\tau}\big|_{\mathcal{N}_i}=\Omega_{t;i} \;\; \forall\, t\in N(\partial B),\,\tau\in\mathbb{I},\,i\in I, \quad \Omega_{t;0,\tau}\big|_{\mathcal{N}_i|_U}=\Omega_{t;i}\big|_{\mathcal{N}_i|_U} \;\; \forall\, t\in B,\,\tau\in\mathbb{I},\,i\in I. \tag{3.19}$$

By the first equality in (3.19), the openness of the nondegeneracy condition, and the compactness of $\mathbb{I}$, there exists a neighborhood $\mathcal{W}$ of $\overline{N(\partial B)}\times V$ in $B\times V$ such that $\Omega_{t;0,\tau}|_{\mathcal{N}_{I'}}$ is nondegenerate over x for all $(t,x)\in\overline{\mathcal{W}}$, $\tau\in\mathbb{I}$, and $I'\in\mathcal{P}^*(I)$. By the second equality in (3.19), the openness of the nondegeneracy condition, and the compactness of $B\times\mathbb{I}$, there exists a neighborhood U' of $\overline{U}\subset V$ such that $\Omega_{t;0,\tau}|_{\mathcal{N}_{I'}}$ is nondegenerate over $x\in\overline{U'}$ for all $t\in B$, $\tau\in\mathbb{I}$, and $I'\in\mathcal{P}^*(I)$. By Corollary 3.5(1), both nondegeneracy statements apply to $\Omega_{t;\mathfrak{s},\tau}$ for all $\mathfrak{s}\in C^\infty(B\times V;\mathbb{R}^{\ge0})$.

By the choices made above, the restriction of the 2-tensor $\pi^*\omega_t+\{\Omega_{t;s,\tau}\}_{\nabla^{(t)}}$ to $T_v\mathcal{N}_{I'}$, for any $v\in\mathcal{N}_{I'}|_{V-U}$ and $I'\in\mathcal{P}^*(I)$, is nondegenerate if

$$\begin{gathered} s\in\mathbb{R}^{\ge0},\ \tau\in\{0,1\}, \quad \text{or} \quad s\ge s_0,\ \tau\in\mathbb{I}, \quad \text{or} \\ s\in\mathbb{R}^{\ge0},\ \tau\in\mathbb{I},\ \big(t,\pi(v)\big)\in\overline{\mathcal{W}}, \quad \text{or} \quad s\in\mathbb{R}^{\ge0},\ \tau\in\mathbb{I},\ \pi(v)\in\overline{U'}. \end{gathered}$$

By the openness of the nondegeneracy condition and the compactness of B, $\mathbb{I}$, $[0,s_0]$, and $V-U$, there thus exists $\epsilon^*\in\mathbb{R}^+$ with the property that $\widehat\omega_v|_{T_v\mathcal{N}_{I'}}$ is nondegenerate whenever $v\in\mathcal{N}_{I'}|_{V-U}$, $I'\in\mathcal{P}^*(I)$, and $\widehat\omega_v$ is a 2-tensor on $T_v\mathcal{N}$ such that

$$\big|\widehat\omega_v-\big(\pi^*\omega_t+\{\Omega_{t;s,\tau}\}_{\nabla^{(t)}}\big)_v\big|<\epsilon^* \tag{3.20}$$

for some $t\in B$ and $s,\tau\in\mathbb{R}$ with

$$s\in[0,s_0],\ \tau\in\{0,1\}, \quad \text{or} \quad s=s_0,\ \tau\in\mathbb{I}, \quad \text{or} \tag{3.21}$$

$$s\in[0,s_0],\ \tau\in\mathbb{I},\ \big(t,\pi(v)\big)\in\overline{\mathcal{W}}, \quad \text{or} \quad s\in\big[0,s_0\big],\ \tau\in\mathbb{I},\ \pi(v)\in\overline{U'}. \tag{3.22}$$

We assume that $\epsilon^*\le1$.

Let $\eta_B\colon B\times V\longrightarrow\mathbb{I}$ and $\eta_V\colon V\longrightarrow\mathbb{I}$ be smooth functions such that

$$\eta_B(t,x)=\begin{cases}0, & \text{if } t\in N(\partial B);\\ 1, & \text{if } (t,x)\notin\mathcal{W};\end{cases} \qquad \eta_V(x)=\begin{cases}0, & \text{if } x\in U;\\ 1, & \text{if } x\notin U'.\end{cases} \tag{3.23}$$

With the notation as in (3.17) and (3.20), define

$$C^*=C_\Omega+C_{\Omega^\circ}+s_0C_{\eta_B\eta_V\Omega^\bullet}\,, \quad \delta=\epsilon^*/2C^*, \quad \varrho=\delta\mathrm{e}^{4s_0/\delta}, \quad \mathcal{N}''=\mathcal{N}(\delta), \quad \widehat{\mathcal{N}}=\mathcal{N}\big(\delta^4/4\varrho^3\big).$$

We assume that $C^*\geq 1$. Let χ be as in Lemma 3.9. For any $\epsilon\in\mathbb{R}^+$, let

$$\chi_\epsilon\colon \mathbb{R}\longrightarrow \mathbb{R}^{\geq 0}, \qquad \chi_\epsilon(r)=\chi\big(\delta,s_0,r/\epsilon\big)\,.$$

By (3.11),

$$\chi_\epsilon(r)=\begin{cases} s_0, & \text{if } r\leq \delta\epsilon; \\ 0, & \text{if } r\geq \varrho\epsilon; \end{cases} \qquad \begin{aligned} &0\leq \chi_\epsilon(r)\leq s_0, \\ &\big|\chi'_\epsilon(r)\big|r\leq \delta. \end{aligned} \tag{3.24}$$

(1) Let $\epsilon_1=\delta/\varrho$. For $v\in\mathcal{N}$, let

$$\begin{aligned} \big\{\Omega^{(1)}_{t,\tau}\big\}_{\nabla^{(t)}}\big|_v &= \{\Omega_t\}_{\nabla^{(t)}}\big|_v+\eta_{\mathbb{I};1}(\tau)\eta_B\big(t,\pi(v)\big)\eta_V\big(\pi(v)\big)\chi_{\epsilon_1}\big(|v|\big)\big\{\Omega^\bullet_t\big\}_{\nabla^{(t)}}\big|_v, \\ \mu^{(1)}_{t,\tau}\big|_v &= \frac{1}{2}\eta_{\mathbb{I};1}(\tau)\eta_B\big(t,\pi(v)\big)\eta_V\big(\pi(v)\big)\chi_{\epsilon_1}\big(|v|\big)\iota_{\zeta_{\mathcal{N}}(v)}\big\{\Omega^\bullet_t\big\}_{\nabla^{(t)}}\big|_v\,. \end{aligned} \tag{3.25}$$

Define a closed 2-form on the total space of $\mathcal{N}$ by

$$\widehat{\omega}^{(1)}_{t,\tau}\equiv \pi^*\omega_t+\frac{1}{2}\mathrm{d}\iota_{\zeta_{\mathcal{N}}}\big\{\Omega^{(1)}_{t,\tau}\big\}_{\nabla^{(t)}}=\widehat{\omega}_t+\mathrm{d}\mu^{(1)}_{t,\tau}; \tag{3.26}$$

the last equality holds by the first definition in (3.13). By (3.16), (3.23), and (3.24),

$$\mu^{(1)}_{t,0}=0\ \ \forall\,t\in B, \qquad \operatorname{supp}\big(\mu^{(1)}_{\cdot,\tau}\big)\subset\big(B-N(\partial B)\big)\times\mathcal{N}(\delta)|_{V-U} \quad \forall\,\tau\in\mathbb{I}, \tag{3.27}$$

$$\mu^{(1)}_{t,\tau}\big|_{\mathcal{N}(\delta\epsilon_1)}=\frac{s_0}{2}\eta_B(t,\cdot)\eta_V\,\iota_{\zeta_{\mathcal{N}}}\big\{\Omega^\bullet_t\big\}_{\nabla^{(t)}}\big|_{\mathcal{N}(\delta\epsilon_1)} \qquad \forall\ t\in B,\ \tau\in[\tfrac{1}{3},1]. \tag{3.28}$$

By (3.25), (3.17), and (3.24),

$$\begin{aligned} \Big|\frac{1}{2}\mathrm{d}\iota_{\zeta_{\mathcal{N}}}\big\{\Omega^{(1)}_{t,\tau}\big\}_{\nabla^{(t)}}-\big\{\Omega^{(1)}_{t,\tau}\big\}_{\nabla^{(t)}}\Big|_v &\leq C^*\big(|v|+\delta\big) && \forall\,v\in\mathcal{N}'|_{V-U}\,, \\ &<\epsilon^* && \forall\,v\in\mathcal{N}(\delta)|_{V-U}\,. \end{aligned} \tag{3.29}$$

By (3.25),

$$\Omega^{(1)}_{t,\tau}=\Omega_{t;s_{t,\tau}(v),0} \qquad \text{with} \quad s_{t,\tau}(v)=\eta_{\mathbb{I};1}(\tau)\eta_B\big(t,\pi(v)\big)\eta_V\big(\pi(v)\big)\chi_{\epsilon_1}\big(|v|\big)\in[0,s_0]\,.$$

Along with (3.26) and (3.29), this implies that

$$\Big|\widehat{\omega}^{(1)}_{t,\tau}-\Big(\pi^*\omega_t+\{\Omega_{t;s_{t,\tau}(v),0}\}_{\nabla^{(t)}}\Big)\Big|_v<\epsilon^* \quad \forall\,(t,\tau)\in B\times\mathbb{I},\ v\in\mathcal{N}(\delta)|_{V-U}\,.$$

By the first case in (3.21), the restriction of $\widehat{\omega}^{(1)}_{t,\tau}$ to $\mathcal{N}(\delta)_{I'}|_{V-U}$ is thus nondegenerate for all $(t,\tau)\in B\times\mathbb{I}$ and $I'\in\mathcal{P}^*(I)$. By the last equality in (3.26) and the second statement in (3.27), this is also the case for the restriction of $\widehat{\omega}^{(1)}_{t,\tau}$ to $(\mathcal{N}'_{I'}-\mathcal{N}(\delta))|_{V-U}$.

(2) Let $\epsilon_2=\delta\epsilon_1/2\varrho$; thus, $\overline{\mathcal{N}(\varrho\epsilon_2)}\subset\mathcal{N}(\delta\epsilon_1)$. For $v\in\mathcal{N}$, let

$$\begin{aligned} \big\{\Omega^{(2)}_{t,\tau}\big\}_{\nabla^{(t)}}\big|_v &= \big\{\Omega_{t;s_0\eta_B\eta_V}\big\}_{\nabla^{(t)}}\big|_v+\frac{1}{s_0}\eta_{\mathbb{I};2}(\tau)\chi_{\epsilon_2}\big(|v|\big)\big\{\Omega^\circ_t\big\}_{\nabla^{(t)}}\big|_v, \\ \mu^{(2)}_{t,\tau}\big|_v &= \frac{1}{2s_0}\eta_{\mathbb{I};2}(\tau)\chi_{\epsilon_2}\big(|v|\big)\iota_{\zeta_{\mathcal{N}}(v)}\big\{\Omega^\circ_t\big\}_{\nabla^{(t)}}\big|_v\,. \end{aligned} \tag{3.30}$$

Define a closed 2-form on the total space of $\mathcal{N}$ by

$$\widehat{\omega}^{(2)}_{t,\tau} \equiv \pi^*\omega_t + \frac{1}{2}\mathrm{d}\iota_{\zeta_{\mathcal{N}}}\{\Omega^{(2)}_{t,\tau}\}_{\nabla^{(t)}} = \widehat{\omega}_t + \frac{s_0}{2}\mathrm{d}\big(\eta_B\eta_V\,\iota_{\zeta_{\mathcal{N}}}\{\Omega^{\bullet}_t\}_{\nabla^{(t)}}\big) + \mathrm{d}\mu^{(2)}_{t,\tau}\,; \tag{3.31}$$

the last equality holds by (3.18) and the first equations in (3.13). By (3.16), the second equation in (3.14), and (3.24),

$$\mu^{(2)}_{t,\tau}=0\ \ \forall\,t\!\in\!B,\,\tau\!\in\![0,\frac{1}{3}],\quad \operatorname{supp}\big(\mu^{(2)}_{\cdot,\tau}\big)\subset\big(B\!-\!N(\partial B)\big)\!\times\!\mathcal{N}(\varrho\epsilon_2)|_{V-U}\ \ \forall\,t\!\in\!B,\,\tau\!\in\!\mathbb{I}, \tag{3.32}$$

$$\mu^{(2)}_{t,\tau}\big|_{\mathcal{N}(\delta\epsilon_2)} = \frac{1}{2}\iota_{\zeta_{\mathcal{N}}}\{\Omega^{\circ}_t\}_{\nabla^{(t)}}\big|_{\mathcal{N}(\delta\varepsilon_2)}\quad \forall\,\tau\!\in\![\frac{2}{3},1]. \tag{3.33}$$

By (3.30), (3.17), and (3.24),

$$\Big|\frac{1}{2}\mathrm{d}\iota_{\zeta_{\mathcal{N}}}\{\Omega^{(2)}_{t,\tau}\}_{\nabla^{(t)}} - \{\Omega^{(2)}_{t,\tau}\}_{\nabla^{(t)}}\Big|_v \le C^*\big(|v|\!+\!\delta\big)\quad \forall\,v\!\in\!\mathcal{N}'|_{V-U}\,. \tag{3.34}$$

By (3.30),

$$\Omega^{(2)}_{t,\tau} = \Omega_{t;s_t(v),\tau'_{t,\tau}(v)}\quad\text{with}\quad s_t(v)=s_0\eta_B\big(t,\pi(v)\big)\eta_V\big(\pi(v)\big),\ \tau'_{t,\tau}(v)=\eta_{\mathbb{I};2}(\tau)\frac{\chi_{\epsilon_2}(|v|)}{s_0}\in\mathbb{I}.$$

Along with (3.31) and (3.34), this implies that

$$\Big|\widehat{\omega}^{(2)}_{t,\tau} - \Big(\pi^*\omega_t + \{\Omega_{t;s_t(v),\tau'_{t,\tau}(v)}\}_{\nabla^{(t)}}\Big)\Big|_v < \epsilon^*\quad \forall\,(t,\tau)\!\in\!B\!\times\!\mathbb{I},\ v\!\in\!\mathcal{N}(\delta)|_{V-U}\,.$$

By (3.23),

$$s_t(v)\in[0,s_0]\ \forall\,v\!\in\!\mathcal{N},\qquad s_t(v)=s_0\ \text{ if }\big(t,\pi(v)\big)\!\notin\!\mathcal{W}\text{ and }\pi(v)\!\notin\!U'.$$

By the last two displayed statements, (3.22), and the second case in (3.21), the restriction of $\widehat{\omega}^{(2)}_{t,\tau}$ to $\mathcal{N}(\delta)_{I'}|_{V-U}$ is then nondegenerate for all $(t,\tau)\!\in\!B\!\times\!\mathbb{I}$ and $I'\in\mathcal{P}^*(I)$. By the last equality in (3.31), the second statement in (3.32), and (3.28), this is also the case for the restriction of $\widehat{\omega}^{(2)}_{t,\tau}$ to $\mathcal{N}(\delta\epsilon_1)_{I'}|_U$.

(3) Let $\varepsilon_3\!=\!\delta\varepsilon_2/2\varrho$; thus, $\overline{\mathcal{N}(\varrho\epsilon_3)}\!\subset\!\mathcal{N}(\delta\epsilon_2)$. We now reduce the diagonal part of

$$\Omega_{t;s_0\eta_B\eta_V,1} \equiv \Omega_t\!+\!s_0\eta_B\eta_V\Omega^{\bullet}_t\!+\!\Omega^{\circ}_t = \Omega'_t\!+\!s_0\eta_B\eta_V\Omega^{\bullet}_t \tag{3.35}$$

back to $\Omega^{\bullet}_t\!=\!\Omega'^{\bullet}_t$. For $v\!\in\!\mathcal{N}$, let

$$\begin{aligned}\big\{\Omega^{(3)}_{t,\tau}\big\}_{\nabla^{(t)}}\big|_v &= \big\{\Omega_{t;s_0\eta_B\eta_V,1}\big\}_{\nabla^{(t)}}\big|_v - \eta_{\mathbb{I};3}(\tau)\eta_B\big(t,\pi(v)\big)\eta_V\big(\pi(v)\big)\chi_{\epsilon_3}\big(|v|\big)\big\{\Omega^{\bullet}_t\big\}_{\nabla^{(t)}}\big|_v,\\ \mu^{(3)}_{t,\tau}\big|_v &= -\frac{1}{2}\eta_{\mathbb{I};3}(\tau)\eta_B\big(t,\pi(v)\big)\eta_V\big(\pi(v)\big)\chi_{\epsilon_3}\big(|v|\big)\,\iota_{\zeta_{\mathcal{N}}(v)}\big\{\Omega^{\bullet}_t\big\}_{\nabla^{(t)}}\big|_v\,.\end{aligned} \tag{3.36}$$

Define a closed 2-form on the total space of $\mathcal{N}$ by

$$\begin{aligned}\widehat{\omega}^{(3)}_{t,\tau} &\equiv \pi^*\omega_t + \frac{1}{2}\mathrm{d}\iota_{\zeta_{\mathcal{N}}}\{\Omega^{(3)}_{t,\tau}\}_{\nabla^{(t)}}\\ &= \widehat{\omega}_t + \frac{1}{2}\mathrm{d}\Big(s_0\eta_B\eta_V\iota_{\zeta_{\mathcal{N}}}\{\Omega^{\bullet}_t\}_{\nabla^{(t)}} + \iota_{\zeta_{\mathcal{N}}}\{\Omega^{\circ}_t\}_{\nabla^{(t)}}\Big) + \mathrm{d}\mu^{(3)}_{t,\tau}\,;\end{aligned} \tag{3.37}$$

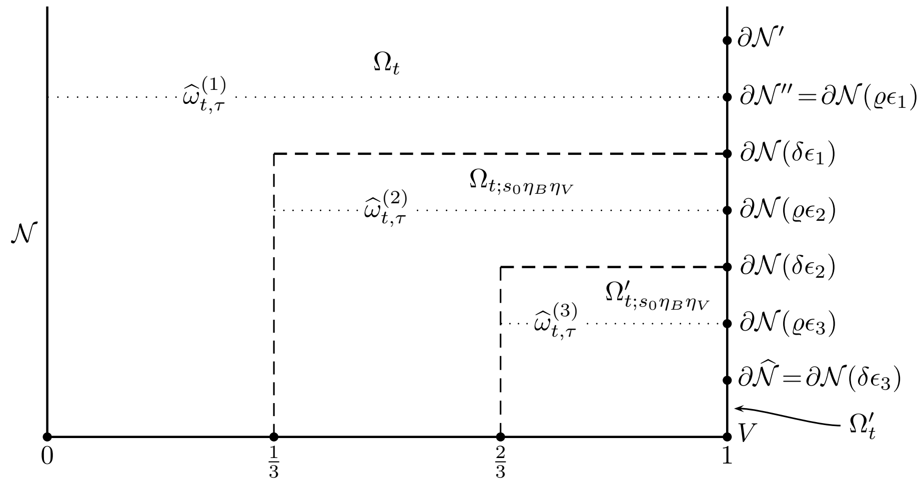

Figure 4: The patched families $(\widehat{\omega}_{t,\tau})_{t\in B,\tau\in\mathbb{I}}$ of closed 2-forms on $\mathcal{N}$ and $(\Omega_{t,\tau})_{t\in B,\tau\in\mathbb{I}}$ of fiberwise 2-forms on $\mathcal{N}$ so that $\widehat{\omega}_{t,\tau}=\pi^*\omega_t+\frac{1}{2}\mathrm{d}\iota_{\zeta_{\mathcal{N}}}\{\Omega_{t,\tau}\}_{\nabla^{(t)}}$ in the indicated regions.

the last equality holds by (3.35) and the first definition in (3.13). By (3.16), (3.23), and (3.24),

$$\mu^{(3)}_{t,\tau}=0 \;\;\forall\, t\in B,\, \tau\in[0,\frac{2}{3}],\quad \mathrm{supp}\big(\mu^{(3)}_{\cdot,\tau}\big)\subset\big(B-N(\partial B)\big)\times\mathcal{N}(\varrho\epsilon_3)|_{V-U}\;\;\forall\,\tau\in\mathbb{I}, \tag{3.38}$$

$$\mu^{(3)}_{t,1}\big|_{\mathcal{N}(\delta\epsilon_3)}=-\frac{s_0}{2}\eta_B(t,\cdot)\eta_V\iota_{\zeta_{\mathcal{N}}}\{\Omega^{\bullet}_t\}_{\nabla^{(t)}}\big|_{\mathcal{N}(\delta\epsilon_3)}\;\;\forall\, t\in B. \tag{3.39}$$

By (3.36), (3.35), (3.17), and (3.24),

$$\Big|\frac{1}{2}\mathrm{d}\iota_{\zeta_{\mathcal{N}}}\{\Omega^{(3)}_{t,\tau}\}_{\nabla^{(t)}}-\{\Omega^{(3)}_{t,\tau}\}_{\nabla^{(t)}}\Big|_v\le C^*\big(|v|+\delta\big)\quad\forall\; v\in\mathcal{N}'|_{V-U}\,. \tag{3.40}$$

By (3.36) and (3.35),

$$\Omega^{(3)}_{t,\tau}=\Omega_{t;s'_{t,\tau}(v),1}\qquad\text{with}\quad s'_{t,\tau}(v)=\eta_B\big(t,\pi(v)\big)\eta_V\big(\pi(v)\big)\big(s_0-\eta_{\mathbb{I};3}(\tau)\chi_{\epsilon_3}\big(|v|\big)\big)\in[0,s_0]\,.$$

Along with (3.37) and (3.40), this implies that

$$\Big|\widehat{\omega}^{(3)}_{t,\tau}-\Big(\pi^*\omega_t+\{\Omega_{t;s'_{t,\tau}(v),1}\}_{\nabla^{(t)}}\Big)\Big|_v<\epsilon^*\quad\forall\,(t,\tau)\in B\times\mathbb{I},\, v\in\mathcal{N}(\delta)|_{V-U}\,.$$

By the first case in (3.21), the restriction of $\widehat{\omega}^{(3)}_{t,\tau}$ to $\mathcal{N}(\delta)_{I'}|_{V-U}$ is then nondegenerate for all $(t,\tau)\in B\times\mathbb{I}$ and $I'\in\mathcal{P}^*(I)$. By the last equality in (3.37), the second statement in (3.38), (3.28), and (3.33), this is also the case for the restriction of $\widehat{\omega}^{(3)}_{t,\tau}$ to $\mathcal{N}(\delta\epsilon_2)_{I'}|_U$.

We define smooth families of 1-forms and 2-forms on the total spaces of $\mathcal{N}$ and $\mathcal{N}_\partial$ by

$$\begin{aligned}\mu_{t,\tau}&=\mu^{(1)}_{t,\tau}+\mu^{(2)}_{t,\tau}+\mu^{(3)}_{t,\tau}, & \big(\mu_{t,\tau;i}\big)_{i\in I}&=\big(\mu_{t,\tau}\big|_{\mathcal{N}_i}\big)_{i\in I}\qquad\forall\, t\in B,\,\tau\in\mathbb{I},\\ \widehat{\omega}_{t,\tau}&=\widehat{\omega}_t+\mathrm{d}\mu_{t,\tau}, & \big(\widehat{\omega}_{t,\tau;i}\big)_{i\in I}&=\big(\widehat{\omega}_{t,\tau}\big|_{\mathcal{N}_i}\big)_{i\in I}\qquad\forall\, t\in B,\,\tau\in\mathbb{I}\,.\end{aligned}$$

By (3.27), (3.32), and (3.38),

$$\mu_{t,0}=0\;\;\forall\, t\in B,\qquad \mathrm{supp}\big(\mu_{\cdot,\tau}\big)\subset\big(B-N(\partial B)\big)\times\big(\mathcal{N}''|_{V-U}\big)\;\;\forall\,\tau\in\mathbb{I}.$$

By (3.32), (3.38), (3.28), and (3.33),

$$\widehat{\omega}_{t,\tau}\big|_v = \begin{cases} \widehat{\omega}^{(1)}_{t,\tau}|_v, & \text{if } (\tau,v)\in\mathbb{I}\times\mathcal{N}'-[\frac{1}{3},1]\times\mathcal{N}(\varrho\epsilon_2);\\ \widehat{\omega}^{(2)}_{t,\tau}|_v, & \text{if } (\tau,v)\in[\frac{1}{3},1]\times\mathcal{N}(\delta\epsilon_1)-[\frac{2}{3},1]\times\mathcal{N}(\varrho\epsilon_3);\\ \widehat{\omega}^{(3)}_{t,\tau}|_v, & \text{if } (\tau,v)\in[\frac{2}{3},1]\times\mathcal{N}(\delta\epsilon_2). \end{cases} \tag{3.41}$$

Along with the observations at the end of each step (1)-(3) of the construction, this implies that $\widehat{\omega}_{t,\tau}|_{\mathcal{N}'_{I'}}$ is nondegenerate for all $I'\in\mathcal{P}^*(I)$ for all $(t,\tau)\in B\times\mathbb{I}$. By (3.41), (3.26), (3.31), and (3.37),

$$\widehat{\omega}_{t,\tau}|_V = \omega_t \qquad \forall\,(t,\tau)\in B\times\mathbb{I}\,.$$

By the last case in (3.41), (3.37), (3.39), and the last equality in (3.13), $\widehat{\omega}_{t,1}|_{\widehat{\mathcal{N}}}=\widehat{\omega}'_t|_{\widehat{\mathcal{N}}}$ for all $t\in B$. Along with the last equation in (3.15), the conclusions in the paragraph imply that the smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in I}$ of 1-forms on $\mathcal{N}_\partial$ satisfies all requirements of the proposition. □

3.4 Deformations of arbitrary structures

We continue with the notation introduced in (3.1) and (3.2). By Lemma 3.10 below, an arbitrary symplectic structure $(\widetilde{\omega}_{t;i})_{i\in I}$ on a neighborhood $\mathcal{N}'_\partial$ of V in $\mathcal{N}_\partial$ can be deformed to a standard one, $(\widehat{\omega}_{t;i})_{i\in I}$ as in (2.9), on a smaller neighborhood of V. As with Proposition 3.6, the forms are kept fixed outside of a neighborhood $\mathcal{N}''_\partial$ of V. By definition, the original symplectic forms $\widetilde{\omega}_{t;i}$ on $\mathcal{N}'_i$ agree along their overlaps, i.e. on $\mathcal{N}'_{i_1i_2}$.

Lemma 3.10. *Let $U\subset V$, I, $\mathcal{N}$, $N(\partial B)\subset B$, and $\mathcal{N}'$ be as in Theorem 3.1. Suppose*

- $(\widetilde{\omega}_{t;i})_{t\in B,i\in I}$ *and* $(\widetilde{\omega}'_{t;i})_{t\in B,i\in I}$ *are smooth families of symplectic structures and of closed 2-forms, respectively, on* $\mathcal{N}'_\partial$ *such that*

$$\begin{gathered}\big(\widetilde{\omega}_{t;i}|_{T\mathcal{N}_i|_V}\big)_{i\in I} = \big(\widetilde{\omega}'_{t;i}|_{T\mathcal{N}_i|_V}\big)_{i\in I} \quad \forall\,t\in B,\\ \big(\widetilde{\omega}_{t;i}\big)_{i\in I} = \big(\widetilde{\omega}'_{t;i}\big)_{i\in I} \quad \forall\,t\in N(\partial B), \qquad \big(\widetilde{\omega}_{t;i}|_{\mathcal{N}'_i|_U}\big)_{i\in I} = \big(\widetilde{\omega}'_{t;i}|_{\mathcal{N}'_i|_U}\big)_{i\in I} \quad \forall\,t\in B;\end{gathered} \tag{3.42}$$

- $K\subset V$ *is a compact subset and* $\mathcal{K}\subset V$ *is an open neighborhood of* K.

Then there exist neighborhoods $\widetilde{\mathcal{N}}\subset\mathcal{N}''$ of $V\subset\mathcal{N}'$ such that $\overline{\mathcal{N}''}\subset\mathcal{N}'$ and a smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in I}$ of 1-forms on $\mathcal{N}_\partial$ such that

$$\big(\widetilde{\omega}_{t,\tau;i}\big)_{i\in I} \equiv \big(\widetilde{\omega}_{t;i}+\mathrm{d}\mu_{t,\tau;i}|_{\mathcal{N}'_i}\big)_{i\in I} \tag{3.43}$$

is a symplectic structure on $\mathcal{N}'_\partial$ for all $(t,\tau)\in B\times\mathbb{I}$ and

$$\begin{gathered}\mu_{t,0;i}=0, \quad \widetilde{\omega}_{t,\tau;i}\big|_{T\mathcal{N}_i|_V} = \widetilde{\omega}_{t;i}\big|_{T\mathcal{N}_i|_V}, \quad \widetilde{\omega}_{t,1;i}\big|_{\widetilde{\mathcal{N}}_i|_K} = \widetilde{\omega}'_{t;i}\big|_{\widetilde{\mathcal{N}}_i|_K},\\ \mathrm{supp}\big(\mu_{\cdot,\tau;i}\big)\subset\big(B-N(\partial B)\big)\times\mathcal{N}''_i|_{\mathcal{K}-U}\end{gathered} \tag{3.44}$$

for all $t\in B$, $\tau\in\mathbb{I}$, and $i\in I$.

Proof. For each $\tau\in\mathbb{R}$, let

$$m_\tau\colon\mathcal{N}\longrightarrow\mathcal{N}, \qquad v\longrightarrow\tau v,$$

be the scalar multiplication map; it preserves the subbundles $\mathcal{N}_i \subset \mathcal{N}$. For each $t \in B$ and $i \in I$, define

$$\varpi_{t;i} = \widetilde{\omega}'_{t;i} - \widetilde{\omega}_{t;i}, \qquad \mu_{t;i}\big|_v = \int_0^1 m_\tau^*\{\varpi_{t;i}(\tau^{-1}\zeta_{\mathcal{N}}, \cdot)\}\mathrm{d}\tau\,.$$

By the second half of the proof of [19, Lemma 3.14],

$$\widetilde{\omega}'_{t;i} = \widetilde{\omega}_{t;i} + \mathrm{d}\mu_{t;i}\,. \tag{3.45}$$

By (3.42),

$$\mu_{t;i}|_V, d\mu_{t;i}\big|_{T\mathcal{N}_i|_V} = 0 \;\;\forall\, t \in B, \quad \mu_{t;i} = 0 \;\;\forall\, t \in N(\partial B), \quad \mu_{t;i}\big|_{\mathcal{N}_i|_U} = 0 \;\;\forall\, t \in B. \tag{3.46}$$

Since $(\widetilde{\omega}_{t;i})_{t\in B, i\in I}$ and $(\widetilde{\omega}'_{t;i})_{t\in B, i\in I}$ are smooth families of 2-forms on $\mathcal{N}'_\partial$, $(\mu_{t;i})_{t\in B, i\in I}$ is a smooth family of 1-forms on $\mathcal{N}'_\partial$.

Let $|\cdot|$ be a norm on $\mathcal{N}$. For $\delta \in \mathbb{R}^+$, let

$$\mathcal{N}(\delta) = \big\{v \in \mathcal{N} : \, |v| < \delta\big\}\,.$$

Since B is compact, we can choose the norm on $\mathcal{N}$ so that $\mathcal{N}(4) \subset \mathcal{N}'$. Choose smooth functions

$$\eta_{\mathbb{R}} : \mathbb{R} \longrightarrow \mathbb{I}, \qquad \eta_V : V \longrightarrow \mathbb{I} \qquad \text{s.t.}$$
$$\eta_{\mathbb{R}}(r) = \begin{cases} 1, & \text{if } r \le 1; \\ 0, & \text{if } r \ge 2; \end{cases} \qquad \eta_V(x) = \begin{cases} 1, & \text{if } x \in K; \\ 0, & \text{if } x \notin \mathcal{K}. \end{cases} \tag{3.47}$$

For $\delta \in (0,1)$, $t \in B$, $\tau \in \mathbb{I}$, and $i \in I$, let

$$\mu^{(\delta)}_{t,\tau;i}(v) = \begin{cases} \tau\eta_{\mathbb{R}}\big(|v|/\delta\big)\eta_V\big(\pi(v)\big)\mu_{t;i}(v), & \text{if } v \in \mathcal{N}'_i; \\ 0, & \text{if } v \in \mathcal{N}_i - \overline{\mathcal{N}(2)}. \end{cases}$$

By (3.45)-(3.47),

$$\begin{gathered} \mu^{(\delta)}_{t,0;i} = 0, \quad d\mu^{(\delta)}_{t,\tau;i}\big|_{T\mathcal{N}_i|_V} = 0, \quad \widetilde{\omega}_{t;i}|_{\mathcal{N}(\delta)_i|_K} + \mathrm{d}\mu^{(\delta)}_{t,1;i}|_{\mathcal{N}(\delta)_i|_K} = \widetilde{\omega}'_{t;i}|_{\mathcal{N}(\delta)_i|_K}, \\ \mathrm{supp}\big(\mu^{(\delta)}_{\cdot,\tau;i}\big) \subset \big(B - N(\partial B)\big) \times \mathcal{N}(2\delta)_i\big|_{\mathcal{K}-U}\,. \end{gathered} \tag{3.48}$$

Thus, the smooth family $(\mu^{(\delta)}_{t,\tau;i})_{t\in B, \tau\in\mathbb{I}, i\in I}$ of 1-forms on $\mathcal{N}_\partial$ satisfies (3.44) with $\widetilde{\mathcal{N}} = \mathcal{N}(\delta)$, $\mathcal{N}'' = \mathcal{N}(2\delta)$, and μ replaced by $\mu^{(\delta)}$.

It remains to verify that (3.43) with μ replaced by $\mu^{(\delta)}$ is a symplectic structure on $\mathcal{N}'_\partial$ for all $(t,\tau) \in B \times \mathbb{I}$ and $\delta \in (0,1)$ sufficiently small. We can assume that $\overline{\mathcal{K}} \subset V$ is compact. Since B is also compact, there exists $\epsilon \in \mathbb{R}^+$ with the property that $\widetilde{\omega}_v|_{T_v\mathcal{N}_{I'}}$ is nondegenerate whenever $v \in \mathcal{N}(2)_{I'}|_{\mathcal{K}}$, $I' \in \mathcal{P}_i(I)$, and $\widetilde{\omega}_v$ is a 2-tensor on $T_v\mathcal{N}_i$ such that

$$\big|\widetilde{\omega}_v - \widetilde{\omega}_{t;i}|_v\big| < \epsilon$$

for some $t \in B$. Since $\varpi_{t;i}|_V = 0$ and $B \times \overline{\mathcal{N}(2)}|_{\overline{\mathcal{K}}}$ is compact, there exists $C \in \mathbb{R}^+$ such that

$$\big|\mathrm{d}\mu^{(\delta)}_{t,\tau;i}\big|_v \le C\big(\delta^{-1}|v|^2 + |v|\big) \le 6C\delta \quad \forall\, v \in \mathcal{N}(2\delta)|_{\mathcal{K}},\ \delta \in (0,1).$$

By the last two inequalities, (3.43) with μ replaced by $\mu^{(\delta)}$ is a symplectic structure on $\mathcal{N}'_\partial|_{\mathcal{K}}$ for all $(t,\tau) \in B \times \mathbb{I}$ and $\delta \in (0,1)$ sufficiently small. It is a symplectic structure on $\mathcal{N}'_\partial|_{V-\mathcal{K}}$ for all $\delta \in (0,1)$ because $\widetilde{\omega}_{t,\tau;i} = \widetilde{\omega}_{t;i}$ over $V - \mathcal{K}$. $\square$

Proof of Theorem 3.1. Let $\{K_\ell^\circ\}_{\ell\in\mathbb{Z}^+}$ be an open cover of V such that the closure K_ℓ of K_ℓ° is compact and contained in $K_{\ell+1}^\circ$ for every $\ell\in\mathbb{Z}^+$. We inductively construct sequences $(\mu_{t,\tau,i}^{(\ell)})_{\ell\in\mathbb{Z}^+}$ of families of 1-forms on $\mathcal{N}_\partial$ and $\widehat{\mathcal{N}}_{(\ell)}\subset\mathcal{N}''_{(\ell)}$ of neighborhoods of $V\subset\mathcal{N}$ so that each $\mu_{t,\tau,i}^{(\ell)}$ is supported in $\mathcal{N}''_{(\ell)}|_{K_{\ell+1}-K_{\ell-1}}$ and for each $\ell^*\in\mathbb{Z}^+$ the sum of $\mu_{t,1,i}^{(\ell)}$ with $\ell\in[\ell^*]$ satisfies the third condition in (3.6) with $\widehat{\mathcal{N}}_i$ replaced by $(\widehat{\mathcal{N}}_{(\ell^*)})_i|_{K_{\ell^*}}$. We then take $\mu_{t,\tau,i}$ and $\widehat{\mathcal{N}}$ to be the sum of all 1-forms $\mu_{t,\tau,i}^{(\ell)}$ and the union of the open sets $\widehat{\mathcal{N}}_{(\ell)}|_{K_\ell^\circ}$, respectively. We use Lemma 3.10 followed by Proposition 3.6 to construct $\mu_{t,\tau,i}^{(\ell)}$ for each $\ell\in\mathbb{Z}^+$.

Define

$$K_0=\emptyset,\quad \widehat{\mathcal{N}}_{(0)}=\mathcal{N}',\quad U_\ell=K_\ell^\circ\cup U\ \ \forall\,\ell\in\mathbb{Z}^{\ge0},\quad \widetilde{\omega}_{t,1;i}^{(0)}=\widetilde{\omega}_{t;i}\ \ \forall\,t\in B,\ i\in I.$$

For each $t\in B$, let ω_t be the symplectic form on V determined by the symplectic structure $(\widetilde{\omega}_{t;i})_{i\in I}$ on $\mathcal{N}'_\partial$ and

$$\big(\Omega_{t;i}\big)_{i\in I},\,(\Omega^\bullet_{t;i})_{i\in I}\in\mathrm{Symp}^+_V(\mathcal{N}_\partial)$$

be the fiberwise symplectic structures on $\mathcal{N}_\partial$ determined by $(\widetilde{\omega}_{t;i})_{i\in I}$ via (1.3) and (3.3).

Suppose $\ell^*\in\mathbb{Z}^+$ and for every $\ell\in[\ell^*-1]$ we have constructed

($\mathcal{N}\mu1$) neighborhoods $\widehat{\mathcal{N}}_{(\ell)}\subset\mathcal{N}''_{(\ell)}$ of $V\subset\mathcal{N}'$ such that $\overline{\mathcal{N}''_{(\ell)}}\subset\widehat{\mathcal{N}}_{(\ell-1)}$,

($\mathcal{N}\mu2$) a smooth family $(\mu_{t,\tau;i}^{(\ell)})_{t\in B,\tau\in\mathbb{I},i\in I}$ of 1-forms on $\mathcal{N}_\partial$ such that

$$\big(\widetilde{\omega}_{t,\tau;i}^{(\ell)}\big)_{i\in I}\equiv\big(\widetilde{\omega}_{t,1;i}^{(\ell-1)}+\mathrm{d}\mu_{t,\tau;i}^{(\ell)}|_{\mathcal{N}'_i}\big)_{i\in I}\tag{3.49}$$

is a symplectic structure on $\mathcal{N}'_\partial$ for all $(t,\tau)\in B\times\mathbb{I}$,

$$\begin{gathered}\mu_{t,0;i}^{(\ell)}=0,\quad \widetilde{\omega}_{t,\tau;i}^{(\ell)}|_V=\omega_t\,,\quad \widetilde{\omega}_{t,1;i}^{(\ell)}\big|_{(\widehat{\mathcal{N}}_{(\ell)})_i|_{K_\ell}}=\widehat{\omega}^\bullet_{t;i}\big|_{(\widehat{\mathcal{N}}_{(\ell)})_i|_{K_\ell}},\\ \mathrm{supp}\big(\mu_{\cdot,\tau;i}^{(\ell)}\big)\subset\big(B-N(\partial B)\big)\times\big(\mathcal{N}''_{(\ell)}\big)_i\big|_{K_{\ell+1}^\circ-U_{\ell-1}}\end{gathered}\tag{3.50}$$

for all $t\in B$, $\tau\in\mathbb{I}$, and $i\in I$, and the family $(\Omega_{t,1;i}^{(\ell)})_{t\in B,i\in I}$ of the fiberwise symplectic structures on $\mathcal{N}_\partial$ determined by $(\widetilde{\omega}_{t,1;i}^{(\ell)})_{i\in I}$ via (1.3) satisfies

$$\big(\Omega_{t,1;i}^{(\ell)\bullet}\big)_{t\in B,i\in I}=\big(\Omega^\bullet_{t;i}\big)_{t\in B,i\in I}\,.\tag{3.51}$$

By (3.49) and induction,

$$\widetilde{\omega}_{t,1;i}^{(\ell^*-1)}=\widetilde{\omega}_{t;i}+\mathrm{d}\sum_{\ell=1}^{\ell^*-1}\mu_{t,1;i}^{(\ell)}\big|_{\mathcal{N}'_i}\qquad\forall\,t\in B,\ i\in I\,.\tag{3.52}$$

By (3.4), the last two properties in (3.50), and induction,

$$\begin{aligned}\big(\widetilde{\omega}_{t,1;i}^{(\ell^*-1)}\big)_{i\in I}&=\big(\widehat{\omega}^\bullet_{t;i}\big)_{i\in I}&&\forall\,t\in N(\partial B),\\ \widetilde{\omega}_{t,1;i}^{(\ell^*-1)}\big|_{(\widehat{\mathcal{N}}_{(\ell-1)})_i|_{U_{\ell-1}}}&=\widehat{\omega}^\bullet_{t;i}\big|_{(\widehat{\mathcal{N}}_{(\ell-1)})_i|_{U_{\ell-1}}}&&\forall\,\ell\in[\ell^*],\ t\in B.\end{aligned}\tag{3.53}$$

Along with the second property in (3.50), this implies that

$$\big(\Omega_{t,1;i}^{(\ell^*-1)}\big)_{i\in I}=\big(\Omega_{t;i}^{\bullet}\big)_{i\in I}\quad\forall\,t\in N(\partial B),\qquad \big(\Omega_{t,1;i}^{(\ell^*-1)}|_{U_{\ell^*-1}}\big)_{i\in I}=\big(\Omega_{t;i}^{\bullet}|_{U_{\ell^*-1}}\big)_{i\in I}\quad\forall\,t\in B. \tag{3.54}$$

Let K° be an open neighborhood of $K_{\ell^*}\subset V$ so that its closure K is contained in $K_{\ell^*+1}^\circ$. For $t\in B$ and $i\in I$, let

$$\widetilde{\omega}'_{t;i}=\pi^*\omega_t|_{\mathcal{N}_i}+\frac{1}{2}\mathrm{d}\iota_{\zeta_{\mathcal{N}}}\{\Omega_{t,1;i}^{(\ell^*-1)}\}_{\nabla^{(t)}}\,.$$

By the $\ell=\ell^*$ case of (3.53), the three conditions in (3.42) with U, $\mathcal{N}'$, and $\widetilde{\omega}_{t;i}$ replaced by U_{ℓ^*-1}, $\widehat{\mathcal{N}}_{(\ell^*-1)}$, and $\widetilde{\omega}_{t,1;i}^{(\ell^*-1)}$, respectively, are satisfied. By Lemma 3.10 applied with $\mathcal{K}=K_{\ell^*+1}^\circ$, there thus exist neighborhoods $\widetilde{\mathcal{N}}\subset\mathcal{N}''_{(\ell^*)}$ of $V\subset\mathcal{N}'$ such that $\overline{\mathcal{N}''_{(\ell^*)}}\subset\widehat{\mathcal{N}}_{(\ell^*-1)}$ and a smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in I}$ of 1-forms on $\mathcal{N}_\partial$ such that

$$\big(\widetilde{\omega}_{t,\tau;i}\big)_{i\in I}\equiv\big(\widetilde{\omega}_{t,1;i}^{(\ell^*-1)}|_{(\widehat{\mathcal{N}}_{(\ell^*-1)})_i}+\mathrm{d}\mu_{t,\tau;i}|_{(\widehat{\mathcal{N}}_{(\ell^*-1)})_i}\big)_{i\in I} \tag{3.55}$$

is a symplectic structure on $(\widehat{\mathcal{N}}_{(\ell^*-1)})_\partial$ for all $(t,\tau)\in B\times\mathbb{I}$ and

$$\begin{gathered}\mu_{t,0;i}=0,\quad \widetilde{\omega}_{t,\tau;i}\big|_{T\mathcal{N}_i|_V}=\widetilde{\omega}_{t,1;i}^{(\ell^*-1)}\big|_{T\mathcal{N}_i|_V},\quad \widetilde{\omega}_{t,1;i}\big|_{\widetilde{\mathcal{N}}_i|_K}=\widetilde{\omega}'_{t;i}\big|_{\widetilde{\mathcal{N}}_i|_K},\\ \operatorname{supp}\big(\mu_{\cdot,\tau;i}\big)\subset\big(B-N(\partial B)\big)\times\big(\mathcal{N}''_{(\ell^*)}\big)_i\big|_{K_{\ell^*+1}^\circ-U_{\ell^*-1}}\end{gathered} \tag{3.56}$$

for all $t\in B$, $\tau\in\mathbb{I}$, and $i\in I$.

Let K'° be an open neighborhood of $K_{\ell^*}\subset V$ so that its closure K' is contained in K°. Choose a smooth function

$$\eta\colon V\longrightarrow\mathbb{I}\qquad\text{s.t.}\quad \eta|_{K_{\ell^*}}=1,\quad \eta|_{V-K'}=0.$$

For $t\in B$ and $i\in I$, define

$$\Omega'_{t;i}=(1-\eta)\Omega_{t,1;i}^{(\ell^*-1)}+\eta\Omega_{t;i}^{\bullet}\,,\quad \widehat{\omega}'_{t;i}=\pi^*\omega_t|_{\mathcal{N}_i}+\frac{1}{2}\mathrm{d}\iota_{\zeta_{\mathcal{N}}}\{\Omega'_{t;i}\}_{\nabla^{(t)}}\,.$$

In particular,

$$\widehat{\omega}'_{t;i}\big|_{\mathcal{N}_i|_{K_{\ell^*}}}=\widehat{\omega}_{t;i}^{\bullet}\big|_{\mathcal{N}_i|_{K_{\ell^*}}}\,. \tag{3.57}$$

By the $\ell=\ell^*-1$ case of (3.51) and (3.54),

$$\begin{gathered}\big(\Omega'^{\bullet}_{t;i}\big)_{i\in I}=\big(\Omega_{t,1;i}^{(\ell^*-1)\bullet}\big)_{i\in I}\quad\forall\,t\in B,\quad \big(\Omega'_{t;i}\big)_{i\in I}=\big(\Omega_{t,1;i}^{(\ell^*-1)}\big)_{i\in I}\quad\forall\,t\in N(\partial B),\\ \big(\Omega'_{t;i}|_{U_{\ell^*-1}\cup(V-K')}\big)_{i\in I}=\big(\Omega_{t,1;i}^{(\ell^*-1)}|_{U_{\ell^*-1}\cup(V-K')}\big)_{i\in I}\quad\forall\,t\in B.\end{gathered} \tag{3.58}$$

Thus, the three conditions in (3.7) with U and $\Omega_{t;i}$ replaced by $U_{\ell^*-1}\cup(V-K')$ and $\Omega_{t,1;i}^{(\ell^*-1)}$, respectively, are satisfied. Since K_{ℓ^*+1} is a compact subset of V, so is

$$V-U_{\ell^*-1}\cup(V-K')=K'-U_{\ell^*-1}\subset K_{\ell^*+1}\,.$$

By Proposition 3.6, there thus exist neighborhoods $\widehat{\mathcal{N}}_{(\ell^*)}\subset\mathcal{N}'''$ of $V\subset\mathcal{N}$ such that $\overline{\mathcal{N}'''}\subset\widetilde{\mathcal{N}}$ and a smooth family $(\mu'_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in I}$ of 1-forms on $\mathcal{N}_\partial$ such that

$$\big(\widehat{\omega}_{t,\tau;i}\big)_{i\in I}\equiv\big((\widetilde{\omega}'_{t;i}+\mathrm{d}\mu'_{t,\tau;i})|_{\mathcal{N}'_i}\big)_{i\in I}$$

is a symplectic structure on $\mathcal{N}_\partial'''$ for all $(t,\tau)\in B\times\mathbb{I}$ and

$$\begin{gathered}\mu'_{t,0;i}=0,\quad \widehat{\omega}_{t,\tau;i}|_V=\omega_t,\quad \widehat{\omega}_{t,1;i}|_{(\widehat{\mathcal{N}}_{(\ell^*)})_i}=\widehat{\omega}'_{t;i}|_{(\widehat{\mathcal{N}}_{(\ell^*)})_i},\\ \mathrm{supp}\big(\mu'_{\cdot,\tau;i}\big)\subset\big(B-N(\partial B)\big)\times\mathcal{N}_i'''|_{K'-U_{\ell^*-1}}\end{gathered}\tag{3.59}$$

for all $t\in B$, $\tau\in\mathbb{I}$, and $i\in I$.

By reparametrizing $\mu_{t,\tau;i}$ and $\mu'_{t,\tau;i}$ as functions of τ, we can assume that

$$\mathrm{supp}\big(\mu'_{t,\cdot;i}\big)\subset\big(\frac{1}{2},1\big],\quad \mathrm{supp}\big(\mu_{t,\cdot;i}-\mu_{t,1;i}\big)\subset\big[0,\frac{1}{2}\big)\quad\forall\,t\in B,\,i\in I.\tag{3.60}$$

The tuple

$$\big(\mu^{(\ell^*)}_{t,\tau;i}\big)_{t\in B,\tau\in\mathbb{I},i\in I}\equiv\big(\mu_{t,\tau;i}+\mu'_{t,\tau;i}\big)_{t\in B,\tau\in\mathbb{I},i\in I}$$

is then a smooth family of 1-forms on $\mathcal{N}_\partial$. By the first and last properties in (3.56) and in (3.59), it satisfies the first and last properties in (3.50) with $\ell=\ell^*$. By (3.60), the last properties in (3.56) and in (3.59), and the third property in (3.56), the closed 2-form on $\mathcal{N}_\partial'$ given by (3.49) with $\ell=\ell^*$ satisfies

$$\widetilde{\omega}^{(\ell^*)}_{t,\tau;i}\big|_v=\begin{cases}\widetilde{\omega}^{(\ell^*-1)}_{t,1;i}|_v, & \text{if } (\tau,v)\in\mathbb{I}\times(\mathcal{N}_i'-\mathcal{N}''_{(\ell^*)}|_{K^\circ_{\ell^*+1}-U_{\ell^*-1}});\\ \widetilde{\omega}_{t,\tau;i}|_v, & \text{if } (\tau,v)\in\mathbb{I}\times(\mathcal{N}''_{(\ell^*)})_i-(\frac{1}{2},1]\times\mathcal{N}'''|_{K'-U_{\ell^*-1}};\\ \widehat{\omega}_{t,\tau;i}|_v, & \text{if } (\tau,v)\in[\frac{1}{2},1]\times\mathcal{N}_i'''|_K.\end{cases}\tag{3.61}$$

Thus, $\widetilde{\omega}^{(\ell^*)}_{t,\tau;i}$ is a symplectic structure on $\mathcal{N}_\partial'$. By (3.61) and the second properties in (3.50) with $\ell=\ell^*-1$, in (3.56), and in (3.59), the second property in (3.50) with $\ell=\ell^*$ is satisfied as well. By (3.61), (3.51) with $\ell=\ell^*-1$, the second property in (3.56), the third property in (3.59), and the first property in (3.58), (3.51) with $\ell=\ell^*$ holds. By the third case in (3.61), the third property in (3.59), and (3.57), the second property in (3.50) with $\ell=\ell^*$ is satisfied.

By the above, we can assume that $(\mathcal{N}\mu1)$ and $(\mathcal{N}\mu2)$ hold for all $\ell\in\mathbb{Z}^+$. We can also assume that

$$\mathrm{supp}\big(\mu^{(\ell)}_{t,\cdot;i}\big)\subset\big(1-2^{1-\ell},1\big],\quad \mathrm{supp}\big(\mu^{(\ell)}_{t,\cdot;i}-\mu^{(\ell)}_{t,1;i}\big)\subset\big[0,1-2^{-\ell}\big)\quad\forall\,\ell\in\mathbb{Z}^+,\,t\in B,\,i\in I,\tag{3.62}$$

i.e. $\mu^{(\ell)}_{t,\tau;i}$ as a function of τ changes only in the interval $(1-2^{1-\ell},1-2^{-\ell})$. Let

$$\mathcal{N}''=\mathcal{N}''_{(1)},\quad \widehat{\mathcal{N}}=\bigcup_{\ell=1}^{\infty}\widehat{\mathcal{N}}_{(\ell)}|_{K^\circ_\ell},\quad \mu_{t,\tau;i}=\sum_{\ell=1}^{\infty}\mu^{(\ell)}_{t,\tau;i}|_{\mathcal{N}_i'}\quad\forall\;t\in B,\,\tau\in\mathbb{I},\,i\in I.$$

The sets $\mathcal{N}''$ and $\widehat{\mathcal{N}}$ are open neighborhoods of $V\subset\mathcal{N}'$ such that $\overline{\mathcal{N}''}\subset\mathcal{N}'$. By the last property in (3.50),

$$\mu^{(\ell)}_{t,\tau;i}\big|_v=0\qquad\forall\,v\in\mathcal{N}_i|_{K_{\ell^*}-K^\circ_{\ell^*-1}},\,\ell\in\mathbb{Z}^+-\{\ell^*-1,\ell^*\},\,\ell^*\in\mathbb{Z}^+\,.$$

Thus, the sum above is well-defined and determines a smooth family of 1-forms on $\mathcal{N}_\partial'$.

By the first, second, and last properties in (3.50), the family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in I}$ satisfies the first, second, and last requirements in (3.6). By the last two properties in (3.50) and (3.52),

$$\begin{split}\widetilde{\omega}_{t,1;i}\big|_{(\widehat{\mathcal{N}}_{(\ell^*)})_i|_{K^\circ_{\ell^*}}}\equiv\bigg(\widetilde{\omega}_{t;i}+\mathrm{d}\sum_{\ell=1}^{\infty}\mu^{(\ell)}_{t,1;i}\bigg)\bigg|_{(\widehat{\mathcal{N}}_{(\ell^*)})_i|_{K^\circ_{\ell^*}}}&=\bigg(\widetilde{\omega}_{t;i}+\mathrm{d}\sum_{\ell=1}^{\ell^*}\mu^{(\ell)}_{t,1;i}\bigg)\bigg|_{(\widehat{\mathcal{N}}_{(\ell^*)})_i|_{K^\circ_{\ell^*}}}\\ &=\widetilde{\omega}^{(\ell^*)}_{t,1;i}\big|_{(\widehat{\mathcal{N}}_{(\ell^*)})_i|_{K^\circ_{\ell^*}}}=\widehat{\omega}^\bullet_{t;i}\big|_{(\widehat{\mathcal{N}}_{(\ell^*)})_i|_{K^\circ_{\ell^*}}}\end{split}$$

for all $\ell^* \in \mathbb{Z}^+$. Thus, the third requirement in (3.6) is also satisfied. If $\tau \in [1-2^{1-\ell^*}, 1-2^{-\ell^*}]$ for some $\ell^* \in \mathbb{Z}^+$, then

$$\begin{aligned}\widetilde{\omega}_{t,\tau;i} \equiv \widetilde{\omega}_{t;i} + \mathrm{d}\sum_{\ell=1}^{\infty} \mu_{t,\tau;i}^{(\ell)}\big|_{\mathcal{N}_i'} &= \widetilde{\omega}_{t;i} + \mathrm{d}\sum_{\ell=1}^{\ell^*-1} \mu_{t,1;i}^{(\ell)}\big|_{\mathcal{N}_i'} + \mathrm{d}\mu_{t,\tau;i}^{(\ell^*)}\big|_{\mathcal{N}_i'} \\ &= \widetilde{\omega}_{t,1;i}^{(\ell^*-1)} + \mathrm{d}\mu_{t,\tau;i}^{(\ell^*)}\big|_{\mathcal{N}_i'} = \widetilde{\omega}_{t,\tau;i}^{(\ell^*)};\end{aligned}$$

see (3.62) and (3.52). Thus, $\widetilde{\omega}_{t,\tau;i}$ is a symplectic structure on $\mathcal{N}_\partial'$ for all $(t,\tau) \in B \times \mathbb{I}$. □

Remark 3.11. By the proof of Theorem 3.1 above, the compactness requirements on $V-U$ in Proposition 3.6 and on K in Lemma 3.10 are not necessary.

4 Tubular neighborhood theorems

We next obtain a stratified version of the usual Tubular Neighborhood Theorem which respects a symplectic form along a symplectic submanifold. Proposition 4.2 below is used in Section 5 to apply the essentially local statement of Theorem 3.1 in the setting of Theorem 2.17. We continue with the notation of Sections 2.2 and 2.3.

Definition 4.1. Let $N \in \mathbb{Z}^+$, $\mathbf{X} \equiv \{X_I\}_{I \in \mathcal{P}^*(N)}$ be a transverse configuration, $I^* \in \mathcal{P}^*(N)$, and $U \subset X_{I^*}$ be an open subset. A regularization for U in $\mathbf{X}$ is a tuple $(\Psi_i)_{i \in I^*}$, where Ψ_i is a regularization for U in X_i in the sense of Definition 2.8, such that

$$\Psi_i\big(\mathcal{N}_{I^*;I} \cap \mathrm{Dom}(\Psi_i)\big) = X_I \cap \mathrm{Im}(\Psi_i) \qquad \forall\, i \in I \subset I^*, \tag{4.1}$$

$$\Psi_{i_1}\big|_{\mathcal{N}_{I^*;i_1 i_2} \cap \mathrm{Dom}(\Psi_{i_1})} = \Psi_{i_2}\big|_{\mathcal{N}_{I^*;i_1 i_2} \cap \mathrm{Dom}(\Psi_{i_2})} \quad \forall\, i_1, i_2 \in I^*. \tag{4.2}$$

Smooth families of regularizations for U in $\mathbf{X}$ are defined analogously to Definition 2.12(2).

Proposition 4.2. *Let $N \in \mathbb{Z}^+$, $\mathbf{X} \equiv \{X_I\}_{I \in \mathcal{P}^*(N)}$ be a transverse configuration such that X_{ij} is a closed submanifold of X_i of codimension 2 for all $i,j \in [N]$ distinct, $I^* \in \mathcal{P}^*(N)$, and $U, U' \subset X_{I^*}$ be open subsets, possibly empty, such that $\overline{U'} \subset U$. Suppose*

- *B is a compact manifold, possibly with boundary,*
- *$N(\partial B), N'(\partial B)$ are neighborhoods of $\partial B \subset B$ such that $\overline{N'(\partial B)} \subset N(\partial B)$,*
- *$(\omega_{t;i})_{t \in B}$ is a smooth family of symplectic structures on $\mathbf{X}$ in the sense of Definition 2.4,*
- *$(\Psi_{I^*;t;i})_{t \in N(\partial B), i \in I^*}$ and $(\Psi_{U;t;i})_{t \in B, i \in I^*}$ are smooth families of regularizations for X_{I^*} and U, respectively, in $\mathbf{X}$ such that*

$$\mathrm{d}_x \Psi_{\star;t;i}\big(\mathcal{N}_{I^*;i}|_x\big) = T_x X_{I^*}^{\omega_{t;i}} \quad \forall\, \star = I^*, U,\ (t,x) \in \begin{cases} N(\partial B) \times X_{I^*}, & \text{if } \star = I^*; \\ B \times U, & \text{if } \star = U; \end{cases} \tag{4.3}$$

$$\big(\Psi_{I^*;t;i}|_{\mathrm{Dom}(\Psi_{I^*;t;i})|_U}\big)_{t \in N(\partial B), i \in I^*} = \big(\Psi_{U;t;i}\big)_{t \in N(\partial B), i \in I^*}. \tag{4.4}$$

Then there exists a smooth family $(\Psi_{t;i})_{t \in B, i \in I^*}$ of regularizations for X_{I^*} in $\mathbf{X}$ such that

$$\mathrm{d}_x \Psi_{t;i}\big(\mathcal{N}_{I^*;i}|_x\big) = T_x X_{I^*}^{\omega_{t;i}} \quad \forall\, t \in B,\ x \in X_{I^*},\ i \in I^*, \tag{4.5}$$

$$\begin{aligned}\big(\Psi_{t;i}\big)_{t \in N'(\partial B), i \in I^*} &= \big(\Psi_{I^*;t;i}\big)_{t \in N'(\partial B), i \in I^*}, \\ \big(\Psi_{t;i}|_{\mathrm{Dom}(\Psi_{t;i})|_{U'}}\big)_{t \in B, i \in I^*} &= \big(\Psi_{U;t;i}|_{\mathrm{Dom}(\Psi_{U;t;i})|_{U'}}\big)_{t \in B, i \in I^*}.\end{aligned} \tag{4.6}$$

4.1 Smooth regularizations for transverse collections

Lemma 4.3 below shows that regularizations in the sense of Definition 2.8 that satisfy the stratification condition (2.14) always exist, if V_I is a closed submanifold. By Lemma 4.4, they can be chosen to extend given regularizations over an open subspace, after slightly shrinking the latter, and to respect a symplectic form along V_I.

Lemma 4.3. *Let X be a manifold and $\{V_i\}_{i\in I}$ be a transverse collection of closed submanifolds of X. Then there exists a smooth map $\exp_I\colon TX|_{V_I}\longrightarrow X$ such that*

$$\begin{aligned} \exp_I\big|_{V_I} = \mathrm{id}, \qquad \mathrm{d}_x \exp_I = \mathrm{id} : T_xX \longrightarrow T_xX \quad \forall\, x\in V_I, \\ \exp_I\big(TV_{I'}|_{V_I}\big) = V_{I'}\cap \mathrm{Im}(\exp_I)\ \forall\, I'\subset I. \end{aligned} \tag{4.7}$$

Proof. Choose a metric g on X so that the orthogonal complements L_i of TV_I in $TV_{I-i}|_{V_I}$ are orthogonal for pairs of different values of $i\in I$. For each $I'\subset I$, let

$$\mathcal{N}_{I;I'} = \bigoplus_{i\in I-I'} L_i \approx \mathcal{N}_{V_{I'}}V_I$$

and $\pi_{I'}\colon \mathcal{N}_{I;\emptyset}\longrightarrow \mathcal{N}_{I;I'}$ be the projection map. There is then a canonical identification

$$T\mathcal{N}_{I;I'}|_{V_I} = TV_{I'}|_{V_I}\,.$$

Let $\exp\colon W\longrightarrow X$, where W is a neighborhood of $X\subset TX$, be the exponential map with respect to the Levi-Civita connection of the metric g.

Denote by $\Psi_0 : \mathcal{N}_{I;\emptyset}\longrightarrow X$ the composition of exp with a diffeomorphism from $\mathcal{N}_{I;\emptyset}$ to a neighborhood of $V_I\subset \mathcal{N}_{I;\emptyset}\cap W$ which restricts to the identity on a smaller neighborhood of $V_I\subset\mathcal{N}_{I;\emptyset}$. Suppose $\ell\in\{1,\ldots,|I|\}$ and we have constructed a smooth map $\Psi_{\ell-1}\colon \mathcal{N}_{I;\emptyset}\longrightarrow X$ such that

$$\begin{aligned} \Psi_{\ell-1}|_{V_I} = \mathrm{id}_{V_I}, \quad \mathrm{d}\Psi_{\ell-1}|_{T\mathcal{N}_{I;\emptyset}|_{V_I}} = \mathrm{id}_{T\mathcal{N}_{I;\emptyset}|_{V_I}} : T\mathcal{N}_{I;\emptyset}|_{V_I} \longrightarrow TX|_{V_I}\,, \\ \Psi_{\ell-1}(\mathcal{N}_{I;I''}) = V_{I''}\cap\mathrm{Im}(\Psi_{\ell-1}) \quad \forall\, I''\subset I \ \text{ s.t. } \ |I''|>|I|-\ell. \end{aligned} \tag{4.8}$$

By the first two statements in (4.8) and the Inverse Function Theorem [29, Theorem 1.30], there exist a neighborhood W of $V_I\subset X$ and a smooth map $\Phi\colon W\longrightarrow \mathcal{N}_{I;\emptyset}$ such that

$$\Psi_{\ell-1}\circ\Phi = \mathrm{id}_W, \quad \Phi\circ\Psi_{\ell-1}|_{\Phi(W)} = \mathrm{id}_{\Phi(W)}\,. \tag{4.9}$$

In particular, $\Phi(V_{I'}\cap W)\subset\mathcal{N}_{I;\emptyset}$ is a smooth submanifold for all $I'\subset I$. By (4.8) and the second equation in (4.9),

$$\begin{aligned} \Phi|_{V_I} = \mathrm{id}_{V_I}, \quad T\Phi(V_{I'}\cap W)\big|_{V_I} = T\mathcal{N}_{I;I'}|_{V_I} \quad \forall\, I'\subset I, \\ \Phi(V_{I''}\cap W) = \mathcal{N}_{I;I''}\cap\mathrm{Im}(\Phi) \quad \forall\, I''\subset I \ \text{ s.t. } \ |I''|>|I|-\ell. \end{aligned} \tag{4.10}$$

By the first two statements in (4.10), for every $I'\subset I$ we can apply the Inverse Function Theorem to the projection

$$\pi_{I'}\colon \Phi(V_{I'}\cap W) \longrightarrow \mathcal{N}_{I;I'}$$

forgetting the components in L_i with $i\in I'$. There thus exist a neighborhood $\mathcal{N}'$ of $V_I\subset\mathcal{N}_{I;\emptyset}$, a neighborhood W' of $V_I\subset W$, and fiber-preserving smooth maps

$$h_{I';i}\colon \mathcal{N}_{I;I'}\cap\mathcal{N}'\longrightarrow L_i,\quad i\in I'\subset I,\qquad \text{s.t.}$$
$$\pi_{I'}(\mathcal{N}')=\mathcal{N}_{I;I'}\cap\mathcal{N}',\quad \big(\mathrm{id}_{\mathcal{N}_{I;I'}},(h_{I';i})_{i\in I'}\big)\circ\pi_{I'}|_{\Phi(V_{I'}\cap W')}=\mathrm{id}_{\Phi(V_{I'}\cap W')}\quad\forall\,I'\subset I. \tag{4.11}$$

By (4.10) and (4.11),

$$\mathrm{d}_x h_{I';i}=0\ \ \forall\,x\in V_I,\qquad h_{I';i}(v)=0\ \ \forall\,v\in\mathcal{N}_{I;I''}\cap\mathcal{N}',\ I''\supset I',\ |I''|>|I|-\ell. \tag{4.12}$$

Let $\mathcal{P}_\ell^c(I)$ denote the collection of subsets $I'\subset I$ with $|I'|=|I|-\ell$. We define a smooth fiber-preserving map

$$\Theta\equiv(\Theta_i)_{i\in I}:\mathcal{N}'\longrightarrow\mathcal{N}_{I;\emptyset}\qquad\text{by}\quad \Theta_i(v)=v_i+\sum_{\substack{I'\in\mathcal{P}_\ell^c(I)\\ i\in I'}}h_{I';i}\big(\pi_{I'}(v)\big).$$

By (4.12) and $\pi_{I''}(\mathcal{N}_{I;I'})=\mathcal{N}_{I;I'\cup I''}$ for every $I''\subset I$,

$$\mathrm{d}\Theta\big|_{T\mathcal{N}_{I;\emptyset}|_{V_I}}=\mathrm{id}\big|_{T\mathcal{N}_{I;\emptyset}|_{V_I}},\qquad \Theta\big|_{\mathcal{N}_{I;I'}\cap\mathcal{N}'}=\big(\mathrm{id}_{\mathcal{N}_{I;I'}},(h_{I';i})_{i\in I'}\big)\big|_{\mathcal{N}_{I;I'}\cap\mathcal{N}'}\ \ \forall\,I'\in\mathcal{P}_\ell^c(I). \tag{4.13}$$

By the Inverse Function Theorem, Θ thus restricts to a diffeomorphism on a neighborhood $\mathcal{N}''$ of $V_I\subset\mathcal{N}'$. By the second statement in (4.13) and (4.11), the diffeomorphism

$$\Psi'_\ell\equiv\Psi_{\ell-1}\circ\Theta\colon\mathcal{N}''\longrightarrow\Psi_{\ell-1}\big(\Theta(\mathcal{N}'')\big)$$

satisfies the last condition in (4.8) with ℓ replaced by $\ell+1$ and $\mathcal{N}_{I;I'}$ by $\mathcal{N}_{I;I'}\cap\mathcal{N}''$. As it also satisfies the first two conditions in (4.8), we can obtain a smooth map $\Psi_\ell\colon\mathcal{N}_{I;\emptyset}\longrightarrow X$ satisfying (4.8) with ℓ replaced by $\ell+1$ by composing Ψ'_ℓ with a diffeomorphism from $\mathcal{N}_{I;\emptyset}$ to a neighborhood of $V_I\subset\mathcal{N}''$ which restricts to the identity on a smaller neighborhood of $V_I\subset\mathcal{N}''$ and preserves lines inside of each fiber of $\mathcal{N}_{I;\emptyset}$.

Thus, there exists a smooth map $\Psi_\ell:\mathcal{N}_{I;\emptyset}\longrightarrow X$ satisfying (4.8) with $\ell=|I|+1$. By composing Ψ_ℓ with the orthogonal projection $TX|_{V_I}\longrightarrow\mathcal{N}_{I;\emptyset}$, we obtain a smooth map $\exp_I$ with the desired properties. □

Lemma 4.4. *Let X be a manifold, $\{V_i\}_{i\in S}$ be a finite transverse collection of closed submanifolds of X of codimension 2, $I\in\mathcal{P}^*(S)$, and $U,U'\subset V_I$ be open subsets, possibly empty, such that $\overline{U'}\subset U$. Suppose*

- *$N'(\partial B)\subset N(\partial B)\subset B$ are as in Proposition 4.2,*
- *$(\omega_t)_{t\in B}$ is a smooth family of symplectic structures on $\{V_i\}_{i\in S}$ in X in the sense of Definition 2.2,*
- *$(\Psi_{I;t})_{t\in N(\partial B)}$ and $(\Psi_{U;t})_{t\in B}$ are smooth families of regularizations for V_I and U, respectively, in X such that*

$$\Psi_{\star;t}\big(\mathcal{N}_{I;I'}\cap\mathrm{Dom}(\Psi_{\star;t})\big)=V_{I'}\cap\mathrm{Im}(\Psi_{\star;t})\ \ \forall\,I'\subset I,\quad \mathrm{d}_x\Psi_{\star;t}\big(\mathcal{N}_XV_I|_x\big)=T_xV_I^{\omega_t}, \tag{4.14}$$

for all $\star=I,U$, $(t,x)\in N(\partial B)\times V_I$ if $\star=I$, and $(t,x)\in B\times U$ if $\star=U$, and

$$\big(\Psi_{I;t}|_{\mathrm{Dom}(\Psi_{I;t})|_U}\big)_{t\in N(\partial B)}=\big(\Psi_{U;t}\big)_{t\in N(\partial B)}\,. \tag{4.15}$$

Then there exists a smooth family $(\Psi_t)_{t\in B}$ *of regularizations for* V_I *in* X *such that*

$$\Psi_t\big(\mathcal{N}_{I;I'}\cap\mathrm{Dom}(\Psi_t)\big)=V_{I'}\cap\mathrm{Im}(\Psi_t)\ \ \forall\, I'\subset I,\quad \mathrm{d}_x\Psi_t\big(\mathcal{N}_XV_I|_x\big)=T_xV_I^{\omega_t}\ \ \forall\, x\in V_I, \tag{4.16}$$

$$\big(\Psi_t\big)_{t\in N'(\partial B)}=\big(\Psi_{I;t}\big)_{t\in N'(\partial B)},\quad \big(\Psi_t|_{\mathrm{Dom}(\Psi_t)|_{U'}}\big)_{t\in B}=\big(\Psi_{U;t}|_{\mathrm{Dom}(\Psi_{U;t})|_{U'}}\big)_{t\in B}. \tag{4.17}$$

Proof. Let $\pi\colon TV_I\longrightarrow V_I$ and $\pi_I\colon\mathcal{N}_XV_I\longrightarrow V_I$ be the projection maps and $\exp_I\colon TX|_{V_I}\longrightarrow X$ be as in Lemma 4.3. Choose an isomorphism

$$\widetilde{\exp}_{V_I}\colon \pi^*\mathcal{N}_XV_I\longrightarrow \big\{\exp_I|_{TV_I}\big\}^*\mathcal{N}_XV_I\subset TV_I\times\mathcal{N}_XV_I$$

of split vector bundles over TV_I restricting to the identity over $V_I\subset TV_I$, i.e.

$$\widetilde{\exp}_{V_I}(x,v)=(x,v)\qquad\forall\,(x,v)\in\big(\pi^*\mathcal{N}_XV_I\big)|_{V_I}\subset TV_I\times\mathcal{N}_XV_I. \tag{4.18}$$

Denote by $\pi_2\colon\{\exp_I|_{TV_I}\}^*\mathcal{N}_XV_I\longrightarrow\mathcal{N}_XV_I$ the projection onto the second component.

Let $t\in B$. We identify $\mathcal{N}_XV_I=\mathcal{N}_{I;\emptyset}$ with the ω_t-orthogonal complement $TV_I^{\omega_t}\subset TX|_{V_I}$ of TV_I via the quotient projection map; it is the direct sum of the ω_t-orthogonal complements of TV_I in $TV_{I-i}|_{TV_I}$ with $i\in I$. Define

$$\widehat\Psi_t'=\exp_I\big|_{TV_I^{\omega_t}}\colon\mathcal{N}_XV_I=TV_I^{\omega_t}\longrightarrow X.$$

By the first two statements in (4.7) and the Tubular Neighborhood Theorem [2, (12.11)], there exists a neighborhood $\mathcal{W}$ of $B\times V_I$ in $B\times\mathcal{N}_XV_I$ such that

$$\widehat\Psi_t\equiv\widehat\Psi_t'|_{W_t}\colon W_t\longrightarrow X,\qquad\text{where}\quad \{t\}\times W_t\equiv\big(\{t\}\times\mathcal{N}_XV_I\big)\cap\mathcal{W},$$

is a regularization for V_I in X for each $t\in B$. By the last two statements in (4.7),

$$\widehat\Psi_t\big(\mathcal{N}_{I;I'}\cap W_t\big)=V_{I'}\cap\mathrm{Im}(\widehat\Psi_t)\ \ \forall\, I'\subset I,\quad \mathrm{d}_x\widehat\Psi_t\big(\mathcal{N}_XV_I|_x\big)=T_xV_I^{\omega_t}\ \ \forall\, x\in V_I. \tag{4.19}$$

Let $\star=I,U$, $t\in N(\partial B)$ if $\star=I$, and $t\in B$ if $\star=U$. Since $\widehat\Psi_t$ and $\Psi_{\star;t}$ are regularizations,

$$\mathrm{d}_x\widehat\Psi_t=\mathrm{d}_x\Psi_{\star;t}\qquad\forall\,x\in\begin{cases}V_I, & \text{if } \star=I;\\ U, & \text{if } \star=U.\end{cases} \tag{4.20}$$

By (4.20) and the Inverse Function Theorem, there exists a neighborhood $\mathcal{W}_\star$ of $N(\partial B)\times V_I$ in $\mathcal{W}$ if $\star=I$ and of $B\times U$ in $\mathcal{W}\cap(B\times\mathcal{N}_XV_I|_U)$ if $\star=U$ independent of t such that the map

$$\Theta_{\star;t}\equiv\widehat\Psi_t^{-1}\circ\Psi_{\star;t}\colon W_{\star;t}\longrightarrow\mathcal{N}_XV_I,\quad\text{where}\quad \{t\}\times W_{\star;t}\equiv\big(\{t\}\times\mathcal{N}_XV_I\big)\cap\mathcal{W}_\star, \tag{4.21}$$

is a well-defined diffeomorphism onto a neighborhood of $V_I\subset\mathcal{N}_XV_I$ if $\star=I$ and of $U\subset\mathcal{N}_XV_I|_U$ for $t\in B$. By (4.20), the first assumption in (4.14), and the second property in (4.19),

$$\Theta_{\star;t}(x)=x,\quad \mathrm{d}_x\Theta_{\star;t}=\mathrm{id},\quad \Theta_{\star;t}\big(\mathcal{N}_{I;I'}\cap W_{\star;t}\big)=\mathcal{N}_{I;I'}\cap\mathrm{Im}\big(\Theta_{\star;t}\big)\ \ \forall\, I'\subset I, \tag{4.22}$$

for all $x\in V_I$ if $\star=I$ and $x\in U$ if $\star=U$. By (4.15),

$$\big(\Theta_{I;t}|_{W_{I;t}\cap W_{U;t}}\big)_{t\in N(\partial B)}=\big(\Theta_{U;t}|_{W_{I;t}\cap W_{U;t}}\big)_{t\in N(\partial B)}. \tag{4.23}$$

With $\star$ and t as above, define

$$\begin{gathered}
W'_{\star;t} = \big\{v\in W_{\star;t}\colon\ \pi_I(\Theta_{\star;t}(v))\in \exp_I(T_{\pi_I(v)}V_I)\big\},\\
\Theta^{\rm hor}_{\star;t}\colon W'_{\star;t}\longrightarrow TV_I \qquad \text{by}\quad \Theta^{\rm hor}_{\star;t}(v)\in T_{\pi_I(v)}V_I, \qquad \exp_I\big(\Theta^{\rm hor}_{\star;t}(v)\big)=\pi_I\big(\Theta_{\star;t}(v)\big),\\
\Theta^{\rm ver}_{\star;t}\colon W'_{\star;t}\longrightarrow \mathcal N_XV_I \qquad \text{by}\quad \Theta^{\rm ver}_{\star;t}(v)\in \mathcal N_XV_I|_{\pi_I(v)}, \quad \pi_2\big(\widetilde{\exp}_{V_I}\big(\Theta^{\rm hor}_{\star;t}(v),\Theta^{\rm ver}_{\star;t}(v)\big)\big)=\Theta_{\star;t}(v).
\end{gathered}$$

For a smooth function $\eta\colon V_I\longrightarrow\mathbb R$, let

$$\begin{gathered}
\Theta_{\star;t,\eta}\colon W'_{\star;t}\longrightarrow \mathcal N_XV_I,\\
\Theta_{\star;t,\eta}(v)=\pi_2\Big(\widetilde{\exp}_{V_I}\big(\big(1-\eta(\pi_I(v))\big)\Theta^{\rm hor}_{\star;t}(v), v+\big(1-\eta(\pi_I(v))\big)\big(\Theta^{\rm ver}_{\star;t}(v)-v\big)\big)\Big).
\end{gathered}$$

By (4.22),

$$\Theta_{\star;t,\eta}(x)=x, \quad \mathrm d_x\Theta_{\star;t,\eta}=\mathrm{id}, \quad \Theta_{\star;t,\eta}\big(\mathcal N_{I;I'}\cap W'_{\star;t}\big)=\mathcal N_{I;I'}\cap \mathrm{Im}\big(\Theta_{\star;t,\eta}\big) \quad \forall\, I'\subset I, \tag{4.24}$$

for all $x\in V_I$ if $\star=I$ and $x\in U$ if $\star=U$. By (4.23),

$$\big(\Theta_{I;t,\eta}|_{W'_{I;t}\cap W'_{U;t}}\big)_{t\in N(\partial B)}=\big(\Theta_{U;t,\eta}|_{W'_{I;t}\cap W'_{U;t}}\big)_{t\in N(\partial B)}. \tag{4.25}$$

The set

$$\mathcal W''_\star\equiv\big\{(t,v)\in B\times\mathcal N_XV_I\colon\ v\in W'_{\star;t},\ \Theta_{\star;t,\tau}(v)\in W_t\ \forall\,\tau\in\mathbb I\big\}$$

is a neighborhood of

$$\begin{aligned}
N(\partial B)\times V_I\subset \bigcup_{t\in N(\partial B)}\{t\}\times\mathrm{Dom}(\Psi_{I;t})&\subset N(\partial B)\times\mathcal N_XV_I \quad &&\text{if } \star=I,\\
B\times U\subset \bigcup_{t\in B}\{t\}\times\mathrm{Dom}(\Psi_{U;t})&\subset B\times\mathcal N_XV_I|_U &&\text{if } \star=U.
\end{aligned}$$

Let $N''(\partial B)\subset B$ and $U'_0\subset U''\subset V_I$ be open subsets such that

$$\overline{N'(\partial B)}\subset N''(\partial B),\quad \overline{N''(\partial B)}\subset N(\partial B), \qquad \overline{U'}\subset U'_0,\ \ \overline{U'_0}\subset U'',\ \ \overline{U''}\subset U.$$

The set

$$\begin{gathered}
\widetilde{\mathcal W}\equiv\Big(\mathcal W-\overline{N''(\partial B)}\times\mathcal N_XV_I-B\times\mathcal N_XV_I|_{\overline{U''}}\Big)\cup\mathcal W''_I\cup\mathcal W''_U\\
\cup\Big(\bigcup_{t\in N'(\partial B)}\{t\}\times\mathrm{Dom}(\Psi_{I;t})\Big)\cup\Big(\bigcup_{t\in B}\{t\}\times\mathrm{Dom}(\Psi_{U;t})|_{U'_0}\Big)
\end{gathered}$$

is then a neighborhood of $B\times V_I$ in $B\times\mathcal N_XV_I$. Choose smooth $[0,1]$-valued functions η_I on B and η_U on V_I such that

$$\eta_I(t)=\begin{cases}0, & \text{if } t\in N'(\partial B);\\ 1, & \text{if } t\notin\overline{N''(\partial B)};\end{cases} \qquad \eta_U(x)=\begin{cases}0, & \text{if } x\in U'_0;\\ 1, & \text{if } x\notin\overline{U''}.\end{cases} \tag{4.26}$$

By (4.26), (4.18), (4.21), and (4.15),

$$\widehat\Psi_t\big(\Theta_{\star;t,\eta_I(t)\eta_U}(v)\big)=\begin{cases}\widehat\Psi_t(v), & \text{if } (t,v)\in\mathcal W''_\star,\ t\notin\overline{N''(\partial B)},\ \pi_I(v)\notin\overline{U''};\\ \Psi_{I;t}(v), & \text{if } (t,v)\in\mathcal W''_\star,\ t\in N'(\partial B),\ v\in\mathrm{Dom}(\Psi_{I;t});\\ \Psi_{U;t}(v), & \text{if } (t,v)\in\mathcal W''_\star,\ v\in\mathrm{Dom}(\Psi_{U;t})|_{U'_0}.\end{cases} \tag{4.27}$$

Define

$$\widetilde{\Psi}\colon \widetilde{\mathcal{W}} \longrightarrow X, \quad \widetilde{\Psi}_t(v) = \begin{cases} \widehat{\Psi}_t(v), & \text{if } t \notin \overline{N''(\partial B)},\ \pi_I(v) \notin \overline{U''}; \\ \widehat{\Psi}_t\big(\Theta_{\star;t,\eta_I(t)\eta_U}(v)\big), & \text{if } (t,v) \in \mathcal{W}''_\star,\ \star = I,U; \\ \Psi_{I;t}(v), & \text{if } t \in N'(\partial B),\ v \in \mathrm{Dom}(\Psi_{I;t}); \\ \Psi_{U;t}(v), & \text{if } v \in \mathrm{Dom}(\Psi_{U;t})|_{U_0'}. \end{cases} \tag{4.28}$$

The domain in the first case above is disjoint from the domains in the last two cases. By (4.25), the definitions of $\widetilde{\Psi}_t(v)$ agree on the overlap of the two domains in the second case. By (4.27), its definition on either of these domains agrees with the definitions in the first, second, and fourth cases on the overlaps. By (4.15), the definitions of $\widetilde{\Psi}_t(v)$ agree on the overlap of the last two cases. Thus, $\widetilde{\Psi}$ is well-defined and smooth. By the last two cases in (4.28), $\widetilde{\Psi}_t$ satisfies (4.17) with $\Psi_t = \widetilde{\Psi}_t$. By the first statements in (4.14) and (4.19) and the last statement in (4.24), $\widetilde{\Psi}_t$ satisfies the first property in (4.16). By the first two cases in (4.28), (4.7), and the first two statements in (4.24),

$$\widetilde{\Psi}_t(x) = x, \quad \mathrm{d}_x\widetilde{\Psi}_t = \mathrm{id}\colon T_x\mathcal{N}_XV_I = T_xV_I \oplus T_xV_I^{\omega_t} \longrightarrow T_xX \qquad \forall\,(t,x) \in B \times V_I. \tag{4.29}$$

This implies that $\widetilde{\Psi}_t$ satisfies the second property in (4.16).

By (4.29) and the Tubular Neighborhood Theorem, there exists a neighborhood

$$\widetilde{\mathcal{W}}' \equiv \bigcup_{t\in B}\{t\}\times W_t'$$

of $B\times V_I$ in $\widetilde{\mathcal{W}}$ such that

$$\widetilde{\Psi}_t|_{W_t'}\colon W_t' \longrightarrow \widetilde{\Psi}_t(W_t')$$

is a diffeomorphism onto an open neighborhood. Let

$$\bigcup_{t\in B}\{t\}\times W_t'' \equiv \bigcup_{t\in B}\{t\}\times\Big(W_t' - \widetilde{\Psi}_t^{-1}\big(\overline{\widetilde{\Psi}_t(\mathrm{Dom}(\widetilde{\Psi}_t)|_{U'})}\big) \cap \mathcal{N}_XV_I|_{V_I-U_0'}\Big) \\ \cup\Big(\bigcup_{t\in N'(\partial B)}\{t\}\times\mathrm{Dom}(\Psi_{I;t})\Big) \cup \Big(\bigcup_{t\in B}\{t\}\times\mathrm{Dom}(\Psi_{U;t})|_{U'}\Big).$$

This is a neighborhood of $B\times V_I$ in $B\times\mathcal{N}_XV_I$ and $\Psi_t \equiv \widetilde{\Psi}_t|_{W_t''}$ is injective, since $\Psi_{I;t}$ and $\Psi_{U;t}$ are. Thus, $(\Psi_t)_{t\in B}$ is a smooth family of regularizations for V_I in X with the desired properties. □

Remark 4.5. The first requirement in (4.16) is non-trivial only for $|I|\geq 2$. By [19, Lemma 3.14] and its proof, the second requirement in (4.16) can be strengthened to the equality of $\Psi_t^*\omega_t$ with a standard 2-form $\widehat{\omega}_t$ on $\mathcal{N}_XV_I$ as in (2.9) over a neighborhood $\mathcal{N}'$ of V_I in $\mathcal{N}_XV_I$ at the cost of dropping the first requirement in (4.16). By Lemma 3.10 and Remark 3.11, this strengthening can also be achieved after deforming ω_t on a neighborhood of V in X while keeping all submanifolds $V_{I'}$ symplectic. It appears that this strengthening can be achieved without weakening some other condition if either $|I|=2$ or the submanifolds $V_i\subset X$ are ω_t-orthogonal. Due to symplectic angle considerations, this strengthening cannot be achieved in the general case without weakening some other condition if $|I|\geq 3$. The approach of this paper, as summarized by the principle on page 2 and the nexus on page 5, is a way around this fundamental obstacle in the setting of singular symplectic divisors and varieties.

4.2 Proof of Proposition 4.2

By Lemma 4.4, for each $i\in I^*$ there exists a smooth family of regularizations $(\Psi_{t;i})_{t\in B}$ for X_{I^*} in X_i such that

$$\Psi_{t;i}\big(\mathcal{N}_{I^*;I}\cap\mathrm{Dom}(\Psi_{t;i})\big)=X_I\cap\mathrm{Im}(\Psi_{t;i})\quad\forall\, i\in I\subset I^*, \tag{4.30}$$

$$\mathrm{d}_x\Psi_{t;i}\big(\mathcal{N}_{I^*;i}|_x\big)=T_xX_{I^*}^{\omega_{t;i}}\quad\forall\, x\in X_{I^*},\, i\in I^*, \tag{4.31}$$

$$\big(\Psi_{t;i}\big)_{t\in N'(\partial B)}=\big(\Psi_{I^*;t;i}\big)_{t\in N'(\partial B)},\quad \big(\Psi_{t;i}|_{\mathrm{Dom}(\Psi_{t;i})|_{U'}}\big)_{t\in B}=\big(\Psi_{U;t;i}|_{\mathrm{Dom}(\Psi_{U;t;i})|_{U'}}\big)_{t\in B}. \tag{4.32}$$

Below we modify the maps $\Psi_{t;i}$ on the intersections of their domains, i.e. neighborhoods of X_{I^*} in $\mathcal{N}_{I^*;i_1i_2}$, in order to make them agree there.

We can assume that $I^*=[\ell^*]$ for some $\ell^*\in\mathbb{Z}^+$. Let $\pi\colon TX_{I^*}\longrightarrow X_{I^*}$ be the projection map and $\exp\colon TX_{I^*}\longrightarrow X_{I^*}$ be a smooth map such that

$$\exp(x)=x,\quad \mathrm{d}_x\exp=\pi_1+\pi_2\colon T_xTX_{I^*}=T_xX_{I^*}\oplus T_xX_{I^*}\longrightarrow T_xX_{I^*}\quad\forall\, x\in X_{I^*}\,.$$

For each $\ell\in[\ell^*]$, choose an isomorphism

$$\widetilde{\exp}_\ell\colon \pi^*\mathcal{N}_{X_{I^*-\ell}}X_{I^*}\longrightarrow \exp^*\mathcal{N}_{X_{I^*-\ell}}X_{I^*}\subset TX_{I^*}\times\mathcal{N}_{X_{I^*-\ell}}X_{I^*}$$

of vector bundles over TX_{I^*} restricting to the identity over $X_{I^*}\subset TX_{I^*}$. Let

$$\pi_2\colon \exp^*\mathcal{N}_{X_{I^*-\ell}}X_{I^*}\longrightarrow\mathcal{N}_{X_{I^*-\ell}}X_{I^*}$$

be the projection to the second component.

Suppose $\ell\in[\ell^*-1]$, $\ell'\in[\ell^*]-[\ell]$, and

$$\Psi_{t;i_1}\big|_{\mathcal{N}_{I^*;i_1i_2}\cap\mathrm{Dom}(\Psi_{t;i_1})}=\Psi_{t;i_2}\big|_{\mathcal{N}_{I^*;i_1i_2}\cap\mathrm{Dom}(\Psi_{t;i_2})}\quad\forall\, t\in B \tag{4.33}$$

if either $i_1\in[\ell-1]$ or $(i_1,i_2)\in[\ell]\times[\ell'-1]$. For each $t\in B$, let

$$W_t=\Psi_{t;\ell}^{-1}\big(\mathrm{Im}(\Psi_{t;\ell'})\big)\subset\mathcal{N}_{I^*;\ell\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell})\,. \tag{4.34}$$

By our assumptions,

$$\Psi_{t;\ell}\big|_{\mathcal{N}_{I^*;\ell\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell})}\colon\mathcal{N}_{I^*;\ell\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell})\longrightarrow X_{\ell\ell'}\qquad\text{and}$$
$$\Psi_{t;\ell'}\big|_{\mathcal{N}_{I^*;\ell\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell'})}\colon\mathcal{N}_{I^*;\ell\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell'})\longrightarrow X_{\ell\ell'}$$

are regularizations for X_{I^*} in $X_{\ell\ell'}$ in the sense of Definition 2.8 satisfying the stratification condition (2.14) with $I=I^*$ and $I'\supset\{\ell,\ell'\}$. Thus,

$$\Theta_t\equiv\Psi_{t;\ell'}^{-1}\circ\Psi_{t;\ell}|_{W_t}\colon W_t\longrightarrow\mathcal{N}_{I^*;\ell\ell'}=\mathcal{N}_{X_{\ell\ell'}}X_{I^*}, \tag{4.35}$$

is a diffeomorphism onto a neighborhood of $X_{I^*}\subset\mathcal{N}_{I^*;\ell\ell'}$ such that

$$\Theta_t(x)=x,\ \ \mathrm{d}_x\Theta_t=\mathrm{id}\quad\forall\, x\in X_{I^*},\quad \Theta_t\big(\mathcal{N}_{I^*;I}\cap W_t\big)=\mathcal{N}_{I^*;I}\cap\mathrm{Im}(\Theta_t)\ \text{ if } \ell,\ell'\in I\subset I^*\,. \tag{4.36}$$

By (4.33),

$$\big(\Theta_t|_{\mathcal{N}_{I^*;i\ell\ell'}\cap W_t}\big)_{t\in B}=\big(\mathrm{id}_{\mathcal{N}_{I^*;i\ell\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell'})}\big)_{t\in B}\quad\forall\ i\in[\ell'-1]-\ell. \tag{4.37}$$

Since $(\Psi_{I^*;t;i})_{i\in I^*}$ and $(\Psi_{U;t;i})_{i\in I^*}$ are regularizations in the sense of Definition 4.1, (4.32) implies that

$$\big(\Theta_t\big)_{t\in N'(\partial B)}=\big(\mathrm{id}_{\mathcal N_{I^*;\ell\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell'})}\big)_{t\in N'(\partial B)},\quad \big(\Theta_t|_{W_t|_{U'}}\big)_{t\in B}=\big(\mathrm{id}_{\mathcal N_{I^*;\ell\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell'})|_{U'}}\big)_{t\in B}. \tag{4.38}$$

Let $\pi_{\ell\ell'}\colon\mathcal N_{I^*;\ell\ell'}\longrightarrow X_{I^*}$ be the projection map. For each $t\in B$, define

$$\begin{gathered}W_t'=\big\{v\in W_t\colon\ \pi_{\ell\ell'}(\Theta_t(v))\in\exp(T_{\pi_{\ell\ell'}(v)}X_{I^*})\big\}\subset\mathcal N_{I^*;\ell\ell'},\\ \Theta_t^{\mathrm{hor}}\colon W_t'\longrightarrow TX_{I^*}\qquad\text{by}\quad \Theta_t^{\mathrm{hor}}(v)\in T_{\pi_{\ell\ell'}(v)}X_{I^*},\quad \exp\big(\Theta_t^{\mathrm{hor}}(v)\big)=\pi_{\ell\ell'}\big(\Theta_t(v)\big),\\ \widetilde\Theta_t\colon\pi_{\ell\ell'}^*\mathcal N_{X_{I^*-\ell}}X_{I^*}\big|_{W_t'}\longrightarrow\mathcal N_{I^*;\ell'}=\mathcal N_{X_{\ell'}}X_{I^*},\quad \widetilde\Theta_t(v,v_\ell)=\big(\Theta_t(v),\pi_2\big(\widetilde{\exp}_\ell(\Theta_t^{\mathrm{hor}}(v),v_\ell)\big)\big).\end{gathered}$$

Let $\widetilde W_{\ell';t}=\widetilde\Theta_t^{-1}(\mathrm{Dom}(\Psi_{t;\ell'}))$ and $\widetilde\Theta_t'=\widetilde\Theta_t|_{\widetilde W_{\ell';t}}$. With identifications as in (2.13), $\widetilde W_{\ell';t}\subset\mathcal N_{I^*;\ell'}$. By (4.36),

$$\widetilde\Theta_t'(x)=x,\quad \mathrm d_x\widetilde\Theta_t'=\mathrm{id}\quad\forall\,x\in X_{I^*},\quad \widetilde\Theta_t'\big(\mathcal N_{I^*;I}\cap\widetilde W_{\ell';t}\big)=\mathcal N_{I^*;I}\cap\mathrm{Im}(\widetilde\Theta_t')\ \text{ if }\ell'\in I\subset I^*. \tag{4.39}$$

By (4.37) and (4.38),

$$\big(\widetilde\Theta_t'|_{\mathcal N_{I^*;i\ell'}\cap\widetilde W_{\ell';t}}\big)_{t\in B}=\big(\mathrm{id}_{\mathcal N_{I^*;i\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell'})}\big)_{t\in B}\quad\forall\,i\in[\ell'-1]-\ell, \tag{4.40}$$

$$\big(\widetilde\Theta_t'\big)_{t\in N'(\partial B)}=\big(\mathrm{id}_{\mathrm{Dom}(\Psi_{t;\ell'})}\big)_{t\in N'(\partial B)},\quad \big(\widetilde\Theta_t'|_{\widetilde W_{\ell';t}|_{U'}}\big)_{t\in B}=\big(\mathrm{id}_{\mathrm{Dom}(\Psi_{t;\ell'})|_{U'}}\big)_{t\in B}. \tag{4.41}$$

By (4.39), the diffeomorphism

$$\Psi_{t;\ell'}'\equiv\Psi_{t;\ell'}\circ\widetilde\Theta_t'\colon\widetilde W_{\ell';t}\longrightarrow X_{\ell'}$$

is a regularization for X_{I^*} in $X_{\ell'}$ for each $t\in B$ satisfying the stratification condition (2.14) with $I=I^*$ and $I'\ni\ell'$. By (4.41) and (4.32),

$$\big(\Psi_{t;\ell'}'\big)_{t\in N'(\partial B)}=\big(\Psi_{I^*;t;\ell'}\big)_{t\in N'(\partial B)},\quad \big(\Psi_{t;\ell'}'|_{\mathrm{Dom}(\Psi_{t;\ell'}')|_{U'}}\big)_{t\in B}=\big(\Psi_{U;t;\ell'}|_{\mathrm{Dom}(\Psi_{U;t;\ell'})|_{U'}}\big)_{t\in B}. \tag{4.42}$$

By (4.40) and (4.33),

$$\Psi_{t;i}\big|_{\mathcal N_{I^*;i\ell'}\cap\mathrm{Dom}(\Psi_{t;i})}=\Psi_{t;\ell'}'\big|_{\mathcal N_{I^*;i\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell'}')}\qquad\forall\,i\in[\ell-1]. \tag{4.43}$$

By (4.34) and (4.35),

$$\begin{gathered}\mathcal N_{I^*;\ell\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell})\supset\mathcal N_{I^*;\ell\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell'}')=\mathcal N_{I^*;\ell\ell'}\cap W_t',\\ \Psi_{t;\ell}\big|_{\mathcal N_{I^*;\ell\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell'}')}=\Psi_{t;\ell'}'\big|_{\mathcal N_{I^*;\ell\ell'}\cap\mathrm{Dom}(\Psi_{t;\ell'}')}.\end{gathered} \tag{4.44}$$

For $i\in[\ell^*]-\ell'$, let $\Psi_{t;i}'=\Psi_{t;i}$.

Choose a neighborhood $\widetilde{\mathcal W}$ of $B\times X_{I^*}$ in $B\times\mathcal N_{I^*}$ such that

$$\big(\{t\}\times\mathcal N_{I^*;i}\big)\cap\widetilde{\mathcal W}\subset\{t\}\times\mathrm{Dom}(\Psi_{t;i}')\qquad\forall\,i\in I^*,\ t\in B.$$

Define $\widetilde{W}_t''\subset\mathcal{N}_{I^*}$ by

$$\bigcup_{t\in B}\{t\}\times\widetilde{W}_t''=\widetilde{\mathcal{W}}\cup\bigcup_{i\in I^*}\bigg(\bigcup_{t\in N'(\partial B)}\mathrm{Dom}(\Psi'_{t;i})\cup\bigcup_{t\in B}\mathrm{Dom}(\Psi'_{t;i})|_{U'}\bigg)\,.$$

For each $t\in B$ and each $i\in[\ell^*]$,

$$\Psi''_{t;i}\equiv\Psi'_{t;i}|_{\mathcal{N}_{I^*;i}\cap\widetilde{W}_t''}:\mathcal{N}_{I^*;i}\cap\widetilde{W}_t''\longrightarrow X_i$$

is a regularization for X_{I^*} in X_i satisfying the stratification condition (2.14) with $I=I^*$ and $I'\ni i$. By (4.30) and the last statement in (4.39), (4.30) with $\Psi_{t;i}$ replaced by $\Psi''_{t;i}$ is satisfied. By (4.31) and the middle statement in (4.39), (4.31) with $\Psi_{t;i}$ replaced by $\Psi''_{t;i}$ is satisfied. By (4.32) and (4.42), (4.32) with $\Psi_{t;i}$ replaced by $\Psi''_{t;i}$ is satisfied. By (4.33), (4.43), and (4.44), these new regularizations satisfy (4.33) with $\Psi_{t;i}$ replaced by $\Psi''_{t;i}$ whenever either $i_1\in[\ell-1]$ or $(i_1,i_2)\in[\ell]\times[\ell']$. This establishes the claim of the proposition by induction.

5 Proof of Theorem 2.17

We prove Theorem 2.17 by induction on the strata of the transverse configuration $\mathbf{X}$. For each $I^*\in\mathcal{P}^*(N)$, we view $\{X_I\}_{I\in\mathcal{P}^*(I^*)}$ as an $|I^*|$-fold transverse configuration. Definition 5.2 introduces a notion of a weak symplectic regularization for any transverse configuration $\mathbf{X}$ over an open subset W of $X_\emptyset$, with $X_\emptyset$ given by (2.5). If W contains all X_I with $I\supsetneq I^*$, a family of such regularizations associated with a family of elements of $\mathrm{Symp}^+(\mathbf{X})$ extends to a family of weak regularizations for $\{X_I\}_{I\in\mathcal{P}^*(I^*)}$ over a neighborhood W_{I^*} of X_{I^*} in $\mathbf{X}$ after deforming the symplectic forms as in (2.29); see Corollary 5.5. Using the operations on regularizations described in Section 5.3, we can combine the original family of weak regularizations for $\mathbf{X}$ over W and the new family of weak regularizations for $\{X_I\}_{I\in\mathcal{P}^*(I^*)}$ over W_{I^*} into a family of weak regularizations over an open subset $\widetilde{W}$ containing all X_I with $I\supset I^*$; see Lemma 5.6. This accomplishes the inductive step in the proof of Theorem 2.17; see Proposition 5.3. By Lemma 5.8 and Corollary 5.9, the difference between a weak regularization for $\mathbf{X}$ and a regularization is insignificant.

We continue to use the notation introduced in Section 2.3 and combine it with the notation introduced in Section 3.1. In particular, for a configuration $\mathbf{X}$ as in Theorem 2.17,

$$\mathcal{N}X_I=\bigoplus_{i\in I}\mathcal{N}_{X_{I-i}}X_I,\quad \mathcal{N}_{I;I'}=\bigoplus_{i\in I-I'}\mathcal{N}_{X_{I-i}}X_I\subset\mathcal{N}X_I,\quad \mathcal{N}\partial X_I=\bigcup_{i\in I}\mathcal{N}_{I;i}\subset\mathcal{N}X_I$$

for all $I'\subset I\subset[N]$ with $|I|\geq 2$. If in addition $\mathcal{N}'\subset\mathcal{N}X_I$,

$$\mathcal{N}'_{I;I'}=\mathcal{N}_{I;I'}\cap\mathcal{N}',\qquad \mathcal{N}'_\partial=\mathcal{N}\partial X_I\cap\mathcal{N}'\,.$$

5.1 Local weak regularizations

We begin with notions of a weak ω-regularization for $\mathbf{X}$ over an open subset of $X_\emptyset$ and of an equivalence of two such regularizations. We then deduce Theorem 2.17 from several technical statements proved in Sections 5.2-5.4.

Definition 5.1. Let $\mathbf{X} \equiv \{X_I\}_{I\in\mathcal{P}^*(N)}$ be a transverse configuration such that X_{ij} is a closed submanifold of X_i of codimension 2 for all $i,j\in[N]$ distinct, $I^*\in\mathcal{P}^*(N)$, $U\subset X_{I^*}$ be an open subset, and $(\omega_i)_{i\in[N]}$ be a symplectic structure on $\mathbf{X}$ in the sense of Definition 2.4. An $(\omega_i)_{i\in[N]}$-regularization for U in $\mathbf{X}$ is a tuple $(\rho_i,\nabla^{(i)},\Psi_i)_{i\in I^*}$ such that $(\Psi_i)_{i\in I^*}$ is a regularization for U in $\mathbf{X}$ in the sense of Definition 4.1 and $((\rho_j,\nabla^{(j)})_{j\in I^*-i},\Psi_i)$ is an ω_i-regularization for U in X_i in the sense of Definition 2.9(1) for each $i\in I^*$.

Definition 5.2. Let $\mathbf{X}$ and $(\omega_i)_{i\in[N]}$ be as in Definition 5.1 and $W\subset X_\emptyset$ be an open subset. A weak $(\omega_i)_{i\in[N]}$-regularization for $\mathbf{X}$ over W is a tuple

$$\mathfrak{R}\equiv(\mathcal{R}_I)_{I\in\mathcal{P}^*(N)}\equiv\big(\rho_{I;i},\nabla^{(I;i)},\Psi_{I;i}\big)_{i\in I\subset[N]} \tag{5.1}$$

such that

- $\mathcal{R}_I$ is an $(\omega_i)_{i\in[N]}$-regularization for $X_I\cap W$ in $\mathbf{X}$ for all $I\in\mathcal{P}^*(N)$,
- the associated isomorphism (2.25) of split vector bundles is a product Hermitian isomorphism and

$$\Psi_{I;i}\big|_{\mathrm{Dom}(\Psi_{I;i})\cap\mathfrak{D}\Psi_{I;I'}^{-1}(\mathrm{Dom}(\Psi_{I';i}))}=\Psi_{I';i}\circ\mathfrak{D}\Psi_{I;I'}\big|_{\mathrm{Dom}(\Psi_{I;i})\cap\mathfrak{D}\Psi_{I;I'}^{-1}(\mathrm{Dom}(\Psi_{I';i}))} \tag{5.2}$$

  for all $i\in I'\subset I\subset[N]$ with $|I'|\ge2$.

An $(\omega_i)_{i\in[N]}$-regularization for $\mathbf{X}$ in the sense of Definition 2.15(1) is a weak $(\omega_i)_{i\in[N]}$-regularization for $\mathbf{X}$ over $W=X_\emptyset$ such that

$$\mathrm{Dom}(\Psi_{I;i})=\mathfrak{D}\Psi_{I;I'}^{-1}(\mathrm{Dom}(\Psi_{I';i}))\qquad\forall\,i\in I'\subset I\subset[N],\ |I'|\ge2,$$

as required by the first condition in (2.17). By Lemma 5.8, a weak $(\omega_i)_{i\in[N]}$-regularization for $\mathbf{X}$ over $W=X_\emptyset$ can be cut down to an $(\omega_i)_{i\in[N]}$-regularization for $\mathbf{X}$. For a smooth family $(\omega_{t;i})_{t\in B,i\in[N]}$ of symplectic structures on $\mathbf{X}$, we define $(\omega_{t;i})_{t\in B,i\in[N]}$-families of regularization for U in $\mathbf{X}$ and of weak regularizations for $\mathbf{X}$ over W analogously to Definition 2.12(2).

Let $W,W^{(1)},W^{(2)}\subset X_\emptyset$ be open subsets and $(\omega^{(1)}_{t;i})_{t\in B,i\in[N]}$ and $(\omega^{(2)}_{t;i})_{t\in B,i\in[N]}$ be two smooth families of symplectic structures on $\mathbf{X}$ such that

$$W\subset W^{(1)}\cap W^{(2)}\qquad\text{and}\qquad\big(\omega^{(1)}_{t;i}|_{X_i\cap W}\big)_{t\in B,i\in[N]}=\big(\omega^{(2)}_{t;i}|_{X_i\cap W}\big)_{t\in B,i\in[N]}.$$

Suppose the tuples

$$\begin{aligned}\big(\mathfrak{R}^{(1)}_t\big)_{t\in B}&\equiv\big(\mathcal{R}^{(1)}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(N)}\equiv\big(\rho^{(1)}_{t;I;i},\nabla^{(1),(t;I;i)},\Psi^{(1)}_{t;I;i}\big)_{t\in B,i\in I\subset[N]},\\ \big(\mathfrak{R}^{(2)}_t\big)_{t\in B}&\equiv\big(\mathcal{R}^{(2)}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(N)}\equiv\big(\rho^{(2)}_{t;I;i},\nabla^{(2),(t;I;i)},\Psi^{(2)}_{t;I;i}\big)_{t\in B,i\in I\subset[N]}\end{aligned} \tag{5.3}$$

are an $(\omega^{(1)}_{t;i})_{t\in B,i\in[N]}$-family of weak regularizations for $\mathbf{X}$ over $W^{(1)}$ and an $(\omega^{(2)}_{t;i})_{t\in B,i\in[N]}$-family of weak regularizations for $\mathbf{X}$ over $W^{(2)}$, respectively. We define

$$\big(\mathfrak{R}^{(1)}_t\big)_{t\in B}\cong_W\big(\mathfrak{R}^{(2)}_t\big)_{t\in B}$$

if there exists an $(\omega^{(1)}_{t;i})_{t\in B,i\in[N]}$-family

$$(\mathfrak{R}_t)_{t\in B}\equiv(\mathcal{R}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}\equiv\big(\rho_{t;I;i},\nabla^{(t;I;i)},\Psi_{t;I;i}\big)_{t\in B,i\in I\subset[N]} \tag{5.4}$$

of weak regularizations for $\mathbf{X}$ over W such that

$$\begin{gathered}(\rho_{t;I;i},\nabla^{(t;I;i)})_{i\in I}=(\rho^{(1)}_{t;I;i},\nabla^{(1),(t;I;i)})_{i\in I}\big|_{X_I\cap W},(\rho^{(2)}_{t;I;i},\nabla^{(2),(t;I;i)})_{i\in I}\big|_{X_I\cap W},\\ \mathrm{Dom}(\Psi_{t;I;i})\subset\mathrm{Dom}(\Psi^{(1)}_{t;I;i}),\mathrm{Dom}(\Psi^{(2)}_{t;I;i}),\quad \Psi_{t;I;i}=\Psi^{(1)}_{t;I;i}\big|_{\mathrm{Dom}(\Psi_{t;I;i})},\Psi^{(2)}_{t;I;i}\big|_{\mathrm{Dom}(\Psi_{t;I;i})}\quad\forall\,i\in I\end{gathered}$$

for all $I\in\mathcal{P}^*(N)$ and $t\in B$. The relation $\cong_W$ is transitive. By Corollary 5.9, two regularizations over $W=X_\emptyset$ that are equivalent as weak regularizations are also equivalent as regularizations.

Proposition 5.3. *Let* $\mathbf{X}$, $N'(\partial B)\subset N(\partial B)\subset B$, *and* $(\omega_{t;i})_{t\in B,i\in[N]}$ *be as in Theorem 2.17. Suppose*

- $I^*\in\mathcal{P}^*(N)$ *and* $X^*_\emptyset,W,W'\subset X_\emptyset$ *are open subsets such that*

$$\overline{W'}\subset W,\quad \overline{X^*_\emptyset}\cap X_{I^*}\subset W'\;\;\textit{if}\;|I^*|\geq 3,\quad X_I\subset W'\;\;\forall\,I\in\mathcal{P}^*(N),\,I\supsetneq I^*,\tag{5.5}$$

- $(\mathfrak{R}_t)_{t\in N(\partial B)}$ *and* $(\mathfrak{R}'_t)_{t\in B}$ *are an* $(\omega_{t;i})_{t\in N(\partial B),i\in[N]}$*-family of weak regularizations for* $\mathbf{X}$ *over* $X_\emptyset$ *and an* $(\omega_{t;i})_{t\in B,i\in[N]}$*-family of weak regularizations for* $\mathbf{X}$ *over* W*, respectively, such that*

$$\big(\mathfrak{R}_t\big)_{t\in N(\partial B)}\cong_W\big(\mathfrak{R}'_t\big)_{t\in N(\partial B)}.\tag{5.6}$$

Then there exist a neighborhood W_{I^*} *of* $X_{I^*}\subset X_\emptyset$*, a smooth family* $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ *of 1-forms on* $X_\emptyset$ *such that*

$$\begin{gathered}\big(\omega_{t,\tau;i}\equiv\omega_{t;i}+\mathrm{d}\mu_{t,\tau;i}\big)_{i\in[N]}\in\mathrm{Symp}^+(\mathbf{X})\;\forall\,t\in B,\,\tau\in\mathbb{I},\\ \mu_{t,0;i}=0\;\forall\,t\in B,\,i\in[N],\quad \mathrm{supp}\big(\mu_{\cdot,\tau;i}\big)\subset\big(B-N'(\partial B)\big)\times\big(X_i-W'\cup X^*_\emptyset\big)\;\forall\,\tau\in\mathbb{I},\,i\in[N],\end{gathered}\tag{5.7}$$

and an $(\omega_{t,1;i})_{t\in B,i\in[N]}$*-family* $(\widetilde{\mathfrak{R}}_t)_{t\in B}$ *of weak regularizations for* $\mathbf{X}$ *over* $W'\cup W_{I^*}$ *such that*

$$\big(\widetilde{\mathfrak{R}}_t\big)_{t\in N'(\partial B)}\cong_{W'\cup W_{I^*}}\big(\mathfrak{R}_t\big)_{t\in N'(\partial B)},\quad \big(\widetilde{\mathfrak{R}}_t\big)_{t\in B}\cong_{W'}\big(\mathfrak{R}'_t\big)_{t\in B}.\tag{5.8}$$

Proof. Let

$$(\mathfrak{R}_t)_{t\in N(\partial B)}=(\mathcal{R}_{t;I})_{t\in N(\partial B),I\in\mathcal{P}^*(N)}\quad\text{and}\quad(\mathfrak{R}'_t)_{t\in B}=(\mathcal{R}'_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}.$$

Choose a neighborhood W'' of $\overline{W'}\subset X_\emptyset$ such that $\overline{W''}\subset W$. By Corollary 5.5 with W' replaced by W'', there exist

- a neighborhood W_{I^*} of $X_{I^*}\subset X_\emptyset$ such that $X_I\cap W_{I^*}\subset W''$ for all $I\in\mathcal{P}(N)-\mathcal{P}(I^*)$,
- a smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ of 1-forms on $X_\emptyset$ satisfying (5.7) with W' replaced by W'',
- an $(\omega_{t,1;i})_{t\in B,i\in[N]}$-family $(\widehat{\mathcal{R}}_{t;I})_{t\in B,I\in\mathcal{P}^*(I^*)}$ of weak regularizations for $\{X_I\}_{I\in\mathcal{P}^*(I^*)}$ over W_{I^*} satisfying (5.34) and (5.35) with W' replaced by W''.

In particular,

$$\omega_{t;i}\big|_{X_i\cap W''}=\omega_{t,1;i}\big|_{X_i\cap W''}\qquad\forall\,t\in B,\,i\in[N].$$

Let W'_{I^*} be a neighborhood of $X_{I^*}\subset X_\emptyset$ such that $\overline{W'_{I^*}}\subset W_{I^*}$ and W''' be a neighborhood of $\overline{W'}\subset X_\emptyset$ such that $\overline{W'''}\subset W''$. We next apply Lemma 5.6 with

$$\begin{gathered}W=W'',\quad W'=W''',\quad(\omega_{t;i})_{t\in B,i\in[N]}=\big(\omega_{t,1;i}\big)_{t\in B,i\in[N]},\\ (\mathcal{R}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}=(\mathcal{R}'_{t;I})_{t\in B,I\in\mathcal{P}^*(N)};\end{gathered}$$

the condition (5.47) holds by (5.35) with W' replaced by W''. Thus, there exists an $(\omega_{t,1;i})_{t\in B,i\in[N]}$-family $(\widetilde{\mathcal{R}}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}$ of weak regularizations for $\mathbf{X}$ over $W'''\cup W'_{I^*}$ such that

$$\big(\widetilde{\mathcal{R}}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(N)}\cong_{W'''}\big(\mathcal{R}'_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(N)},\quad \big(\widetilde{\mathcal{R}}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(I^*)}\cong_{W'_{I^*}}\big(\widehat{\mathcal{R}}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(I^*)}. \tag{5.9}$$

The first equivalence above implies the second equivalence in (5.8).

By (5.9), (5.6), and (5.34),

$$\begin{gathered}\big(\widetilde{\mathcal{R}}_{t;I}\big)_{t\in N'(\partial B),I\in\mathcal{P}^*(N)}\cong_{W'''}\big(\mathcal{R}_{t;I}\big)_{t\in N'(\partial B),I\in\mathcal{P}^*(N)},\\ \big(\widetilde{\mathcal{R}}_{t;I}\big)_{t\in N'(\partial B),I\in\mathcal{P}^*(I^*)}\cong_{W'_{I^*}}\big(\mathcal{R}_{t;I}\big)_{t\in N'(\partial B),I\in\mathcal{P}^*(I^*)}.\end{gathered}$$

Let W''_{I^*} be a neighborhood of $X_{I^*}\subset X_\emptyset$ such that $\overline{W''_{I^*}}\subset W'_{I^*}$. Applying Corollary 5.7 with

$$\begin{gathered}W_{I^*}=W'_{I^*},\quad W'_{I^*}=W''_{I^*},\quad W=W''',\quad B=N'(\partial B),\\ (\omega_{t;i})_{t\in B,i\in[N]}=\big(\omega_{t,1;i}\big)_{t\in B,i\in[N]},\quad \mathcal{R}^{(1)}_{t;I}=\mathcal{R}_{t;I},\quad \mathcal{R}^{(2)}_{t;I}=\widetilde{\mathcal{R}}_{t;I},\end{gathered}$$

we obtain the first equivalence in (5.8) with W_{I^*} replaced by W''_{I^*}. □

Proof of Theorem 2.17. Choose a total order $>$ on subsets $I\subset[N]$ so that $I>I^*$ whenever $I\supsetneq I^*$. Suppose $I^*\subset[N]$ with $|I^*|\ge2$ and we have constructed

- a neighborhood $W^>_{I^*}$ of
$$X^>_{I^*}\equiv\bigcup_{I>I^*}X_I\subset X_\emptyset,$$
- a neighborhood $N^>_{I^*}(\partial B)$ of $\overline{N'(\partial B)}\subset N(\partial B)$,
- a smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ of 1-forms on $X_\emptyset$ such that
$$\begin{gathered}\big(\omega_{t,\tau;i}\equiv\omega_{t;i}+\mathrm{d}\mu_{t,\tau;i}\big)_{i\in[N]}\in\mathrm{Symp}^+(\mathbf{X})\quad\forall\,t\in B,\,\tau\in\mathbb{I},\\ \mu_{t,0;i}=0\quad\forall\,t\in B,\,i\in[N],\quad\mathrm{supp}\big(\mu_{\cdot,\tau;i}\big)\subset\big(B-N^>_{I^*}(\partial B)\big)\times(X_i-X_i^*)\quad\forall\,\tau\in\mathbb{I},\,i\in[N],\end{gathered}\tag{5.10}$$
- an $(\omega_{t,1;i})_{t\in B,i\in[N]}$-family $(\mathfrak{R}'_t)_{t\in B}$ of weak regularizations for $\mathbf{X}$ over $W^>_{I^*}$ such that
$$\big(\mathfrak{R}'_t\big)_{t\in N^>_{I^*}(\partial B)}\cong_{W^>_{I^*}}\big(\mathfrak{R}_t\big)_{t\in N^>_{I^*}(\partial B)}.\tag{5.11}$$

Let W' be a neighborhood of $X^>_{I^*}\subset X_\emptyset$ and $N^{\ge}_{I^*}(\partial B)$ be a neighborhood of $\overline{N'(\partial B)}\subset N(\partial B)$ such that

$$\overline{W'}\subset W^>_{I^*}\quad\text{and}\quad\overline{N^{\ge}_{I^*}(\partial B)}\subset N^>_{I^*}(\partial B).$$

We apply Proposition 5.3 with

$$\begin{gathered}X^*_\emptyset=\bigcup_{i\in[N]}X^*_i,\quad W=W^>_{I^*},\quad N(\partial B)=N^>_{I^*}(\partial B),\quad N'(\partial B)=N^{\ge}_{I^*}(\partial B),\\ (\omega_{t;i})_{t\in B,i\in[N]}=(\omega_{t,1;i})_{t\in B,i\in[N]}.\end{gathered}$$

Thus, there exist

- a neighborhood W_{I^*} of $X_{I^*}\subset X_\emptyset$,
- a smooth family $(\mu'_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ of 1-forms on $X_\emptyset$ such that

$$\big(\omega'_{t,\tau;i}\equiv\omega_{t,1;i}+\mathrm{d}\mu'_{t,\tau;i}\big)_{i\in[N]}\in\mathrm{Symp}^+(\mathbf{X})\quad\forall\,t\in B,\,\tau\in\mathbb{I},$$
$$\mu'_{t,0;i}=0\quad\forall\,t\in B,\,i\in[N],\quad\mathrm{supp}\big(\mu'_{\cdot,\tau;i}\big)\subset\big(B-N^{\geq}_{I^*}(\partial B)\big)\times\big(X_i-W'\cup X^*_i\big)\quad\forall\,\tau\in\mathbb{I},\,i\in[N],$$

- an $(\omega'_{t,1;i})_{t\in B,i\in[N]}$-family $(\widetilde{\mathfrak{R}}_t)_{t\in B}$ of weak regularizations for $\mathbf{X}$ over $W^{\geq}_{I^*}\equiv W'\cup W_{I^*}$ so that (5.8) holds with $N'(\partial B)$ replaced by $N^{\geq}_{I^*}(\partial B)$.

We concatenate the families $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ and $(\mu_{t,1;i}+\mu'_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ of 1-forms on $\mathbf{X}$ into a new smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ such that (5.10) holds with $N^{>}_{I^*}(\partial B)$ replaced by $N^{\geq}_{I^*}(\partial B)$. By the first equivalence in (5.8) with $N'(\partial B)$ replaced by $N^{\geq}_{I^*}(\partial B)$, (5.11) holds with $N^{>}_{I^*}(\partial B)$ and $W^{>}_{I^*}$ replaced by $N^{\geq}_{I^*}(\partial B)$ and $W^{\geq}_{I^*}$, respectively.

By the downward induction on $\mathcal{P}^*(N)$ with respect to $<$, we thus obtain a family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ of 1-forms on $\mathbf{X}$ satisfying (2.29) and an $(\omega_{t,1;i})_{t\in B,i\in[N]}$-family $(\mathfrak{R}'_t)_{t\in B}$ of weak regularizations for $\mathbf{X}$ over $X_\emptyset$ such that

$$\big(\mathfrak{R}'_t\big)_{t\in N'(\partial B)}\cong_{X_\emptyset}\big(\mathfrak{R}_t\big)_{t\in N'(\partial B)}\,.$$

By Lemma 5.8, these weak regularizations can be cut down to an $(\omega_{t,1;i})_{t\in B,i\in[N]}$-family $(\widetilde{\mathfrak{R}}_t)_{t\in B}$ of regularizations for $\mathbf{X}$. In particular,

$$(\widetilde{\mathfrak{R}}_t)_{t\in N'(\partial B)}\cong_{X_\emptyset}(\mathfrak{R}'_t)_{t\in N'(\partial B)}\cong_{X_\emptyset}\big(\mathfrak{R}_t\big)_{t\in N'(\partial B)}\,.$$

By Corollary 5.9, this implies (2.30). □

5.2 Extending weak regularizations

Lemma 5.4 below is the main step in the proof of Proposition 5.3 which provides for extensions of weak regularizations. This lemma implements the deformations for symplectic forms on split vector bundles obtained in Theorem 3.1 via Proposition 4.2.

Lemma 5.4. *Let $\mathbf{X}$, $N'(\partial B)\subset N(\partial B)\subset B$, $(\omega_{t;i})_{t\in B,i\in[N]}$, I^*, $X^*_\emptyset$, and $W'\subset W$ be as in Proposition 5.3. Suppose*

$$\big(\rho_{I^*;t;i},\nabla^{(I^*;t;i)},\Psi_{I^*;t;i}\big)_{t\in N(\partial B),i\in I^*}\quad\textit{and}\quad\big(\rho_{W;t;i},\nabla^{(W;t;i)},\Psi_{W;t;i}\big)_{t\in B,i\in I^*}$$

are $(\omega_{t;i})_{t\in B}$-families of regularizations for X_{I^*} and $X_{I^*}\cap W$, respectively, in $\mathbf{X}$ such that

$$\begin{split}\big(\rho_{I^*;t;i}|_{X_{I^*}\cap W},\nabla^{(I^*;t;i)}|_{X_{I^*}\cap W},\Psi_{I^*;t;i}|_{\mathrm{Dom}(\Psi_{I^*;t;i})|_{X_{I^*}\cap W}}\big)_{t\in N(\partial B),i\in I^*}\\=\big(\rho_{W;t;i},\nabla^{(W;t;i)},\Psi_{W;t;i}\big)_{t\in N(\partial B),i\in I^*}\,.\end{split}\tag{5.12}$$

Then there exist a smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ of 1-forms on $X_\emptyset$ satisfying (5.7) and an $(\omega_{t,1;i})_{t\in B,i\in[N]}$-family $(\rho_{t;i},\nabla^{(t;i)},\Psi_{t;i})_{t\in B,i\in I^*}$ of regularizations for X_{I^*} in $\mathbf{X}$ such that

$$\begin{split}&\big(\rho_{t;i},\nabla^{(t;i)},\Psi_{t;i}\big)_{t\in N'(\partial B),i\in I^*}=\big(\rho_{I^*;t;i},\nabla^{(I^*;t;i)},\Psi_{I^*;t;i}\big)_{t\in N'(\partial B),i\in I^*},\\&\big((\rho_{t;i},\nabla^{(t;i)})|_{X_{I^*}\cap W'},\Psi_{t;i}|_{\mathrm{Dom}(\Psi_{t;i})|_{X_{I^*}\cap W'}}\big)_{t\in B,i\in I^*}\\&\qquad=\big((\rho_{W;t;i},\nabla^{(W;t;i)})|_{X_{I^*}\cap W'},\Psi_{W;t;i}|_{\mathrm{Dom}(\Psi_{W;t;i})|_{X_{I^*}\cap W'}}\big)_{t\in B,i\in I^*}.\end{split}\tag{5.13}$$

Proof. For each $i\in I^*$, let

$$L_i=\mathcal{N}_{X_{I^*-i}}X_{I^*}\longrightarrow X_{I^*}\,,\qquad \mathcal{N}_{I^*;i}=\bigoplus_{j\in I^*-i}L_j\subset\bigoplus_{j\in I^*}L_j=\mathcal{N}X_{I^*}\,.$$

If in addition $t\in B$, define

$$\omega_{t;I^*}=\omega_{t;i}|_{X_{I^*}},\qquad \Omega^\bullet_{t;i}=\bigoplus_{j\in I^*-i}\omega_{t;i}|_{L_j}\,.$$

Choose a neighborhood W'' of $\overline{W'}\subset X_\emptyset$ such that $\overline{W''}\subset W$.

Since $(\omega_{t;i})_{i\in[N]}\in\mathrm{Symp}(\mathbf{X})$, $\omega_{t;i}|_{L_i}$ is symplectic for every $i\in I^*$. For each $i\in I^*$, choose a smooth family $(\rho_{t;i},\nabla^{(t;i)})_{t\in B}$ of $\omega_{t;i}|_{L_i}$-compatible Hermitian structures on L_i such that

$$\begin{split}\big(\rho_{t;i},\nabla^{(t;i)}\big)_{t\in N'(\partial B)}&=\big(\rho_{I^*;t;i},\nabla^{(I^*;t;i)}\big)_{t\in N'(\partial B)},\\ \big((\rho_{t;i},\nabla^{(t;i)})|_{X_{I^*}\cap W''}\big)_{t\in B}&=\big((\rho_{W;t;i},\nabla^{(W;t;i)})|_{X_{I^*}\cap W''}\big)_{t\in B}\,;\end{split}\tag{5.14}$$

this is possible to do by (5.12). For each $t\in B$, denote by $(\widehat{\omega}^\bullet_{t;i})_{i\in I^*}$ the closed 2-form on $\mathcal{N}_\partial X_{I^*}$ induced by $\omega_{t;I^*}$, the diagonal fiberwise 2-form $(\Omega^\bullet_{t;i})_{i\in I^*}$ on $\mathcal{N}_\partial X_{I^*}$, and $(\nabla^{(t;i)})_{i\in I^*}$ as in (2.9). By (2.11) and (5.14),

$$\begin{split}\big(\Psi^*_{I^*;t;i}\omega_{t;i}\big)_{t\in N'(\partial B),i\in I^*}&=\big(\widehat{\omega}^\bullet_{t;i}|_{\mathrm{Dom}(\Psi_{I^*;t;i})}\big)_{t\in N'(\partial B),i\in I^*},\\ \big(\Psi^*_{W;t;i}\omega_{t;i}|_{\mathrm{Dom}(\Psi_{W;t;i})|_{X_{I^*}\cap W''}}\big)_{t\in B,i\in I^*}&=\big(\widehat{\omega}^\bullet_{t;i}|_{\mathrm{Dom}(\Psi_{W;t;i})|_{X_{I^*}\cap W''}}\big)_{t\in B,i\in I^*}.\end{split}\tag{5.15}$$

By (2.11) and (2.9),

$$\begin{split}\mathrm{d}_x\Psi_{I^*;t;i}\big(\mathcal{N}_{I^*;i}|_x\big)&=T_xX_{I^*}^{\omega_{t;i}}\qquad\forall\,(t,x)\in N(\partial B)\times X_{I^*},\\ \mathrm{d}_x\Psi_{W;t;i}\big(\mathcal{N}_{I^*;i}|_x\big)&=T_xX_{I^*}^{\omega_{t;i}}\qquad\forall\,(t,x)\in B\times(X_{I^*}\cap W).\end{split}\tag{5.16}$$

We first apply Proposition 4.2 with

$$U=X_{I^*}\cap W,\qquad U'=X_{I^*}\cap W'',\quad \big(\Psi_{U;t;i}\big)_{t\in B,i\in I^*}=\big(\Psi_{W;t;i}\big)_{t\in B,i\in I^*}\,;$$

the conditions (4.3) and (4.4) are satisfied by (5.16) and (5.12), respectively. There thus exists a smooth family $(\Psi_{t;i})_{t\in B,i\in I^*}$ of regularizations for X_{I^*} in $\mathbf{X}$ in the sense of Definition 4.1 such that

$$\mathrm{d}_x\Psi_{t;i}\big(\mathcal{N}_{I^*;i}|_x\big)=T_xX_{I^*}^{\omega_{t;i}}\quad\forall\,t\in B,\ x\in X_{I^*},\ i\in I^*,\tag{5.17}$$

$$\begin{split}\big(\Psi_{t;i}\big)_{t\in N'(\partial B),i\in I^*}&=\big(\Psi_{I^*;t;i}\big)_{t\in N'(\partial B),i\in I^*},\\ \big(\Psi_{t;i}|_{\mathrm{Dom}(\Psi_{t;i})|_{X_{I^*}\cap W''}}\big)_{t\in B,i\in I^*}&=\big(\Psi_{W;t;i}|_{\mathrm{Dom}(\Psi_{W;t;i})|_{X_{I^*}\cap W''}}\big)_{t\in B,i\in I^*}.\end{split}\tag{5.18}$$

The requirement (5.13) holds by (5.14) and (5.18).

For $t\in B$ and $i\in I^*$, let $\widetilde{\omega}_{t;i}=\Psi^*_{t;i}\omega_{t;i}$. By the last condition in Definition 2.8 and (5.17), $(\widehat{\omega}^\bullet_{t;i})_{t\in B,i\in I^*}$ is the smooth family of diagonalized 2-forms on $\mathcal{N}_\partial X_{I^*}$ determined by $(\widetilde{\omega}_{t;i})_{t\in B,i\in I^*}$ and $(\nabla^{(t;i)})_{t\in B,i\in I^*}$ in the terminology of Theorem 3.1. By (5.18) and (5.15),

$$\begin{split}\big(\widetilde{\omega}_{t;i}\big)_{t\in N'(\partial B),i\in I^*}&=\big(\widehat{\omega}^\bullet_{t;i}|_{\mathrm{Dom}(\Psi_{t;i})}\big)_{t\in N'(\partial B),i\in I^*},\\ \big(\widetilde{\omega}_{t;i}|_{\mathrm{Dom}(\Psi_{t;i})|_{X_{I^*}\cap W''}}\big)_{t\in B,i\in I^*}&=\big(\widehat{\omega}^\bullet_{t;i}|_{\mathrm{Dom}(\Psi_{t;i})|_{X_{I^*}\cap W''}}\big)_{t\in B,i\in I^*}.\end{split}\tag{5.19}$$

By the compactness of B, there exists a neighborhood $\mathcal{N}'$ of $X_{I^*}\subset\mathcal{N}X_{I^*}$ such that $\mathcal{N}'_i\subset\mathrm{Dom}(\Psi_{t;i})$ for all $i\in I^*$ and $t\in B$.

Suppose $|I^*|\geq 3$. Since

$$\overline{W}'\subset W'',\quad \overline{X^*_\emptyset}\cap X_{I^*}\subset W',\quad X_{I^*}\cap X_I=X_{I^*\cup I}\subset W'\quad\forall\, I\in\mathcal{P}(N)-\mathcal{P}(I^*),$$

we can shrink $\mathcal{N}'$ so that

$$\mathcal{N}'\cap\Psi_{t;i}^{-1}\big(\overline{W}'\big)\subset\mathcal{N}'_i\big|_{X_{I^*}\cap W''},\quad \mathcal{N}'\cap\Psi_{t;i}^{-1}\big(\overline{X^*_\emptyset}\big)\subset\mathcal{N}'_i|_{X_{I^*}\cap W'},\tag{5.20}$$

$$\mathcal{N}'\cap\Psi_{t;i}^{-1}\big(X_I\big)\subset\mathcal{N}'_i\big|_{X_{I^*}\cap W'}\quad\forall\, I\in\mathcal{P}(N)-\mathcal{P}(I^*)\tag{5.21}$$

for all $i\in I^*$ and $t\in B$. We next apply Theorem 3.1 with

$$V=X_{I^*},\quad I=I^*,\quad U=X_{I^*}\cap W'',\quad N(\partial B)=N'(\partial B),\quad \big(\widetilde{\omega}_{t;i}\big)_{t\in B,i\in I}=\big(\widetilde{\omega}_{t;i}|_{\mathcal{N}'_i}\big)_{t\in B,i\in I^*};$$

the requirements in (3.4) are satisfied by (5.19). Thus, there exist neighborhoods $\widehat{\mathcal{N}},\mathcal{N}''$ of $X_{I^*}\subset\mathcal{N}'$ such that $\overline{\mathcal{N}''}\subset\mathcal{N}'$ and a smooth family $(\mu'_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in I^*}$ of 1-forms on $\mathcal{N}'_\partial$ such that

$$\big(\widetilde{\omega}_{t,\tau;i}\big)_{i\in I^*}\equiv\big(\widetilde{\omega}_{t;i}+\mathrm{d}\mu'_{t,\tau;i}\big)_{i\in I^*}\tag{5.22}$$

is a symplectic structure on $\mathcal{N}'_\partial$ for all $(t,\tau)\in B\times\mathbb{I}$ and

$$\mu'_{t,0;i}=0,\quad \widetilde{\omega}_{t,1;i}|_{\widehat{\mathcal{N}}_i}=\widehat{\omega}^\bullet_{t;i}|_{\widehat{\mathcal{N}}_i},\quad \mathrm{supp}\big(\mu'_{\cdot,\tau;i}\big)\subset\big(B-N'(\partial B)\big)\times\mathcal{N}''|_{X_{I^*}-W''}\tag{5.23}$$

for all $t\in B$, $\tau\in\mathbb{I}$, and $i\in I^*$.

For each $i\in I^*$, define a smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I}}$ of 1-forms on X_i by

$$\mu_{t,\tau;i}\big|_x=\begin{cases}0, & \text{if } x\in X_i-\Psi_{t;i}(\overline{\mathcal{N}''_i});\\ \mu'_{t,\tau;i}|_{\Psi_{t;i}^{-1}(x)}\circ\mathrm{d}_x\Psi_{t;i}^{-1}, & \text{if } x\in\Psi_{t;i}(\mathcal{N}'_i).\end{cases}$$

By (5.23),

$$\mu_{t,0;i}=0,\quad \Psi_{t;i}^*\big\{\omega_{t;i}+\mathrm{d}\mu_{t,1;i}\big\}\big|_{\widehat{\mathcal{N}}_i}=\widehat{\omega}^\bullet_{t;i}\big|_{\widehat{\mathcal{N}}_i},\tag{5.24}$$

$$\mathrm{supp}\big(\mu_{\cdot,\tau;i}\big)\subset\big(B-N'(\partial B)\big)\times\big(\Psi_{t;i}(\mathcal{N}''_i)-\Psi_{t;i}\big(\mathcal{N}'_i|_{X_{I^*}\cap W''}\big)\big)$$

for all $t\in B$, $\tau\in\mathbb{I}$, and $i\in I^*$. By the last statement, (5.20), and (5.21),

$$\mathrm{supp}\big(\mu_{\cdot,\tau;i}\big)\subset\big(B-N'(\partial B)\big)\times\big(X_i-W'\cup X^*_\emptyset\big)\quad\forall\,\tau\in\mathbb{I},\, i\in I^*,\tag{5.25}$$

$$\mu_{t,\tau;i}|_{X_I}=0\quad\forall\, I\in\mathcal{P}(N)-\mathcal{P}(I^*),\, t\in B,\, \tau\in\mathbb{I},\, i\in I^*.\tag{5.26}$$

For $i\notin I^*$, we set $\mu_{t,\tau;i}=0$ for all $(t,\tau)\in B\times\mathbb{I}$. Since $(\mu'_{t,\tau;i})_{i\in I^*}$ is a 1-form on $\mathcal{N}'_\partial$, (5.26) implies that

$$\mu_{t,\tau;i_1}|_{X_{i_1i_2}}=\mu_{t,\tau;i_2}|_{X_{i_1i_2}}\qquad\forall\, i_1,i_2\in[N].$$

Since $(\widetilde{\omega}_{t,\tau;i})_{i\in I^*}$ is a symplectic structure on $\mathcal{N}'_\partial$ for all $(t,\tau)\in B\times\mathbb{I}$, we conclude that the tuple $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ is a smooth family of 1-forms on $X_\emptyset$ satisfying (5.7). By the second statement in (5.24), $((\rho_{t;j},\nabla^{(t;j)})_{j\in I^*-i},\Psi_{t;i})$ is an $\omega_{t,1;i}$-regularization for X_{I^*} in X_i in the sense of Definition 2.9(1) for all $t\in B$ and $i\in I^*$.

Suppose $|I^*|=2$. Denote by $\zeta_{I^*}\equiv\zeta_{\mathcal{N}X_{I^*}}$ the radial vector field on the total space of $\mathcal{N}X_{I^*}$ as defined above (2.9). For each $\tau\in\mathbb{R}$, let

$$m_\tau\colon \mathcal{N}X_{I^*}\longrightarrow \mathcal{N}X_{I^*}, \qquad v\longrightarrow \tau v,$$

be the scalar multiplication map; it preserves the subbundles $\mathcal{N}_{I^*;i}\subset\mathcal{N}X_{I^*}$. For $t\in B$ and $i\in I^*$, define a 2-form and a 1-form on $\mathcal{N}_i'$ by

$$\varpi_{t;i}=\widehat{\omega}^{\bullet}_{t;i}-\widetilde{\omega}_{t;i}, \qquad \mu_{t;i}\big|_v=\int_0^1 m_\tau^*\big\{\varpi_{t;i}(\tau^{-1}\zeta_{I^*},\cdot)\big\}\mathrm{d}\tau\,.$$

Since $\varpi_{t;i}$ vanishes on $T\mathcal{N}_{I^*;i}|_{X_I^*}$, the integrand above extends smoothly over $\tau=0$. We also note that

$$\mu_{t;i}\big|_{X_{I^*}}=0\ \ \forall\,t\in B, \quad \mu_{t;i}=0\ \ \forall\,t\in N'(\partial B), \quad \mu_{t;i}\big|_{\mathcal{N}_i'|_{X_{I^*}\cap W''}}=0\ \ \forall\,t\in B. \tag{5.27}$$

The first equality above is immediate from the definition of $\mu_{t;i}$, while the other two follow from (5.19).

Shrinking $\mathcal{N}'$ if necessary, we can assume that the restrictions of the closed 2-forms $\widetilde{\omega}_{t;i}+\tau\varpi_{t;i}$ to $\mathcal{N}_i'$ are nondegenerate for all $(t,\tau)\in B\times\mathbb{I}$ and $i\in I^*$. Let $\xi_{t,\tau;i}$ be the vector field on $\mathcal{N}_i'$ given by

$$\big\{\widetilde{\omega}_{t;i}+\tau\varpi_{t;i}\big\}\big(\xi_{t,\tau;i},\cdot\big)=\mu_{t;i}(\cdot);$$

it corresponds to the negative of the vector field X_τ below [19, (3.7)]. By (5.27),

$$\xi_{t,\tau;i}\big|_{X_{I^*}}=0\ \ \forall\,t\in B, \quad \xi_{t,\tau;i}=0\ \ \forall\,t\in N'(\partial B), \quad \xi_{t,\tau;i}|_{\mathcal{N}_i'|_{X_{I^*}\cap W''}}=0\ \ \forall\,t\in B. \tag{5.28}$$

By the compactness of B, there exists a neighborhood $\mathcal{N}''$ of $X_{I^*}\subset\mathcal{N}'$ such that the time 1 flow $\psi_{t;i}$ of $\xi_{t,\tau;i}$ is defined on $\mathcal{N}''$ (and takes values in $\mathcal{N}'$). By (5.28),

$$\begin{gathered}\psi_{t;i}\big|_{X_{I^*}}=\mathrm{id}_{X_{I^*}}\ \ \forall\,t\in B,\\ \psi_{t;i}=\mathrm{id}_{\mathcal{N}_i''}\ \ \forall\,t\in N'(\partial B), \quad \psi_{t;i}|_{\mathcal{N}_i''|_{X_{I^*}\cap W''}}=\mathrm{id}_{\mathcal{N}_i''|_{X_{I^*}\cap W''}}\ \ \forall\,t\in B.\end{gathered} \tag{5.29}$$

By the proof of [19, Lemma 3.14],

$$\psi_{t;i}^*\widetilde{\omega}_{t;i}\big|_{\mathcal{N}_i''}=\widehat{\omega}^{\bullet}_{t;i}\big|_{\mathcal{N}_i''} \qquad \forall\,t\in B,\ i\in I\,. \tag{5.30}$$

For each $i\in I^*$, the set

$$\mathcal{W}\equiv B\times\mathcal{N}_i''\cup\bigg(\bigcup_{t\in N'(\partial B)}\{t\}\times\mathrm{Dom}(\Psi_{t;i})\bigg)\cup\bigg(\bigcup_{t\in B}\{t\}\times\mathrm{Dom}(\Psi_{t;i})\big|_{X_{I^*}\cap W''}\bigg)$$

is an open neighborhood of $B\times X_{I^*}$ in $B\times\mathcal{N}_{I^*;i}$. Let

$$\Theta_{\cdot;i}\colon\mathcal{W}\longrightarrow\mathcal{N}_{I^*;i}, \quad \Theta_{t;i}(v)=\begin{cases}\psi_{t;i}(v), & \text{if } v\in\mathcal{N}_i'',\\ v, & \text{if } t\in N'(\partial B),\\ v, & \text{if } v\in\mathrm{Dom}(\Psi_{t;i})\big|_{X_{I^*}\cap W''}.\end{cases}$$

By (5.29), (5.30), and (5.19), this map is well-defined and

$$\begin{aligned}\Theta_{t;i}\big|_{X_{I^*}} = \mathrm{id}_{X_{I^*}} \quad \forall\, t\in B, \quad \Theta_{t;i}^*\widetilde{\omega}_{t;i}\big|_{\mathrm{Dom}(\Theta_{t;i})} = \widehat{\omega}_{t;i}^{\bullet}\big|_{\mathrm{Dom}(\Theta_{t;i})},\\ \Theta_{t;i} = \mathrm{id}_{\mathrm{Dom}(\Psi_{t;i})} \quad \forall\, t\in N'(\partial B), \quad \psi_{t;i}|_{\mathrm{Dom}(\Psi_{t;i})|_{X_{I^*}\cap W''}} = \mathrm{id}_{\mathrm{Dom}(\Psi_{t;i})|_{X_{I^*}\cap W''}} \quad \forall\, t\in B.\end{aligned} \tag{5.31}$$

For each $t\in B$, $\Theta_{t;i}$ is a diffeomorphism onto an open subset of $\mathrm{Dom}(\Psi_{t;i})$. Since $|I^*|=2$, the tuple $(\Psi_{t;i}\circ\Theta_{t;i})_{t\in B,i\in I^*}$ is thus a smooth family of regularizations for X_{I^*} in $\mathbf{X}$ satisfying (5.18). By the second statement in (5.31), $((\rho_{t;j},\nabla^{(t;j)})_{j\in I^*-i},\Psi_{t;i}\circ\Theta_{t;i})$ is an $\omega_{t;i}$-regularization for X_{I^*} in X_i for all $t\in B$ and $i\in I^*$. □

Corollary 5.5. *Let $\mathbf{X}$, $N'(\partial B)\subset N(\partial B)\subset B$, $(\omega_{t;i})_{t\in B,i\in[N]}$, I^*, $X^*_\emptyset$, and $W'\subset W$ be as in Proposition 5.3. Suppose $(\mathcal{R}_{t;I})_{t\in N(\partial B),I\in\mathcal{P}^*(N)}$ and $(\mathcal{R}'_{t;I})_{t\in B,I\in\mathcal{P}^*(I^*)}$ are an $(\omega_{t;i})_{t\in N(\partial B),i\in[N]}$-family of weak regularizations for $\mathbf{X}$ over $X_\emptyset$ and an $(\omega_{t;i})_{t\in B,i\in I^*}$-family of weak regularizations for $\{X_I\}_{I\in\mathcal{P}^*(I^*)}$ over W, respectively, such that*

$$\big(\mathcal{R}_{t;I}\big)_{t\in N(\partial B),I\in\mathcal{P}^*(I^*)} \cong_W \big(\mathcal{R}'_{t;I}\big)_{t\in N(\partial B),I\in\mathcal{P}^*(I^*)}. \tag{5.32}$$

Then there exist a neighborhood W_{I^*} of $X_{I^*}\subset X_\emptyset$ such that

$$X_I\cap W_{I^*}\subset W' \qquad \forall\ I\in\mathcal{P}(N)-\mathcal{P}(I^*), \tag{5.33}$$

a smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ of 1-forms on $X_\emptyset$ satisfying (5.7), and an $(\omega_{t,1;i})_{t\in B,i\in[N]}$-family $(\widehat{\mathcal{R}}_{t;I})_{t\in B,I\in\mathcal{P}^*(I^*)}$ of weak regularizations for $\{X_I\}_{I\in\mathcal{P}^*(I^*)}$ over W_{I^*} such that

$$\big(\widehat{\mathcal{R}}_{t;I}\big)_{t\in N'(\partial B),I\in\mathcal{P}^*(I^*)} \cong_{W_{I^*}} \big(\mathcal{R}_{t;I}\big)_{t\in N'(\partial B),I\in\mathcal{P}^*(I^*)}, \tag{5.34}$$

$$\big(\widehat{\mathcal{R}}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(I^*)} \cong_{W'\cap W_{I^*}} \big(\mathcal{R}'_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(I^*)}. \tag{5.35}$$

Proof. Let

$$\begin{aligned}(\mathcal{R}_{t;I})_{t\in N(\partial B),I\in\mathcal{P}^*(N)} &= \big(\rho_{t;I;i},\nabla^{(t;I;i)},\Psi_{t;I;i}\big)_{t\in N(\partial B),i\in I\subset[N]},\\ (\mathcal{R}'_{t;I})_{t\in B,I\in\mathcal{P}^*(I^*)} &= \big(\rho'_{t;I;i},\nabla'^{(t;I;i)},\Psi'_{t;I;i}\big)_{t\in B,i\in I\subset I^*}.\end{aligned}$$

Choose open subsets $W'',W'''\subset X_\emptyset$ and $N''(\partial B)\subset B$ such that

$$\overline{W'}\subset W'', \quad \overline{W''}\subset W''', \quad \overline{W'''}\subset W, \quad \overline{N'(\partial B)}\subset N''(\partial B), \quad \overline{N''(\partial B)}\subset N(\partial B).$$

By (5.32) and the compactness of B, there exist a neighborhood $\mathcal{N}^\circ$ of $X_{I^*}\subset\mathcal{N}X_{I^*}$ such that

$$\begin{aligned}\mathcal{N}_i^\circ\subset\mathrm{Dom}(\Psi_{t;I^*;i}) \quad \forall\, t\in N''(\partial B),\, i\in I^*, \quad \mathcal{N}_i^\circ\subset\mathrm{Dom}(\Psi'_{t;I^*;i}) \quad \forall\, t\in B,\, i\in I^*,\\ \big(\Psi_{t;I^*;i}|_{\mathcal{N}_i^\circ|_{X_{I^*}\cap W'''}}\big)_{t\in N''(\partial B),i\in I^*} = \big(\Psi'_{t;I^*;i}|_{\mathcal{N}_i^\circ|_{X_{I^*}\cap W'''}}\big)_{t\in N''(\partial B),i\in I^*}.\end{aligned} \tag{5.36}$$

We apply Lemma 5.4 with

$$\begin{aligned}N(\partial B)=N''(\partial B), \quad W=W''', \quad W'=W'',\\ \big(\rho_{I^*;t;i},\nabla^{(I^*;t;i)},\Psi_{I^*;t;i}\big)_{t\in N(\partial B),i\in I^*} = \big(\rho_{t;I^*;i},\nabla^{(t;I^*;i)},\Psi_{t;I^*;i}|_{\mathcal{N}_i^\circ}\big)_{t\in N''(\partial B),i\in I^*},\\ \big(\rho_{W;t;i},\nabla^{(W;t;i)},\Psi_{W;t;i}\big)_{t\in B,i\in I^*} = \big(\rho'_{t;I^*;i},\nabla'^{(t;I^*;i)},\Psi'_{t;I^*;i}|_{\mathcal{N}_i^\circ}\big)_{t\in B,i\in I^*};\end{aligned}$$

the requirement (5.12) is satisfied due to (5.32) and (5.36). There thus exist a smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ of 1-forms on $X_\emptyset$ satisfying (5.7) with W' replaced by $W''\supset W'$ and an $(\omega_{t,1;i})_{t\in B,i\in[N]}$-family $(\rho_{t;i},\nabla^{(t;i)},\Psi_{t;i})_{t\in B,i\in I^*}$ of regularizations for X_{I^*} in $\mathbf{X}$ such that

$$\begin{split}&\big(\rho_{t;i},\nabla^{(t;i)},\Psi_{t;i}\big)_{t\in N'(\partial B),i\in I^*}=\big(\rho_{t;I^*;i},\nabla^{(t;I^*;i)},\Psi_{t;I^*;i}|_{\mathcal{N}_i^\circ}\big)_{t\in N'(\partial B),i\in I^*},\\&\big((\rho_{t;i},\nabla^{(t;i)})|_{X_{I^*}\cap W''},\Psi_{t;i}|_{\mathrm{Dom}(\Psi_{t;i})|_{X_{I^*}\cap W''}}\big)_{t\in B,i\in I^*}\\&\qquad\qquad=\big((\rho'_{t;i},\nabla'^{(t;i)})|_{X_{I^*}\cap W''},\Psi'_{t;I^*;i}|_{\mathcal{N}_i^\circ|_{X_{I^*}\cap W''}}\big)_{t\in B,i\in I^*}.\end{split}\tag{5.37}$$

Since B is compact, there exists a neighborhood W_{I^*} of $X_{I^*}\subset X_\emptyset$ such that

$$B\times W_{I^*}\subset\bigcup_{t\in B}\bigg(\{t\}\times\bigcup_{i\in I^*}\mathrm{Im}\big(\Psi_{t;i}\big)\bigg).\tag{5.38}$$

Since

$$\overline{W}'\subset W''\qquad\text{and}\qquad X_{I^*}\cap X_I=X_{I^*\cup I}\subset W'\quad\forall\,I\in\mathcal{P}(N)-\mathcal{P}(I^*),$$

we can shrink W_{I^*} so that

$$W_{I^*}\cap W'\subset\bigcup_{i\in I^*}\Psi_{t;i}\big(\mathcal{N}_i^\circ|_{X_{I^*}\cap W''}\big)\quad\forall\,t\in B\tag{5.39}$$

and that (5.33) holds.

For $t\in B$ and $i\in I\subset I^*$ with $|I|\geq 2$, let

$$\widehat{\mathcal{N}}_{t;I^*;I}=\mathcal{N}_I^\circ\cap\Psi_{t;i}^{-1}\big(W_{I^*}\big)\subset\mathcal{N}_{I^*;I}\subset\mathcal{N}X_{I^*},\quad \Psi_{t;I}=\Psi_{t;i}\big|_{\widehat{\mathcal{N}}_{t;I^*;I}}\colon\widehat{\mathcal{N}}_{t;I^*;I}\longrightarrow X_I\cap W_{I^*}\subset X_i\,;$$

the diffeomorphism $\Psi_{t;I}$ is independent of the choice of $i\in I$ by (4.2). Let

$$\mathfrak{D}\Psi_{t;I}\colon\pi_{I^*;I}^*\mathcal{N}_{I^*;I^*-I}\big|_{\widehat{\mathcal{N}}_{t;I^*;I}}\longrightarrow\mathcal{N}X_I\big|_{X_I\cap W_{I^*}}\tag{5.40}$$

be the isomorphism of split vector bundles covering $\Psi_{t;I}$ as in (2.25) with $I'\subset I$ replaced by $I\subset I^*$. Analogously to (2.21), we identify $\pi^*_{I^*;I}\mathcal{N}_{I^*;I^*-I}$ with $\mathcal{N}X_{I^*}$ so that

$$\widehat{\mathcal{N}}_{t;I^*;i}\equiv\Psi_{t;i}^{-1}\big(W_{I^*}\big)\subset\pi^*_{I^*;I}\mathcal{N}_{I^*;I^*-I}\qquad\forall\,i\in I\,.$$

For $i\in I\subset I^*$, define

$$\big(\widehat\rho_{t;I;i},\widehat\nabla^{(t;I;i)}\big)=\big\{\mathfrak{D}\Psi_{t;I}^{-1}\big\}^*\pi^*_{I^*;I}\big(\rho_{t;i},\nabla^{(t;i)}\big),\tag{5.41}$$

$$\widehat{\mathcal{N}}_{t;I;i}=\mathfrak{D}\Psi_{t;I}\big(\widehat{\mathcal{N}}_{t;I^*;i}|_{\widehat{\mathcal{N}}_{t;I^*;I}}\big),\quad\widehat\Psi_{t;I;i}=\Psi_{t;i}\circ\mathfrak{D}\Psi_{t;I}^{-1}\big|_{\widehat{\mathcal{N}}_{t;I;i}}\colon\widehat{\mathcal{N}}_{t;I;i}\longrightarrow W_{I^*}\subset X_i\,.\tag{5.42}$$

Since $(\Psi_{t;i})_{i\in I^*}$ is a regularization for X_{I^*} in $\mathbf{X}$ and (5.40) is an isomorphism of split vector bundles, the tuple $(\widehat\Psi_{t;I;i})_{i\in I}$ is a regularization for $X_I\cap W_{I^*}$ in $\mathbf{X}$ in the sense of Definition 4.1 for all $I\subset I^*$ and $t\in B$. Since $((\rho_{t;j},\nabla^{(t;j)})_{j\in I^*-i},\Psi_{t;i})$ is an $(\omega_{t,1;i})_{i\in[N]}$-regularization for X_{I^*} in $\mathbf{X}$ for all $t\in B$ and $i\in I^*$, $((\widehat\rho_{t;I;j},\widehat\nabla^{(t;I;j)})_{j\in I-i},\widehat\Psi_{t;I;i})$ is an $(\omega_{t,1;i})_{i\in[N]}$-regularization for $X_I\cap W_{I^*}$ in $\mathbf{X}$ for all $t\in B$ and $i\in I\subset I^*$.

By the first equation in (5.42),

$$\mathfrak{D}\Psi_{t;I}^{-1}\big(\widehat{\mathcal{N}}_{t;I;i}\cap\mathcal{N}_{I;I'}\big)\subset\widehat{\mathcal{N}}_{t;I^*;I'},\quad \mathfrak{D}\Psi_{t;I}^{-1}\big(\pi_{I;I'}^*\mathcal{N}_{I;I-I'}\big|_{\widehat{\mathcal{N}}_{t;I;i}\cap\mathcal{N}_{I;I'}}\big)\subset\pi_{I^*;I'}^*\mathcal{N}_{I^*;I^*-I'}\big|_{\widehat{\mathcal{N}}_{t;I^*;I'}},$$

whenever $i\in I'\subset I\subset I^*$. By the second equation in (5.42),

$$\begin{gathered}\mathfrak{D}\widehat{\Psi}_{t;I;I'}=\mathfrak{D}\Psi_{t;I'}\circ\mathfrak{D}\Psi_{t;I}^{-1}\colon\pi_{I;I'}^*\mathcal{N}_{I;I-I'}\big|_{\widehat{\mathcal{N}}_{t;I;i}\cap\mathcal{N}_{I;I'}}\longrightarrow\mathcal{N}X_{I'}\big|_{X_{I'}\cap W_{I^*}},\\ \mathfrak{D}\widehat{\Psi}_{t;I;I'}^{-1}\big(\widehat{\mathcal{N}}_{t;I';i}\big)\subset\widehat{\mathcal{N}}_{t;I;i}\,,\quad \widehat{\Psi}_{t;I;i}=\Psi_{t;I';i}\circ\mathfrak{D}\widehat{\Psi}_{t;I;I'}\colon\mathfrak{D}\widehat{\Psi}_{t;I;I'}^{-1}\big(\widehat{\mathcal{N}}_{t;I';i}\big)\longrightarrow X_i\,,\end{gathered}$$

whenever $i\in I'\subset I\subset I^*$ and $|I'|\ge2$. Along with (5.41), this implies that the tuple $(\widehat{\mathcal{R}}_{t;I})_{I\in\mathcal{P}^*(I^*)}$ satisfies the second bullet condition in Definition 5.2 with $[N]$ replaced by I^* for every $t\in B$. Thus,

$$\big(\widehat{\mathcal{R}}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(I^*)}\equiv\big(\widehat{\rho}_{t;I;i},\widehat{\nabla}^{(t;I;i)},\widehat{\Psi}_{t;I;i}\big)_{i\in I\subset I^*,t\in B}$$

is an $(\omega_{t,1;i})_{t\in B,i\in I^*}$-family of weak regularizations for $\{X_I\}_{I\in\mathcal{P}^*(I^*)}$ over W_{I^*}.

By (5.38) and (5.39),

$$X_i\cap W_{I^*}\subset\mathrm{Im}(\Psi_{t;i})\ \ \forall\,t\in N'(\partial B),\quad X_i\cap W'\cap W_{I^*}\subset\Psi_{t;i}\big(\mathrm{Dom}(\Psi_{t;i})|_{X_{I^*}\cap W''}\big)\ \ \forall\,t\in B,\tag{5.43}$$

whenever $i\in I^*$. Along with the first part of the second bullet condition in Definition 5.2, this implies that

$$\begin{aligned}\big(\rho_{t;I;i},\nabla^{(t;I;i)}\big)\big|_{X_I\cap W_{I^*}}&=\big\{\mathfrak{D}\Psi_{t;I^*;I}^{-1}\big\}^*\pi_{I^*;I}^*\big(\rho_{t;I^*;i},\nabla^{(t;I^*;i)}\big) &&\forall\,t\in N'(\partial B),\\ \big(\rho'_{t;I;i},\nabla'^{(t;I;i)}\big)\big|_{X_I\cap W'\cap W_{I^*}}&=\big\{\mathfrak{D}\Psi_{t;I^*;I}^{-1}\big\}^*\pi_{I^*;I}^*\big(\rho'_{t;I^*;i},\nabla'^{(t;I^*;i)}\big)\big|_{X_I\cap W'\cap W_{I^*}} &&\forall\,t\in B,\end{aligned}$$

whenever $i\in I\subset I^*$ and $|I|\ge2$. Combining the last four equations with (5.41) and (5.37), we obtain

$$\begin{aligned}\big(\widehat{\rho}_{t;I;i},\widehat{\nabla}^{(t;I;i)}\big)_{t\in N'(\partial B),i\in I\subset I^*}&=\big((\rho_{t;I;i},\nabla^{(t;I;i)})|_{X_I\cap W_{I^*}}\big)_{t\in N'(\partial B),i\in I\subset I^*},\\ \big((\widehat{\rho}_{t;I;i},\widehat{\nabla}^{(t;I;i)})|_{X_I\cap W'\cap W_{I^*}}\big)_{t\in B,i\in I\subset I^*}&=\big((\rho'_{t;I;i},\nabla'^{(t;I;i)})|_{X_I\cap W'\cap W_{I^*}}\big)_{t\in B,i\in I\subset I^*}.\end{aligned}\tag{5.44}$$

By (5.43) and (5.37),

$$\begin{aligned}&\mathfrak{D}\Psi_{t;I^*;I}^{-1}\big(\widehat{\mathcal{N}}_{t;I;i}\cap\mathrm{Dom}(\Psi_{t;I;i})\big)=\mathfrak{D}\Psi_{t;I}^{-1}\big(\widehat{\mathcal{N}}_{t;I;i}\cap\mathrm{Dom}(\Psi_{t;I;i})\big)\subset\widehat{\mathcal{N}}_{t;I^*;i}\subset\mathrm{Dom}(\Psi_{t;i})\quad\forall\,t\in N'(\partial B),\\ &\mathfrak{D}\Psi'^{-1}_{t;I^*;I}\big(\widehat{\mathcal{N}}_{t;I;i}|_{X_I\cap W'\cap W_{I^*}}\cap\mathrm{Dom}(\Psi'_{t;I;i})\big)=\mathfrak{D}\Psi_{t;I}^{-1}\big(\widehat{\mathcal{N}}_{t;I;i}|_{X_I\cap W'\cap W_{I^*}}\cap\mathrm{Dom}(\Psi'_{t;I;i})\big)\\ &\qquad\qquad\subset\widehat{\mathcal{N}}_{t;I^*;i}\big|_{X_{I^*}\cap W''}\subset\mathrm{Dom}(\Psi_{t;i})\big|_{X_{I^*}\cap W''}\quad\forall\,t\in B,\end{aligned}$$

whenever $i\in I\subset I^*$ and $|I|\ge2$. Along with the second part of the second bullet condition in Definition 5.2, this implies that

$$\begin{aligned}\Psi_{t;I;i}\big|_{\widehat{\mathcal{N}}_{t;I;i}\cap\mathrm{Dom}(\Psi_{t;I;i})}&=\Psi_{t;I^*;i}\circ\mathfrak{D}\Psi_{t;I^*;I}^{-1}\big|_{\widehat{\mathcal{N}}_{t;I;i}\cap\mathrm{Dom}(\Psi_{t;I;i})} &&\forall\,t\in N'(\partial B),\\ \Psi'_{t;I;i}\big|_{\widehat{\mathcal{N}}_{t;I;i}|_{X_I\cap W'\cap W_{I^*}}\cap\mathrm{Dom}(\Psi'_{t;I;i})}&=\Psi'_{t;I^*;i}\circ\mathfrak{D}\Psi'^{-1}_{t;I^*;I}\big|_{\widehat{\mathcal{N}}_{t;I;i}|_{X_I\cap W'\cap W_{I^*}}\cap\mathrm{Dom}(\Psi'_{t;I;i})} &&\forall\,t\in B.\end{aligned}$$

Combining the last four equations with (5.42) and (5.37), we obtain

$$\begin{aligned}\big(\widehat{\Psi}_{t;I;i}|_{\widehat{\mathcal{N}}_{t;I;i}\cap\mathrm{Dom}(\Psi_{t;I;i})}\big)_{t\in N'(\partial B),i\in I\subset I^*}&=\big(\Psi_{t;I;i}|_{\widehat{\mathcal{N}}_{t;I;i}\cap\mathrm{Dom}(\Psi_{t;I;i})}\big)_{t\in N'(\partial B),i\in I\subset I^*},\\ \big(\widehat{\Psi}_{t;I;i}|_{\widehat{\mathcal{N}}_{t;I;i}|_{X_I\cap W'\cap W_{I^*}}\cap\mathrm{Dom}(\Psi'_{t;I;i})}\big)_{t\in B,i\in I\subset I^*}&=\big(\Psi'_{t;I;i}|_{\widehat{\mathcal{N}}_{t;I;i}|_{X_I\cap W'\cap W_{I^*}}\cap\mathrm{Dom}(\Psi'_{t;I;i})}\big)_{t\in B,i\in I\subset I^*}.\end{aligned}\tag{5.45}$$

By (5.44) and (5.45), $(\widehat{\mathcal{R}}_{t;I})_{t\in B,I\in\mathcal{P}^*(I^*)}$ satisfies (5.34) and (5.35). $\square$

5.3 Merging weak regularizations and equivalences

By Lemma 5.6 below, two weak regularizations for $\mathbf{X}$ over open subsets of $X_\emptyset$ that are equivalent over their intersection can be pasted together over the union of slightly smaller open subsets. By Corollary 5.7, two weak regularizations that are equivalent over each of two open subsets are also equivalent over the union of slightly smaller open subsets.

Lemma 5.6. *Let $\mathbf{X}$, B, $(\omega_{t;i})_{t\in B,i\in[N]}$, I^*, and $W'\subset W$ be as in Proposition 5.3. Suppose $W_{I^*},W'_{I^*}\subset X_\emptyset$ are open subsets such that*

$$\overline{W'_{I^*}}\subset W_{I^*}, \qquad X_I\cap\overline{W'_{I^*}}\subset W \quad \forall\, I\in\mathcal{P}(N)-\mathcal{P}(I^*), \tag{5.46}$$

and $(\mathcal{R}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}$ and $(\widehat{\mathcal{R}}_{t;I})_{t\in B,I\in\mathcal{P}^*(I^*)}$ are an $(\omega_{t;i})_{t\in B,i\in[N]}$-family of weak regularizations for $\mathbf{X}$ over W and an $(\omega_{t;i})_{t\in B,i\in I^*}$-family of weak regularizations for $\{X_I\}_{I\in\mathcal{P}^*(I^*)}$ over W_{I^*}, respectively, such that

$$\big(\mathcal{R}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(I^*)}\cong_{W\cap W_{I^*}}\big(\widehat{\mathcal{R}}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(I^*)}. \tag{5.47}$$

Then there exists an $(\omega_{t;i})_{t\in B,i\in[N]}$-family $(\widetilde{\mathcal{R}}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}$ of weak regularizations for $\mathbf{X}$ over $W'\cup W'_{I^*}$ such that

$$\big(\widetilde{\mathcal{R}}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(N)}\cong_{W'}\big(\mathcal{R}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(N)},\quad \big(\widetilde{\mathcal{R}}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(I^*)}\cong_{W'_{I^*}}\big(\widehat{\mathcal{R}}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(I^*)}. \tag{5.48}$$

Proof. Let

$$\begin{aligned}(\mathcal{R}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}&\equiv\big(\rho_{t;I;i},\nabla^{(t;I;i)},\Psi_{t;I;i}\big)_{t\in B,i\in I\subset[N]}, & \mathcal{N}_{t;I;i}&=\mathrm{Dom}(\Psi_{t;I;i})\subset\mathcal{N}_{I;i},\\ (\widehat{\mathcal{R}}_{t;I})_{t\in B,I\in\mathcal{P}^*(I^*)}&\equiv\big(\widehat\rho_{t;I;i},\widehat\nabla^{(t;I;i)},\widehat\Psi_{t;I;i}\big)_{t\in B,i\in I\subset I^*}, & \widehat{\mathcal{N}}_{t;I;i}&=\mathrm{Dom}(\widehat\Psi_{t;I;i})\subset\mathcal{N}_{I;i}.\end{aligned} \tag{5.49}$$

By (5.47), there exists an $(\omega_{t;i})_{t\in B,i\in I^*}$-family

$$(\mathcal{R}'_{t;I})_{t\in B,I\in\mathcal{P}^*(I^*)}\equiv\big(\rho'_{t;I;i},\nabla'^{(t;I;i)},\Psi'_{t;I;i}\big)_{t\in B,i\in I\subset I^*}$$

of weak regularizations for $\{X_I\}_{I\in\mathcal{P}^*(I^*)}$ over $W\cap W_{I^*}$ such that

$$(\rho'_{t;I;i},\nabla'^{(t;I;i)})=(\rho_{t;I;i},\nabla^{(t;I;i)})\big|_{X_I\cap W\cap W_{I^*}},(\widehat\rho_{t;I;i},\widehat\nabla^{(t;I;i)})\big|_{X_I\cap W\cap W_{I^*}}\quad\forall\, i\in I\subset I^*, \tag{5.50}$$

$$\mathcal{N}'_{t;I;i}\equiv\mathrm{Dom}(\Psi'_{t;I;i})\subset\mathcal{N}_{t;I;i},\widehat{\mathcal{N}}_{t;I;i},\quad \Psi'_{t;I;i}=\Psi_{t;I;i}|_{\mathcal{N}'_{t;I;i}},\widehat\Psi_{t;I;i}|_{\mathcal{N}'_{t;I;i}}\quad\forall\, i\in I\subset I^*. \tag{5.51}$$

Define

$$\begin{gathered}W^\circ=W'-\overline{W'\cap W'_{I^*}},\qquad W^\circ_{I^*}=W'_{I^*}-\overline{W'\cap W'_{I^*}},\qquad W_\cap=W\cap W_{I^*}\cap(W'\cup W'_{I^*}),\\ \mathcal{N}^\circ_{t;I;i}=\Psi_{t;I;i}^{-1}(W^\circ)\big|_{X_I\cap W^\circ},\quad \widehat{\mathcal{N}}^\circ_{t;I;i}=\widehat\Psi_{t;I;i}^{-1}(W^\circ_{I^*})\big|_{X_I\cap W^\circ_{I^*}},\quad \mathcal{N}^\cap_{t;I;i}=\Psi'^{-1}_{t;I;i}(W_\cap)\big|_{X_I\cap W_\cap}.\end{gathered}$$

By the first assumption in (5.5) and (5.46),

$$W^\circ\cap W^\circ_{I^*}=\emptyset,\qquad W'\cup W'_{I^*}=W^\circ\cup W^\circ_{I^*}\cup W_\cap, \tag{5.52}$$

$$X_I\cap(W'\cup W'_{I^*})=X_I\cap\big(W^\circ\cup W_\cap\big)\subset X_I\cap W\quad\forall\, I\in\mathcal{P}(N)-\mathcal{P}(I^*). \tag{5.53}$$

For $t\in B$ and $i\in I\subset[N]$ with $|I|\geq 2$, let

$$\widetilde{\mathcal{N}}_{t;I;i}=\begin{cases}\mathcal{N}^{\circ}_{t;I;i}\cup\mathcal{N}^{\cap}_{t;I;i}, & \text{if } I\in\mathcal{P}(N)-\mathcal{P}(I^*);\\ \mathcal{N}^{\circ}_{t;I;i}\cup\widehat{\mathcal{N}}^{\circ}_{t;I;i}\cup\mathcal{N}^{\cap}_{t;I;i}, & \text{if } I\in\mathcal{P}^*(I^*).\end{cases}$$

By the second statement in (5.52) and (5.53), $\widetilde{\mathcal{N}}_{t;I;i}$ is a neighborhood of $X_I\cap(W'\cup W'_{I^*})$ in $\mathcal{N}_{I;i}|_{X_I\cap(W'\cup W'_{I^*})}$.

With $t\in B$ and $i\in I$ as above, define

$$\widetilde{\Psi}_{t;I;i}\colon\widetilde{\mathcal{N}}_{t;I;i}\longrightarrow X_i,\qquad \widetilde{\Psi}_{t;I;i}(v)=\begin{cases}\Psi_{t;I;i}(v), & \text{if } v\in\mathcal{N}^{\circ}_{t;I;i}\cup\mathcal{N}^{\cap}_{t;I;i};\\ \widehat{\Psi}_{t;I;i}(v), & \text{if } v\in\widehat{\mathcal{N}}^{\circ}_{t;I;i},\ I\in\mathcal{P}^*(I^*).\end{cases}\tag{5.54}$$

By the first statement in (5.52) and (5.51), these definitions agree on the overlap $\mathcal{N}^{\cap}_{t;I;i}\cap\widehat{\mathcal{N}}^{\circ}_{t;I;i}$. Since

$$\widetilde{\Psi}_{t;I;i}\big(\mathcal{N}^{\circ}_{t;I;i}\big)\cap\widetilde{\Psi}_{t;I;i}\big(\widehat{\mathcal{N}}^{\circ}_{t;I;i}\big)\subset W^{\circ}\cap W^{\circ}_{I^*}=\emptyset\tag{5.55}$$

by the first statement in (5.52) and the maps $\Psi_{t;I;i}$ and $\widehat{\Psi}_{t;I;i}$ are injective, (5.51) implies that the map $\widetilde{\Psi}_{t;I;i}$ is injective as well. Since the tuple $(\Psi_{t;I;i})_{i\in I}$ is a regularization for $X_I\cap W$ in $\mathbf{X}$ for every $I\in\mathcal{P}^*(N)$ and $(\widehat{\Psi}_{t;I;i})_{i\in I}$ is a regularization for $X_I\cap W_{I^*}$ in $\mathbf{X}$ for every $I\in\mathcal{P}^*(I^*)$, we conclude that $(\widetilde{\Psi}_{t;I;i})_{i\in I}$ is a regularization for $X_I\cap(W'\cup W'_{I^*})$ in $\mathbf{X}$ for every $I\in\mathcal{P}^*(N)$. By (5.55) and (5.51), these regularizations satisfy (5.2).

We define a metric $\widetilde{\rho}_{t;I;i}$ and a connection $\widetilde{\nabla}^{(t;I;i)}$ on the vector bundle $\mathcal{N}_{X_{I-i}}X_I|_{X_I\cap(W'\cup W'_{I^*})}$ by

$$\big(\widetilde{\rho}_{t;I;i},\widetilde{\nabla}^{(t;I;i)}\big)_x=\begin{cases}(\rho_{t;I;i},\nabla^{(t;I;i)})_x, & \text{if } x\in X_I\cap(W^{\circ}\cup W_{\cap});\\ (\widehat{\rho}_{t;I;i},\widehat{\nabla}^{(t;I;i)})_x, & \text{if } x\in X_I\cap W^{\circ}_{I^*},\ I\in\mathcal{P}^*(I^*).\end{cases}\tag{5.56}$$

By the first statement in (5.52) and (5.50), these definitions agree on the overlap $X_I\cap W_{\cap}\cap W^{\circ}_{I^*}$. Since the tuples

$$\big((\rho_{t;I;j},\nabla^{(t;I;j)})_{j\in I-i},\Psi_{t;I;i}\big)\qquad\text{and}\qquad\big((\widehat{\rho}_{t;I;j},\widehat{\nabla}^{(t;I;j)})_{j\in I-i},\widehat{\Psi}_{t;I;i}\big)$$

are an $\omega_{t;i}$-regularization for $X_I\cap W$ in X_i whenever $i\in I\subset[N]$ and an $\omega_{t;i}$-regularization for $X_I\cap W_{I^*}$ in X_i whenever $i\in I\subset I^*$, respectively, we conclude that the tuple $((\widetilde{\rho}_{t;I;j},\widetilde{\nabla}^{(t;I;j)})_{j\in I-i},\widetilde{\Psi}_{t;I;i})$ is an $\omega_{t;i}$-regularization for $X_I\cap(W'\cup W'_{I^*})$ in X_i whenever $i\in I\subset[N]$. By (5.55), the maps (5.54) and the pairs (5.56) satisfy the first part of the second bullet condition in Definition 5.2.

By the last two paragraphs, the tuple

$$\big(\widetilde{\mathcal{R}}_{t;I}\big)_{I\in\mathcal{P}^*(N)}\equiv\big(\widetilde{\rho}_{t;I;i},\widetilde{\nabla}^{(t;I;i)},\widetilde{\Psi}_{t;I;i}\big)_{i\in I\subset[N]}\tag{5.57}$$

is an $(\omega_{t;i})_{i\in[N]}$-family of weak regularizations for $\mathbf{X}$ over $W'\cup W'_{I^*}$. By (5.54) and (5.56), it satisfies (5.48). □

Corollary 5.7. *Let $\mathbf{X}$, B, $(\omega_{t;i})_{t\in B,i\in[N]}$, I^*, $W'\subset W$, and $W'_{I^*}\subset W_{I^*}$ be as in Lemma 5.6. If $(\mathcal{R}^{(1)}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}$ and $(\mathcal{R}^{(2)}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}$ are $(\omega_{t;i})_{t\in B,i\in[N]}$-families of weak regularizations for $\mathbf{X}$ over $W\cup W_{I^*}$ such that*

$$\big(\mathcal{R}^{(1)}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(N)}\cong_W\big(\mathcal{R}^{(2)}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(N)},\quad\big(\mathcal{R}^{(1)}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(I^*)}\cong_{W_{I^*}}\big(\mathcal{R}^{(2)}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(I^*)},\tag{5.58}$$

then

$$\big(\mathcal{R}^{(1)}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(N)} \cong_{W'\cup W'_{I^*}} \big(\mathcal{R}^{(2)}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(N)}\,.$$

Proof. Let $\mathcal{R}^{(1)}_{t;I}$ and $\mathcal{R}^{(2)}_{t;I}$ be as in (5.3). By the first assumption in (5.58), there exists an $(\omega_{t;i})_{t\in B,i\in[N]}$-family

$$\big(\mathcal{R}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(N)} \equiv \big(\rho_{t;I;i},\nabla^{(t;I;i)},\Psi_{t;I;i}\big)_{t\in B,i\in I\subset[N]} \tag{5.59}$$

of weak regularizations for $\mathbf{X}$ over W such that

$$\begin{gathered}\big(\rho_{t;I;i},\nabla^{(t;I;i)}\big)=\big(\rho^{(1)}_{t;I;i},\nabla^{(1),(t;I;i)}\big)\big|_{X_I\cap W},\big(\rho^{(2)}_{t;I;i},\nabla^{(2),(t;I;i)}\big)\big|_{X_I\cap W},\\ \mathrm{Dom}\big(\Psi_{t;I;i}\big)\subset\mathrm{Dom}\big(\Psi^{(1)}_{t;I;i}\big),\mathrm{Dom}\big(\Psi^{(2)}_{t;I;i}\big),\quad \Psi_{t;I;i}=\Psi^{(1)}_{t;I;i}\big|_{\mathrm{Dom}(\Psi_{t;I;i})},\Psi^{(2)}_{t;I;i}\big|_{\mathrm{Dom}(\Psi_{t;I;i})}\end{gathered} \tag{5.60}$$

for all $t\in B$ and $i\in I\subset[N]$ with $|I|\ge 2$. By the second assumption in (5.58), there exists an $(\omega_{t;i})_{t\in B,i\in I^*}$-family

$$\big(\widehat{\mathcal{R}}_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(I^*)} \equiv \big(\widehat\rho_{t;I;i},\widehat\nabla^{(t;I;i)},\widehat\Psi_{t;I;i}\big)_{t\in B,i\in I\subset I^*} \tag{5.61}$$

of weak regularizations for $\{X_I\}_{I\in\mathcal{P}(I^*)}$ over W_{I^*} such that

$$\begin{gathered}\big(\widehat\rho_{t;I;i},\widehat\nabla^{(t;I;i)}\big)=\big(\rho^{(1)}_{t;I;i},\nabla^{(1),(t;I;i)}\big)\big|_{X_I\cap W_{I^*}},\big(\rho^{(2)}_{t;I;i},\nabla^{(2),(t;I;i)}\big)\big|_{X_I\cap W_{I^*}},\\ \mathrm{Dom}\big(\widehat\Psi_{t;I;i}\big)\subset\mathrm{Dom}\big(\Psi^{(1)}_{t;I;i}\big),\mathrm{Dom}\big(\Psi^{(2)}_{t;I;i}\big),\quad \widehat\Psi_{t;I;i}=\Psi^{(1)}_{t;I;i}\big|_{\mathrm{Dom}(\widehat\Psi_{t;I;i})},\Psi^{(2)}_{t;I;i}\big|_{\mathrm{Dom}(\widehat\Psi_{t;I;i})}\end{gathered} \tag{5.62}$$

for all $t\in B$ and $i\in I\subset I^*$ with $|I|\ge 2$.

By (5.60) and (5.62),

$$\begin{gathered}\big(\rho_{t;I;i},\nabla^{(t;I;i)}\big)\big|_{X_I\cap W\cap W_{I^*}}=\big(\widehat\rho_{t;I;i},\widehat\nabla^{(t;I;i)}\big)\big|_{X_I\cap W\cap W_{I^*}},\\ \Psi_{t;I;i}\big|_{\mathrm{Dom}(\Psi_{t;I;i})\cap\mathrm{Dom}(\widehat\Psi_{t;I;i})|_{X_I\cap W\cap W_{I^*}}}=\widehat\Psi_{t;I;i}\big|_{\mathrm{Dom}(\Psi_{t;I;i})\cap\mathrm{Dom}(\widehat\Psi_{t;I;i})|_{X_I\cap W\cap W_{I^*}}}\end{gathered}$$

for all $t\in B$ and $i\in I\subset I^*$ with $|I|\ge 2$. Thus, the families (5.59) and (5.61) of weak regularizations satisfy (5.47). The proof of Lemma 5.6 provides an $(\omega_{t;i})_{t\in B,i\in[N]}$-family (5.57) of weak regularizations for $\mathbf{X}$ over $W'\cup W'_{I^*}$ such that

$$\begin{gathered}\big(\widetilde\rho_{t;I;i},\widetilde\nabla^{(t;I;i)}\big)_x=\begin{cases}(\rho_{t;I;i},\nabla^{(t;I;i)})_x, & \text{if } x\in X_I\cap W';\\ (\rho_{t;I;i},\nabla^{(t;I;i)})_x, & \text{if } x\in X_I\cap W'_{I^*},\ I\in\mathcal{P}(N)-\mathcal{P}(I^*);\\ (\widehat\rho_{t;I;i},\widehat\nabla^{(t;I;i)})_x, & \text{if } x\in X_I\cap W'_{I^*},\ I\in\mathcal{P}^*(I^*);\end{cases}\\ \mathrm{Dom}\big(\widetilde\Psi_{t;I;i}\big)\subset\begin{cases}\mathrm{Dom}(\Psi_{t;I;i}), & \text{if } I\in\mathcal{P}(N)-\mathcal{P}(I^*);\\ \mathrm{Dom}(\Psi_{t;I;i})\big|_{X_I\cap W'}\cup\mathrm{Dom}(\widehat\Psi_{t;I;i})|_{X_I\cap W'_{I^*}}, & \text{if } I\in\mathcal{P}^*(I^*);\end{cases}\\ \widetilde\Psi_{t;I;i}(v)=\begin{cases}\Psi_{t;I;i}(v), & \text{if } v\in\mathrm{Dom}(\widetilde\Psi_{t;I;i})\big|_{X_I\cap W'};\\ \Psi_{t;I;i}(v), & \text{if } v\in\mathrm{Dom}(\widetilde\Psi_{t;I;i})\big|_{X_I\cap W'_{I^*}},\ I\in\mathcal{P}(N)-\mathcal{P}(I^*);\\ \widehat\Psi_{t;I;i}(v), & \text{if } v\in\mathrm{Dom}(\widetilde\Psi_{t;I;i})|_{X_I\cap W'_{I^*}},\ I\in\mathcal{P}^*(I^*).\end{cases}\end{gathered}$$

Along with (5.60) and (5.62), these identities imply that

$$\begin{gathered}\big(\widetilde\rho_{t;I;i},\widetilde\nabla^{(t;I;i)}\big)=\big(\rho^{(1)}_{t;I;i},\nabla^{(1),(t;I;i)}\big)\big|_{X_I\cap(W'\cup W'_{I^*})},\big(\rho^{(2)}_{t;I;i},\nabla^{(2),(t;I;i)}\big)\big|_{X_I\cap(W'\cup W'_{I^*})},\\ \mathrm{Dom}\big(\widetilde\Psi_{t;I;i}\big)\subset\mathrm{Dom}\big(\Psi^{(1)}_{t;I;i}\big),\mathrm{Dom}\big(\Psi^{(2)}_{t;I;i}\big),\quad \widetilde\Psi_{t;I;i}=\Psi^{(1)}_{t;I;i}\big|_{\mathrm{Dom}(\widetilde\Psi_{t;I;i})},\Psi^{(2)}_{t;I;i}\big|_{\mathrm{Dom}(\widetilde\Psi_{t;I;i})}\end{gathered}$$

for all $t\in B$ and $i\in I\subset[N]$. This establishes the claim. □

5.4 From weak regularizations to regularizations

We show below that the first requirement in (2.17) is not material, provided the second requirement in (2.17) is appropriately modified. By Lemma 5.8 below, a weak regularization for $\mathbf{X}$ over $X_\emptyset$ can be cut down to a regularization for $\mathbf{X}$. By Corollary 5.9, two regularizations for $\mathbf{X}$ that are equivalent as weak regularizations over $X_\emptyset$ are also equivalent as regularizations.

Lemma 5.8. *Let $\mathbf{X}$, B, and $(\omega_{t;i})_{t\in B,i\in[N]}$ be as in Theorem 2.17 and $(\mathcal{R}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}$ be an $(\omega_{t;i})_{t\in B,i\in[N]}$-family of weak regularizations for $\mathbf{X}$ over $X_\emptyset$ as in (5.49). Then there exists a collection of neighborhoods*

$$\bigcup_{t\in B}\{t\}\times\mathcal{N}'_{t;I}\subset B\times\mathcal{N}X_I$$

of $B\times X_I$ with $I\in\mathcal{P}^*(N)$ and $|I|\ge 2$ such that $\mathcal{N}'_{t;I}\cap\mathcal{N}_{I;i}\subset\mathcal{N}_{t;I;i}$ for all $i\in I\subset[N]$ with $|I|\ge 2$ and the tuple

$$\big(\mathcal{R}'_{t;I}\big)_{t\in B,I\in\mathcal{P}^*(N)}\equiv\big(\rho_{t;I;i},\nabla^{(t;I;i)},\Psi_{t;I;i}|_{\mathcal{N}'_{t;I}\cap\mathcal{N}_{I;i}}\big)_{t\in B,i\in I\subset[N]} \tag{5.63}$$

is an $(\omega_{t;i})_{t\in B,i\in[N]}$-family of regularizations for $\mathbf{X}$ in the sense of Definition 2.15(2).

Proof. For each $I\in\mathcal{P}^*(N)$ with $|I|\ge 2$, let $\pi_I:\mathcal{N}X_I\longrightarrow X_I$ be the bundle projection map. If in addition $t\in B$, define

$$\Psi_{t;I}:\bigcup_{i\in I}\mathcal{N}_{t;I;i}\longrightarrow X_\emptyset,\qquad \Psi_{t;I}(v)=\Psi_{t;I;i}(v)\quad\forall\,v\in\mathcal{N}_{t;I;i},\,i\in I;$$

by (4.2), $\Psi_{t;I}(v)$ is well-defined for $v\in\mathcal{N}_{t;I;i_1}\cap\mathcal{N}_{t;I;i_2}$. Let

$$\rho_{t;I}:\mathcal{N}X_I\longrightarrow\mathbb{R},\qquad \rho_{t;I}\big((v_i)_{i\in I}\big)=\max\big\{\rho_{t;I;i}(v_i):\,i\in I\big\},$$

to be the square metric on $\mathcal{N}X_I$.

For $I\in\mathcal{P}^*(N)$ with $|I|\ge 2$, let

$$\bigcup_{t\in B}\{t\}\times\mathcal{N}_{t;I}\subset B\times\mathcal{N}X_I$$

be a neighborhood of $B\times X_I$ such that $\mathcal{N}_{t;I}\cap\mathcal{N}_{I;i}\subset\mathcal{N}_{t;I;i}$ for all $t\in B$ and $i\in I$. Define

$$\mathcal{N}^\circ_{t;I}=\bigcap_{\substack{I'\subset I\\|I'|\ge2}}\mathfrak{D}\Psi^{-1}_{t;I;I'}(\mathcal{N}_{t;I'}),\quad \mathcal{N}^\circ_{t;I;\partial}=\mathcal{N}^\circ_{t;I}\cap\mathcal{N}\partial X_I\qquad\forall\,t\in B. \tag{5.64}$$

By (5.2),

$$\Psi_{t;I}\big|_{\mathcal{N}^\circ_{t;I;\partial}}=\Psi_{t;I'}\circ\mathfrak{D}\Psi_{t;I;I'}\big|_{\mathcal{N}^\circ_{t;I;\partial}}\quad\forall\;I'\subset I\subset[N],\,|I'|\ge2. \tag{5.65}$$

By (5.64) and (5.65),

$$\mathfrak{D}\Psi_{t;I;I'}\big(\mathcal{N}^\circ_{t;I}\big)\subset\mathcal{N}^\circ_{t;I'}\quad\forall\;I'\subset I\subset[N],\,|I'|\ge2. \tag{5.66}$$

If in addition $\varepsilon\in C^\infty(B\times X_I;\mathbb{R}^+)$, define

$$\mathcal{N}_{t;I}(\varepsilon)=\big\{v\in\mathcal{N}X_I:\,\rho_{t;I}(v)<\varepsilon\big(t,\pi_I(v)\big)\big\},\quad \mathcal{N}_{t;I;\partial}(\varepsilon)=\mathcal{N}_{t;I}(\varepsilon)\cap\mathcal{N}\partial X_I.$$

In particular,

$$\mathcal{N}^\circ X_I\equiv\bigcup_{t\in B}\{t\}\times\mathcal{N}^\circ_{t;I},\quad \mathcal{N}X_I(\varepsilon)\equiv\bigcup_{t\in B}\{t\}\times\mathcal{N}_{t;I}(\varepsilon)\subset B\times\mathcal{N}X_I$$

are neighborhoods of $B\times X_I$ in $B\times\mathcal{N}X_I$.

We show below that there exist functions $\varepsilon_I\in C^\infty(B\times X_I;\mathbb{R}^+)$ with $I\subset[N]$, $|I|\geq 2$, such that

$$\overline{\mathcal{N}_{t;I}(2^{|I|}\varepsilon_I)}\subset\mathcal{N}_{t;I}^\circ, \tag{5.67}$$

$$\varepsilon_{I'}\big(t,\Psi_{t;I}(v)\big)=\varepsilon_I\big(t,\pi_I(v)\big)\quad\forall\,v\in\mathcal{N}_{t;I}\big(2^{|I'|}\varepsilon_I\big)\cap\mathcal{N}_{I;I'} \tag{5.68}$$

for all $t\in B$ and $I'\subset I\subset[N]$ with $|I'|\geq 2$. We take $\mathcal{N}'_{t;I}=\mathcal{N}_{t;I}(\varepsilon_I)$. By (5.65) and (5.67),

$$\Psi_{t;I}\big|_{\mathcal{N}'_{t;I}\cap\mathcal{N}_\partial X_I}=\Psi_{t;I'}\circ\mathfrak{D}\Psi_{t;I;I'}\big|_{\mathcal{N}'_{t;I}\cap\mathcal{N}_\partial X_I}.$$

Since $\mathfrak{D}\Psi_{t;I;I'}$ is a product Hermitian isomorphism,

$$\begin{split}\mathfrak{D}\Psi_{t;I;I'}\big(\mathcal{N}_{t;I}(\varepsilon_I)\big)&=\bigcup_{v\in\mathcal{N}_{t;I}(\varepsilon_I)\cap\mathcal{N}_{I;I'}}\big\{w\in\mathcal{N}X_{I'}|_{\Psi_{t;I}(v)}\colon\,\rho_{t;I'}(w)<\varepsilon_I(\pi_I(v))\big\}\\&=\mathcal{N}_{t;I'}(\varepsilon_{I'})\big|_{\Psi_{t;I}(\mathcal{N}_{t;I}(\varepsilon_I)\cap\mathcal{N}_{I;I'})};\end{split} \tag{5.69}$$

the last equality holds by (5.68). Combining (5.69) and (4.1), we conclude that

$$\mathfrak{D}\Psi_{t;I;I'}(\mathcal{N}'_{t;I})=\mathcal{N}'_{t;I'}\big|_{X_{I'}\cap\Psi_{t;I}(\mathcal{N}'_{t;I}\cap\mathcal{N}_{I;I'})}\quad\forall\;I'\subset I\subset[N],\;|I'|\geq 2,\;t\in B\,.$$

Along with the assumption that $(\mathcal{R}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}$ is an $(\omega_{t;i})_{t\in B,i\in[N]}$-family of weak regularizations for $\mathbf{X}$ over $X_\emptyset$, this implies that (5.63) is an $(\omega_{t;i})_{t\in B,i\in[N]}$-family of regularizations for $\mathbf{X}$.

In the remainder of this proof, we inductively construct functions $\varepsilon_I\in C^\infty(B\times X_I;\mathbb{R}^+)$ satisfying (5.67) and (5.68). By (5.65) and (5.67), (5.68) for all $I'\subset I\subset[N]$ with $|I'|\geq 2$ is equivalent to (5.68) for such I',I with $|I-I'|=1$. For each $\ell\in\mathbb{Z}$, let

$$\mathcal{P}^{=\ell}(N),\mathcal{P}^{>\ell}(N)\subset\mathcal{P}(N)$$

denote the collections of subsets of cardinality ℓ and of cardinality greater than ℓ, respectively.

Suppose $\ell\in\{2,\ldots,N\}$ and we have chosen ε_I for all $I\in\mathcal{P}^{>\ell}(N)$ so that (5.67) and (5.68) are satisfied by all elements of $\mathcal{P}^{>\ell}(N)$,

$$\Psi_{t;I_1}\big(\mathcal{N}_{t;I_1;\partial}(2^{\ell+1}\varepsilon_{I_1})\big)\cap\Psi_{t;I_2}\big(\mathcal{N}_{t;I_2;\partial}(2^{\ell+1}\varepsilon_{I_2})\big)\subset\Psi_{t;I_1\cup I_2}\big(\mathcal{N}_{t;I_1\cup I_2;\partial}(2^{\ell+1}\varepsilon_{I_1\cup I_2})\big) \tag{5.70}$$

for all $I_1,I_2\in\mathcal{P}^{>\ell}(N)$, and

$$\overline{\Psi_{t;I_1}\big(\mathcal{N}_{t;I_1;\partial}(2^{\ell+1}\varepsilon_{I_1})\big)}\cap X_{I_2}\subset\overline{\Psi_{t;I_1\cup I_2}\big(\mathcal{N}_{t;I_1\cup I_2;\partial}(2^{\ell+1}\varepsilon_{I_1\cup I_2})\big)} \tag{5.71}$$

whenever $I_1\in\mathcal{P}^{>\ell}(N)$ and $I_2\in\mathcal{P}(N)$. Furthermore, (5.70) and (5.71) hold with $2^{\ell+1}$ and the inclusions replaced by $C\in[0,2^{\ell+1}]$ and the equalities.

For $t\in B$ and $I^*\subsetneq I\subset[N]$ with $I^*\in\mathcal{P}^{=\ell}(N)$, let

$$\begin{split}W_{t;I^*;I}=\big\{\Psi_{t;I}(u,v)\colon(u,v)\in(\mathcal{N}_{I;I^*}\oplus\mathcal{N}_{I;I-I^*})\cap\mathcal{N}\partial X_I,\;\rho_{t;I}(u)<2^{\ell+1}\varepsilon_I(\pi_I(u)),\\ \rho_{t;I}(v)<2^\ell\varepsilon_I(\pi_I(v))\big\}\subset X_\emptyset.\end{split}$$

By (5.67) and (2.14),

$$\overline{\Psi_{t;I}\big(\mathcal{N}_{t;I;\partial}(2^{\ell+1}\varepsilon_I)\big)-W_{t;I^*;I}}\cap X_{I^*}=\emptyset$$

for all $I^*\subsetneq I\subset[N]$ with $I^*\in\mathcal{P}^{=\ell}(N)$. Along with (5.71), this implies that

$$\overline{\Psi_{t;I}\big(\mathcal{N}_{t;I;\partial}(2^{\ell+1}\varepsilon_I)\big)-W_{t;I^*;I^*\cup I}}\cap X_{I^*}=\emptyset \tag{5.72}$$

for all $I^*\in\mathcal{P}^{=\ell}(N)$ and $I\in\mathcal{P}^{>\ell}(N)$. By (5.70), (5.67) with I replaced by $I_1\cup I_2$, and (5.65) with (I,I') replaced by $(I_1\cup I_2,I_1^*)$, $(I_1\cup I_2,I_2)$, and $(I_1\cup I_2,I_1^*\cup I_2)$,

$$W_{t;I_1^*;I_1}\cap\Psi_{t;I_2}\big(\mathcal{N}_{t;I_2;\partial}(2^{\ell}\varepsilon_{I_2})\big)\subset\Psi_{t;I_1^*\cup I_2}\big(\mathcal{N}_{t;I_1^*\cup I_2;\partial}(2^{\ell}\varepsilon_{I_1^*\cup I_2})\big) \tag{5.73}$$

for all $I_1^*\subsetneq I_1\subset[N]$ with $I_1^*\in\mathcal{P}^{=\ell}(N)$ and $I_2\in\mathcal{P}^{>\ell}(N)$. By (5.70), (5.67) with I replaced by $I_1\cup I_2$, and (5.65) with (I,I') replaced by $(I_1\cup I_2,I_1^*)$, $(I_1\cup I_2,I_2^*)$, and $(I_1\cup I_2,I_1^*\cup I_2^*)$,

$$W_{t;I_1^*;I_1}\cap W_{t;I_2^*;I_2}\subset\Psi_{t;I_1^*\cup I_2^*}\big(\mathcal{N}_{t;I_1^*\cup I_2^*;\partial}(2^{\ell}\varepsilon_{I_1^*\cup I_2^*})\big) \tag{5.74}$$

for all $I_1^*\subsetneq I_1\subset[N]$ and $I_2^*\subsetneq I_2\subset[N]$ with $I_1^*,I_2^*\in\mathcal{P}^{=\ell}(N)$ and $I_1^*\neq I_2^*$. By (5.71), (5.67) with I replaced by $I_1\cup I_2$, (5.65) with (I,I') replaced by $(I_1\cup I_2,I_1^*)$ and $(I_1\cup I_2,I_1^*\cup I_2)$, and (4.1) with (I^*,I) replaced by $(I_1\cup I_2,I_2)$,

$$\overline{W_{t;I_1^*;I_1}}\cap X_{I_2}\subset\overline{\Psi_{t;I_1^*\cup I_2}\big(\mathcal{N}_{t;I_1^*\cup I_2;\partial}(2^{\ell}\varepsilon_{I_1^*\cup I_2})\big)} \tag{5.75}$$

for all $I_1^*\subsetneq I_1\subset[N]$ with $I_1^*\in\mathcal{P}^{=\ell}(N)$ and $I_2\in\mathcal{P}^*(N)-\mathcal{P}^*(I_1^*)$.

For each $I^*\in\mathcal{P}^{=\ell}(N)$, let

$$W_{t;I^*}=X_\emptyset-\bigcup_{I^*\subsetneq I\subset[N]}\overline{\Psi_{t;I}\big(\mathcal{N}_{t;I;\partial}(2^{\ell}\varepsilon_I)\big)}-\bigcup_{I\in\mathcal{P}^{>\ell}(N)}\overline{\Psi_{t;I}\big(\mathcal{N}_{t;I;\partial}(2^{\ell+1}\varepsilon_I)\big)-W_{t;I^*;I^*\cup I}}-\bigcup_{\substack{I\subset[N]\\ I\not\subset I^*}}X_I,$$

$$X'_{t;I^*}=X_{I^*}-\bigcup_{I^*\subsetneq I\subset[N]}W_{t;I^*;I},\quad \mathcal{X}'_{I^*}=\bigcup_{t\in B}\{t\}\times X'_{t;I^*},\quad \mathcal{W}_{I^*}=\bigcup_{t\in B}\{t\}\times W_{t;I^*}.$$

Since $\Psi_{t;I_1}$ depends continuously on t, $\mathcal{X}'_{I^*}$ is a closed subset of $B\times X_\emptyset$ and $\mathcal{W}_{I^*}$ is an open subset. By (5.73), (5.74), and the definition of $W_{t;I^*}$,

$$W_{t;I_1^*}\cap\Psi_{t;I_2}\big(\mathcal{N}_{t;I_2;\partial}(2^{\ell}\varepsilon_{I_2})\big)=\emptyset,\quad W_{t;I_1^*}\cap W_{t;I_2^*;I_2}=\emptyset,\quad W_{t;I_1^*}\cap X_{I_2}=\emptyset \tag{5.76}$$

for all $I_1^*\in\mathcal{P}^{=\ell}(N)$, $I_2\in\mathcal{P}^{>\ell}(N)$ in the first case, $I_2^*\subsetneq I_2\subset[N]$ with $I_2^*\in\mathcal{P}^{=\ell}(N)$ and $I_2^*\neq I_1^*$ in the second case, and $I_2\in\mathcal{P}(N)-\mathcal{P}(I_1^*)$ in the third case. By (5.72), $\mathcal{X}'_{I^*}\subset\mathcal{W}_{I^*}$. Since the closed sets $\mathcal{X}'_{I^*}$ are disjoint, there exist open subsets

$$\mathcal{W}'_{I^*}\equiv\bigcup_{t\in B}\{t\}\times W'_{t;I^*}\subset B\times X_\emptyset\quad\forall\,I^*\in\mathcal{P}^{=\ell}(N)\qquad\text{s.t.}$$

$$\mathcal{W}'_{I_1^*}\cap\mathcal{W}'_{I_2^*}=\emptyset\ \ \forall\,I_1^*,I_2^*\in\mathcal{P}^{=\ell}(N),\,I_1^*\neq I_2^*,\quad \mathcal{X}'_{I^*}\subset\mathcal{W}'_{I^*},\ \overline{\mathcal{W}'_{I^*}}\subset\mathcal{W}_{I^*}\ \ \forall\,I^*\in\mathcal{P}^{=\ell}(N). \tag{5.77}$$

For each $I^*\in\mathcal{P}^{=\ell}(N)$, define

$$\widetilde{\mathcal{W}}_{I^*}\equiv\bigcup_{t\in B}\{t\}\times\widetilde{W}_{t;I^*}=\mathcal{W}'_{I^*}\cup\bigcup_{t\in B}\bigcup_{I^*\subsetneq I\subset[N]}\{t\}\times W_{t;I^*;I},$$

$$\mathcal{N}'_{t;I^*;\partial}=\Psi_{t;I^*}^{-1}\big(\widetilde{W}_{t;I^*}\big),\qquad \mathcal{N}'_{\partial}X_{I^*}\equiv\bigcup_{t\in B}\{t\}\times\mathcal{N}'_{t;I^*;\partial}.$$

By (5.67) and (5.65),

$$\big\{v\in\mathcal{N}_\partial X_{I^*}|_{\Psi_{t;I}(u)}: u\in\mathcal{N}_{t;I;\partial}(2^{\ell+1}\varepsilon_I)\cap\mathcal{N}_{I;I^*},\ \rho_{t;I^*}(v)<2^\ell\varepsilon_I(t,\pi_I(u))\big\}\subset\mathcal{N}'_{t;I^*;\partial} \tag{5.78}$$

for all $t\in B$ and $I^*\subsetneq I\subset[N]$. By the last assumption in (5.77), the first statement in (5.76), and (5.73),

$$\widetilde{W}_{t;I_1^*}\cap\Psi_{t;I_2}\big(\mathcal{N}_{t;I_2;\partial}(2^\ell\varepsilon_{I_2})\big)\subset\Psi_{t;I_1^*\cup I_2}\big(\mathcal{N}_{t;I_1^*\cup I_2;\partial}(2^\ell\varepsilon_{I_1^*\cup I_2})\big) \tag{5.79}$$

for all $I_1^*\in\mathcal{P}^{=\ell}(N)$ and $I_2\in\mathcal{P}^{>\ell}(N)$. By the first and last assumptions in (5.77), the second statements in (5.76), and (5.74),

$$\widetilde{W}_{t;I_1^*}\cap\widetilde{W}_{t;I_2^*}\subset\Psi_{t;I_1^*\cup I_2^*}\big(\mathcal{N}_{t;I_1^*\cup I_2^*;\partial}(2^\ell\varepsilon_{I_1^*\cup I_2^*})\big) \tag{5.80}$$

for all $I_1^*,I_2^*\in\mathcal{P}^{=\ell}(N)$ with $I_1^*\neq I_2^*$. By the last assumption in (5.77), the last statement in (5.76), and (5.75),

$$\overline{\widetilde{W}_{t;I_1^*}}\cap X_{I_2}\subset\overline{\Psi_{t;I_1^*\cup I_2}\big(\mathcal{N}_{t;I_1^*\cup I_2;\partial}(2^\ell\varepsilon_{I_1^*\cup I_2})\big)} \tag{5.81}$$

for all $I_1^*\in\mathcal{P}^{=\ell}(N)$ and $I_2\in\mathcal{P}^*(N)-\mathcal{P}^*(I_1^*)$.

Since $\widetilde{\mathcal{W}}_{I^*}$ is a neighborhood of $B\times X_{I^*}$ in $B\times X_\emptyset$, $\mathcal{N}'_\partial X_{I^*}$ is a neighborhood of $B\times X_{I^*}$ in $B\times\mathcal{N}_\partial X_{I^*}$. Thus, there exists an open subset

$$\mathcal{N}'X_{I^*}\equiv\bigcup_{t\in B}\{t\}\times\mathcal{N}'_{t;I^*}\subset B\times\mathcal{N}X_{I^*}\qquad\text{s.t.}\quad \mathcal{N}'_{t;I^*;\partial}=\mathcal{N}_\partial X_{I^*}\cap\mathcal{N}'_{t;I^*}\quad\forall\,t\in B.$$

Choose $\varepsilon'_{I^*}\in C^\infty(B\times X_{I^*};\mathbb{R}^+)$ so that

$$\overline{\mathcal{N}X_{I^*}(2^\ell\varepsilon'_{I^*})}\subset\mathcal{N}^\circ X_{I^*}\cap\mathcal{N}'X_{I^*}\,. \tag{5.82}$$

Let

$$X_{t;I^*;\partial}=X_{I^*}\cap\bigcup_{\substack{I\in\mathcal{P}^{=(\ell+1)}(N)\\ I^*\subset I}}\Psi_{t;I}\big(\mathcal{N}_{t;I;\partial}(2^{\ell+1}\varepsilon_I)\big),\qquad \mathcal{X}_{I^*;\partial}=\bigcup_{t\in B}\{t\}\times X_{t;I^*;\partial},$$

$$X'_{t;I^*}=X_{I^*}-\bigcup_{\substack{I\in\mathcal{P}^{=(\ell+1)}(N)\\ I^*\subset I}}\overline{\Psi_{t;I}\big(\mathcal{N}_{t;I;\partial}(2^\ell\varepsilon_I)\big)},\qquad \mathcal{X}'_{I^*}=\bigcup_{t\in B}\{t\}\times X'_{t;I^*}.$$

Since $\Psi_{t;I}$ depends continuously on t, $\mathcal{X}'_{I^*}$ is an open subset of $B\times X_{I^*}$. Let $\{\eta_{I^*;\partial},\eta'_{I^*}\}$ be a partition of unity on $B\times X_{I^*}$ subordinate to the open cover $\{\mathcal{X}_{I^*;\partial},\mathcal{X}'_{I^*}\}$ of $B\times X_{I^*}$.

Define

$$\varepsilon_{I^*;\partial}\colon\mathcal{X}_{I^*;\partial}\longrightarrow\mathbb{R}^+\qquad\text{by}$$
$$\varepsilon_{I^*;\partial}\big(t,\Psi_{t;I}(u)\big)=\varepsilon_I\big(t,\pi_I(u)\big)\quad\forall\,u\in\mathcal{N}_{t;I;\partial}(2^{\ell+1}\varepsilon_I)\cap\mathcal{N}_{I;I^*},\ I\in\mathcal{P}^{=(\ell+1)}(N),\ I^*\subset I.$$

By (5.70), (5.67), (5.65), and (5.68), these definitions agree on the overlaps. Let

$$\varepsilon_{I^*}=\eta_{I^*;\partial}\varepsilon_{I^*;\partial}+\eta'_{I^*}\varepsilon'_{I^*}\colon B\times X_{I^*}\longrightarrow\mathbb{R}^+.$$

We next observe that

$$\overline{\mathcal{N}_{t;I^*}(2^{|\ell|}\varepsilon_{I^*})} \subset \mathcal{N}^\circ_{t;I^*}, \quad \mathcal{N}_{t;I^*;\partial}(2^\ell\varepsilon_{I^*}) \subset \mathcal{N}'_{t;I^*} \quad \forall\, t\in B,\, I^*\in\mathcal{P}^{=\ell}(N)\,. \tag{5.83}$$

By (5.82), this is the case for the fibers over $\mathcal{X}'_{I^*}-\mathcal{X}_{I^*;\partial}$. The second inclusion in (5.83) for the fibers over $\mathcal{X}_{I^*;\partial}-\mathcal{X}'_{I^*}$ is a special case of (5.78). The first inclusion in (5.83) for these fibers follows from (5.67) and (5.66) with $I'=I^*$. If $(t,x)\in\mathcal{X}'_{I^*}\cap\mathcal{X}_{I^*;\partial}$, then

$$\varepsilon_{I^*}(t,x) \le \varepsilon'_{I^*}(t,x) \qquad \text{or} \qquad \varepsilon_{I^*}(t,x) \le \varepsilon_{I^*;\partial}(t,x).$$

Either of these cases implies (5.83).

By the first inclusion in (5.83), ε_{I^*} satisfies (5.67) with $I=I^*$. Since $\varepsilon_{I^*}=\varepsilon_{I^*;\partial}$ on $\mathcal{X}_{I^*;\partial}-\mathcal{X}'_{I^*}$, ε_{I^*} satisfies (5.68) with $I'=I^*$ and $|I|=\ell+1$ and thus for all $I\supset I^*$. By the second inclusion in (5.83),

$$\Psi_{t;I^*}\big(\mathcal{N}_{t;I^*;\partial}(2^\ell\varepsilon_{I^*})\big) \subset \widetilde{W}_{t;I^*}\,. \tag{5.84}$$

By (5.81), ε_{I^*} thus satisfies (5.71) with $I_1=I^*$ and $2^{\ell+1}$ replaced by 2^ℓ. By (5.84) and (5.79), ε_{I^*} satisfies (5.70) with $I_1=I^*$, $|I_2|>\ell$, and $2^{\ell+1}$ replaced by 2^ℓ. By (5.84) and (5.80), (5.70) with $2^{\ell+1}$ replaced by 2^ℓ is satisfied whenever $|I_1|,|I_2|=\ell$. By the downward induction on $|I|$, we thus obtain functions $\varepsilon_I\in C^\infty(B\times X_I;\mathbb{R}^+)$ satisfying (5.67) and (5.68), as well as (5.70) and (5.71). □

Corollary 5.9. *Let $\mathbf{X}$, B, and $(\omega_{t;i})_{t\in B,i\in[N]}$ be as in Theorem 2.17 and $(\mathfrak{R}_t^{(1)})_{t\in B}$ and $(\mathfrak{R}_t^{(2)})_{t\in B}$ be $(\omega_{t;i})_{t\in B,i\in[N]}$-families of regularizations for $\mathbf{X}$ that are equivalent as families of weak regularizations for $\mathbf{X}$ over $X_\emptyset$, i.e.*

$$(\mathfrak{R}_t^{(1)})_{t\in B} \cong_{X_\emptyset} (\mathfrak{R}_t^{(2)})_{t\in B}. \tag{5.85}$$

Then they are equivalent as families of regularizations for $\mathbf{X}$ as in (2.28).

Proof. Let $(\mathfrak{R}_t^{(1)})_{t\in B}$ and $(\mathfrak{R}_t^{(2)})_{t\in B}$ be as in (5.3). By (5.85), there exists an $(\omega_{t;i})_{t\in B,i\in[N]}$-family $(\mathfrak{R}_t)_{t\in B}$ of weak regularizations for $\mathbf{X}$ over $X_\emptyset$ which satisfies the conditions below (5.4). By Lemma 5.8, it can be cut down to an $(\omega_{t;i})_{t\in B,i\in[N]}$-family $(\mathfrak{R}'_t)_{t\in B}$ of regularizations for $\mathbf{X}$. Since the latter still satisfies the conditions below (5.4), we obtain (2.28). □

Remark 5.10. Lemmas 5.6 and 5.8, Corollaries 5.7 and 5.9, and their proofs apply in the smooth category as well (as opposed to the symplectic category). For a smooth regularization, we need only Riemannian metrics $\rho_{t;I;i}$ on the real rank 2 vector bundles $\mathcal{N}_{X_{I-i}}X_I$ which are preserved by the differentials $\mathfrak{D}\Psi_{t;I;I'}$.

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