Normal Crossings Degenerations of Symplectic Manifolds

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Abstract

We use local Hamiltonian torus actions to degenerate a symplectic manifold to a normal crossings symplectic variety in a smooth one-parameter family. This construction, motivated in part by the Gross-Siebert and B. Parker's programs, contains a multifold version of the usual (two-fold) symplectic cut construction and in particular splits a symplectic manifold into several symplectic manifolds containing normal crossings symplectic divisors with shared irreducible components in one step.

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1 Introduction

Flat one-parameter families of degenerations are an important tool in algebraic geometry and raise considerable interest in related areas of symplectic topology. The Gross-Siebert program [9] for a direct proof of mirror symmetry has highlighted in particular the significance of log smooth degenerations to log smooth algebraic varieties. A central part of this program is the study of Gromov-Witten invariants (which are fundamentally symplectic topology invariants) under such degenerations. It is undertaken from an algebro-geometric perspective in [1, 4, 10]. The almost complex analogue of the log smooth category provided by the exploded manifold category of [16] underlines a similar study of Gromov-Witten invariants in [17]. Log smooth varieties include varieties with normal crossings (or NC) singularities, i.e. singularities of the form $z_1\ldots z_N=0$ in complex coordinates. Purely symplectic topology notions of NC symplectic variety and of smooth one-parameter family of degenerations to such a variety are introduced in [5, 6]. The main construction of this paper uses a collection of compatible Hamiltonian torus actions, i.e. an N-fold cutting configuration in the sense of Definition 1.2, to degenerate a symplectic manifold to an NC symplectic variety in a smooth one-parameter family.

The now classical symplectic cut construction of [11] decomposes a symplectic manifold (X,ω) into two symplectic manifolds, (X_-,ω_-) and (X_+,ω_+), using an S^1-action with a Hamiltonian h on an open subset W of X. This construction cuts X into closed subsets $U^{\leqslant}$ and $U^{\geqslant}$ along a separating real hypersurface $\widetilde{V}=h^{-1}(a)$ and collapses their boundary $\widetilde{V}$ to a smooth symplectic divisor $V=\widetilde{V}/S^1$ inside (X_-,ω_-) and (X_+,ω_+). The associated "wedge"

$$X_\emptyset \equiv X_-\cup_V X_+$$

is an SC symplectic variety associated with a 2-fold simple crossings (or SC) symplectic configuration in the sense of Definition 2.5.

The main construction of this paper, described in Section 4, contains a multifold version of the construction of [11]. We use an N-fold cutting configuration $\mathscr{C}$ in particular to decompose a symplectic manifold (X,ω) into N symplectic manifolds (X_i,ω_i) at once. This construction cuts X into closed subsets $U_i^{\leqslant}$ along separating real hypersurfaces $U_{ij}^{\leqslant}=h_{ij}^{-1}(0)$ for Hamiltonians h_{ij} generating S^1-actions on open subsets W_{ij} of X. These subsets $U_i^{\leqslant}$ have boundary and corners

$$U_I^{\leqslant}\equiv \bigcap_{j\in I-i} U_{ij}^{\leqslant}\subset U_i^{\leqslant}\,, \qquad \{i\}\subsetneq I\subset\{1,\ldots,N\}.$$

We collapse these $U_I^{\leqslant}$ to symplectic submanifolds $X_I\subset X_i$ with $I\ni i$ which form an SC symplectic divisor in X_i in the sense of Definition 2.1. The entire collection $\{X_I\}$ determines an SC symplectic variety $X_\emptyset$. This output of our main construction corresponds to the two middle statements in

Theorem 1 in Section 3.3. The output described by the remainder of Theorem 1 and by Theorem 2 endows the basic output of Theorem 1 with a rich geometric structure desirable for a range of applications; this structure is summarized below.

The normal bundles of the symplectic divisor V in the symplectic manifolds (X_-,ω_-) and (X_+,ω_+) arising from a symplectic cut construction of [11] are canonically dual. The symplectic sum construction of [8] thus determines a deformation equivalence class of nearly regular symplectic fibrations

$$\big(\mathcal{Z},\omega_{\mathcal{Z}},\pi\colon \mathcal{Z}\longrightarrow\mathbb{C}\big) \qquad \text{s.t.} \quad \pi^{-1}(0)=X_{\emptyset}\subset\mathcal{Z}\,; \tag{1.1}$$

see Definition 2.6. If $\widetilde{V}$ is compact, the symplectic deformation equivalence class of a fiber $\mathcal{Z}_\lambda\equiv\pi^{-1}(\lambda)$ of π is independent of the choice of $\lambda\in\mathbb{C}^*$ sufficiently small. It is then called a symplectic sum $X_-\#_V X_+$ of X_- and X_+ and is symplectically deformation equivalent to (X,ω). The constructions of [11] and [8] together thus provide a symplectic topology analogue of the algebro-geometric notion of smooth one-parameter family of degenerations of a smooth algebraic variety to two smooth algebraic varieties joined along a smooth algebraic divisor.

The main construction of this paper uses an N-fold cutting configuration $\mathscr{C}$ for (X,ω) to produce a symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$ which contains the tuple $(X_i,\omega_i)_{i=1}^N$ of the symplectic manifolds cut out from X by $\mathscr{C}$ as an SC symplectic divisor. The cutting configuration $\mathscr{C}$ also determines an N-fold Hamiltonian configuration for $(\mathcal{Z},\omega_{\mathcal{Z}})$ in the sense of Definition 1.1 and a deformation equivalence class of maps as in (1.1) that restrict to nearly regular symplectic fibrations on sufficiently small neighborhoods $\mathcal{Z}'$ of $X_\emptyset$. One implication for the well-known $N=2$ case is that the domain of the map π in (1.1) can be taken inside of a symplectic manifold *completely* determined by the data (W,ϕ,h,a) of the symplectic cut construction of [11]. If X is compact, then the fiber of $\pi|_{\mathcal{Z}'}$ over every sufficiently small value $\lambda\in\mathbb{R}^+$ is canonically isomorphic to the original symplectic manifold (X,ω) with the cutting configuration $\mathscr{C}$; see Theorem 2. The full output of the constructions of Sections 4 and 5, described in Theorems 1 and 2, thus provides a symplectic topology analogue of the algebro-geometric notion of smooth one-parameter family of degenerations of a smooth algebraic variety to an NC algebraic variety.

By [6, Proposition 5.1], a fibration $\pi\colon\mathcal{Z}'\longrightarrow\mathbb{C}$ as above determines a homotopy class of trivializations of the normal bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ of the singular locus X_∂ of $X_\emptyset$. We show in [7] that the multifold symplectic sum/smoothing construction of [6] and the multifold symplectic cut/degeneration construction of Sections 4 and 5 are mutual inverses as operations between the deformation equivalences classes of compact SC symplectic varieties with trivializations of the normal bundle of the singular locus and of compact symplectic manifolds with cutting configurations. This can be seen explicitly in the basic local setting of Section 6.2.

For $N\in\mathbb{Z}^+$, we define

$$[N]=\big\{1,\ldots,N\big\},\quad \mathcal{P}^*(N)=\big\{I\subset[N]\colon I\neq\emptyset\big\},\quad (S^1)^N_\bullet=\Big\{(\mathrm{e}^{\mathrm{i}\theta_i})_{i\in[N]}\in(S^1)^N\colon \prod_{i\in[N]}\mathrm{e}^{\mathrm{i}\theta_i}=1\Big\}.$$

The Lie algebra of the codimension 1 subtorus $(S^1)^N_\bullet\subset(S^1)^N$ and its dual are given by

$$\mathfrak{t}_{N;\bullet}=\big\{(r_i)_{i\in[N]}\in\mathbb{R}^N\colon \sum_{i\in[N]}r_i=0\big\} \quad\text{and}\quad \mathfrak{t}^*_{N;\bullet}=\mathbb{R}^N/\big\{(a,\ldots,a)\in\mathbb{R}^N\colon a\in\mathbb{R}\big\}, \tag{1.2}$$

respectively. For $I\subset[N]$, we identify $(S^1)^I$ with the subgroup

$$\big\{(\mathrm{e}^{\mathrm{i}\theta_i})_{i\in[N]}\in(S^1)^N:\, \mathrm{e}^{\mathrm{i}\theta_i}=1~\forall\,i\in[N]-I\big\}$$

of $(S^1)^N$ in the natural way and let

$$(S^1)^I_\bullet\equiv(S^1)^N_\bullet\cap(S^1)^I\,.$$

Denote by $\mathfrak{t}_{I;\bullet}\subset\mathfrak{t}_{N;\bullet}$ the Lie algebra of $(S^1)^I_\bullet$ and by $\mathfrak{t}^*_{I;\bullet}$ its dual. For $i,j\in I\subset[N]$, the homomorphism

$$\mathfrak{t}^*_{I;\bullet}=\mathbb{R}^I/\big\{(a,\ldots,a)\in\mathbb{R}^I:\ a\in\mathbb{R}\big\}\longrightarrow\mathbb{R},\qquad \eta\equiv\big[(a_k)_{k\in I}\big]\longrightarrow\eta_{ij}\equiv a_j-a_i\,,$$

is well-defined. We write $(\eta)_i<(\eta)_j$ (resp. $(\eta)_i\leqslant(\eta)_j$, $(\eta)_i=(\eta)_j$) if $0<\eta_{ij}$ (resp. $0\leqslant\eta_{ij}$, $0=\eta_{ij}$).

Definition 1.1. Let $N\in\mathbb{Z}^+$ and (X,ω) be a symplectic manifold. An N-fold Hamiltonian configuration for (X,ω) is a tuple

$$\mathscr{C}\equiv\big(U_I,\phi_I,\mu_I\big)_{I\in\mathcal{P}^*(N)},\tag{1.3}$$

where $(U_I)_{I\in\mathcal{P}^*(N)}$ is an open cover of X and ϕ_I is a Hamiltonian $(S^1)^I_\bullet$-action on U_I with moment map μ_I, such that

(*a*) $U_I\cap U_J=\emptyset$ unless $I\subset J$ or $J\subset I$;

(*b*) $\mu_J(x)|_{\mathfrak{t}_{I;\bullet}}=\mu_I(x)$ for all $x\in U_I\cap U_J$ and $I\subset J\subset[N]$;

(*c*) $(\mu_J(x))_i<(\mu_J(x))_j$ for all $x\in U_I\cap U_J$, $i\in I\subset J\subset[N]$, and $j\in J-I$.

Definition 1.2. An N-fold cutting configuration for (X,ω) is an N-fold Hamiltonian configuration as in (1.3) such that the restriction of the $(S^1)^I_\bullet$-action ϕ_I to $(S^1)^{I_0}_\bullet$ is free on the preimage of $0\in\mathfrak{t}^*_{I_0;\bullet}$ under the moment map

$$\mu_{I_0;I}\colon\big\{x\in U_I\colon\big(\mu_I(x)\big)_i<\big(\mu_I(x)\big)_j~\forall\,i\in I_0,\,j\in I-I_0\big\}\longrightarrow\mathfrak{t}^*_{I_0;\bullet},\quad \mu_{I_0;I}(x)=\mu_I(x)\big|_{\mathfrak{t}_{I_0;\bullet}},\tag{1.4}$$

for all $I_0\subset I\subset[N]$ with $I_0\neq\emptyset$.

We specify our conventions concerning moment maps for Hamiltonian actions on symplectic manifolds and identifications of Lie algebras in Section 3.1. An N-fold Hamiltonian configuration is determined by the 2^N-N-1 actions ϕ_I by the non-trivial subtori $(S^1)^I_\bullet\subset(S^1)^N$ and their moment maps μ_I on open subsets U_I of X. As described in Section 3.2, such a collection can alternatively be specified by $\binom{N}{2}$ Hamiltonian S^1-actions ϕ_{ij} and their Hamiltonians h_{ij} on (generally) larger open subsets of X. The usual symplectic cut construction of [11] is the $N=2$ case of this alternative description, which identifies $(S^1)^2_\bullet$ with S^1 by projection to one of the components of $(S^1)^2$. Simple examples of N-fold Hamiltonian and cutting configurations are described in Section 6.2. The output of the constructions of Sections 4 and 5 for the cutting configurations of Section 6.2 can be readily identified; see Proposition 6.4.

If the domain $U_{[N]}$ of the action $\phi_{[N]}$ by the largest subtorus $(S^1)^N_\bullet\subset(S^1)^N$ is the entire manifold X, the remaining actions ϕ_I and moment maps μ_I are the restrictions of $\phi_{[N]}$ and $\mu_{[N]}$, respectively, to U_I. As described in Section 6.1, the symplectic manifolds $(\mathcal{Z},\omega_{\mathcal{Z}})$ and (X_i,ω_i) are then obtained through a single application of the symplectic reduction of [13, 12]. There is also a natural nearly regular fibration π as in (1.1) defined on the entire manifold $\mathcal{Z}$. If in addition X is compact, $(\mathcal{Z},\omega_{\mathcal{Z}},\pi)$ can be replaced by a compact symplectic manifold $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$ for each $a\in\mathbb{R}^+$ sufficiently large and an S^1-equivariant nearly regular symplectic fibration

$$\widehat{\pi}\colon \widehat{\mathcal{Z}}_a \longrightarrow \mathbb{P}^1 \qquad \text{s.t.} \quad \widehat{\pi}^{-1}(0) = X_\emptyset \subset \widehat{\mathcal{Z}}_a\,; \tag{1.5}$$

see Section 6.3. In Sections 6.4 and 6.5, we relate on the setup for and the output of the main constructions of this paper to the moment polytopes arising from the Atiyah-Guillemin-Sternberg Convexity Theorem; see Theorem 3 and Proposition 6.10.

The notions of normal crossings symplectic singularities and of smoothings of such singularities introduced in [5] and [6], respectively, are recalled in Section 2. Section 3.3 contains the main statements of this paper, Theorems 1 and 2; they describe the output of our multifold symplectic cut/degeneration construction. The SC symplectic configuration determined by a cutting configuration and the symplectic manifold containing the associated SC symplectic variety are constructed in Section 4; this establishes Theorem 1. Section 5 endows a neighborhood of this symplectic variety with the structure of a one-parameter family of smoothings and establishes Theorem 2.

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2 Notation and terminology

For $I\subset[N]$, define

$$\mathcal{P}_I(N) = \big\{J\in\mathcal{P}^*(N)\colon\, I\subset J\big\}.$$

For $i\in[N]$, we write $\mathcal{P}_{\{i\}}(N)$ as $\mathcal{P}_i(N)$. For $i,j\in[N]$ distinct, we write $\mathcal{P}_{\{i,j\}}(N)$ as $\mathcal{P}_{ij}(N)$.

2.1 Normal crossings symplectic varieties

We now recall the notions of simple crossings (or SC) symplectic divisor and variety introduced, described in more detail, and illustrated with examples in [5, Section 2.1].

Let X be a (smooth) manifold. For any submanifold $V\subset X$, let

$$\mathcal{N}_X V \equiv \frac{TX|_V}{TV} \longrightarrow V$$

denote the normal bundle of V in X. For a collection $\{V_i\}_{i\in S}$ of submanifolds of X and $I\subset S$, let

$$V_I \equiv \bigcap_{i\in I} V_i \subset X\,.$$

Such a collection is called transverse if any subcollection $\{V_i\}_{i\in I}$ of these submanifolds intersects transversely, i.e. the homomorphism

$$T_xX \oplus \bigoplus_{i\in I} T_xV_i \longrightarrow \bigoplus_{i\in I} T_xX, \qquad \big(v,(v_i)_{i\in I}\big) \longrightarrow (v+v_i)_{i\in I}\,, \tag{2.1}$$

is surjective for every $x\in V_I$. Each subspace $V_I\subset X$ is then a submanifold of X.

If X is an oriented manifold, a transverse collection $\{V_i\}_{i\in S}$ of oriented submanifolds of X of even codimensions induces an orientation on each submanifold $V_I\subset X$ with $|I|\geqslant 2$, which we call the intersection orientation of V_I. If V_I is zero-dimensional, it is a discrete collection of points in X and the homomorphism (2.1) is an isomorphism at each point $x\in V_I$; the intersection orientation of V_I at $x\in V_I$ then corresponds to a plus or minus sign, depending on whether this isomorphism is orientation-preserving or orientation-reversing. For convenience, we call the original orientations of $X=V_\emptyset$ and $V_i=V_{\{i\}}$ the intersection orientations of these submanifolds V_I of X with $|I|<2$.

Suppose (X,ω) is a symplectic manifold and $\{V_i\}_{i\in S}$ is a transverse collection of submanifolds of X such that each V_I is a symplectic submanifold of (X,ω). Each V_I then carries an orientation induced by $\omega|_{V_I}$, which we call the ω-orientation. If V_I is zero-dimensional, it is automatically a symplectic submanifold of (X,ω); the ω-orientation of V_I at each point $x\in V_I$ corresponds to the plus sign by definition. By the previous paragraph, the ω-orientations of X and V_i with $i\in I$ also induce intersection orientations on all V_I. By definition, the intersection and symplectic orientations of V_I agree if $|I|<2$.

Definition 2.1. Let (X,ω) be a symplectic manifold. An SC symplectic divisor in (X,ω) is a finite transverse collection $\{V_i\}_{i\in S}$ of closed submanifolds of X of codimension 2 such that V_I is a symplectic submanifold of (X,ω) for every $I\subset S$ and the intersection and ω-orientations of V_I agree.

Definition 2.2. Let X be a manifold and $\{V_i\}_{i\in S}$ be a finite transverse collection of closed submanifolds of X of codimension 2. A symplectic structure on $\{V_i\}_{i\in S}$ in X is a symplectic form ω on X such that V_I is a symplectic submanifold of (X,ω) for all $I\subset S$.

For X and $\{V_i\}_{i\in S}$ as in Definition 2.2, we denote by $\mathrm{Symp}^+(X,\{V_i\}_{i\in S})$ the space of all symplectic structures ω on $\{V_i\}_{i\in S}$ in X such that $\{V_i\}_{i\in S}$ is an SC symplectic divisor in (X,ω).

Definition 2.3. Let $N\in\mathbb{Z}^+$. An N-fold transverse configuration is a tuple $\{X_I\}_{I\in\mathcal{P}^*(N)}$ of manifolds such that $\{X_{ij}\}_{j\in[N]-i}$ is a transverse collection of submanifolds of X_i for each $i\in[N]$ and

$$X_{\{ij_1,\ldots,ij_k\}}\equiv\bigcap_{m=1}^{k}X_{ij_m}=X_{ij_1\ldots j_k}\qquad\forall\ j_1,\ldots,j_k\in[N]-i.$$

Definition 2.4. Let $N\in\mathbb{Z}^+$ and $\mathbf{X}\equiv\{X_I\}_{I\in\mathcal{P}^*(N)}$ be an N-fold transverse configuration such that X_{ij} is a closed submanifold of X_i of codimension 2 for all $i,j\in[N]$ distinct. A symplectic structure on $\mathbf{X}$ is a tuple

$$(\omega_i)_{i\in[N]}\in\prod_{i=1}^{N}\mathrm{Symp}\big(X_i,\{X_{ij}\}_{j\in[N]-i}\big)$$

such that $\omega_{i_1}|_{X_{i_1i_2}}=\omega_{i_2}|_{X_{i_1i_2}}$ for all $i_1,i_2\in[N]$.

For $\mathbf{X}\equiv\{X_I\}_{I\in\mathcal{P}^*(N)}$ as in Definition 2.3, define

$$X_\emptyset=\Big(\bigsqcup_{i=1}^{N}X_i\Big)\Big/\!\sim,\quad X_i\ni x\sim x\in X_j\ \ \forall\ x\in X_{ij}\subset X_i,X_j,\ i\neq j\,, \tag{2.2}$$

$$X_\partial\equiv\bigcup_{\substack{I\subset[N]\\|I|=2}}X_I\subset X_\emptyset\,. \tag{2.3}$$

For $\mathbf{X}$ as in Definition 2.4, denote by $\mathrm{Symp}^+(\mathbf{X})$ the space of all symplectic structures $(\omega_i)_{i\in[N]}$ on $\mathbf{X}$ such that $\{X_{ij}\}_{j\in[N]-i}$ is an SC symplectic divisor in (X_i,ω_i) for each $i\in[N]$.

Definition 2.5. Let $N\in\mathbb{Z}^+$. An N-fold simple crossings (or SC) symplectic configuration is a tuple

$$\mathbf{X}=\big((X_I)_{I\in\mathcal{P}^*(N)},(\omega_i)_{i\in[N]}\big) \tag{2.4}$$

such that $\{X_I\}_{I\in\mathcal{P}^*(N)}$ is an N-fold transverse configuration, X_{ij} is a closed submanifold of X_i of codimension 2 for all $i,j\in[N]$ distinct, and $(\omega_i)_{i\in[N]}\in\mathrm{Symp}^+(\mathbf{X})$. The SC symplectic variety associated to such a tuple $\mathbf{X}$ is the pair $(X_\emptyset,(\omega_i)_{i\in[N]})$.

The basic local example of an SC symplectic variety is the union of the N coordinate hyperplanes in $\mathbb{C}^N$ with the restrictions of the standard symplectic form on $\mathbb{C}^N$, i.e. the $\lambda=0$ case of the hypersurface X in (6.26). This is the SC symplectic variety associated to the SC symplectic configuration (6.23) and is central fiber of the one-parameter family (6.24) of degenerations of the smooth symplectic manifolds X in (6.26) with $\lambda\neq0$ and arises from the basic local symplectic cutting configuration of Section 6.2.

2.2 Divisors, line bundles, and smoothability

Suppose (X,ω) is a symplectic manifold and $V\subset X$ is a smooth symplectic divisor, i.e. $|S|=1$ in the notation of Definition 2.1. The normal bundle of V in X,

$$\mathcal{N}_XV\equiv\frac{TX|_V}{TV}\approx TV^\omega\equiv\big\{v\in T_xX:\ x\in V,\ \omega(v,w)=0\ \forall\,w\in T_xV\big\}\longrightarrow V, \tag{2.5}$$

then inherits a fiberwise symplectic form $\omega|_{\mathcal{N}_XV}$ from ω. The space of complex structures on the fibers of (2.5) compatible with (resp. tamed by) $\omega|_{\mathcal{N}_XV}$ is non-empty and contractible; we call such complex structures ω-compatible (resp. ω-tame). Fix an identification Ψ of a tubular neighborhood D^ϵ_XV of V in $\mathcal{N}_XV$ with a tubular neighborhood of V in X (i.e. a regularization of V in X in the sense of [5, Definition 2.8]) and an ω-tame complex structure $\mathfrak{i}$ on $\mathcal{N}_XV$. Let

$$\begin{gathered}\mathcal{O}_X(V)=\big(\Psi^{-1\,*}\pi^*_{\mathcal{N}_XV}\mathcal{N}_XV|_{\Psi(D^\epsilon_XV)}\sqcup(X-V)\times\mathbb{C}\big)/\sim,\\ \Psi^{-1\,*}\pi^*_{\mathcal{N}_XV}\mathcal{N}_XV|_{\Psi(D^\epsilon_XV)}\ni\big(\Psi(v),v,cv\big)\sim\big(\Psi(v),c\big)\in(X-V)\times\mathbb{C}.\end{gathered} \tag{2.6}$$

This is a complex line bundle over X. The space of pairs $(\Psi,\mathfrak{i})$ involved in explicitly constructing this line bundle is contractible.

Suppose $\mathbf{X}$ is an SC symplectic configuration as in (2.4). If $i,j,k\in[N]$ are distinct, the inclusion $(X_{jk},X_{ijk})\longrightarrow(X_j,X_{ij})$ induces an isomorphism

$$\mathcal{N}_{X_{jk}}X_{ijk}\equiv\frac{TX_{jk}|_{X_{ijk}}}{TX_{ijk}}\longrightarrow\frac{TX_j|_{X_{ijk}}}{TX_{ij}|_{X_{ijk}}}\equiv\mathcal{N}_{X_j}X_{ij}\big|_{X_{ijk}}$$

of rank 2 oriented real vector bundles over X_{ijk}; see the last third of [6, Section 2.1]. In particular, the rank 2 oriented real vector bundles $\mathcal{N}_{X_j}X_{ij}|_{X_{ijk}}$ and $\mathcal{N}_{X_k}X_{ik}|_{X_{ijk}}$ are canonically identified with $\mathcal{N}_{X_{jk}}X_{ijk}$. We can thus choose a collection

$$\Psi_{ij;j}:\mathcal{N}'_{ij;j}\longrightarrow X_j,\qquad i,j\in[N],\ i\neq j,$$

of identifications of tubular neighborhoods of X_{ij} in $\mathcal{N}_{X_j}X_{ij}$ and in X_j and a collection of ω_j-tame complex structures $\mathfrak{i}_{ij;j}$ on the vector bundles $\mathcal{N}_{ij;j}$ so that

$$\Psi_{ij;j}\big|_{\mathcal{N}'_{ij;j}\cap\mathcal{N}_{X_{jk}}X_{ijk}} = \Psi_{ik;k}\big|_{\mathcal{N}'_{ik;k}\cap\mathcal{N}_{X_{jk}}X_{ijk}} \quad\text{and}\quad \mathfrak{i}_{ij;j}\big|_{\mathcal{N}_{X_{jk}}X_{ijk}} = \mathfrak{i}_{ik;k}\big|_{\mathcal{N}_{X_{jk}}X_{ijk}} \tag{2.7}$$

for all $i,j,k\in[N]$ with $k,j\neq i$.

For $i,j\in[N]$ distinct, let $\mathcal{O}_{X_j}(X_{ij})$ be the complex line bundle over X_j constructed as in (2.6) using the identification $\Psi_{ij;j}$ and the complex structure $\mathfrak{i}_{ij;j}$. By (2.7), there are canonical identifications

$$\mathcal{O}_{X_j}(X_{ij})\big|_{X_{jk}} = \mathcal{O}_{X_{jk}}(X_{ijk}) = \mathcal{O}_{X_k}(X_{ik})\big|_{X_{jk}}$$

for all $i,j,k\in[N]$ with $j,k\neq i$. For each $i\in[N]$,

$$\begin{gathered}\mathcal{O}_{X_i^c}(X_i) \equiv \Big(\bigsqcup_{j\in[N]-\{i\}}\mathcal{O}_{X_j}(X_{ij})\Big)\Big/\!\sim \longrightarrow X_i^c \equiv \bigcup_{j\in[N]-\{i\}}X_j \subset X_\emptyset\,, \\ \mathcal{O}_{X_j}(X_{ij})\big|_{X_{jk}} \ni u \sim u \in \mathcal{O}_{X_k}(X_{ik})\big|_{X_{jk}} \quad \forall\, i,j,k\in[N],\ j,k\neq i,\end{gathered} \tag{2.8}$$

is thus a well-defined complex line bundle. Let $\mathcal{O}_{X_\partial}(X_i)=\mathcal{O}_{X_i^c}(X_i)|_{X_\partial}$. We call the complex line bundle

$$\mathcal{O}_{X_\partial}(X_\emptyset) \equiv \bigotimes_{i=1}^{N}\mathcal{O}_{X_\partial}(X_i) \tag{2.9}$$

the normal bundle of the singular locus X_∂ in $X_\emptyset$. The space of the collections of pairs $(\Psi_{ij;j},\mathfrak{i}_{ij;j})$ involved in explicitly constructing this line bundle is contractible.

The notions of smooth families of varieties and of smoothings of singular varieties play important roles in algebraic geometry. Analogues of these notions in the category of SC symplectic varieties are introduced in [6].

Definition 2.6 ([6, Definition 2.6])**.** If $(\mathcal{Z},\omega_{\mathcal{Z}})$ is a symplectic manifold and $\Delta\subset\mathbb{C}$ is a disk centered at the origin, a smooth surjective map $\pi\colon\mathcal{Z}\longrightarrow\Delta$ is a nearly regular symplectic fibration if

- $\mathcal{Z}_0\equiv\pi^{-1}(0) = X_1\cup\ldots\cup X_N$ for some SC symplectic divisor $\{X_i\}_{i\in[N]}$ in $(\mathcal{Z},\omega_{\mathcal{Z}})$,
- π is a submersion outside of the submanifolds X_I with $|I|\geqslant 2$,
- for every $\lambda\in\Delta-\{0\}$, the restriction ω_λ of $\omega_{\mathcal{Z}}$ to $\mathcal{Z}_\lambda\equiv\pi^{-1}(\lambda)$ is nondegenerate.

We call a nearly regular symplectic fibration as in Definition 2.6 a one-parameter family of smoothings of the SC variety $(X_\emptyset,(\omega_i)_{i\in[N]})$ associated to the SC symplectic configuration (2.4) with

$$X_I = \bigcap_{i\in I}X_i \subset \mathcal{Z}_0 \subset \mathcal{Z} \qquad \forall\ I\in\mathcal{P}^*(N)\,.$$

By [6, Theorem 2.7], *some* SC symplectic variety $(X_\emptyset,(\omega'_i)_{i\in[N]})$ deformation equivalent to $(X_\emptyset,(\omega_i)_{i\in[N]})$ admits a one-parameter family of smoothings if and only if the line bundle (2.9) admits a trivialization (or equivalently its Chern class vanishes). Furthermore, the germ of the deformation equivalence class of such a smoothing is determined by a homotopy class of trivializations of (2.9).

If the SC symplectic variety $X_\emptyset$ associated with an SC symplectic configuration $\mathbf{X}$ is compact, we call the fiber of a one-parameter family of smoothings of $X_\emptyset$ over a point in a small punctured disk Δ^* around the origin a symplectic sum for $\mathbf{X}$. Each such fiber comes with a natural cutting configuration. The concern of Corollary 3.2 with the trivializations of the line bundle (2.9) is in preparation for showing in [7] that the symplectic cut/degeneration construction of this paper and the symplectic sum/smoothing construction of [6] are mutual inverses as morphisms between appropriate categories.

2.3 Families of symplectic varieties

We also need family versions of Definitions 2.4 and 2.6, especially over the interval $\mathbb{I}\equiv[0,1]$.

Definition 2.7. Let B be a manifold, possibly with boundary.

(1) A smooth map $\pi_{\mathfrak{X}}:\mathfrak{X}\longrightarrow B$ is a family of manifolds over B if $\partial\mathfrak{X}=\pi_{\mathfrak{X}}^{-1}(\partial B)$ and $\pi_{\mathfrak{X}}$ is a submersion.

(2) A family of symplectic manifolds over B is a tuple $(\mathfrak{X},\omega_{\mathfrak{X}},\pi_{\mathfrak{X}})$, where $\pi_{\mathfrak{X}}\colon\mathfrak{X}\longrightarrow B$ is a family of manifolds over B and $\omega_{\mathfrak{X}}$ is a 2-form on $\mathfrak{X}$ such that

$$\big(\mathfrak{X}_t,\omega_t\big)\equiv\big(\pi_{\mathfrak{X}}^{-1}(t),\omega|_{\mathfrak{X}_t}\big)$$

is a symplectic manifold for every $t\in B$.

Definition 2.8. Let $N\in\mathbb{Z}^+$ and B be a manifold, possibly with boundary.

(1) A family of N-fold transverse configurations over B is a tuple $\{\pi_I:\mathfrak{X}_I\longrightarrow B\}_{I\in\mathcal{P}^*(N)}$, where $\{\mathfrak{X}_I\}_{I\in\mathcal{P}^*(N)}$ is an N-fold transverse configuration and $\pi_I:\mathfrak{X}_I\longrightarrow B$ is a family of manifolds over B for each $I\in\mathcal{P}^*(N)$ such that $\pi_I=\pi_i|_{\mathfrak{X}_I}$ for all $i\in I\subset[N]$.

(2) A family of N-fold SC symplectic configurations over B is a tuple

$$\big((\pi_I\colon\mathfrak{X}_I\longrightarrow B)_{I\in\mathcal{P}^*(N)},(\omega_i)_{i\in[N]}\big),$$

where $\{\pi_I\colon\mathfrak{X}_I\longrightarrow B\}_{I\in\mathcal{P}^*(N)}$ is a family of N-fold transverse configurations over B and ω_i is a 2-form on $\mathfrak{X}_i$ for each $i\in[N]$ such that

$$\big((\mathfrak{X}_{t;I})_{I\in\mathcal{P}^*(N)},(\omega_{t;i})_{i\in[N]}\big)\equiv\big((\pi_I^{-1}(t))_{I\in\mathcal{P}^*(N)},(\omega_i)_{i\in[N]}\big)$$

is an N-fold SC symplectic configuration for every $t\in B$.

We call a family $(\omega_t)_{t\in B}$ of 2-forms on a manifold X smooth if the induced 2-form on $B\times X$ given by

$$\omega_{t,x}(v,w)=\begin{cases}\omega_t|_x(v,w), & \text{if } v,w\in T_xX;\\ 0, & \text{if } v\in T_tB;\end{cases}$$

is smooth. Let $\mathbf{X}\equiv\{X_I\}_{I\in\mathcal{P}^*(N)}$ be an N-fold transverse configuration,

$$\big(\omega_{t;i}\big)_{i\in[N]}\in\mathrm{Symp}^+(\mathbf{X})\qquad\forall\ t\in B,$$

and ω_i be the 2-form on $B\times X_i$ induced by the family $\big(\omega_{t;i}\big)_{t\in B}$. If the family $(\omega_{t;i})_{t\in B}$ of 2-forms on X_i is smooth for every $i\in[N]$ and $\pi_I: B\times X_I \longrightarrow X_I$ is the component projection map, then the tuple

$$\big((\pi_I\colon \mathfrak{X}_I \equiv B\times X_I \longrightarrow B)_{I\in\mathcal{P}^*(N)}, (\omega_i)_{i\in[N]}\big)$$

is a family of N-fold SC symplectic configurations over B.

Suppose the fibers of the submersions π_I in Definition 2.8(1) are compact. These projections are then locally trivial, i.e. for every $t_0\in B$ there exist a neighborhood U of t_0 in B and a diffeomorphism

$$\Xi_I\colon \mathfrak{X}_I|_U \equiv \pi_I^{-1}(U) \longrightarrow U\times\pi_I^{-1}(t_0)$$

such that $\pi_I|_{\mathfrak{X}_I|_U}=\pi_U\circ\Xi_I$, where

$$\pi_U\colon U\times\pi_I^{-1}(t_0)\longrightarrow U$$

is the component projection map. By the same reasoning as in the proof of [5, Proposition 4.2], the diffeomorphisms Ξ_I can be chosen so that

$$\Xi_I = \Xi_i|_{\pi_I^{-1}(U)} \qquad \forall\ i\in I\subset[N].$$

This implies that any family of compact N-fold transverse configurations is locally trivial. Thus, we can identify a family of N-fold SC symplectic configurations over $\mathbb{I}$ with a smooth path $(\omega_{t;i})_{i\in[N]}$ in $\mathrm{Symp}^+(\mathbf{X})$ for some N-fold transverse configuration $\mathbf{X}$.

Given a family $\{\pi_I\colon \mathfrak{X}_I\longrightarrow B\}_{I\in\mathcal{P}^*(N)}$ of transverse configurations, we define fiber bundles

$$\pi_\emptyset\colon \mathfrak{X}_\emptyset \longrightarrow B \qquad \text{and} \qquad \pi_\partial\colon \mathfrak{X}_\partial\longrightarrow B$$

similarly to (2.2) and (2.3). For each $t\in B$, let

$$\mathfrak{X}_{t;\emptyset}=\pi_\emptyset^{-1}(t) \qquad \text{and} \qquad \mathfrak{X}_{t;\partial}=\pi_\partial^{-1}(t).$$

The precise definition of the total space of the complex line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ in (2.9) depends on the choices of identifications $\Psi_{ij;j}$ of neighborhoods of X_{ij} in X_j and in X_j and of the ω_j-tame complex structures $\mathfrak{i}_{ij;j}$ on (the fibers of) $\mathcal{N}_{X_j}X_{ij}$ that satisfy (2.7), respectively. For a family of SC symplectic configurations as in Definition 2.8(2), such choices can be made continuously with respect to $t\in B$. We then obtain a complex line bundle

$$\mathcal{O}_{\mathfrak{X}_\partial}(\mathfrak{X}_\emptyset)\longrightarrow\mathfrak{X}_\partial \qquad \text{s.t.} \qquad \mathcal{O}_{\mathfrak{X}_\partial}(\mathfrak{X}_\emptyset)\big|_{\mathfrak{X}_{t;\partial}}=\mathcal{O}_{\mathfrak{X}_{t;\partial}}(\mathfrak{X}_{t;\emptyset}) \quad \forall\ t\in B\,. \tag{2.10}$$

We call a family $(\hbar_t)_{t\in B}$ of homotopy classes of trivializations of $\mathcal{O}_{\mathfrak{X}_{t;\partial}}(\mathfrak{X}_{t;\emptyset})$ continuous if for each $t_0\in B$ there exist a neighborhood U of t_0 in B and a trivialization Φ of $\mathcal{O}_{\mathfrak{X}_\partial}(\mathfrak{X}_\emptyset)|_{\pi_\partial^{-1}(U)}$ such that $\Phi|_{\mathfrak{X}_{t;\partial}}\in\hbar_t$ for every $t\in U$.

Suppose $\mathbf{X}$ is an N-fold transverse configuration as in Definition 2.4,

$$\big(\omega_{0;i}\big)_{i\in[N]}, \big(\omega_{1;i}\big)_{i\in[N]}\in\mathrm{Symp}^+(\mathbf{X})$$

lie in the same topological component of $\mathrm{Symp}^+(\mathbf{X})$, and the associated line bundle (2.9) admits a trivialization. Let $\hbar_0$ and $\hbar_1$ be homotopy classes of trivializations of the line bundles $\mathcal{O}_{X_\partial}(X_\emptyset)$ corresponding to the SC symplectic configurations determined by $(\omega_{0;i})_{i\in[N]}$ and $(\omega_{1;i})_{i\in[N]}$, respectively. We define $\hbar_0=\hbar_1$ if there exist a smooth path $(\omega_{t;i})_{i\in[N]}$ in $\mathrm{Symp}^+(\mathbf{X})$ and a continuous family $(\hbar_t)_{t\in B}$ of homotopy classes of trivializations of the corresponding line bundles $\mathcal{O}_{\mathfrak{X}_{t;\partial}}(\mathfrak{X}_{t;\emptyset})$.

Definition 2.9. Let B be a manifold, possibly with boundary.

(1) A family of nearly regular symplectic fibrations over B is a tuple $(\mathscr{Z},\omega_{\mathscr{Z}},\pi,\pi_{\mathscr{Z}})$, where $(\mathscr{Z},\omega_{\mathscr{Z}},\pi_{\mathscr{Z}})$ is a family of symplectic manifolds over B and $\pi:\mathscr{Z}\longrightarrow\mathbb{C}$ is a smooth map such that

$$\big(\mathscr{Z}_t,\omega_{\mathscr{Z};t},\pi_t\big)\equiv\big(\pi_{\mathscr{Z}}^{-1}(t),\omega_{\mathscr{Z}}|_{\mathscr{Z}_t},\pi|_{\mathscr{Z}_t}\big) \tag{2.11}$$

is a nearly regular symplectic fibration for every $t\in B$.

(2) Let $(\mathfrak{X}_{t;\emptyset})_{t\in B}$ be the family of SC symplectic varieties associated with a family of N-fold SC symplectic configurations over B as in Definition 2.8(2). A family of one-parameter families of smoothings of $(\mathfrak{X}_{t;\emptyset})_{t\in B}$ is a family $(\mathscr{Z},\omega_{\mathscr{Z}},\pi,\pi_{\mathscr{Z}})$ of nearly regular symplectic fibrations over B such that (2.11) is a one-parameter family of smoothings of $\mathfrak{X}_{t;\emptyset}$ for every $t\in B$.

Suppose $(\mathscr{Z},\omega_{\mathscr{Z}},\pi_{\mathscr{Z}})$ is a family of symplectic manifolds over B, $(\mathfrak{X}_{t;\emptyset})_{t\in B}$ is as in Definition 2.9(2), and $\mathfrak{X}_{t;\emptyset}\subset\mathscr{Z}_t$ is an SC symplectic divisor for every $t\in B$. We call a family $(\mathfrak{h}_t)_{t\in B}$ of germs of homotopy classes of one-parameter families $(\mathscr{Z}_t',\omega_{\mathscr{Z}}|_{\mathscr{Z}_t'},\pi_t)$ of smoothings of $\mathfrak{X}_{t;\emptyset}$ with $\mathscr{Z}_t'\subset\mathscr{Z}_t$ continuous if for each $t_0\in B$ there exist a neighborhood U of t_0 in B and a family $(\mathscr{Z}',\omega_{\mathscr{Z}}|_{\mathscr{Z}'},\pi,\pi_{\mathscr{Z}}|_{\mathscr{Z}'})$ of one-parameter families of smoothings of $(\mathfrak{X}_{t;\emptyset})_{t\in U}$ with $\mathscr{Z}'\subset\mathscr{Z}$ such that $(\mathscr{Z}_t',\omega_{\mathscr{Z}}|_{\mathcal{Z}_t'},\pi_t)\in\mathfrak{h}_t$ for every $t\in B$.

3 Main setup and output

We review basic notions from Hamiltonian symplectic geometry, specifying our conventions, in Section 3.1. In Section 3.2, we study the notions of Hamiltonian and cutting configurations of Definitions 1.1 and 1.2. Simple local examples of such configurations are described in Section 6.2. Section 6.4 provides a plethora of Hamiltonian and cutting configurations for symplectic manifolds with Hamiltonian torus actions. Theorems 1 and 2, stated in Section 3.3 and proved in Sections 4 and 5, describe the output determined by a cutting configuration via the multifold symplectic cut/degeneration construction of this paper.

3.1 Torus actions and moment maps

The characteristic vector field of a (smooth) S^1-action

$$\phi\colon S^1\times X\longrightarrow X$$

on a manifold X is the vector field ζ_ϕ on X given by

$$\zeta_\phi(x)=\frac{\mathrm{d}}{\mathrm{d}\theta}\phi(\mathrm{e}^{\mathrm{i}\theta};x)\bigg|_{\theta=0}\in T_xX\qquad\forall\,x\in X\,.$$

If (X,ω) is a symplectic manifold, a Hamiltonian for an S^1-action ϕ on X as above is a smooth map $h\colon X\longrightarrow\mathbb{R}$ such that
$$-\mathrm{d}h=\iota_{\zeta_\phi}\omega\equiv\omega(\zeta_\phi,\cdot);$$
such a function h is S^1-invariant. For example, a Hamiltonian for the S^1-action
$$\phi\colon S^1\times\mathbb{C}^N\longrightarrow\mathbb{C}^N,\qquad \phi(\mathrm{e}^{\mathrm{i}\theta};z_1,\ldots,z_N)=\big(z_1,\ldots,z_{i-1},\mathrm{e}^{\mathrm{i}\theta}z_i,z_{i+1},\ldots,z_N\big),$$
on $\mathbb{C}^N$ with the standard symplectic form
$$\omega_{\mathbb{C}^N}\equiv\mathrm{d}x_1\wedge\mathrm{d}y_1+\ldots+\mathrm{d}x_N\wedge\mathrm{d}y_N$$
is given by
$$h\colon\mathbb{C}^N\longrightarrow\mathbb{R},\qquad h(z_1,\ldots,z_N)=\frac{1}{2}|z_i|^2\,.$$
An S^1-action ϕ on (X,ω) is called Hamiltonian if a Hamiltonian h for ϕ exists. In such a case, h is well-defined up to a constant (on each connected component of X).

For a k-torus $\mathbb{T}\approx(S^1)^k$, we denote by $\mathfrak{t}$ its Lie algebra and by $\mathfrak{t}^*$ the dual of $\mathfrak{t}$. A smooth map
$$\phi\colon\mathbb{T}\times X\longrightarrow X\tag{3.1}$$
and an element $\xi\in\mathfrak{t}$ determine a vector field ξ_ϕ on X by
$$\xi_\phi(x)=\mathrm{d}_{(\mathrm{id},x)}\phi(\xi,0)\in T_xX\qquad\forall\,x\in X.$$
If (X,ω) is a symplectic manifold, a moment map for a $\mathbb{T}$-action ϕ on X as in (3.1) is a $\mathbb{T}$-invariant smooth map
$$\mu\colon X\longrightarrow\mathfrak{t}^*\qquad\text{s.t.}\quad-\mathrm{d}\big\langle\mu(\cdot),\xi\big\rangle=\iota_{\xi_\phi}\omega\quad\forall\,\xi\in\mathfrak{t}\,.$$
An $\mathbb{T}$-action ϕ on (X,ω) is called Hamiltonian if a moment map μ for ϕ exists. In such a case, μ is well-defined up to a constant (on each connected component of X). A Hamiltonian $\mathbb{T}$-pair for a symplectic manifold (X,ω) is a pair (ϕ,μ) consisting of a Hamiltonian action ϕ of a torus $\mathbb{T}$ on (X,ω) and a moment map μ for this action.

We identify the Lie algebra $\mathfrak{t}_1$ of S^1 and its dual $\mathfrak{t}_1^*$ with $\mathbb{R}$ by the dual homomorphisms
$$\mathbb{R}\longrightarrow\mathfrak{t}_1,\quad r\longrightarrow\xi_r\equiv\frac{\mathrm{d}}{\mathrm{d}\theta}\big(\mathrm{e}^{\mathrm{i}r\theta}\big)\Big|_{\theta=0}\in T_{\mathrm{id}}S^1,\qquad \mathfrak{t}_1^*\longrightarrow\mathbb{R},\quad\eta\longrightarrow\big\langle\eta,\xi_1\big\rangle\in\mathbb{R}.\tag{3.2}$$
In particular, $\zeta_\phi=(\xi_1)_\phi$. The composition of the second homomorphism above with a moment map μ for an S^1-action ϕ on (X,ω) is a Hamiltonian for this action. An isomorphism $\mathbb{T}\approx(S^1)^k$ and the identifications (3.2) determine identifications
$$\mathfrak{t}\approx\mathbb{R}^k\qquad\text{and}\qquad\mathfrak{t}^*\approx\mathbb{R}^k\,.$$
A $\mathbb{T}$-action ϕ on X then corresponds to a tuple $(\phi_i)_{i\in[k]}$ of commuting S^1-actions on X. A moment map μ for a $\mathbb{T}$-action ϕ on (X,ω) corresponds to a tuple $(h_i)_{i\in[k]}$ of Hamiltonians for the S^1-actions ϕ_i preserved by all S^1-actions, i.e.
$$h_i\big(\phi_j(\mathrm{e}^{\mathrm{i}\theta};x)\big)=h_i(x)\qquad\forall\ (\mathrm{e}^{\mathrm{i}\theta};x)\in S^1\times X,\ i,j\in[k].$$

3.2 Hamiltonian and cutting configurations

A Hamiltonian configuration (1.3) can alternatively be described in terms of compatible S^1-actions as follows. For $i,j\in\mathbb{Z}$, define

$$\mathfrak{s}_{ij}=\begin{cases}1, & \text{if } i>j;\\ 0, & \text{if } i=j;\\ -1, & \text{if } i<j.\end{cases}$$

For $i\in I\subset[N]$, the homomorphism

$$\varrho_{I;i}\colon (S^1)^{I-i}\longrightarrow (S^1)^I, \qquad \big(\varrho_{I;i}\big((\mathrm{e}^{\mathrm{i}\theta_j})_{j\in I-i}\big)\big)_k=\begin{cases}\prod\limits_{j\in I-i}\mathrm{e}^{-\mathrm{i}\mathfrak{s}_{ij}\theta_j}, & \text{if } k=i;\\ \mathrm{e}^{\mathrm{i}\mathfrak{s}_{ik}\theta_k}, & \text{if } k\in I-i;\end{cases} \tag{3.3}$$

is an isomorphism onto $(S^1)^I_\bullet$. The induced homomorphism on the duals of the Lie algebras,

$$\big\{\mathrm{d}_{\mathrm{id}}\varrho_{I;i}\big\}^*\colon \mathbb{R}^I\longrightarrow\mathbb{R}^{I-i}, \qquad (a_j)_{j\in I}\longrightarrow\big(\mathfrak{s}_{ij}(a_j-a_i)\big)_{j\in I}, \tag{3.4}$$

descends to an isomorphism from $\mathfrak{t}^*_{I;\bullet}$ to $\mathbb{R}^{I-i}$. Via the isomorphisms (3.3) and (3.4), the tuple $(\phi_I,\mu_I)_{I\in\mathcal{P}^*(N)}$ in (1.3) corresponds to smooth maps

$$\phi_{ij}=\phi_{ji}\colon S^1\times\bigcup_{I\in\mathcal{P}_{ij}(N)}U_I\longrightarrow\bigcup_{I\in\mathcal{P}_{ij}(N)}U_I \quad\text{and}\quad h_{ij}=h_{ji}\colon\bigcup_{I\in\mathcal{P}_{ij}(N)}U_I\longrightarrow\mathbb{R} \tag{3.5}$$

with $i,j\in[N]$ distinct so that ϕ_{ij} are commuting S^1-actions with Hamiltonians h_{ij} preserved by all these actions and satisfying

$$\mathfrak{s}_{ij}h_{ij}|_{U_I}+\mathfrak{s}_{jk}h_{jk}|_{U_I}=\mathfrak{s}_{ik}h_{ik}|_{U_I}$$

for all $i,j,k\in I\subset[N]$.

Since $(S^1)^I_\bullet$ is a torus of dimension $|I|-1$, the "actions" $\phi_{\{i\}}$ with $i\in[N]$ in (1.3) are trivial. In the $N=1$ case of Definition 1.2, the open cover consists of the single set $U_1=X$. There are no torus actions then; the output of Sections 4 and 5 is then simply

$$(X_1,\omega_1)=(X,\omega), \quad (\mathcal{Z},\omega_{\mathcal{Z}})=\big(X\times\mathbb{C},\pi_1^*\omega+\pi_2^*\omega_{\mathbb{C}}\big), \quad \pi=\pi_2\colon\mathcal{Z}\longrightarrow\mathbb{C}.$$

The $N=2$ case corresponds to the setting in the symplectic cut construction of [11] with a separating hypersurface

$$\widetilde{V}=U_{12;12}\equiv\mu_{12}^{-1}(0)\subset U_{12}.$$

In this case, U_{12} is an open neighborhood of $\widetilde{V}$ with an action of $(S^1)^2_\bullet\approx S^1$. The open subsets $U_1,U_2\subset X$ can be taken to be the unions of the topological components of $X-\widetilde{V}$ so that

$$(\mu_{12}(x))_1<(\mu_{12}(x))_2 \ \ \forall\,x\in U_1\cap U_{12} \qquad\text{and}\qquad (\mu_{12}(x))_2<(\mu_{12}(x))_1 \ \ \forall\,x\in U_2\cap U_{12};$$

see Definition 1.1(c). This implies that $\widetilde{V}$ is closed in X (and not just in U_{12}).

Let $\mathscr{C}$ be an N-fold Hamiltonian configuration for (X,ω) as in (1.3). We call a subset $Y\subset X$ $\mathscr{C}$-invariant if

$$\phi_I\big((S^1)^I_\bullet\times(Y\cap U_I)\big)=Y\cap U_I \qquad \forall\ I\in\mathcal{P}^*(N).$$

For such a subset, let

$$\mathscr{C}|_Y = \big(Y\cap U_I, \phi_I|_{(S^1)^I_\bullet\times(Y\cap U_I)}, \mu_I|_{Y\cap U_I}\big)_{I\in\mathcal{P}^*(N)}.$$

If in addition Y is a symplectic submanifold of (X,ω), then $\mathscr{C}|_Y$ is an N-fold Hamiltonian configuration for $(Y,\omega|_Y)$; we call it the restriction of $\mathscr{C}$ to Y. If $\mathscr{C}$ is a cutting configuration, then so is $\mathscr{C}|_Y$.

If $(U_I)_{I\in\mathcal{P}^*(N)}$ is a collection of subsets of X, another collection $(U'_I)_{I\in\mathcal{P}^*(N)}$ of subsets of X refines $(U_I)_{I\in\mathcal{P}^*(N)}$ if $U'_I\subset U_I$ for all $I\in\mathcal{P}^*(N)$. We call such a refinement proper if $\overline{U'_I}\subset U_I$ for all $I\in\mathcal{P}^*(N)$. If $\mathscr{C}$ is an N-fold Hamiltonian configuration for (X,ω) as in (1.3), we call a collection $(U'_I)_{I\in\mathcal{P}^*(N)}$ of subsets of X refining $(U_I)_{I\in\mathcal{P}^*(N)}$ $\mathscr{C}$-invariant if

$$\phi_I\big((S^1)^I_\bullet\times U'_I\big) = U'_I \qquad \forall\, I\in\mathcal{P}^*(N).$$

If in addition each U'_I is open and the union of these subsets is $X'\subset X$, then

$$\mathscr{C}' \equiv \big(U'_I, \phi_I|_{(S^1)^I_\bullet\times U'_I}, \mu_I|_{U'_I}\big)_{I\in\mathcal{P}^*(N)} \tag{3.6}$$

is an N-fold Hamiltonian configuration for $(X',\omega|_{X'})$; we call it the restriction of $\mathscr{C}$ to $(U'_I)_{I\in\mathcal{P}^*(N)}$. If $\mathscr{C}$ is a cutting configuration, then so is $\mathscr{C}'$. Restricting to an open refinement covering X has no effect on the cut symplectic manifolds (X_i,ω_i), their submanifolds (X_I,ω_I), the symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$, or the deformation equivalence class of the fibration $\pi\colon\mathcal{Z}'\longrightarrow\mathbb{C}$ constructed in Sections 4 and 5, but refines the open cover $\{X_I^\circ\}_{I\in\mathcal{P}_i(N)}$ of each X_i and the open cover $\{\mathcal{Z}_I^\circ\}_{I\in\mathcal{P}^*(N)}$ of $\mathcal{Z}$ determined by $\mathscr{C}$.

According to Definition 1.1(c),

$$(\mu_J(x))_i<(\mu_J(x))_j \qquad \forall\, x\in U_I\cap U_J,\ i\in I\subset J\subset[N],\ j\in J-I.$$

We call an N-fold Hamiltonian configuration as in (1.3) maximal if

$$\big\{x\in U_J\colon\ (\mu_J(x))_i<(\mu_J(x))_j\ \forall\, i\in I,\ j\in J-I\big\}\subset U_I \tag{3.7}$$

whenever $I,J\in\mathcal{P}^*(N)$ and $I\subset J$. The condition (3.7) is automatically satisfied if $I=J$. The $N=1,2$ configurations described above and the configuration (6.4) with a Hamiltonian $(S^1)^N_\bullet$-action on the entire symplectic manifold (X,ω) are maximal. Special cases of the latter include the basic local N-fold configuration (6.27) and the configuration (6.63) for a symplectic manifold with a Hamiltonian action of an abstract torus. For a maximal Hamiltonian configuration, the set of conditions of Definition 1.2 indexed by the pairs $I_0\subset I$ reduces to its subset with $I_0=I$. In light of the second statement in Lemma 4.1, maximal cutting configurations give rise to open covers $\{X_I^\circ\}_{I\in\mathcal{P}_i(N)}$ of the cut symplectic manifolds X_i in the spirit of toric and tropical geometries (i.e. similar to Zariski open sets in algebraic geometry). For such configurations, each X_I° contains the submanifold X_I outside of the submanifolds X_J with $J\supsetneq I$ (the real codimension of X_J in X_I is $2(|J|-|I|)$).

A maximal N-fold Hamiltonian configuration can be obtained from any given N-fold Hamiltonian configuration (1.3) by taking

$$U'_I = U_I\cup\bigcup_{J\in\mathcal{P}_I(N)}\big\{x\in U_J\colon\ (\mu_J(x))_i<(\mu_J(x))_j\ \forall\, i\in I,\ j\in J-I\big\},$$

$$\phi'_I(g;x)=\phi_J(g;x),\quad \mu'_I(x)=\mu_J(x)\big|_{\mathfrak{t}_{I;\bullet}} \qquad \forall\ g\in(S^1)^I_\bullet,\ x\in U'_I\cap U_J,\ J\in\mathcal{P}_I(N).$$

By (a) and (b) in Definition 1.1, $\phi'_I(g;x)$ and $\mu'_I(x)$ are independent of the possible choices of J. Since the moment map μ_J is $(S^1)^I_\bullet$-invariant whenever $J \in \mathcal{P}_I(N)$, the image of ϕ'_I is contained in U'_I. Thus, ϕ'_I is an $(S^1)^I_\bullet$-action on U'_I with moment map μ'_I. It is immediate that the new collection

$$\mathscr{C}' \equiv \big(U'_I, \phi'_I, \mu'_I\big)_{I\in\mathcal{P}^*(N)}$$

satisfies Definition 1.1(b); verifying (a) and (c) in Definition 1.1 and (3.7) is a bit tedious, but straightforward. If $\mathscr{C}$ is a cutting configuration, then so is $\mathscr{C}'$. Thus, every Hamiltonian (resp. cutting) configuration is a restriction of a maximal Hamiltonian (resp. cutting) configuration. We also note that if all torus actions ϕ_I are free, then so are the torus actions ϕ'_I.

The open sets U_I in a maximal cutting configuration $\mathscr{C}$ can be inductively shrunk so that the restricted actions in (3.6) are free. Suppose $I^* \in \mathcal{P}^*(N)$, the actions ϕ_I are free for all $I \supsetneq I^*$, and (3.7) holds for all $I \subsetneq I^*$. Since the ϕ_{I^*}-action on $\mu_{I^*}^{-1}(0) = \mu_{I^*;I^*}^{-1}(0)$ is free, it is also free on a ϕ_{I^*}-invariant neighborhood U'_{I^*} of $\mu_{I^*}^{-1}(0)$ in U_{I^*}. By the inductive assumption regarding (3.7),

$$U_{I^*} - U'_{I^*} \subset \bigcup_{\emptyset\neq I\subsetneq I^*} U_I\,.$$

Thus, replacing U_{I^*} with U'_{I^*} accomplishes the inductive step. Proceeding in this way, we obtain a $\mathscr{C}$-invariant open cover $(U'_I)_{I\in\mathcal{P}^*(N)}$ of X refining $(U_I)_{I\in\mathcal{P}^*(N)}$ so that the actions ϕ_I are free on U'_I.

In summary, any N-fold cutting configuration $\mathscr{C}$ for (X,ω) can replaced by a maximal N-fold cutting configuration $\mathscr{C}'$ with free torus actions without any effect on the output of the symplectic cut construction: the symplectic manifolds (X_i,ω_i) with $i\in[N]$, their submanifolds (X_I,ω_I) with $I\in\mathcal{P}_i(N)$, the symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$, or the deformation equivalence class of the fibration $\pi\colon \mathcal{Z}'\longrightarrow\mathbb{C}$.

3.3 Induced degenerations

Suppose $\pi_{\mathfrak{X}}\colon \mathfrak{X}\longrightarrow B$ is a family of manifolds as in Definition 2.7(1), $U_t\subset\mathfrak{X}_t$ is an open subset for each $t\in B$, and Y is a smooth manifold. We call a family $(\phi_t\colon U_t\longrightarrow Y)_{t\in B}$ smooth if

$$\bigcup_{t\in B} U_t \longrightarrow Y, \qquad (t,x)\longrightarrow \phi_t(x),$$

is a smooth map from an open subspace of $\mathfrak{X}$.

Definition 3.1. Let $N\in\mathbb{Z}^+$, B be a manifold, possibly with boundary, and $(\mathfrak{X},\omega_{\mathfrak{X}},\pi_{\mathfrak{X}})$ be a family of symplectic manifolds over B as in Definition 2.7(2). A family of N-fold Hamiltonian (resp. cutting) configurations for $(\mathfrak{X},\omega_{\mathfrak{X}},\pi_{\mathfrak{X}})$ is a tuple

$$(\mathscr{C}_t)_{t\in B} \equiv \big(U_{t;I}, \phi_{t;I}, \mu_{t;I}\big)_{I\in\mathcal{P}^*(N),t\in B} \tag{3.8}$$

such that $\mathscr{C}_t$ is an N-fold Hamiltonian (resp. cutting) configuration for $(\mathfrak{X}_t,\omega_t)$ for each $t\in B$ and the families of maps

$$\Big(\phi_{t;I}\colon (S^1)^I_\bullet\times U_{t;I}\longrightarrow \mathfrak{X}\Big)_{t\in B} \quad\text{and}\quad \Big(\mu_{t;I}\colon U_{t;I}\longrightarrow \mathfrak{t}^*_{I;\bullet}\Big)_{t\in B}$$

are smooth for all $I\in\mathcal{P}^*(N)$.

As described in Section 2.3, a smooth path $(\omega_t)_{t\in B}$ of symplectic forms on a manifold X determines a family $(\mathfrak{X},\omega_{\mathfrak{X}},\pi_{\mathfrak{X}})$ of symplectic manifolds over B. We call a family of N-fold Hamiltonian (resp. cutting) configurations for such a tuple $(\mathfrak{X},\omega_{\mathfrak{X}},\pi_{\mathfrak{X}})$ an $(\omega_t)_{t\in B}$-family of N-fold Hamiltonian (resp. cutting) configurations for X. We call N-fold Hamiltonian (resp. cutting) configurations $\mathscr{C}_0$ for (X,ω_0) and $\mathscr{C}_1$ for (X,ω_1) deformation equivalent if there are a path $(\omega_t)_{t\in\mathbb{I}}$ of symplectic forms on X and an $(\omega_t)_{t\in\mathbb{I}}$-family $(\mathscr{C}_t)_{t\in\mathbb{I}}$ of N-fold Hamiltonian (resp. cutting) configurations for X. For example, the restriction $\mathscr{C}'$ of a Hamiltonian or cutting configuration $\mathscr{C}$ as in (1.3) to a $\mathscr{C}$-invariant open cover $(U'_I)_{I\in\mathcal{P}^*(N)}$ of X refining $(U_I)_{I\in\mathcal{P}^*(N)}$ is deformation equivalent to $\mathscr{C}$. In this case, we can define $U_{t;I}\subset\{t\}\times X$ by

$$\bigcup_{t\in\mathbb{I}}U_{t;I}\equiv\big[0,1/2)\times U_I\cup[0,1]\times U'_I\subset\mathbb{I}\times U_I$$

and take $\phi_{t;I}$ and $\mu_{t;I}$ to be the restrictions of ϕ_I and μ_I, respectively.

Let $\mathscr{C}$ be an N-fold cutting configuration for (X,ω) as in (1.3). For $i\in I\subset[N]$, we define

$$U_{i;I}^{\leqslant}=\big\{x\in U_I:\ (\mu_I(x))_i\leqslant(\mu_I(x))_j\ \forall\,j\in I\big\}; \tag{3.9}$$

in particular, $U_{i;i}^{\leqslant}=U_i$. Since the ϕ_I-action is Hamiltonian, the subset $U_{i;I}^{\leqslant}\subset U_I$ is ϕ_I-invariant. Let

$$U_i^{\leqslant}=\bigcup_{I\in\mathcal{P}_i(N)}U_{i;I}^{\leqslant}, \tag{3.10}$$

$$\partial U_i^{\leqslant}=\big\{x\in U_i^{\leqslant}:\ (\mu_I(x))_i=(\mu_I(x))_j\ \text{for some}\ I\in\mathcal{P}_i(N),\ j\in I-i\ \text{s.t.}\ x\in U_{i;I}^{\leqslant}\big\}.$$

By Lemmas 4.2 and 4.3, the subsets $U_i^{\leqslant},\partial U_i^{\leqslant}\subset X$ are closed. Since the sets $U_{i;I}^{\leqslant}$ with $i\in I$ cover U_I and the sets U_I with $I\in\mathcal{P}^*(N)$ cover X, the collection $(U_i^{\leqslant})_{i\in[N]}$ covers X.

In the $N=2$ case of [11, Section 1.1], a cutting configuration for a symplectic manifold (X,ω) produces two symplectic manifolds, (X_1,ω_1) and (X_2,ω_2), with a common symplectic divisor X_{12}. They are obtained by cutting X into the closed subsets $U_1^{\leqslant}$ and $U_2^{\leqslant}$ along the zero set of the only (non-trivial) moment map μ_{12} and collapsing their boundary

$$\partial U_1^{\leqslant}=\partial U_2^{\leqslant}=U_{12;12}$$

by the single S^1-action ϕ_{12}. We show in Section 4 that this construction extends to an arbitrary N-fold cutting configuration in the sense of Definition 1.2 and produces N symplectic manifolds (X_i,ω_i) with common symplectic divisors X_{ij} which together form an SC symplectic divisor inside of another symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$. The symplectic manifolds (X_i,ω_i) are obtained by cutting X into the closed subspaces $U_i^{\leqslant}$ along the zero sets of the moment maps μ_I and collapsing their boundaries and corners $\partial U_i^{\leqslant}$ by the $(S^1)^I_{\bullet}$-actions ϕ_I.

Theorem 1. *Suppose $N\in\mathbb{Z}^+$, B is a manifold, possibly with boundary, and $(\mathfrak{X},\omega_{\mathfrak{X}},\pi_{\mathfrak{X}})$ is a family of symplectic manifolds over B as in Definition 2.7(2). A family $(\mathscr{C}_t)_{t\in B}$ of (fiberwise) N-fold cutting configurations for $(\mathfrak{X},\omega_{\mathfrak{X}},\pi_{\mathfrak{X}})$ as in (3.8) determines*

(1) a family $(\mathscr{Z},\omega_{\mathscr{Z}},\pi_{\mathscr{Z}})$ of symplectic manifolds over B,

(2) a family $(\pi_I:\mathfrak{X}_I\longrightarrow B,\omega_I)_{I\in\mathcal{P}^*(N)}$ of N-fold SC symplectic configurations over B as in Definition 2.8 such that $\mathfrak{X}_\emptyset\subset\mathscr{Z}$ and $\mathfrak{X}_{t;\emptyset}\subset\mathscr{Z}_t$ is an SC symplectic divisor for every $t\in B$,

(3) a continuous map $q_\emptyset\colon \mathfrak{X}\longrightarrow\mathfrak{X}_\emptyset$ *such that* $q_\emptyset(\partial U_{t;i}^{\leqslant})=\mathfrak{X}_{t;i}\cap\mathfrak{X}_{t;\partial}$ *and*

$$q_\emptyset\colon \big(U_{t;i}^{\leqslant}-\partial U_{t;i}^{\leqslant},\omega_t|_{U_{t;i}^{\leqslant}-\partial U_{t;i}^{\leqslant}}\big)\longrightarrow\big(\mathfrak{X}_{t;i}-\mathfrak{X}_{t;\partial},\omega_{\mathscr{X}}|_{\mathfrak{X}_{t;i}-\mathfrak{X}_{t;\partial}}\big)$$

is a symplectomorphism for every $t\in B$ *and* $i\in[N]$*,*

(4) a family

$$(\mathscr{C}_{\mathscr{Z};t})_{t\in B}\equiv\big(U_{\mathcal{Z};t;I},\phi_{\mathcal{Z};t;I},\mu_{\mathcal{Z};t;I}\big)_{I\in\mathcal{P}^*(N),t\in B}$$

of (fiberwise) N-fold Hamiltonian configurations for $(\mathscr{Z},\omega_{\mathscr{Z}},\pi_{\mathscr{Z}})$ *which restricts to a family of cutting configurations over* $\mathscr{Z}-\mathfrak{X}_\partial$*.*

If $\mathfrak{X}_t$ *is compact, then so is* $\mathfrak{X}_{t;I}$ *for every* $I\in\mathcal{P}^*(N)$*.*

Theorem 2. *Let* N*,* B*,* $(\mathfrak{X},\omega_{\mathfrak{X}},\pi_{\mathfrak{X}})$*,* $(\mathscr{C}_t)_{t\in B}$*,* $(\mathfrak{X}_{t;\emptyset})_{t\in B}$*,* $(\mathscr{Z},\omega_{\mathscr{Z}},\pi_{\mathscr{Z}})$*, and* $(\mathscr{C}_{\mathscr{Z};t})_{t\in B}$ *be as in Theorem 1. The family* $(\mathscr{C}_t)_{t\in B}$ *determines a continuous family* $(\mathfrak{h}_t)_{t\in B}$ *of (fiberwise) germs of deformation equivalence classes of one-parameter families* $(\mathscr{Z}'_t,\omega_{\mathscr{Z}}|_{\mathscr{Z}'_t},\pi_t)$ *of smoothings of* $\mathfrak{X}_{t;\emptyset}$ *with* $\mathscr{Z}'_t\subset\mathscr{Z}_t$*. For every such germ* $\mathfrak{h}_t$*, there are a representative* $\pi_t:\mathscr{Z}'_t\longrightarrow\mathbb{C}$ *and a* $\mathscr{C}_{\mathscr{Z};t}$*-invariant open cover* $(U'_{\mathscr{Z};t;I})_{I\in\mathcal{P}^*(N)}$ *of* $\mathscr{Z}'_t$ *refining* $(U_{\mathscr{Z};t;I})_{I\in\mathcal{P}^*(N)}$ *such that for every* $\lambda\in\mathbb{C}$ *the restriction* $\mathscr{C}'_{\mathscr{Z};t}$ *of* $\mathscr{C}_{\mathscr{Z};t}$ *to* $(U'_{\mathscr{Z};t;I})_{I\in\mathcal{P}^*(N)}$ *restricts to a Hamiltonian configuration* $\mathscr{C}'_{\mathscr{Z};t}|_{\mathscr{Z}_{t;\lambda}}$ *for*

$$\big(\mathscr{Z}_{t;\lambda},\omega_{\mathscr{Z};t;\lambda}\big)\equiv\big(\pi_t^{-1}(\lambda),\omega_{\mathscr{Z}}|_{\pi_t^{-1}(\lambda)}\big);$$

this restriction is a cutting configuration if $\lambda\neq0$*. If* $\mathfrak{X}_t$ *is compact, such a germ can be chosen so that for every* $\lambda\in\mathbb{R}^+$ *sufficiently small* $\mathscr{C}'_{\mathscr{Z};t}|_{\mathscr{Z}_{t;\lambda}}$ *is canonically isomorphic to the restriction of the original cutting configuration* $\mathscr{C}_t$ *for* $(\mathfrak{X}_t,\omega_t)$ *to an open cover* $(U'_{t;I})_{I\in\mathcal{P}^*(N)}$ *of* $\mathfrak{X}_t$ *refining* $(U_{t;I})_{I\in\mathcal{P}^*(N)}$*.*

By the first statement in Theorem 2 and the family analogue of [6, Proposition 5.1], a family $(\mathscr{C}_t)_{t\in B}$ of N-fold cutting configurations determines a homotopy class of trivializations of the complex line bundle (2.10) and thus a continuous family $(\hbar_t)_{t\in B}$ of homotopy classes of trivializations of the normal bundles $\mathcal{O}_{\mathfrak{X}_{t;\partial}}(\mathfrak{X}_{t;\emptyset})$ of the singular locus $\mathfrak{X}_{t;\partial}$ of $\mathfrak{X}_{t;\emptyset}$. Given a single cutting configuration $\mathscr{C}$ (i.e. B is a point in Theorems 1 and 2), we denote by $\mathbf{X}(\mathscr{C})$ the associated N-fold SC symplectic configuration and by $\hbar(\mathscr{C})$ the corresponding homotopy class of trivializations of the line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$.

Corollary 3.2. *Let* $N\in\mathbb{Z}^+$*,* $\mathscr{C}_0$ *be an* N*-fold cutting configuration for* (X,ω_0)*, and* $\mathscr{C}_1$ *be an* N*-fold cutting configuration for* (X,ω_1)*. If* $\mathscr{C}_0$ *and* $\mathscr{C}_1$ *are deformation equivalent, then the* N*-fold SC symplectic configurations* $\mathbf{X}(\mathscr{C}_0)$ *and* $\mathbf{X}(\mathscr{C}_1)$ *lie in the same path component of* $\mathrm{Symp}^+(\mathbf{X})$ *for some* N*-fold transverse configuration* $\mathbf{X}$ *and* $\hbar(\mathscr{C}_0)=\hbar(\mathscr{C}_1)$*.*

This corollary follows immediately from Theorems 1 and 2, the family analogue of [6, Proposition 5.1], and the triviality of families of SC symplectic configurations over $\mathbb{I}$ (see Section 2.3). Corollary 3.2 implies that a deformation equivalence class of cutting configurations determines a deformation equivalence class of SC symplectic varieties $X_\emptyset$ with a homotopy class of trivializations of the normal bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ in (2.9) of the singular locus $X_\partial\subset X_\emptyset$.

4 Proof of Theorem 1

Let $\mathscr{C}$ be an N-fold cutting configuration on (X,ω) as in Definition 1.2. Section 4.1 provides basic topological characterizations of natural spaces determined by $\mathscr{C}$ that appear throughout the rest of this paper. In Section 4.2, we use the symplectic reduction technique of [13, 12] to construct symplectic manifolds $(\mathcal{Z}_I^\circ,\varpi_I^\circ)$ out of the open subsets U_I of X so that

$$\dim_{\mathbb{R}} \mathcal{Z}_I^\circ = \dim_{\mathbb{R}} X + 2\,. \tag{4.1}$$

We glue these global quotients in Section 4.3 into a single symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$. As shown in Section 4.4, $(\mathcal{Z},\omega_{\mathcal{Z}})$ contains closed symplectic submanifolds $(X_I=\mathcal{Z}_I,\omega_I)$ with $I\in\mathcal{P}^*(N)$ so that

$$\mathbf{X}(\mathscr{C}) \equiv \big((X_I)_{I\in\mathcal{P}^*(N)},(\omega_i)_{i\in[N]}\big)$$

is an N-fold SC symplectic configuration in the sense of Definition 2.5 and the corresponding SC symplectic variety $X_\emptyset$ is embedded into $(\mathcal{Z},\omega_{\mathcal{Z}})$ as an SC symplectic divisor. A natural N-fold Hamiltonian configuration $\mathscr{C}_{\mathcal{Z}}$ for $(\mathcal{Z},\omega_{\mathcal{Z}})$, which restricts to a cutting configuration on the complement of the singular locus X_∂ of $X_\emptyset$, is constructed in Section 4.5.

As the constructions of Sections 4.2-4.5 involve no choices, they produce a family

$$\big((\pi_I:\mathfrak{X}_I\longrightarrow B)_{I\in\mathcal{P}^*(N)},(\omega_i)_{i\in[N]}\big)$$

of N-fold SC symplectic configurations over B and a family $(\mathscr{C}_{\mathscr{Z};t})_{t\in B}$ of N-fold Hamiltonian configurations for a family $(\mathscr{Z},\omega_{\mathscr{Z}},\pi_{\mathscr{Z}})$ of symplectic manifolds over B when applied to a family of N-fold cutting configurations. This establishes Theorem 1.

4.1 Topological preliminaries

The topological observations of Lemmas 4.1-4.5 below concern subspaces of X, including $U_{i;I}^{\leqslant}$ and $U_i^{\leqslant}$ defined in (3.9) and (3.10), respectively, as well as subspaces of $X\times\mathbb{C}$. For $I_0\in\mathcal{P}^*(N)$ and $I\in\mathcal{P}_{I_0}(N)$, let

$$\begin{aligned} U_{I_0;I}^{<} = \mu_{I_0;I}^{-1}(0) = \big\{x\in U_I:\, &(\mu_I(x))_i=(\mu_I(x))_j \;\forall\, i,j\in I_0,\\ &(\mu_I(x))_i<(\mu_I(x))_j \;\forall\, i\in I_0,\, j\in I-I_0\big\}. \end{aligned}$$

For $i\in I\subset[N]$, let

$$\mathcal{U}_{i;I}^{\leqslant} = \big\{(x,z)\in U_I\times\mathbb{C}:\; (\mu_I(x))_i-\frac{1}{2}|z|^2\leqslant(\mu_I(x))_j \;\forall\, j\in I\big\}. \tag{4.2}$$

While the components $(\mu_I(x))_i\in\mathbb{R}$ of $\mu_I(x)\in\mathfrak{t}_{I;\bullet}^*$ are not well-defined, the numbers

$$\big(\mu_I(x)\big)_{ij} \equiv \big(\mu_I(x)\big)_j-\big(\mu_I(x)\big)_i \in \mathbb{R} \tag{4.3}$$

are well-defined; thus, so is the condition in (4.2). For $i\in[N]$, let

$$\mathcal{U}_i^{\leqslant} = \bigcup_{I\in\mathcal{P}_i(N)} \mathcal{U}_{i;I}^{\leqslant} \subset X\times\mathbb{C}\,.$$

The proofs of Lemmas 4.1-4.5 make no use of the properties of Definition 1.2 and thus apply to any Hamiltonian configuration $\mathscr{C}$ of Definition 1.1.

Lemma 4.1. *The collection $\{U^{\leqslant}_{i;I}\}_{i\in I\subset[N]}$ covers X. If $\mathscr{C}$ is a maximal cutting configuration, then the collection $\{U^{<}_{I;I}\}_{I\in\mathcal{P}^*(N)}$ covers X.*

Proof. Let $x\in X$ and $I\in\mathcal{P}^*(N)$ be such that $x\in U_I$. The condition $(\mu_I(x))_i\leqslant(\mu_I(x))_j$ defines a reflexive transitive relation $\leqslant$ on I such that either $i\leqslant j$ or $j\geqslant i$ for all $i,j\in I$. Thus, there exists $i\in I$ such that $(\mu_I(x))_i\leqslant(\mu_I(x))_j$ for all $j\in I$ and so $x\in U^{\leqslant}_{i;I}$. Let

$$I_0=\big\{i_0\in I\colon\ (\mu_I(x))_i=(\mu_I(x))_{i_0}\big\}\ni i.$$

By the choice of $i\in I$, $(\mu_I(x))_i<(\mu_I(x))_j$ for all $j\in I-I_0$ and so

$$x\in\big\{x'\in U_I\colon\ (\mu_I(x))_{i_0}<(\mu_I(x))_j\ \forall\, i_0\in I_0,\ j\in I-I_0\big\}.$$

If $\mathscr{C}$ is maximal, (3.7) with (I,J) replaced by (I_0,I) then implies that $x\in U^{<}_{I_0;I_0}$. □

Lemma 4.2. *For every $i\in[N]$, the subspace $U^{\leqslant}_i\subset X$ is closed.*

Proof. Let $(x_k)_{k=1}^{\infty}$ be a sequence in $U^{\leqslant}_i$ converging to some $x\in X$. Since X is first countable, it is sufficient to show that $x\in U^{\leqslant}_i$. By Definition 1.1(a), for each $k\in\mathbb{Z}^+$ there is a unique maximal $I_k\in\mathcal{P}_i(N)$ such that $x_k\in U^{\leqslant}_{i;I_k}$. Passing to a subsequence, we can assume that $I_k\equiv I$ for all $k\in\mathbb{Z}^+$ and for some fixed $I\in\mathcal{P}_i(N)$. Thus,

$$\big(\mu_I(x_k)\big)_{ij}\geqslant 0\qquad\forall\,k\in\mathbb{Z}^+,\ j\in I. \tag{4.4}$$

Let J be the maximal element of $\mathcal{P}^*(N)$ such that $x\in U_J$. Since U_J is open, $x_k\in U_I\cap U_J$ for all $k\in\mathbb{Z}^+$ sufficiently large. By Definition 1.1(a), either $I\subsetneq J$ or $J\subset I$.

Suppose first $I\subsetneq J$. By (b) and (c) in Definition 1.1 and (4.4),

$$\big(\mu_J(x_k)\big)_{ij}=\big(\mu_I(x_k)\big)_{ij}\geqslant 0\quad\forall\,j\in I,\qquad\big(\mu_J(x_k)\big)_{ij}>0\quad\forall\,j\in J-I \tag{4.5}$$

for all $k\in\mathbb{Z}^+$ sufficiently large. Thus, $x_k\in U^{\leqslant}_{i;J}$; this contradicts the maximality assumption on I above.

Suppose $J\subset I$ instead. If $i\notin J$, then $i\in I-J$ and

$$\big(\mu_I(x_k)\big)_j<\big(\mu_I(x_k)\big)_i\qquad\forall\ j\in J$$

for all k sufficiently large by Definition 1.1(c) with the roles of (i,I) and (j,J) interchanged; this contradicts (4.4). By Definition 1.1(b) and (4.4),

$$\big(\mu_J(x_k)\big)_{ij}=\big(\mu_I(x_k)\big)_{ij}\geqslant 0\quad\forall\,x_k\in U_I\cap U_J,\ j\in J\,.$$

By the continuity of the functions $(\mu_J(\cdot))_{ij}$, this implies that

$$\big(\mu_J(x)\big)_{ij}=\lim_{k\longrightarrow\infty}\big(\mu_J(x_k)\big)_{ij}\geqslant 0\qquad\forall\ j\in J.$$

We conclude that $x\in U^{\leqslant}_{i;J}\subset U^{\leqslant}_i$. □

For $I_0\in\mathcal{P}^*(N)$ and $I\in\mathcal{P}_{I_0}(N)$, define

$$U^{\leqslant}_{I_0;I}=\big\{x\in U_I\colon (\mu_I(x))_i\leqslant(\mu_I(x))_j\ \forall\, i\in I_0,\, j\in I\big\}. \tag{4.6}$$

By (3.9),

$$U^{\leqslant}_{I_0;I}=\big\{x\in U^{\leqslant}_{i;I}\colon (\mu_I(x))_i=(\mu_I(x))_j\ \forall\, j\in I_0\big\}$$

for any $i\in I_0$.

Lemma 4.3. *For every $I_0\in\mathcal{P}^*(N)$, the subspace*

$$U^{\leqslant}_{I_0}\equiv\bigcup_{I\in\mathcal{P}_{I_0}(N)}U^{\leqslant}_{I_0;I}\subset X$$

is closed in X. For all $I_0,J_0\in\mathcal{P}^*(N)$, $U^{\leqslant}_{I_0}\cap U^{\leqslant}_{J_0}=U^{\leqslant}_{I_0\cup J_0}$.

Proof. Let $i\in I_0$. Since

$$U^{\leqslant}_{I_0}=\mathcal{U}^{\leqslant}_{i,I_0}\cap\big(X\times\{0\}\big)\qquad\text{and}\qquad U^{\leqslant}_{i}=\mathcal{U}^{\leqslant}_{i}\cap\big(X\times\{0\}\big),$$

the first claim follows from Lemmas 4.2 and 4.5.

It is immediate that $U^{\leqslant}_{I_0},U^{\leqslant}_{J_0}\supset U^{\leqslant}_{I_0\cup J_0}$. Suppose $x\in U^{\leqslant}_{I_0;I}\cap U^{\leqslant}_{J_0;J}$. Thus, $I_0\subset I$, $J_0\subset J$,

$$\big(\mu_I(x)\big)_i\leqslant\big(\mu_I(x)\big)_j\ \ \forall\, i\in I_0,\, j\in I,\quad \big(\mu_J(x)\big)_i\leqslant\big(\mu_J(x)\big)_j\ \ \forall\, i\in J_0,\, j\in J, \tag{4.7}$$

and either $I\subset J$ or $I\supset J$ by Definition 1.1(a). We can assume that $I\subset J$. By (b) and (c) in Definition 1.1 and the first statement in (4.7),

$$\big(\mu_J(x)\big)_i\leqslant\big(\mu_J(x)\big)_j\ \ \forall\, i\in I_0,\, j\in J.$$

Combining this with the second statement in (4.7), we obtain

$$\big(\mu_J(x)\big)_i\leqslant\big(\mu_J(x)\big)_j\ \ \forall\, i\in I_0\cup J_0,\, j\in J.$$

Thus, $x\in U^{\leqslant}_{I_0\cup J_0;J}\subset U^{\leqslant}_{I_0\cup J_0}$. □

Lemma 4.4. *For all $i\in I\subset[N]$, the subspace $\mathcal{U}^{\leqslant}_{i;I}\subset\mathcal{U}^{\leqslant}_{i}$ is open in $\mathcal{U}^{\leqslant}_{i}$ and*

$$\mathcal{U}^{\leqslant}_{i}\cap\big(U_I\times\mathbb{C}\big)=\mathcal{U}^{\leqslant}_{i;I}\,.$$

Proof. The first claim follows from the second. Suppose

$$(x,z)\in\mathcal{U}^{\leqslant}_{i;J}\cap\big(U_I\times\mathbb{C}\big)$$

for some $J\in\mathcal{P}_i(N)$ and thus

$$\big(\mu_J(x)\big)_{ij}+\frac{1}{2}|z|^2\geqslant 0\qquad\forall\, j\in J. \tag{4.8}$$

By Definition 1.1(a), either $I\subset J$ or $J\subset I$. If $I\subset J$, Definition 1.1(b) implies that (4.8) holds with J replaced by I and so $(x,z)\in\mathcal{U}^{\leqslant}_{i;I}$. If $J\subset I$, (b) and (c) in Definition 1.1 imply that (4.5) holds for $J=I_0$ and $x_k=x$. From this, we again conclude that (4.8) holds with I and by J interchanged and so $(x,z)\in\mathcal{U}^{\leqslant}_{i;I}$. This establishes the second claim of the lemma. □

For all $i\in I\subset[N]$ and $I_0\subset I$, define

$$\mathcal{U}_{i,I_0;I}=\big\{(x,z)\in U_I\times\mathbb{C}\colon\ (\mu_I(x))_i-\frac{1}{2}|z|^2=(\mu_I(x))_j\ \forall\,j\in I_0\big\},\quad \mathcal{U}_{i,I_0;I}^{\leqslant}=\mathcal{U}_{i,I_0;I}\cap\mathcal{U}_{i;I}^{\leqslant}\,. \tag{4.9}$$

Lemma 4.5. *For all $i\in[N]$ and $I_0\subset[N]$, the subspace*

$$\mathcal{U}_{i,I_0}^{\leqslant}\equiv\bigcup_{I\in\mathcal{P}_i(N)\cap\mathcal{P}_{I_0}(N)}\mathcal{U}_{i,I_0;I}^{\leqslant}\subset\mathcal{U}_i^{\leqslant}$$

is closed.

Proof. Let $(x_k,z_k)_{k=1}^{\infty}$ be a sequence in $\mathcal{U}_{i,I_0}^{\leqslant}$ converging to some $(x,z)\in\mathcal{U}_i^{\leqslant}$. Similarly to the proof of Lemma 4.2, we can assume $(x_k,z_k)_{k=1}^{\infty}\in\mathcal{U}_{i,I_0;I}^{\leqslant}$ for all $k\in\mathbb{Z}^+$ and for some fixed $I\in\mathcal{P}_i(N)\cap\mathcal{P}_{I_0}(N)$. Thus,

$$\big(\mu_I(x_k)\big)_{ij}+\frac{1}{2}|z_k|^2\geqslant 0\quad\forall\,j\in I,\qquad \big(\mu_I(x_k)\big)_{ij}+\frac{1}{2}|z_k|^2=0\quad\forall\,j\in I_0. \tag{4.10}$$

We also assume that $I\in\mathcal{P}_i(N)\cap\mathcal{P}_{I_0}(N)$ is the maximal element with this property.

Let J be the maximal element of $\mathcal{P}_i(N)$ such that $x\in U_J$. By the same reasoning as in the second paragraph of the proof of Lemma 4.2, $J\subset I$. Thus, $i\in J\subset I$ and $x_k\in U_J$ for all k sufficiently large. By Definition 1.1(b) and the first statement in (4.10),

$$\big(\mu_J(x_k)\big)_{ij}+\frac{1}{2}|z_k|^2=\big(\mu_I(x_k)\big)_{ij}+\frac{1}{2}|z_k|^2\geqslant 0\quad\forall\,k\in\mathbb{Z}^+,\ j\in J.$$

By the continuity of the functions $(\mu_J(\cdot))_{ij}$, this statement implies that $(x,z)\in\mathcal{U}_{i;J}^{\leqslant}$. If $I_0\not\subset J$, then

$$(\mu_I(x_k))_i<(\mu_I(x_k))_{i_0}\qquad\forall\ i_0\in I_0-J\subset I-J$$

for all k sufficiently large by Definition 1.1(c); this contradicts the second statement in (4.10). Thus, $I_0\subset J$ and

$$\big(\mu_J(x_k)\big)_{ij}+\frac{1}{2}|z_k|^2=\big(\mu_I(x_k)\big)_{ij}+\frac{1}{2}|z_k|^2=0\quad\forall\,k\in\mathbb{Z}^+,\ j\in I_0.$$

By the continuity of the functions $(\mu_J(\cdot))_{ij}$, this statement implies that $(x,z)\in\mathcal{U}_{i,I_0;J}^{\leqslant}\subset\mathcal{U}_{i,I_0}^{\leqslant}$. □

4.2 Symplectic reduction

For $I_0\subset I\subset[N]$ and $F=\mathbb{R},\mathbb{C}$, we identify

$$F_{I_0}^N\equiv\big\{(z_i)_{i\in[N]}\in F^N\colon z_i=0\ \forall\,i\in I_0\big\}\quad\text{and}\quad F_{I_0}^I\equiv\big\{(z_i)_{i\in I}\in F^I\colon z_i=0\ \forall\,i\in I_0\big\}$$

with $F^{[N]-I_0}$ and F^{I-I_0}, respectively. For $i\in I\subset[N]$, let

$$\mathbb{C}_i^N=\mathbb{C}_{\{i\}}^N\qquad\text{and}\qquad\mathbb{C}_i^I=\mathbb{C}_{\{i\}}^I.$$

We denote by $\omega_{\mathbb{C}^I}$ and $\omega_{\mathbb{C}_{I_0}^I}$ the standard symplectic forms on $\mathbb{C}^I$ and $\mathbb{C}_{I_0}^I$, respectively. Thus,

$$\omega_{\mathbb{C}^{I-I_0}}=\omega_{\mathbb{C}^I}\big|_{\mathbb{C}_{I_0}^I}=\omega_{\mathbb{C}^N}\big|_{\mathbb{C}^{I-I_0}}$$

under the above identifications.

The standard $(S^1)^N$-action on $\mathbb{C}^N$ and a moment map for this action are given by

$$\begin{aligned}\phi_{\mathbb{C}^N}\colon (S^1)^N\times\mathbb{C}^N &\longrightarrow \mathbb{C}^N, & \phi_{\mathbb{C}^N}\big((\mathrm{e}^{\mathrm{i}\theta_i})_{i\in[N]};(z_i)_{i\in[N]}\big) &= \big(\mathrm{e}^{\mathrm{i}\theta_i}z_i\big)_{i\in[N]},\\ \mu_{\mathbb{C}^N}\colon \mathbb{C}^N &\longrightarrow \mathbb{R}^N, & \mu_{\mathbb{C}^N}\big((z_i)_{i\in[N]}\big) &= \frac{1}{2}\big(|z_i|^2\big)_{i\in[N]}.\end{aligned} \tag{4.11}$$

For $I\in\mathcal{P}^*(N)$, let

$$\phi_{\mathbb{C}^I;\bullet}\colon (S^1)^I_\bullet\times\mathbb{C}^I\longrightarrow\mathbb{C}^I \qquad\text{and}\qquad \mu_{\mathbb{C}^I;\bullet}\colon\mathbb{C}^I\longrightarrow\mathfrak{t}^*_{I;\bullet}$$

be the restriction of this action and the induced moment map, respectively.

With notation as in (1.3), let

$$\pi_1,\pi_2\colon U_I\times\mathbb{C}^I\longrightarrow U_I,\mathbb{C}^I$$

be the projection maps and

$$\widetilde{\omega}_I=\pi_1^*\omega+\pi_2^*\omega_{\mathbb{C}^I}$$

be the product symplectic form. We lift (ϕ_I,μ_I) to a Hamiltonian $(S^1)^I_\bullet$-pair for $(U_I\times\mathbb{C}^I,\widetilde{\omega}_I)$ by

$$\widetilde{\phi}_I\colon (S^1)^I_\bullet\times\big(U_I\times\mathbb{C}^I\big)\longrightarrow U_I\times\mathbb{C}^I, \qquad \widetilde{\phi}_I(g;x,z)=\big(\phi_I(g;x),\phi_{\mathbb{C}^I;\bullet}(g^{-1};z)\big), \tag{4.12}$$

$$\widetilde{\mu}_I\colon U_I\times\mathbb{C}^I\longrightarrow\mathfrak{t}^*_{I;\bullet}, \qquad \widetilde{\mu}_I(x,z)=\mu_I(x)-\mu_{\mathbb{C}^I;\bullet}(z). \tag{4.13}$$

The action $\widetilde{\phi}_I$ then preserves the symplectic submanifolds $U\times\mathbb{C}^I_{I_0}\subset U\times\mathbb{C}^I$ with $I_0\subset I$. Let

$$\widetilde{\mathcal{Z}}^\circ_I\equiv\widetilde{\mu}_I^{-1}(0)=\big\{\big(x,(z_i)_{i\in I}\big)\in U_I\times\mathbb{C}^I\colon (\mu_I(x))_i-\frac{1}{2}|z_i|^2=(\mu_I(x))_j-\frac{1}{2}|z_j|^2\ \forall\,i,j\in I\big\}. \tag{4.14}$$

Similarly to the situation at the beginning of Section 4.1, the condition on the elements of $U_I\times\mathbb{C}^I$ above is independent of the choice of representative for $\mu_I(x)$ in $\mathbb{R}^I$ and is thus well-defined.

Lemma 4.6. *Let $\mathscr{C}$ be a cutting configuration as in (1.3). For all $I\in\mathcal{P}^*(N)$ and $I_0\subset I$, $0\in\mathfrak{t}^*_{I;\bullet}$ is a regular value of the restriction of $\widetilde{\mu}_I$ to $U_I\times\mathbb{C}^I_{I_0}$. For all $I\in\mathcal{P}^*(N)$, the restriction of the $\widetilde{\phi}_I$-action to $\widetilde{\mathcal{Z}}^\circ_I$ is free.*

Proof. Since $(\widetilde{\phi}_I,\widetilde{\mu}_I)$ restricts to a Hamiltonian pair on $U_I\times\mathbb{C}^I_{I_0}$, the first claim is implied by the second; see [3, Section 23.2.1]. Let $\widetilde{x}=(x,(z_i)_{i\in I})$ be an element of $U_I\times\mathbb{C}^I$ such that $z_i=0$ if and only if $i\in I_0$. If

$$\widetilde{\phi}_I\big((\mathrm{e}^{\mathrm{i}\theta_i})_{i\in I};x,(z_i)_{i\in I}\big)=\big(x,(z_i)_{i\in I}\big),$$

then $\mathrm{e}^{\mathrm{i}\theta_i}=1$ for every $i\in I-I_0$. Thus,

$$\big(\mathrm{e}^{\mathrm{i}\theta_i}\big)_{i\in[N]}\in(S^1)^{I_0}_\bullet, \qquad \phi_I\big((\mathrm{e}^{\mathrm{i}\theta_i})_{i\in I};x\big)=x. \tag{4.15}$$

If $|I_0|\leqslant 1$, $\mathrm{e}^{\mathrm{i}\theta_i}=1$ for all $i\in[N]$ by the first statement in (4.15). If $I_0\neq\emptyset$, then $x\in\mu^{-1}_{I_0;I}(0)$ by (4.14) and (1.4). By (4.15) and the assumption of Definition 1.2, this implies that $\mathrm{e}^{\mathrm{i}\theta_i}=1$ for all $i\in[N]$ and establishes the second claim. □

For $I \in \mathcal{P}^*(N)$, define

$$\mathcal{Z}_I^\circ = \widetilde{\mathcal{Z}}_I^\circ/(S^1)^I_\bullet\,. \tag{4.16}$$

For example, $\mathcal{Z}_{\{i\}}^\circ = U_{\{i\}} \times \mathbb{C}$. Let

$$q_{\mathcal{Z};I} \colon \widetilde{\mathcal{Z}}_I^\circ \longrightarrow \mathcal{Z}_I^\circ$$

be the projection map. By Lemma 4.6 and the Symplectic Reduction Theorem [3, Theorem 23.1], $\widetilde{\mathcal{Z}}_I^\circ$ is a smooth submanifold of $U_I \times \mathbb{C}^I$, $\mathcal{Z}_I^\circ$ is a smooth manifold, and there is a unique symplectic form ϖ_I on $\mathcal{Z}_I^\circ$ such that

$$q_{\mathcal{Z};I}^*\varpi_I = \widetilde{\omega}_I\big|_{\widetilde{\mathcal{Z}}_I^\circ}\,. \tag{4.17}$$

Thus, $(\mathcal{Z}_I^\circ, \varpi_I)$ is a symplectic manifold satisfying (4.1).

For $I \in \mathcal{P}^*(N)$ and $I_0 \subset I$, the subspace

$$\widetilde{\mathcal{Z}}_{I_0;I}^\circ \equiv \widetilde{\mathcal{Z}}_I^\circ \cap \big(U_I \times \mathbb{C}^I_{I_0}\big) \subset U_I \times \mathbb{C}^I$$

is preserved by the $\widetilde{\phi}_I$-action. Let

$$\mathcal{Z}_{I_0;I}^\circ = \widetilde{\mathcal{Z}}_{I_0;I}^\circ/(S^1)^I_\bullet = q_{\mathcal{Z};I}\big(\widetilde{\mathcal{Z}}_{I_0;I}^\circ\big) \subset \mathcal{Z}_I^\circ, \quad q_{\mathcal{Z};I_0;I} = q_{\mathcal{Z};I}|_{\widetilde{\mathcal{Z}}_{I_0;I}^\circ} \colon \widetilde{\mathcal{Z}}_{I_0;I}^\circ \longrightarrow \mathcal{Z}_{I_0;I}^\circ\,. \tag{4.18}$$

By Lemma 4.6 and the Symplectic Reduction Theorem, $\widetilde{\mathcal{Z}}_{I_0;I}^\circ$ is a smooth submanifold of $U_I \times \mathbb{C}^I_{I_0}$, $\mathcal{Z}_{I_0;I}^\circ$ is a smooth manifold, and there is a unique symplectic form $\varpi_{I_0;I}$ on $\mathcal{Z}_{I_0;I}^\circ$ such that

$$q_{\mathcal{Z};I_0;I}^*\varpi_{I_0;I} = \widetilde{\omega}_I\big|_{\widetilde{\mathcal{Z}}_{I_0;I}^\circ}\,. \tag{4.19}$$

Thus, $(\mathcal{Z}_{I_0;I}^\circ, \varpi_{I_0;I})$ is a symplectic manifold of real dimension $\dim_{\mathbb{R}} X + 2 - 2|I_0|$.

For $I \in \mathcal{P}^*(N)$ and $I_0' \subset I_0 \subset I$, let

$$\widetilde{\pi}_{I_0;I_0';I}^\circ \colon \widetilde{\mathcal{N}}_{I_0;I_0';I}^\circ = \widetilde{\mathcal{Z}}_{I_0;I}^\circ \times \mathbb{C}^{I_0}_{I_0'} \longrightarrow \widetilde{\mathcal{Z}}_{I_0;I}^\circ, \tag{4.20}$$

$$\pi_{I_0;I_0';I}^\circ \colon \mathcal{N}_{I_0;I_0';I}^\circ = \widetilde{\mathcal{N}}_{I_0;I_0';I}^\circ/(S^1)^I_\bullet \longrightarrow \widetilde{\mathcal{Z}}_{I_0;I}^\circ/(S^1)^I_\bullet = \mathcal{Z}_{I_0;I}^\circ\,, \tag{4.21}$$

with the quotients taken by the restriction of the $(S^1)^I_\bullet$-action $\widetilde{\phi}_I$ to

$$\widetilde{\mathcal{Z}}_{I_0;I}^\circ \times \mathbb{C}^{I_0}_{I_0'} \subset U_I \times \mathbb{C}^I\,.$$

Since the $\widetilde{\phi}_I$-action on $\widetilde{\mathcal{Z}}_{I_0;I}^\circ$ is free, (4.21) is a complex vector bundle.

Lemma 4.7. *Suppose $\mathscr{C}$ is a cutting configuration as in (1.3) and $I \in \mathcal{P}^*(N)$. For all $I_0' \subset I_0 \subset I$, $(\mathcal{Z}_{I_0;I}^\circ, \varpi_{I_0;I})$ is a symplectic submanifold of $(\mathcal{Z}_{I_0';I}^\circ, \varpi_{I_0';I})$ with the oriented normal bundle canonically isomorphic to (4.21).*

Proof. By (4.17)-(4.19),

$$q_{\mathcal{Z};I_0;I}^*\big(\varpi_I|_{\mathcal{Z}_{I_0;I}^\circ}\big) = \big(q_{\mathcal{Z};I}^*\varpi_I\big)\big|_{\widetilde{\mathcal{Z}}_{I_0;I}^\circ} = \widetilde{\omega}_I\big|_{\widetilde{\mathcal{Z}}_{I_0;I}^\circ} = q_{\mathcal{Z};I_0;I}^*\varpi_{I_0;I}\,. \tag{4.22}$$

By the uniqueness part of [3, Theorem 23.1] and (4.22), $\varpi_{I_0;I} = \varpi_I|_{\mathcal{Z}_{I_0;I}^\circ}$. Thus, $(\mathcal{Z}_{I_0;I}^\circ, \varpi_{I_0;I})$ is a symplectic submanifold of $(\mathcal{Z}_I^\circ, \varpi_I)$. Since $\mathcal{Z}_{I_0;I}^\circ \subset \mathcal{Z}_{I_0';I}^\circ$ whenever $I_0 \supset I_0'$, it follows that

$(\mathcal{Z}_{I_0;I}^\circ, \varpi_{I_0;I})$ is a symplectic submanifold of $(\mathcal{Z}_{I_0';I}^\circ, \varpi_{I_0';I})$.

For all $I_0'\subset I_0\subset I$, $\widetilde{\mathcal{Z}}_{I_0';I}^\circ\subset U_I\times\mathbb{C}^I_{I_0'}$. The projection $\mathbb{C}^I_{I_0'}\longrightarrow\mathbb{C}^{I_0}_{I_0'}$ induces a vector bundle homomorphism

$$\mathcal{N}_{\widetilde{\mathcal{Z}}_{I_0';I}^\circ}\widetilde{\mathcal{Z}}_{I_0;I}^\circ \equiv \frac{T\widetilde{\mathcal{Z}}_{I_0';I}^\circ|_{\widetilde{\mathcal{Z}}_{I_0;I}^\circ}}{T\widetilde{\mathcal{Z}}_{I_0;I}^\circ} \longrightarrow \widetilde{\mathcal{Z}}_{I_0;I}^\circ\times\mathbb{C}^{I_0}_{I_0'} \equiv \widetilde{\mathcal{N}}_{I_0;I_0';I}^\circ\,. \tag{4.23}$$

By Lemma 4.6, the homomorphism (4.23) is surjective and thus an isomorphism for dimensional reasons. Since it is $\widetilde{\phi}_I$-equivariant, it descends to a vector bundle isomorphism

$$\mathcal{N}_{\mathcal{Z}_{I_0';I}^\circ}\mathcal{Z}_{I_0;I}^\circ \equiv \frac{T\mathcal{Z}_{I_0';I}^\circ|_{\mathcal{Z}_{I_0;I}^\circ}}{T\mathcal{Z}_{I_0;I}^\circ} \longrightarrow \mathcal{N}_{I_0;I_0';I}^\circ \tag{4.24}$$

over $\mathcal{Z}_{I_0;I}^\circ$. The fiberwise symplectic form on the left-hand side of (4.24) corresponds to the symplectic form on the right-hand side of (4.24) induced by the standard symplectic form on $\mathbb{C}^{I_0}_{I_0'}$; the latter is compatible with the complex orientation of $\mathbb{C}^{I_0}_{I_0'}$. Thus, the isomorphism (4.24) is orientation-preserving. $\square$

4.3 The ambient space

We will glue the symplectic manifolds $(\mathcal{Z}_I^\circ, \varpi_I)$ with $I\in\mathcal{P}^*(N)$ into a single symplectic manifold $(\mathcal{Z}, \omega_{\mathcal{Z}})$. For $i\in I, J\subset[N]$, define

$$\varrho_{i;J,I}\colon (S^1)^I_\bullet \longrightarrow (S^1)^J_\bullet, \quad \big(\varrho_{i;J,I}\big((\mathrm{e}^{\mathrm{i}\theta_k})_{k\in I}\big)\big)_l = \begin{cases} \mathrm{e}^{\mathrm{i}\theta_i}\prod\limits_{k\in I-J}\mathrm{e}^{\mathrm{i}\theta_k}, & \text{if } l=i;\\ \mathrm{e}^{\mathrm{i}\theta_l}, & \text{if } l\in I\cap J-i;\\ 1, & \text{if } l\in J-I.\end{cases} \tag{4.25}$$

For $I, J\in\mathcal{P}^*(N)$, let

$$\widetilde{\mathcal{Z}}_{I,J}^\circ = \widetilde{\mathcal{Z}}_I^\circ \cap \big((U_I\cap U_J)\times\mathbb{C}^I\big)\,. \tag{4.26}$$

By Definition 1.1(*a*), $\widetilde{\mathcal{Z}}_{I,J}^\circ=\emptyset$ unless $I\subset J$ or $I\supset J$. By Definition 1.1(*c*) and (4.14),

$$\big(\mu_J(x)\big)_{ij}\equiv\big(\mu_J(x)\big)_j-\big(\mu_J(x)\big)_i > 0 \qquad \forall\ \big(x,(z_k)_{k\in I}\big)\in\widetilde{\mathcal{Z}}_{I,J}^\circ,\ i\in I\subset J,\ j\in J-I; \tag{4.27}$$

$$z_j \neq 0 \qquad \forall\ \big(x,(z_k)_{k\in I}\big)\in\widetilde{\mathcal{Z}}_{I,J}^\circ,\ \emptyset\neq J\subset I,\ j\in I-J. \tag{4.28}$$

If $i\in J\subset I\subset[N]$, define

$$\varphi_{I,J;i}\colon \widetilde{\mathcal{Z}}_{I,J}^\circ \longrightarrow (S^1)^I_\bullet, \quad \big(\varphi_{I,J;i}\big(x,(z_k)_{k\in I}\big)\big)_l = \begin{cases} \prod\limits_{k\in I-J}\frac{\overline{z_k}}{|z_k|}, & \text{if } l=i;\\ 1, & \text{if } l\in J-i;\\ \frac{z_l}{|z_l|}, & \text{if } l\in I-J.\end{cases}$$

If $J\subset I\subset[N]$ and $i,j\in J$ are distinct, define

$$\varphi_{I,J;i,j}\colon \widetilde{\mathcal{Z}}_{I,J}^\circ \longrightarrow (S^1)^J_\bullet, \quad \big(\varphi_{I,J;i,j}\big(x,(z_k)_{k\in I}\big)\big)_l = \begin{cases} \prod\limits_{k\in I-J}\frac{\overline{z_k}}{|z_k|}, & \text{if } l=i;\\ \prod\limits_{k\in I-J}\frac{z_k}{|z_k|}, & \text{if } l=j;\\ 1, & \text{if } l\in J-\{i,j\}.\end{cases}$$

By (4.28), both maps are well-defined ($z_k \neq 0$ for all $k \in I - J$). We take $\varphi_{I,J;i,i}$ to be the map taking $\widetilde{\mathcal{Z}}^\circ_{I,J}$ to the identity in $(S^1)^J_\bullet$.

For $i \in I$, $J \subset [N]$, we define

$$\widetilde{\Theta}_{J,I;i}\colon \widetilde{\mathcal{Z}}^\circ_{I,J} \longrightarrow \widetilde{\mathcal{Z}}^\circ_{J,I},$$
$$\widetilde{\Theta}_{J,I;i}\big(x,(z_k)_{k\in I}\big) = \begin{cases} \big(x,(z_k)_{k\in I}, \big(\sqrt{2(\mu_J(x))_{ik} + |z_i|^2}\big)_{k\in J-I}\big), & \text{if } I \subset J; \\ \widetilde{\phi}_I\big(\varphi_{I,J;i}\big(x,(z_k)_{k\in I}\big); x,(z_k)_{k\in J}\big), & \text{if } I \supset J. \end{cases} \tag{4.29}$$

By (4.14) and (4.27), this map is well-defined in either case. It is $(\widetilde{\phi}_J, \widetilde{\phi}_I)$-equivariant with respect to the homomorphism (4.25) and

$$\widetilde{\Theta}^*_{J,I;i}\big(\widetilde{\omega}_J|_{\widetilde{\mathcal{Z}}^\circ_{J,I}}\big) = \widetilde{\omega}_I|_{\widetilde{\mathcal{Z}}^\circ_{I,J}}. \tag{4.30}$$

Furthermore,

$$\begin{aligned} \widetilde{\Theta}_{I,J;i}\big(\widetilde{\Theta}_{J,I;i}(x,z)\big) &= \begin{cases} (x,z), & \text{if } I \subset J; \\ \widetilde{\phi}_I\big(\varphi_{I,J;i}(x,z); x,z\big), & \text{if } I \supset J; \end{cases} \\ \widetilde{\Theta}_{K,I;i}|_{\widetilde{\mathcal{Z}}^\circ_{I,J}\cap\widetilde{\mathcal{Z}}^\circ_{I,K}} &= \widetilde{\Theta}_{K,J;i}\circ\widetilde{\Theta}_{J,I;i}|_{\widetilde{\mathcal{Z}}^\circ_{I,J}\cap\widetilde{\mathcal{Z}}^\circ_{I,K}} \quad \forall\, i \in I \subset J \subset K \subset [N]. \end{aligned} \tag{4.31}$$

For $i, j \in I, J$,

$$\widetilde{\Theta}_{J,I;i}(x,z) = \begin{cases} \widetilde{\Theta}_{J,I;j}(x,z), & \text{if } I \subset J; \\ \widetilde{\phi}_J\big(\varphi_{I,J;i,j}(x,z); \widetilde{\Theta}_{J,I;j}(x,z)\big), & \text{if } I \supset J; \end{cases} \tag{4.32}$$

the claim in the first case above follows from (4.14).

For $I, J \in \mathcal{P}^*(N)$, let

$$\mathcal{Z}^\circ_{I,J} = q_{\mathcal{Z};I}\big(\widetilde{\mathcal{Z}}^\circ_{I,J}\big) \subset \mathcal{Z}^\circ_I. \tag{4.33}$$

By the $(\widetilde{\phi}_J, \widetilde{\phi}_I)$-equivariance of the smooth map $\widetilde{\Theta}_{J,I;i}$ with $i \in I, J$, it descends to a smooth map

$$\Theta_{J,I}\colon \mathcal{Z}^\circ_{I,J} \longrightarrow \mathcal{Z}^\circ_{J,I}\,. \tag{4.34}$$

By (4.32), $\Theta_{J,I}$ is independent of the choice of $i \in I, J$. By (4.31),

$$\Theta_{K,I}|_{\mathcal{Z}^\circ_{I,J}\cap\mathcal{Z}^\circ_{I,K}} = \Theta_{K,J}\circ\Theta_{J,I}|_{\mathcal{Z}^\circ_{I,J}\cap\mathcal{Z}^\circ_{I,K}} \tag{4.35}$$

whenever $I \subset J \subset K$ or some permutation of this relation holds.

We define

$$\mathcal{Z} = \Big(\bigsqcup_{I\in\mathcal{P}^*(N)} \mathcal{Z}^\circ_I\Big)\Big/\!\sim, \qquad \mathcal{Z}^\circ_{I,J} \ni x \sim \Theta_{J,I}(x) \in \mathcal{Z}^\circ_{J,I} \quad \forall\, I, J \in \mathcal{P}^*(N). \tag{4.36}$$

For $i \in [N]$, let

$$\mathcal{Z}^\star_i = \Big(\bigsqcup_{I\in\mathcal{P}_i(N)} \mathcal{Z}^\circ_I\Big)\Big/\!\sim, \qquad \mathcal{Z}^\circ_{I,J} \ni x \sim \Theta_{J,I}(x) \in \mathcal{Z}^\circ_{J,I} \quad \forall\, I, J \in \mathcal{P}_i(N). \tag{4.37}$$

Lemma 4.8. *The quotient projection map*

$$q_{\mathcal{Z}}\colon \bigsqcup_{I\in\mathcal{P}^*(N)}\mathcal{Z}_I^\circ \longrightarrow \mathcal{Z} \tag{4.38}$$

is open and its restriction to each subspace $\mathcal{Z}_I^\circ$ is injective. For each $i\in[N]$, $\mathcal{Z}_i^\star$ is an open subspace of $\mathcal{Z}$.

Proof. By (4.35), the relations $\sim$ in (4.36) and (4.37) are equivalence relations and thus each $\mathcal{Z}_i^\star$ is a subspace of $\mathcal{Z}$. Since $\Theta_{I,I}$ is the identity on $\mathcal{Z}_{I,I}^\circ=\mathcal{Z}_I^\circ$, (4.36) implies that the map (4.38) is injective on each subspace $\mathcal{Z}_I^\circ$.

Since the maps (4.34) are homeomorphisms between open subsets of the domain of (4.38), the latter map is open. Since the preimage of $\mathcal{Z}_i^\star$ under the restriction of (4.38) to each $\mathcal{Z}_I^\circ$ is the union of the open subsets $\mathcal{Z}_{I,J}^\circ\subset\mathcal{Z}_I^\circ$ with $J\in\mathcal{P}_i(N)\cap\mathcal{P}_I(N)$, $\mathcal{Z}_i^\star\subset\mathcal{Z}$ is open. □

For $I,J\in\mathcal{P}^*(N)$ and $I_0\subset I,J$, the overlap map (4.34) takes $\mathcal{Z}_{I_0;I}^\circ\cap\mathcal{Z}_{I,J}^\circ$ to $\mathcal{Z}_{I_0;J}^\circ\cap\mathcal{Z}_{J,I}^\circ$. Let

$$\mathcal{Z}_{I_0}=\bigg(\bigsqcup_{I\in\mathcal{P}^*(N)\cap\mathcal{P}_{I_0}(N)}\mathcal{Z}_{I_0;I}^\circ\bigg)\Big/\!\sim, \qquad \mathcal{Z}_{I_0;I}^\circ\cap\mathcal{Z}_{I,J}^\circ\ni x\sim\Theta_{J,I}(x)\in\mathcal{Z}_{I_0;J}^\circ\cap\mathcal{Z}_{J,I}^\circ. \tag{4.39}$$

For $i\in[N]$, let

$$\mathcal{Z}_{i,I_0}^\star=\bigg(\bigsqcup_{I\in\mathcal{P}_i(N)\cap\mathcal{P}_{I_0}(N)}\mathcal{Z}_{I_0;I}^\circ\bigg)\Big/\!\sim, \qquad \mathcal{Z}_{I_0;I}^\circ\cap\mathcal{Z}_{I,J}^\circ\ni x\sim\Theta_{J,I}(x)\in\mathcal{Z}_{I_0;J}^\circ\cap\mathcal{Z}_{J,I}^\circ.$$

Since the relations $\sim$ in (4.36) and (4.37) are equivalence relations, $\mathcal{Z}_{I_0}$ is a subspace of $\mathcal{Z}$ and $\mathcal{Z}_{i,I_0}^\star$ is a subspace of $\mathcal{Z}_i^\star$. Furthermore,

$$\mathcal{Z}_\emptyset=\mathcal{Z},\quad \mathcal{Z}_{i,\emptyset}^\star=\mathcal{Z}_i^\star\ \ \forall\,i\in[N],\quad \mathcal{Z}_{i,I_0}^\star=\mathcal{Z}_{I_0}\cap\mathcal{Z}_i^\star\subset\mathcal{Z}\ \ \forall\,i\in[N],\ I_0\subset[N]. \tag{4.40}$$

For $i\in[N]$ and $I_0\in\mathcal{P}^*(N)$, let

$$X_i=\mathcal{Z}_{\{i\}},\qquad X_{I_0}=\mathcal{Z}_{I_0}\,. \tag{4.41}$$

By (4.39) and (4.37), $X_i\subset\mathcal{Z}_i^\star$. If $i\in I_0$, then $X_{I_0}\subset X_i$.

For $i\in I\subset[N]$, let $\mathcal{U}_{i;I}^{\leqslant}\subset U_I\times\mathbb{C}$ be as in (4.2) and define

$$g_{i;I}\colon\mathcal{U}_{i;I}^{\leqslant}\longrightarrow\mathbb{C}^I\quad\text{by}\quad \big(g_{i;I}(x,z)\big)_j=\begin{cases}z, & \text{if } j=i;\\ \sqrt{2(\mu_I(x))_{ij}+|z|^2}, & \text{if } j\in I-i.\end{cases}$$

Thus, the map

$$\mathcal{U}_{i;I}^{\leqslant}\longrightarrow\widetilde{\mathcal{Z}}_{I;i}^{\geqslant}\equiv\big\{\big(x,(z_j)_{j\in I}\big)\in\widetilde{\mathcal{Z}}_I^\circ\colon z_j\in\mathbb{R}^{\geqslant0}\ \forall\,j\in I-i\big\},\quad (x,z)\longrightarrow\big(x,g_{i;I}(x,z)\big), \tag{4.42}$$

is a homeomorphism. If $i\in I_0\subset I$, the restriction

$$g_{I_0;I}\colon U_{I_0;I}^{\leqslant}=\big\{(x,0)\in\mathcal{U}_{i;I}^{\leqslant}\colon(\mu_I(x))_i=(\mu_I(x))_j\ \forall\,j\in I_0\big\}\longrightarrow\big(\mathbb{R}^{\geqslant0}\big)^{I-I_0}\subset\mathbb{C}^I,$$
$$g_{I_0;I}(x)=\Big(\sqrt{2(\mu_I(x))_{ij}}\Big)_{j\in I-I_0},$$

of $g_{i;I}$ is independent of the choice of $i\in I_0$.

Lemma 4.9. *For every $i\in[N]$, the map*

$$q_i:\mathcal{U}_i^{\leqslant}\longrightarrow\mathcal{Z}_i^{\star},\quad (x,z)\longrightarrow q_{\mathcal{Z}}\big(q_{\mathcal{Z};I}\big(x,g_{i;I}(x,z)\big)\big)\quad\forall\ (x,z)\in\mathcal{U}_{i;I}^{\leqslant},\ I\in\mathcal{P}_i(N), \tag{4.43}$$

is well-defined, continuous, surjective, and closed. For $I_0\in\mathcal{P}_i(N)$, it restricts to the surjective closed map

$$q_{I_0}:U_{I_0}^{\leqslant}\longrightarrow X_{I_0},\quad x\longrightarrow q_{\mathcal{Z}}\big(q_{\mathcal{Z};I_0;I}\big(x,g_{I_0;I}(x)\big)\big)\quad\forall\ x\in U_{I_0;I}^{\leqslant},\ I\in\mathcal{P}_{I_0}(N), \tag{4.44}$$

which does not depend on the choice of $i\in I_0$.

Proof. If $(x,z)\in\mathcal{U}_{i;I}^{\leqslant}\cap\mathcal{U}_{i;J}^{\leqslant}$, then either $I\subset J$ or $J\subset I$ by Definition 1.1(*a*). In either case,

$$\Theta_{J,I}\big(q_{\mathcal{Z};I}\big(x,g_{i;I}(x,z)\big)\big)=q_{\mathcal{Z};J}\big(x,g_{i;J}(x,z)\big)$$

by (4.37) and (4.29). Thus, the map (4.43) is well-defined. By Lemma 4.4, $\mathcal{U}_{i;I}^{\leqslant}\subset\mathcal{U}_i^{\leqslant}$ is open. The map (4.43) is continuous because its restriction to each of these open subspaces is continuous.

For every $y\in\mathcal{Z}_i^{\star}$, there are $I\in\mathcal{P}_i(N)$ and

$$\widetilde{x}\equiv\big(x,(z_j)_{j\in I}\big)\in\widetilde{\mathcal{Z}}_I^{\circ}\qquad\text{s.t.}\quad y=q_{\mathcal{Z}}\big(q_{\mathcal{Z};I}(\widetilde{x})\big)\,.$$

Thus,

$$\big(\mu_I(x)\big)_{ij}+\frac{1}{2}|z_i|^2=\frac{1}{2}|z_j|^2\geqslant 0\qquad\forall\ j\in I \tag{4.45}$$

and so $(x,z_i)\in\mathcal{U}_{i;I}^{\leqslant}$. For each $j\in I-i$, let $\mathrm{e}^{\mathrm{i}\theta_j}\in S^1$ be such that $\mathrm{e}^{-\mathrm{i}\theta_j}z_j\in\mathbb{R}^{\geqslant 0}$ and

$$\mathrm{e}^{\mathrm{i}\theta_i}=\prod_{j\in I-i}\mathrm{e}^{-\mathrm{i}\theta_j}\in S^1\,.$$

Then, $(\mathrm{e}^{\mathrm{i}\theta_j})_{j\in I}$ is an element of $(S^1)^I_{\bullet}$. Since the ϕ_I-action is Hamiltonian and $(x,z_i)\in\mathcal{U}_{i;I}^{\leqslant}$,

$$(x',z')\equiv\big(\phi_I\big((\mathrm{e}^{\mathrm{i}\theta_j})_{j\in I};x\big),\mathrm{e}^{-\mathrm{i}\theta_i}z_i\big)\in\mathcal{U}_{i;I}^{\leqslant}\quad\text{and}\quad\widetilde{\phi}_I\big((\mathrm{e}^{\mathrm{i}\theta_j})_{j\in I};\widetilde{x}\big)=\big(x',g_{i;I}(x',z')\big).$$

Thus,

$$y=q_{\mathcal{Z}}\big(q_{\mathcal{Z};I}(\widetilde{x})\big)=q_{\mathcal{Z}}\big(q_{\mathcal{Z};I}\big(\widetilde{\phi}_I((\mathrm{e}^{\mathrm{i}\theta_j})_{j\in I};\widetilde{x})\big)\big)=q_{\mathcal{Z}}\big(q_{\mathcal{Z};I}\big(x',g_{i;I}(x',z')\big)\big)\,.$$

This establishes the surjectivity of the map (4.43).

For $I_0\in\mathcal{P}_i(N)$, the restriction of the map (4.43) to the subspace

$$\mathcal{U}_{i,I_0}^{\leqslant}=\bigcup_{I\in\mathcal{P}_{I_0}(N)}\mathcal{U}_{i,I_0;I}^{\leqslant}=\bigcup_{I\in\mathcal{P}_{I_0}(N)}U_{I_0;I}^{\leqslant}=U_{I_0}^{\leqslant}$$

is the map (4.44). Since the map $g_{I_0;I}$ does not depend on the choice of $i\in I_0$, neither does the map (4.44). If $I\supset I_0$ and $y\in X_{I_0}$ are as in the previous paragraph, then $z_j=0$ for every $j\in I_0$. By (4.45), this implies that $\widetilde{x}\in U_{I_0}^{\leqslant}$ and thus

$$x'\equiv\phi_I\big((\mathrm{e}^{\mathrm{i}\theta_j})_{j\in I};x\big)\in U_{I_0}^{\leqslant}\quad\text{and}\quad y=q_{\mathcal{Z}}\big(q_{\mathcal{Z};I_0;I}\big(x',g_{I_0;I}(x')\big)\big).$$

This establishes the surjectivity of the map (4.44).

For $i\in I\subset[N]$ and $\mathcal{W}\subset X\times\mathbb{C}$, let

$$\mathcal{W}_{i;I}\equiv\big\{\widetilde{\phi}_I\big(g;x,g_{i;I}(x,z)\big): g\in(S^1)^I_\bullet,\ (x,z)\in\mathcal{U}^{\leqslant}_{i;I}\cap\mathcal{W}\big\}\subset\widetilde{\mathcal{Z}}^\circ_I.$$

By the second statement of Lemma 4.4,

$$\mathcal{U}^{\leqslant}_i\cap\big(U_I\times\mathbb{C}\big)\cap\mathcal{W}=\mathcal{U}^{\leqslant}_{i;I}\cap\mathcal{W}\,.$$

Thus, the preimage of $q_i(\mathcal{U}^{\leqslant}_i\cap\mathcal{W})$ in $\widetilde{\mathcal{Z}}^\circ_I$ under $q_{\mathcal{Z}}\circ q_{\mathcal{Z};I}$ is $\mathcal{W}_{i;I}$. Since the map (4.42) is a homeomorphism and the group $(S^1)^I_\bullet$ is compact, $\mathcal{W}_{i;I}$ is closed in

$$\widetilde{\phi}_I\big((S^1)^I_\bullet\times\widetilde{\mathcal{Z}}^{\geqslant}_{I;i}\big)=\widetilde{\mathcal{Z}}^\circ_I$$

if $\mathcal{W}$ is closed in $X\times\mathbb{C}$. Thus, the map (4.43) is closed. Since $q_i^{-1}(X_{I_0})=U^{\leqslant}_{I_0}$ for $I_0\in\mathcal{P}_i(N)$, the map (4.44) is closed as well. □

Corollary 4.10. *The topological spaces $\mathcal{Z}^\star_i$ with $i\in[N]$ and $\mathcal{Z}$ are Hausdorff. They inherit symplectic forms ϖ_i and $\omega_{\mathcal{Z}}$ from the symplectic manifolds $(\mathcal{Z}^\circ_I,\varpi_I)$ with $i\in I\subset[N]$.*

Proof. By Lemma 4.9, (4.43) is a closed quotient map. Since its domain is metrizable (being a subspace of a manifold), it is normal. By [15, Lemma 73.3], $\mathcal{Z}^\star_i$ is thus a normal (and in particular Hausdorff) topological space.

Suppose $x,y\in\mathcal{Z}$ are two distinct points. Let $I,J\in\mathcal{P}^*(N)$, $\widetilde{x}\in\mathcal{Z}^\circ_I$, and $\widetilde{y}\in\mathcal{Z}^\circ_J$ be such that $x=q_{\mathcal{Z}}(\widetilde{x})$ and $y=q_{\mathcal{Z}}(\widetilde{y})$. If $I\not\subset J$ and $I\not\supset J$, then

$$q_{\mathcal{Z}}\big(\mathcal{Z}^\circ_I\big)\cap q_{\mathcal{Z}}\big(\mathcal{Z}^\circ_J\big)=\emptyset$$

by (4.36) and Definition 1.1(a). By Lemma 4.8, $q_{\mathcal{Z}}(\mathcal{Z}^\circ_I)$ and $q_{\mathcal{Z}}(\mathcal{Z}^\circ_J)$ are open in $\mathcal{Z}$. If $i\in I\subset J$, then $x,y\in\mathcal{Z}^\star_i$. Since $\mathcal{Z}^\star_i$ is Hausdorff, there exist disjoint open neighborhoods U_x of x and U_y of y in $\mathcal{Z}^\star_i$. By Lemma 4.8, U_x and U_y are open in $\mathcal{Z}$. Thus, $\mathcal{Z}$ is Hausdorff.

Since the identification maps $\Theta_{I,J}$ are diffeomorphisms between open subspaces of manifolds and the restriction of (4.38) to each $\mathcal{Z}^\circ_I$ is a homeomorphism onto its image, the smooth structure on the domain of (4.38) descends to a smooth structure on $\mathcal{Z}$. By (4.17) and (4.30),

$$q^*_{\mathcal{Z};I}\Theta^*_{J,I}\big(\varpi_J|_{\mathcal{Z}^\circ_{J,I}}\big)=\widetilde{\Theta}^*_{J,I}q^*_{\mathcal{Z};J}\big(\varpi_J|_{\mathcal{Z}^\circ_{J,I}}\big)=\widetilde{\Theta}^*_{J,I}\big(\widetilde{\omega}_J|_{\widetilde{\mathcal{Z}}^\circ_{J,I}}\big)=\widetilde{\omega}_I\big|_{\widetilde{\mathcal{Z}}^\circ_{I,J}}=q^*_{\mathcal{Z};I}\big(\varpi_I|_{\mathcal{Z}^\circ_{I,J}}\big)\,. \tag{4.46}$$

By the uniqueness part of [3, Theorem 23.1], (4.46) implies that

$$\Theta^*_{J,I}\big(\varpi_J|_{\mathcal{Z}^\circ_{J,I}}\big)=\varpi_I\big|_{\mathcal{Z}^\circ_{I,J}}\,.$$

Thus, $q_{\mathcal{Z}}$ induces a symplectic structure on $\mathcal{Z}$. □

4.4 The SC symplectic divisor

By Lemmas 4.8 and 4.9 and Corollary 4.10, $(\mathcal{Z},\omega_{\mathcal{Z}})$ is a symplectic manifold obtained by collapsing the boundary and corners of the subspaces $\mathcal{U}_i^{\leqslant}\subset X\times\mathbb{C}$ to form open symplectic manifolds $\mathcal{Z}_i^\star$ with $i\in[N]$ and gluing the latter together along the common open subspaces $\mathcal{Z}_I^\circ$ with $I\in\mathcal{P}_i(N)$. We next describe the image of the boundary and corners of $\mathcal{U}_i^{\leqslant}$ under the collapsing map as an SC symplectic divisor in $(\mathcal{Z},\omega_{\mathcal{Z}})$ in the sense of Definition 2.1.

Lemma 4.11. *Let $I_0\subset[N]$. The topological spaces $\mathcal{Z}_{i,I_0}^\star$ with $i\in[N]$ and $\mathcal{Z}_{I_0}$ are closed symplectic submanifolds of $(\mathcal{Z}_i^\star,\varpi_i)$ and of $(\mathcal{Z},\omega_{\mathcal{Z}})$, respectively, of codimension $2|I_0|$. If $I_0\neq\emptyset$ and X is compact, then $X_{I_0}=\mathcal{Z}_{I_0}$ is compact.*

Proof. By (4.18) and the surjectivity of the map (4.43), $\mathcal{Z}_{i,I_0}^\star\subset\mathcal{Z}_i^\star$ is the image of the restriction of this map to $\mathcal{U}_{i,I_0}^{\leqslant}$. By Lemma 4.5, $\mathcal{U}_{i,I_0}^{\leqslant}\subset\mathcal{U}_i^{\leqslant}$ is a closed subspace. Since (4.43) is a closed map, $\mathcal{Z}_{i,I_0}^\star\subset\mathcal{Z}_i^\star$ is thus a closed subspace. By the third identity in (4.40), the intersection of $\mathcal{Z}_{I_0}$ with each $\mathcal{Z}_i^\star$ is thus closed in $\mathcal{Z}_i^\star$. By Lemma 4.8, $\{\mathcal{Z}_i^\star\}_{i\in[N]}$ is an open cover of $\mathcal{Z}$. Therefore, $\mathcal{Z}_{I_0}$ is closed in $\mathcal{Z}$. Since $\mathcal{Z}_i^\star$ and $\mathcal{Z}$ are Hausdorff by Corollary 4.10, so are $\mathcal{Z}_{i,I_0}^\star\subset\mathcal{Z}_i^\star$ and $\mathcal{Z}_{I_0}\subset\mathcal{Z}$.

By Lemma 4.7, $(\mathcal{Z}_{I_0;I}^\circ,\varpi_{I_0;I})$ is a symplectic submanifold of $(\mathcal{Z}_I^\circ,\varpi_I)$ for every $I\in\mathcal{P}_{I_0}(N)$. Since the symplectic form $\omega_{\mathcal{Z}}$ on $\mathcal{Z}$ is induced by the symplectic forms ϖ_I on $\mathcal{Z}_I^\circ$, the symplectic form ω_{I_0} on $X_{I_0}=\mathcal{Z}_{I_0}$ induced by the symplectic forms $\varpi_{I_0;I}$ on $\mathcal{Z}_{I_0;I}^\circ$ is the restriction of $\omega_{\mathcal{Z}}$.

Suppose $I_0\neq\emptyset$ and X is compact. By Lemma 4.3, $U_{I_0}^{\leqslant}\subset X$ is then also compact. Since the map (4.44) is continuous and surjective, it follows that $X_{I_0}=\mathcal{Z}_{I_0}$ is compact as well. □

As in the proof of Lemma 4.11, we denote by ω_{I_0} the symplectic form on $\mathcal{Z}_{I_0}$ induced by the symplectic forms $\varpi_{I_0;I}$ on $\mathcal{Z}_{I_0;I}^\circ$. In particular,

$$\big(X_i,\omega_i\big)\equiv\big(\mathcal{Z}_{\{i\}},\omega_{\{i\}}\big)=\big(\mathcal{Z}_{\{i\}},\omega_{\mathcal{Z}}|_{\mathcal{Z}_{\{i\}}}\big)$$

is a symplectic manifold for each $i\in[N]$.

Let $i\in I\subset[N]$ and $I_0'\subset I_0\subset I$. With notation as in (4.20) and (4.26), we define

$$\widetilde\Theta_{J,I;i}\colon\widetilde{\mathcal{N}}_{I_0;I_0';I}^\circ\big|_{\widetilde{\mathcal{Z}}_{I_0;I}^\circ\cap\widetilde{\mathcal{Z}}_{I,J}^\circ}\longrightarrow\widetilde{\mathcal{N}}_{I_0;I_0';J}^\circ\big|_{\widetilde{\mathcal{Z}}_{I_0;J}^\circ\cap\widetilde{\mathcal{Z}}_{J,I}^\circ}$$

by the formula in (4.29) with $|z_i|^2$ replaced by 0 if $i\in I_0-I_0'$. This diffeomorphism is still $(\widetilde\phi_J,\widetilde\phi_I)$-equivariant with respect to the homomorphism (4.25), satisfies the analogues of (4.30)-(4.32) and

$$\widetilde\pi_{I_0;I_0';J}^\circ\circ\widetilde\Theta_{J,I;i}=\widetilde\Theta_{J,I;i}\circ\widetilde\pi_{I_0;I_0';I}^\circ\big|_{\widetilde{\mathcal{Z}}_{I_0;I}^\circ\cap\widetilde{\mathcal{Z}}_{I,J}^\circ},$$

and is complex linear on the fibers of $\widetilde\pi_{I_0;I_0';I}^\circ$. Thus, it descends to a diffeomorphism

$$\Theta_{J,I}\colon\mathcal{N}_{I_0;I_0';I}^\circ\big|_{\mathcal{Z}_{I_0;I}^\circ\cap\mathcal{Z}_{I,J}^\circ}\longrightarrow\mathcal{N}_{I_0;I_0';J}^\circ\big|_{\mathcal{Z}_{I_0;J}^\circ\cap\mathcal{Z}_{J,I}^\circ}$$

which is independent of $i\in I$, satisfies the analogue of (4.35) and

$$\pi_{I_0;I_0';J}^\circ\circ\Theta_{J,I}=\Theta_{J,I}\circ\pi_{I_0;I_0';I}^\circ\big|_{\mathcal{Z}_{I_0;I}^\circ\cap\mathcal{Z}_{I,J}^\circ},\tag{4.47}$$

and is complex linear on the fibers of $\pi^{\circ}_{I_0;I_0';I}$. Define

$$\mathcal{N}_{I_0;I_0'} = \Big(\bigsqcup_{I\in\mathcal{P}_{I_0}(N)}\mathcal{N}^{\circ}_{I_0;I_0';I}\Big)\Big/\!\sim, \qquad \text{where}$$
$$\mathcal{N}^{\circ}_{I_0;I_0';I}\big|_{\mathcal{Z}^{\circ}_{I_0;I}\cap\mathcal{Z}^{\circ}_{I,J}} \ni \big[\widetilde{x},z\big] \sim \Theta_{J,I}\big([\widetilde{x},z]\big) \in \mathcal{N}^{\circ}_{I_0;I_0';J}\big|_{\mathcal{Z}^{\circ}_{I_0;J}\cap\mathcal{Z}^{\circ}_{J,I}}.$$

By (4.47), the projections $\pi^{\circ}_{I_0;I_0';I}$ induce a complex vector bundle

$$\pi_{I_0;I_0'}\colon \mathcal{N}_{I_0;I_0'} \longrightarrow \mathcal{Z}_{I_0}\,. \tag{4.48}$$

Lemma 4.12. *For all $I_0'\subset I_0\subset I$, $(\mathcal{Z}_{I_0},\omega_{I_0})$ is a symplectic submanifold of $(\mathcal{Z}_{I_0'},\omega_{I_0'})$ with the oriented normal bundle canonically isomorphic to (4.48).*

Proof. By Lemma 4.11, $(\mathcal{Z}_{I_0},\omega_{I_0})$ and $(\mathcal{Z}_{I_0'},\omega_{I_0'})$ are symplectic submanifolds of $(\mathcal{Z},\omega_{\mathcal{Z}})$. Since $\mathcal{Z}_{I_0}\subset\mathcal{Z}_{I_0'}$, $(\mathcal{Z}_{I_0},\omega_{I_0})$ is a symplectic submanifold of $(\mathcal{Z}_{I_0'},\omega_{I_0'})$.

Since the collection of all isomorphisms (4.24) respects the patching maps $\Theta_{J,I}$ for the manifold $\mathcal{Z}_{I_0'}$ and for the bundle (4.48), the isomorphisms (4.24) induce a vector bundle isomorphism

$$\mathcal{N}_{\mathcal{Z}_{I_0'}}\mathcal{Z}_{I_0} \equiv \frac{T\mathcal{Z}_{I_0'}|_{\mathcal{Z}_{I_0}}}{T\mathcal{Z}_{I_0}} \longrightarrow \mathcal{N}_{I_0;I_0'}$$

of oriented vector bundles over $\mathcal{Z}_{I_0}$. This isomorphism is orientation-preserving because the isomorphisms (4.24) and the identifications $\Theta_{J,I}$ are. □

Corollary 4.13. *The collection $\{X_i\}_{i\in[N]}$ is an SC symplectic divisor in $(\mathcal{Z},\omega_{\mathcal{Z}})$ with the associated N-fold symplectic configuration $\mathbf{X}\equiv((X_I)_{I\in\mathcal{P}^*(N)},(\omega_i)_{i\in[N]})$. Furthermore, the map*

$$q_{\emptyset}\colon X \longrightarrow X_{\emptyset}\subset\mathcal{Z}, \qquad q_{\emptyset}(x) = q_{\{i\}}(x) \quad \forall\, x\in U_i^{\leqslant},\, i\in[N], \tag{4.49}$$

with $q_{\{i\}}$ as in (4.44), is well-defined, continuous, and surjective. For each $i\in[N]$, it takes $\partial U_i^{\leqslant}$ onto $X_i\cap X_\partial$ and restricts to a symplectomorphism

$$\big(U_i^{\leqslant}-\partial U_i^{\leqslant},\omega|_{U_i^{\leqslant}-\partial U_i^{\leqslant}}\big) \longrightarrow \big(X_i-X_\partial,\omega_i|_{X_i-X_\partial}\big). \tag{4.50}$$

Proof. By Lemma 4.11, (X_i,ω_i) is a closed symplectic submanifold of $(\mathcal{Z},\omega_{\mathcal{Z}})$ for every $i\in[N]$. By construction,

$$X_I = \mathcal{Z}_I = \bigcap_{i\in I}X_i \subset \mathcal{Z} \qquad \forall\ I\subset[N].$$

By Lemma 4.12, the real codimension of X_i in $\mathcal{Z}$ is 2 for every $i\in[N]$ and the homomorphism

$$\mathcal{N}_{\mathcal{Z}}X_I \equiv \frac{T\mathcal{Z}|_{X_I}}{TX_I} \longrightarrow \bigoplus_{i\in I}\frac{T\mathcal{Z}|_{X_I}}{TX_i|_{X_I}} \equiv \bigoplus_{i\in I}\mathcal{N}_{\mathcal{Z}}X_i\big|_{X_I} \tag{4.51}$$

induced by the natural inclusions is an orientation-preserving isomorphism for all $I\subset[N]$. Thus, $\{X_i\}_{i\in[N]}$ is a transverse (in fact orthogonal) collection of closed symplectic submanifolds of $\mathcal{Z}$ so that the symplectic orientation of each X_I (which orients the left-hand side of (4.51)) agrees with its intersection orientation (which is determined by the orientation of the right-hand side of (4.51)).

Therefore, $\{X_i\}_{i\in[N]}$ is an SC symplectic divisor in $(\mathcal{Z},\omega_{\mathcal{Z}})$.

By Lemmas 4.1 and 4.2, the closed subsets $U_i^{\leqslant}\subset X$ cover X. By Lemmas 4.9 and 4.3, $q_{\{i\}}$ is continuous on $U_i^{\leqslant}$ and $q_{\{i\}}=q_{\{j\}}$ on $U_i^{\leqslant}\cap U_j^{\leqslant}=U_{ij}^{\leqslant}$. Thus, (4.49) is a well-defined continuous map. By Lemma 4.9, it takes $U_i^{\leqslant}$ onto X_i for every $i\in[N]$ and $U_{I_0}^{\leqslant}$ onto X_{I_0} for every $I\in\mathcal{P}^*(N)$. Thus, $q_\emptyset$ restricts to surjective continuous maps

$$q_\emptyset\colon \partial U_i^{\leqslant}\equiv\bigcup_{\substack{I_0\in\mathcal{P}_i(N)\\|I_0|\geqslant2}}U_{I_0}^{\leqslant}\longrightarrow\bigcup_{\substack{I_0\in\mathcal{P}_i(N)\\|I_0|\geqslant2}}X_{I_0}=X_i\cap X_\partial,\quad q_\emptyset\colon U_i^{\leqslant}-\partial U_i^{\leqslant}\longrightarrow X_i-X_\partial\,.$$

The latter map is described by its restrictions to the open subsets $U_{i;I}^{<}$ with $I\in\mathcal{P}_i(N)$:

$$q_\emptyset\colon U_{i;I}^{<}\equiv\big\{x\in U_I\colon(\mu_I(x))_i<(\mu_I(x))_j\ \forall\,j\in I-i\big\},\quad q_\emptyset(x)=q_{\mathcal{Z}}\big(q_{\mathcal{Z};i;I}\big(x,g_{i;I}(x)\big)\big),$$
$$\text{where}\qquad g_{i;I}(x)=\Big(\sqrt{2(\mu_I(x))_{ij}}\Big)_{j\in I-i}\in(\mathbb{R}^+)^{I-i}\,.$$

Since the restriction of (4.42) to $U_{i;I}^{<}$ is a homeomorphism onto

$$\widetilde{X}_{i;I}^{>}\equiv\widetilde{\mathcal{Z}}_{\{i\};I}^{\circ}\cap\widetilde{\mathcal{Z}}_{I;i}^{\geqslant}-\bigcup_{\substack{i\in I_0\subset I\\|I_0|\geqslant2}}\widetilde{\mathcal{Z}}_{I_0;I}^{\circ}=\widetilde{\mathcal{Z}}_{\{i\};I}^{\circ}\cap\widetilde{\mathcal{Z}}_{I;i}^{\geqslant}-q_{\mathcal{Z};i;I}^{-1}\big(q_{\mathcal{Z}}^{-1}(X_\partial)\big)\subset U_{i;I}^{<}\times(\mathbb{R}^+)^{I-i}$$

and $\widetilde{X}_{i;I}^{>}$ is a slice for the $\widetilde\phi_I$-action on $\widetilde{\mathcal{Z}}_{\{i\};I}^{\circ}-q_{\mathcal{Z};i;I}^{-1}(q_{\mathcal{Z}}^{-1}(X_\partial))$, the restriction of $q_\emptyset$ to $U_{i;I}^{<}$ is injective. Since $U_{i;I}^{<}$ is ϕ_I-invariant, (4.39) then implies that the map (4.50) is injective. Since restriction of $q_\emptyset$ to $U_{i;I}^{<}$ is a composition of smooth maps, the map (4.50) is smooth. Since the argument of every component of the map

$$g_{i;I}\colon U_{i;I}^{<}\longrightarrow\mathbb{C}^{I-i}$$

is fixed, (4.19) implies that

$$\begin{aligned}\big\{q_\emptyset|_{U_{i;I}^{<}}\big\}^*\omega_i&=\big\{\mathrm{id}\times g_{i;I}\big\}^*q_{\mathcal{Z};i;I}^*\varpi_{i;I}=\big\{\mathrm{id}\times g_{i;I}\big\}^*\pi_1^*\omega+\big\{\mathrm{id}\times g_{i;I}\big\}^*\pi_2^*\omega_{\mathbb{C}^{I-i}}\\&=\omega|_{U_{i;I}^{<}}+g_{i;I}^*\omega_{\mathbb{C}^{I-i}}=\omega|_{U_{i;I}^{<}}\,.\end{aligned}$$

Thus, the map (4.50) is a symplectomorphism. □

4.5 The Hamiltonian configuration

We next describe an N-fold Hamiltonian configuration

$$\mathscr{C}_{\mathcal{Z}}\equiv\big(U_{\mathcal{Z};I},\phi_{\mathcal{Z};I},\mu_{\mathcal{Z};I}\big)_{I\in\mathcal{P}^*(N)}\tag{4.52}$$

for $(\mathcal{Z},\omega_{\mathcal{Z}})$. For each $I\in\mathcal{P}^*(N)$, let $U_{\mathcal{Z};I}=q_{\mathcal{Z}}(\mathcal{Z}_I^\circ)$. By Lemma 4.8, $U_{\mathcal{Z};I}$ is open in $\mathcal{Z}$. By (4.36), $(U_{\mathcal{Z};I})_{I\in\mathcal{P}^*(N)}$ is a cover of $\mathcal{Z}$. Since $U_I\cap U_J=\emptyset$ unless $I\subset J$ or $I\subset J$, (4.36) implies that the same is the case for $U_{\mathcal{Z};I}$ and $U_{\mathcal{Z};J}$.

For $I\in\mathcal{P}^*(N)$, we define a Hamiltonian $(S^1)^I_\bullet$- pair for $(U_I\times\mathbb{C}^I,\widetilde\omega_I)$ by

$$\widetilde\phi_{\mathcal{Z};I}\colon(S^1)^I_\bullet\times\big(U_I\times\mathbb{C}^I\big)\longrightarrow U_I\times\mathbb{C}^I,\qquad\widetilde\phi_{\mathcal{Z};I}\big(g;x,z\big)=\big(\phi_I(g;x),z\big);\tag{4.53}$$
$$\widetilde\mu_{\mathcal{Z};I}\colon U_I\times\mathbb{C}^I\longrightarrow\mathfrak{t}_{I;\bullet}^*,\qquad\widetilde\mu_{\mathcal{Z};I}(x,z)=\mu_I(x).\tag{4.54}$$

By (b) and (c) in Definition 1.1,

$$\begin{aligned}\widetilde{\mu}_{\mathcal{Z};J}(x,z)|_{\mathfrak{t}_{I;\bullet}} &= \widetilde{\mu}_{\mathcal{Z};I}(x,z) && \forall\,(x,z)\in(U_I\cap U_J)\times\mathbb{C}^I,\ I\subset J\subset[N],\\ \big(\widetilde{\mu}_{\mathcal{Z};J}(x)\big)_i &< \big(\widetilde{\mu}_{\mathcal{Z};J}(x)\big)_j && \forall\,(x,z)\in(U_I\cap U_J)\times\mathbb{C}^I,\ i\in I\subset J\subset[N],\ j\in J-I,\end{aligned} \tag{4.55}$$

respectively.

Since the ϕ_I-action is Hamiltonian, (4.53) restricts to an $(S^1)^I_\bullet$-action

$$\widetilde{\phi}_{\mathcal{Z};I}\colon (S^1)^I_\bullet\times\widetilde{\mathcal{Z}}_I^\circ\longrightarrow\widetilde{\mathcal{Z}}_I^\circ\,. \tag{4.56}$$

Since the $\widetilde{\phi}_I$-action commutes with this action and preserves the moment map $\widetilde{\mu}_{\mathcal{Z};I}$, (4.56) and (4.54) determine a Hamiltonian pair

$$\phi^\circ_{\mathcal{Z};I}\colon (S^1)^I_\bullet\times\mathcal{Z}_I^\circ\longrightarrow\mathcal{Z}_I^\circ \qquad\text{and}\qquad \mu^\circ_{\mathcal{Z};I}\colon\mathcal{Z}_I^\circ\longrightarrow\mathfrak{t}^*_{I;\bullet} \tag{4.57}$$

for $(\mathcal{Z}_I^\circ,\varpi_I)$. Since the restriction of (4.38) to $\mathcal{Z}_I^\circ$ is a symplectomorphism onto $U_{\mathcal{Z};I}$, $\phi^\circ_{\mathcal{Z};I}$ and $\mu^\circ_{\mathcal{Z};I}$ induce smooth maps

$$\phi_{\mathcal{Z};I}\colon (S^1)^I_\bullet\times U_{\mathcal{Z};I}\longrightarrow U_{\mathcal{Z};I}, \qquad \mu_{\mathcal{Z};I}\colon U_{\mathcal{Z};I}\longrightarrow\mathfrak{t}^*_{I;\bullet}, \tag{4.58}$$

so that $\phi_{\mathcal{Z};I}$ is a Hamiltonian $(S^1)^I_\bullet$-action with moment map $\mu_{\mathcal{Z};I}$.

Lemma 4.14. *The tuple (4.52) described above is an N-fold Hamiltonian configuration for $(\mathcal{Z},\omega_{\mathcal{Z}})$.*

Proof. As explained above, $(U_{\mathcal{Z};I})_{I\in\mathcal{P}^*(N)}$ is an open cover of $\mathcal{Z}$ satisfying Definition 1.1(a). By (4.55), the moment maps $\mu_{\mathcal{Z};I}$ satisfy (b) and (c) in Definition 1.1. $\square$

For $I\in\mathcal{P}^*(N)$, we define

$$\widetilde{\mathcal{Z}}^\circ_{\partial;I}=\bigcup_{\substack{I_0\subset I\\|I_0|=2}}\widetilde{\mathcal{Z}}^\circ_{I_0;I}\subset\widetilde{\mathcal{Z}}^\circ_I\subset U_I\times\mathbb{C}^I\,.$$

If in addition $I_0\subset I$, let

$$\begin{aligned}\widetilde{\mu}_{\mathcal{Z};I_0;I}\colon \big\{\widetilde{x}\in\widetilde{\mathcal{Z}}^\circ_I\colon \big(\widetilde{\mu}_{\mathcal{Z};I}(\widetilde{x})\big)_i<\big(\widetilde{\mu}_{\mathcal{Z};I}(\widetilde{x})\big)_j\ \forall\,i\in I_0,\ j\in I-I_0\big\}&\longrightarrow\mathfrak{t}^*_{I_0;\bullet},\\ \widetilde{\mu}_{\mathcal{Z};I_0;I}(\widetilde{x})=\widetilde{\mu}_{\mathcal{Z};I}(\widetilde{x})\big|_{\mathfrak{t}_{I_0;\bullet}}&\,.\end{aligned} \tag{4.59}$$

The commuting torus actions (4.12) and (4.53) induce an $(S^1)^{I_0}_\bullet\times(S^1)^I_\bullet$-action on $U_I\times\mathbb{C}^I$:

$$(g_0,g)\cdot\widetilde{x}=\widetilde{\phi}_{\mathcal{Z};I}\big(g_0;\widetilde{\phi}_I(g;\widetilde{x})\big). \tag{4.60}$$

Since the ϕ_I-action is Hamiltonian, the action (4.60) preserves $\widetilde{\mathcal{Z}}^\circ_{\partial;I}$ and $\widetilde{\mu}^{-1}_{\mathcal{Z};I_0;I}(0)$.

Lemma 4.15. *For all $I_0\in\mathcal{P}^*(N)$ and $I\in\mathcal{P}_{I_0}(N)$, the restriction of the torus action (4.60) to $\widetilde{\mu}^{-1}_{\mathcal{Z};I_0;I}(0)-\widetilde{\mathcal{Z}}^\circ_{\partial;I}$ is free.*

Proof. Let $\widetilde{x}=(x,(z_j)_{j\in I})$ be an element of $\widetilde{\mu}^{-1}_{\mathcal{Z};I_0;I}(0)-\widetilde{\mathcal{Z}}^\circ_{\partial;I}$. If the action (4.60) by some (g_0,g) fixes $\widetilde{x}$, then $g=\mathrm{id}$ because $z_j=0$ for at most one element $j\in I$. By (4.59), (4.54), and (1.4),

$$\widetilde{\mu}_{\mathcal{Z};I_0;I}(\widetilde{x})=\mu_{I_0;I}(x)\,.$$

Thus, $g_0=\mathrm{id}$ by Definition 1.2. This establishes the second claim. $\square$

Corollary 4.16. *The restriction of the N-fold Hamiltonian configuration (4.52) to $\mathcal{Z}-X_\partial$ is an N-fold cutting configuration.*

Proof. In light of Lemma 4.14, it remains to show that the restriction satisfies the conditions of Definition 1.2. Fix $I_0\in\mathcal{P}^*(N)$ and $I\in\mathcal{P}_{I_0}(N)$. We denote by $\mu_{\mathcal{Z};I_0;I}$ the analogue of the map (1.4) for the Hamiltonian configuration (4.52) and by $\phi_{\mathcal{Z};I_0;I}$ the restriction of the $(S^1)^I_\bullet$-action $\phi_{\mathcal{Z};I}$ to $(S^1)^{I_0}_\bullet$. Let

$$\mathcal{Z}^\circ_{\partial;I}=q_{\mathcal{Z};I}\big(\widetilde{\mathcal{Z}}^\circ_{\partial;I}\big)\subset\mathcal{Z}^\circ_I,\qquad \phi^\circ_{\mathcal{Z};I_0;I}=\phi^\circ_{\mathcal{Z};I}\big|_{(S^1)^{I_0}_\bullet\times\mathcal{Z}^\circ_I}\colon (S^1)^{I_0}_\bullet\times\mathcal{Z}^\circ_I\longrightarrow\mathcal{Z}^\circ_I,$$
$$\mu^\circ_{\mathcal{Z};I_0;I}\colon\big\{x\in\mathcal{Z}^\circ_I\colon \big(\mu^\circ_{\mathcal{Z};I}(x)\big)_i<\big(\mu^\circ_{\mathcal{Z};I}(x)\big)_j\ \forall\, i\in I_0,\ j\in I-I_0\big\}\longrightarrow\mathfrak{t}^*_{I_0;\bullet},\quad \mu^\circ_{\mathcal{Z};I_0;I}(x)=\mu^\circ_{\mathcal{Z};I}(x)\big|_{\mathfrak{t}_{I_0;\bullet}}.$$

Thus,
$$q_{\mathcal{Z}}\circ\phi^\circ_{\mathcal{Z};I_0;I}=\phi_{\mathcal{Z};I_0;I}\circ\big\{\mathrm{id}\times q_{\mathcal{Z}}\big\}\qquad\text{and}\qquad \mu^\circ_{\mathcal{Z};I_0;I}=\mu_{\mathcal{Z};I_0;I}\circ q_{\mathcal{Z}}\,.$$
Since
$$q_{\mathcal{Z}}\colon \mathrm{Dom}\big(\mu^\circ_{\mathcal{Z};I_0;I}\big)-\mathcal{Z}^\circ_{\partial;I}\longrightarrow\mathrm{Dom}\big(\mu_{\mathcal{Z};I_0;I}\big)-X_\partial$$
is a diffeomorphism, it is sufficient to show that the restriction of the $\phi^\circ_{\mathcal{Z};I_0;I}$-action to $\mu^{\circ\,-1}_{\mathcal{Z};I_0;I}(0)$ is free. Since $\mathcal{Z}^\circ_I$ is the quotient of $\widetilde{\mathcal{Z}}^\circ_I$ by the $\widetilde\phi_I$-action and
$$q_{\mathcal{Z}}\circ\widetilde\phi_{\mathcal{Z};I}\big|_{(S^1)^{I_0}_\bullet\times\widetilde{\mathcal{Z}}^\circ_I}=\phi^\circ_{\mathcal{Z};I_0;I}\circ\big\{\mathrm{id}\times q_{\mathcal{Z};I}\big\},$$
this follows from Lemma 4.15. □

5 Proof of Theorem 2

By Corollary 4.10, an N-fold cutting configuration $\mathscr{C}$ determines an SC symplectic configuration $\mathbf{X}(\mathscr{C})$ and a symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$ containing the SC symplectic variety $X_\emptyset$ associated with $\mathbf{X}(\mathscr{C})$ as an SC symplectic divisor. We show in Section 5.2 that $\mathscr{C}$ also gives rise to a one-parameter family $\pi\colon\mathcal{Z}'\longrightarrow\mathbb{C}$ of smoothings of $X_\emptyset$ so that $\mathcal{Z}'$ is a neighborhood of $X_\emptyset$ in $\mathcal{Z}$. This family of smoothings satisfies the last two claims of Theorem 2. Unlike the constructions of Sections 4.2-4.5, the construction of Sections 5.2 involves choices. However, these choices are deformation equivalent and the resulting one-parameter family of smoothings is well-defined up to deformations. For a family of N-fold cutting configurations, the choices involved in the construction of Sections 5.2 can be made systemically on sufficiently small neighborhoods of all cutting configurations and thus result in a continuously varying family of deformation equivalence classes of one-parameter families of smoothings. This establishes Theorem 2.

5.1 Geometric preliminaries

We begin with a lemma which enables us to refine open covers as in Definition 1.1. We then establish several local statements that are patched together in Section 5.2 to construct a one-parameter family of smoothings of $X_\emptyset$.

Lemma 5.1. *Let $\mathscr{C}$ be an N-fold Hamiltonian configuration for (X,ω) as in (1.3). There exists a $\mathscr{C}$-invariant open cover $(U'_I)_{I\in\mathcal{P}^*(N)}$ of X properly refining $(U_I)_{I\in\mathcal{P}^*(N)}$. For every open cover $(U'_I)_{I\in\mathcal{P}^*(N)}$ of X properly refining $(U_I)_{I\in\mathcal{P}^*(N)}$, there exists a $\mathscr{C}$-invariant open cover $(U''_I)_{I\in\mathcal{P}^*(N)}$ of X properly refining $(U_I)_{I\in\mathcal{P}^*(N)}$ and properly refined by $(U'_I)_{I\in\mathcal{P}^*(N)}$.*

Proof. We modify *Step 1* in the proof of [15, Theorem 36.1] to take into account the torus actions using the following observation. Suppose $I\in\mathcal{P}^*(N)$, $W\subset U_I$ is an open subset such that $\phi_I((S^1)^I_\bullet\times W)=W$, and $A\subset X$ is a closed subset such that $A\subset W$. Since X is normal, there exists an open subset $W'\subset X$ such that $A\subset W'$ and the closure $\overline{W'}$ of W' in X is contained in W. Since the ϕ_I-action is continuous, the subspace

$$W''\equiv\phi_I\big((S^1)^I_\bullet\times W'\big)\subset U_I$$

is open in U_I and thus in X. Since the group $(S^1)^I_\bullet$ is compact and $\phi_I((S^1)^I_\bullet\times W)=W$,

$$\overline{W''}=\phi_I\big((S^1)^I_\bullet\times\overline{W'}\big)\subset W\,,$$

where the closure is taken in X. Thus, $W''\subset X$ is an open subset such that

$$A\subset W'',\qquad \overline{W''}\subset W,\quad\text{and}\quad \phi_I\big((S^1)^I_\bullet\times W''\big)=W''. \tag{5.1}$$

For the remainder of this proof, we fix a total order $<$ on the subsets $I\subset[N]$ so that $I<I^*$ whenever $|I|>|I^*|$.

Suppose $I^*\in\mathcal{P}^*(N)$ and $(U'_I)_{I\in\mathcal{P}^*(N)}$ is a $\mathscr{C}$-invariant open cover of X refining $(U_I)_{I\in\mathcal{P}^*(N)}$ such that

$$\overline{U'_I}\subset U_I\qquad\forall\ I<I^*. \tag{5.2}$$

Since $(U'_I)_{I\in\mathcal{P}^*(N)}$ is an open cover of X, the closed subset

$$A\equiv X-\bigcup_{\substack{I\in\mathcal{P}^*(N)\\ I\neq I^*}}U'_I$$

is contained in $U'_{I^*}\subset U_{I^*}$. By the previous paragraph with $I=I^*$ and $W=U_{I^*}$, there exists an open subset $W''\subset X$ satisfying (5.1). Replacing the open subset U'_{I^*} with W'', we obtain a $\mathscr{C}$-invariant open cover $(U'_I)_{I\in\mathcal{P}^*(N)}$ of X refining $(U_I)_{I\in\mathcal{P}^*(N)}$ such that the inclusion in (5.2) holds for all $I\leqslant I^*$. Continuing in this way, we obtain a $\mathscr{C}$-invariant open cover $(U'_I)_{I\in\mathcal{P}^*(N)}$ of X properly refining $(U_I)_{I\in\mathcal{P}^*(N)}$.

We next establish the second claim. Suppose $I^*\in\mathcal{P}^*(N)$ and $(U''_I)_{I\in\mathcal{P}^*(N)}$ is a $\mathscr{C}$-invariant open cover of X refining $(U_I)_{I\in\mathcal{P}^*(N)}$ and properly refined by $(U'_I)_{I\in\mathcal{P}^*(N)}$ such that

$$\overline{U''_I}\subset U_I\qquad\forall\ I<I^*. \tag{5.3}$$

By the observation in the first paragraph applied with $I=I^*$, $A=\overline{U'_{I^*}}$, and $W=U_{I^*}$, there exists an open subset $W''\subset X$ satisfying (5.1). Replacing the open subset U''_{I^*} with W'', we obtain a $\mathscr{C}$-invariant open cover $(U''_I)_{I\in\mathcal{P}^*(N)}$ of X refining $(U_I)_{I\in\mathcal{P}^*(N)}$ and properly refined by $(U'_I)_{I\in\mathcal{P}^*(N)}$ such that the inclusion in (5.3) holds for all $I\leqslant I^*$. Continuing in this way, we obtain a $\mathscr{C}$-invariant open cover $(U''_I)_{I\in\mathcal{P}^*(N)}$ of X properly refining $(U_I)_{I\in\mathcal{P}^*(N)}$ and properly refined by $(U'_I)_{I\in\mathcal{P}^*(N)}$. $\square$

For a $\mathscr{C}$-invariant open cover $(U'_I)_{I\in\mathcal{P}^*(N)}$ of X properly refining $(U_I)_{I\in\mathcal{P}^*(N)}$, we denote by

$$\widetilde{\mathcal{Z}}'^{\circ}_I\equiv\widetilde{\mathcal{Z}}^{\circ}_I\cap\big(U'_I\times\mathbb{C}^I\big),\quad \mathcal{Z}'^{\circ}_I\subset\mathcal{Z}^{\circ}_I,\quad \widetilde{\mathcal{Z}}'^{\circ}_{I,J}\subset\widetilde{\mathcal{Z}}'^{\circ}_I\cap\widetilde{\mathcal{Z}}^{\circ}_{I,J}\subset\widetilde{\mathcal{Z}}^{\circ}_I,\quad \mathcal{Z}'^{\circ}_{I,J}\subset\mathcal{Z}'^{\circ}_I\cap\mathcal{Z}^{\circ}_{I,J}\subset\mathcal{Z}^{\circ}_I\,,$$

the spaces as in (4.14), (4.16), (4.26), and (4.33) corresponding to the restriction $\mathscr{C}'$ of $\mathscr{C}$ to $(U'_I)_{I\in\mathcal{P}^*(N)}$ defined by (3.6). For each $I\in\mathcal{P}^*(N)$, the torus action $\phi^{\circ}_{\mathcal{Z};I}$ in (4.57) preserves the subspaces $\mathcal{Z}'^{\circ}_I$ and $\mathcal{Z}'^{\circ}_{I,J}$ of $\mathcal{Z}^{\circ}_I$.

Lemma 5.2. *Let $(U'_I)_{I\in\mathcal{P}^*(N)}$ be a $\mathscr{C}$-invariant open cover of X properly refining $(U_I)_{I\in\mathcal{P}^*(N)}$. There exists a tuple $(f_I\colon \mathcal{Z}'^{\circ}_I \longrightarrow \mathbb{R}^+)_{I\in\mathcal{P}^*(N)}$ of smooth functions such that each f_I is $\phi^{\circ}_{\mathcal{Z};I}$-invariant,*

$$f_I\big([x,(z_j)_{j\in I}]\big) = f_J\big(\Theta_{J,I}\big([x,(z_j)_{j\in I}]\big)\big)\prod_{j\in J-I}\sqrt{2(\mu_J(x))_{ij}+|z_i|^2} \qquad \forall\,\big[x,(z_j)_{j\in I}\big]\in\mathcal{Z}'^{\circ}_{I,J},\ i\in I\subset J\subset[N], \tag{5.4}$$

with $(\mu_J(x))_{ij}$ as in (4.3).

Proof. Choose a total order $<$ on the subsets $I\subset[N]$ so that $I<I^*$ whenever $|I|>|I^*|$. Suppose $I^*\in\mathcal{P}^*(N)$ and we have constructed

- a $\mathscr{C}$-invariant open cover $(U''_I)_{I\in\mathcal{P}^*(N)}$ of X refining $(U_I)_{I\in\mathcal{P}^*(N)}$ and properly refined by the open cover $(U'_I)_{I\in\mathcal{P}^*(N)}$,
- smooth functions $f_I\colon \mathcal{Z}''^{\circ}_I\longrightarrow\mathbb{R}^+$ for all $I<I^*$ such that each f_I is $\phi^{\circ}_{\mathcal{Z};I}$-invariant and the equality in (5.4) holds for all $[x,(z_j)_{j\in I}]\in\mathcal{Z}''^{\circ}_{I,J}$ and $i\in I\subset J$ with $I<I^*$.

By Lemma 5.1, there exists a $\mathscr{C}$-invariant open cover $(U'''_I)_{I\in\mathcal{P}^*(N)}$ of X properly refining $(U''_I)_{I\in\mathcal{P}^*(N)}$ and properly refined by $(U'_I)_{I\in\mathcal{P}^*(N)}$. Let

$$W'' = \bigcup_{I^*\subsetneq J\subset[N]}\mathcal{Z}''^{\circ}_{I^*,J}, \qquad W''' = \bigcup_{I^*\subsetneq J\subset[N]}\mathcal{Z}'''^{\circ}_{I^*,J}.$$

We define $f\colon W''\longrightarrow\mathbb{R}^+$ by choosing $i\in I^*$ and setting

$$f\big([x,(z_j)_{j\in I^*}]\big) = f_J\big(\Theta_{J,I^*}\big([x,(z_j)_{j\in I^*}]\big)\big)\prod_{j\in J-I^*}\sqrt{2(\mu_J(x))_{ij}+|z_i|^2} \qquad \forall\,\big[x,(z_j)_{j\in I^*}\big]\in\mathcal{Z}''^{\circ}_{I^*,J},\ I^*\subsetneq J\subset[N]. \tag{5.5}$$

By (4.14) and Definition 1.1(c), f is well-defined on each $\mathcal{Z}''^{\circ}_{I^*,J}$ and is independent of the choice of $i\in I^*$. By (a) and (b) in Definition 1.1, (4.35), and the last inductive assumption, f is well-defined on the overlaps. Since each map Θ_{J,I^*} is $(\phi^{\circ}_{\mathcal{Z};J},\phi^{\circ}_{\mathcal{Z};I^*})$-equivariant with respect to the homomorphism (4.25) with $I=I^*$ and each map f_J is $\phi^{\circ}_{\mathcal{Z};J}$-invariant, the map f is $\phi^{\circ}_{\mathcal{Z};I^*}$-invariant.

Since $\overline{U'''_I}\subset U''_I$ for all $I\in\mathcal{P}^*(N)$, the closure of $\widetilde{\mathcal{Z}}'''^{\circ}_{I^*,J}$ in $\widetilde{\mathcal{Z}}''^{\circ}_{I^*}$ is contained in $\widetilde{\mathcal{Z}}''^{\circ}_{I^*,J}$ for all $J\in\mathcal{P}_{I^*}(N)$. Thus, the closure $\overline{W'''}$ of W''' in $\mathcal{Z}''^{\circ}_{I^*}$ is contained in W''. Therefore, there exists a smooth function

$$f_{I^*}\colon \mathcal{Z}''^{\circ}_{I^*}\longrightarrow\mathbb{R}^+ \qquad \text{s.t.} \quad f_{I^*}\big|_{W'''} = f\,. \tag{5.6}$$

Since the group $(S^1)^{I^*}_{\bullet}$ is compact, we can make f_{I^*} $\phi^{\circ}_{\mathcal{Z};I^*}$-invariant by averaging it over the group action. Since f is $\phi^{\circ}_{\mathcal{Z};I^*}$-invariant, this does not change f_{I^*} over W''' and so (5.6) still holds. By (5.6) and (5.5), the equality in (5.4) with $I=I^*$ holds for all $[x,(z_j)_{j\in I}]\in\mathcal{Z}'''^{\circ}_{I^*,J}$ and $J\in\mathcal{P}_{I^*}(N)$.

The open cover $(U'''_I)_{I\in\mathcal{P}^*(N)}$ of X and the tuple $(f_I)_{I\leqslant I^*}$ of smooth functions satisfy the inductive assumptions of the bullet points with U''_I replaced by U'''_I for all $I\leqslant I^*$. Continuing in this way, we obtain a $\mathscr{C}$-invariant open cover $(U''_I)_{I\in\mathcal{P}^*(N)}$ of X refining $(U_I)_{I\in\mathcal{P}^*(N)}$ and properly refined by $(U'_I)_{I\in\mathcal{P}^*(N)}$ and a tuple $(f_I\colon\mathcal{Z}''^{\circ}_I\longrightarrow\mathbb{R}^+)_{I\in\mathcal{P}^*(N)}$ of smooth functions so that each f_I is $\phi^{\circ}_{\mathcal{Z};I}$-invariant and the equality in (5.4) holds for all $[x,(z_j)_{j\in I}]\in\mathcal{Z}''^{\circ}_{I,J}$ and $i\in I\subset J\subset[N]$. Restricting each f_I to $\mathcal{Z}'^{\circ}_I$, we obtain a desired tuple of functions. □

For $I\in\mathcal{P}^*(N)$, we define

$$\widetilde{\mathcal{Z}}^\circ_{0;I}=\bigcup_{\emptyset\neq I_0\subset I}\widetilde{\mathcal{Z}}^\circ_{I_0;I}\subset\widetilde{\mathcal{Z}}^\circ_I,\qquad \widetilde{\mathcal{Z}}^\circ_{\partial;I}=\bigcup_{\substack{I_0\subset I\\ |I_0|=2}}\widetilde{\mathcal{Z}}^\circ_{I_0;I}\subset\widetilde{\mathcal{Z}}^\circ_{0;I}\,.$$

Lemma 5.3. *Suppose $I\in\mathcal{P}^*(N)$, $\widetilde{f}_I\colon\widetilde{\mathcal{Z}}^\circ_I\longrightarrow\mathbb{R}^+$ is a smooth function, and*

$$\widetilde{\pi}_{\mathcal{Z};I}\colon\widetilde{\mathcal{Z}}^\circ_I\longrightarrow\mathbb{C},\qquad \widetilde{\pi}_{\mathcal{Z};I}\big(x,(z_j)_{j\in I}\big)=\widetilde{f}_I(x,(z_j)_{j\in I})\prod_{j\in I}z_j\,. \tag{5.7}$$

Then there exists an open subset $\widetilde{W}_I\subset\widetilde{\mathcal{Z}}^\circ_I$ containing $\widetilde{\mathcal{Z}}^\circ_{0;I}$ such that the homomorphism

$$\mathrm{d}_{\widetilde{x}}\widetilde{\pi}_{\mathcal{Z};I}\colon\ker\mathrm{d}_{\widetilde{x}}\pi_1\big|_{T_{\widetilde{x}}\widetilde{\mathcal{Z}}^\circ_I}\longrightarrow\mathbb{C}, \tag{5.8}$$

where $\pi_1\colon U_I\times\mathbb{C}^I\longrightarrow U_I$ is the projection, is surjective for all $\widetilde{x}\in\widetilde{W}_I-\widetilde{\mathcal{Z}}^\circ_{\partial;I}$.

Proof. Choose $i\in I$ and define

$$h_{i;I}\colon\mathbb{C}^I\longrightarrow\mathbb{R}^{I-i},\qquad h_{i;I}\big((z_j)_{j\in I}\big)=\frac{1}{2}\big(|z_j|^2-|z_i|^2\big)_{j\in I-i}\,.$$

Thus,

$$\ker\mathrm{d}_{\widetilde{x}}\pi_1\big|_{T_{\widetilde{x}}\widetilde{\mathcal{Z}}^\circ_I}=\ker\mathrm{d}_z h_{i;I}\subset\mathbb{C}^I\subset T_xX\oplus T_z\mathbb{C}^I=T_{\widetilde{x}}\big(U_I\times\mathbb{C}^I\big)\quad\forall\,\widetilde{x}\equiv(x,z)\in\widetilde{\mathcal{Z}}^\circ_I. \tag{5.9}$$

Let $\widetilde{x}=(x,z)$ be an element of $\widetilde{\mathcal{Z}}^\circ_I-\widetilde{\mathcal{Z}}^\circ_{\partial;I}$ and $z=(z_j)_{j\in I}$. Since $\widetilde{x}\notin\widetilde{\mathcal{Z}}^\circ_{\partial;I}$, $z_j=0$ for at most one element $j\in I$.

Suppose $z_{j^*}=0$ for some $j^*\in I$. By (5.9), $\ker\mathrm{d}_{\widetilde{x}}\pi_1|_{T_{\widetilde{x}}\widetilde{\mathcal{Z}}^\circ_I}$ then contains the component $\mathbb{C}^{\{j^*\}}\subset\mathbb{C}^I$ and

$$\mathrm{d}_{\widetilde{x}}\widetilde{\pi}_{\mathcal{Z};I}=\bigg(\widetilde{f}_I(\widetilde{x})\prod_{j\in I-j^*}z_j\bigg)\mathrm{Id}\colon\mathbb{C}^{\{j^*\}}\longrightarrow\mathbb{C}\,.$$

This homomorphism is surjective if $z_j\neq0$ for all $j\in I-j^*$.

From now on, we assume that $z_j\neq0$ for all $j\in I$. Denote by

$$\mathbb{R}^I\subset\mathbb{C}^I\subset T_{\widetilde{x}}\big(U_I\times\mathbb{C}^I\big)$$

the radial tangent directions. Let

$$\mathbb{R}^I_{\widetilde{x}}=\big(\ker\mathrm{d}_{\widetilde{x}}\pi_1\big|_{T_{\widetilde{x}}\widetilde{\mathcal{Z}}^\circ_I}\big)\cap\mathbb{R}^I$$

and $V_{\widetilde{x}}\subset T_{\widetilde{x}}\widetilde{\mathcal{Z}}^\circ_I$ be the span of $\mathbb{R}^I_{\widetilde{x}}$ and of the angular tangent space in the i-th $\mathbb{C}$-factor $\mathbb{C}^{\{i\}}\subset\mathbb{C}$. We will show that the homomorphism

$$\mathrm{d}_{\widetilde{x}}\widetilde{\pi}_{\mathcal{Z};I}\colon V_{\widetilde{x}}\longrightarrow\mathbb{C}$$

is an isomorphism if $\widetilde{x}$ lies in a sufficiently small neighborhood $\widetilde{W}_I \subset \widetilde{\mathcal{Z}}_I^\circ$ of $\widetilde{\mathcal{Z}}_{0;I}^\circ$.

Define

$$\begin{aligned} \pi_{\mathbb{C}^I}\colon \mathbb{C}^I \longrightarrow \mathbb{C}, \qquad & \pi_{\mathbb{C}^I}^{\mathbb{R}}\colon \mathbb{C}^I \longrightarrow \mathbb{R}, \qquad & H_{i;I}\colon \mathbb{C}^I \longrightarrow \mathbb{R}^I \equiv \mathbb{R}^{I-i}\times\mathbb{R}, \\ \pi_{\mathbb{C}^I}\big((z_j)_{j\in I}\big) = \prod_{j\in I} z_j, \qquad & \pi_{\mathbb{C}^I}^{\mathbb{R}}\big((z_j)_{j\in I}\big) = \prod_{j\in I}|z_j|, \qquad & H_{i;I}(z) = \big(h_{i;I}(z), \pi_{\mathbb{C}}^{\mathbb{R}}(z)\big). \end{aligned}$$

The determinant of the restriction of $\mathrm{d}_z H_{i;I}$ to the radial tangent directions $\mathbb{R}^I \subset T_z\mathbb{C}^I$ is given by

$$\det\big(\mathrm{d}_z H_{i;I}\colon \mathbb{R}^I \longrightarrow \mathbb{R}^I\big) = \sum_{j\in I}\prod_{k\in I-j}|z_k|^2 .$$

This implies that there exists a universal constant δ_I (dependent only on $|I|$) such that

$$\big|\mathrm{d}_z \pi_{\mathbb{C}^I}^{\mathbb{R}}\colon \mathbb{R}^I_{\widetilde{x}} \longrightarrow \mathbb{R}\big| \geqslant \delta_I \sum_{j\in I}\prod_{k\in I-j}|z_k|$$

with respect to the standard norms on $\mathbb{R}^I_{\widetilde{x}} \subset \mathbb{C}^I$ and $\mathbb{R}\subset\mathbb{C}$. Since the differential of the angular component of $\pi_{\mathbb{C}^I}$ along the angular direction in $V_{\widetilde{x}}$ satisfies the same bound, it follows that the homomorphism

$$\mathrm{d}_{\widetilde{x}}\pi_{\mathbb{C}^I} + D\colon V_{\widetilde{x}} \longrightarrow \mathbb{C}$$

is an isomorphism whenever $D\colon V_{\widetilde{x}} \longrightarrow \mathbb{C}$ is a homomorphism such that

$$\|D\| < \delta_I \sum_{j\in I}\prod_{k\in I-j}|z_k|.$$

By the definition of $\widetilde{\pi}_{\mathcal{Z};I}$,

$$\mathrm{d}_{\widetilde{x}}\widetilde{\pi}_{\mathcal{Z};I} = \widetilde{f}_I(\widetilde{x})\,\mathrm{d}_z\pi_{\mathbb{C}^I} + \bigg(\prod_{k\in I} z_k\bigg)\mathrm{d}_{\widetilde{x}}\widetilde{f}_I.$$

Thus, the restriction of $\mathrm{d}_{\widetilde{x}}\widetilde{\pi}_{\mathcal{Z};I}$ to $V_{\widetilde{x}}$ is an isomorphism if

$$\frac{\|\mathrm{d}_{\widetilde{x}}\widetilde{f}_I\colon V_{\widetilde{x}} \longrightarrow \mathbb{C}\|}{\widetilde{f}_I(\widetilde{x})}\big|z_j\big| < \delta_I \qquad \text{for some} \ \ j\in I.$$

This specifies an open subset $\widetilde{W}_I \subset \widetilde{\mathcal{Z}}_I^\circ$ containing $\widetilde{\mathcal{Z}}_{0;I}^\circ$ so that the homomorphism (5.8) is surjective on $\widetilde{W}_I - \widetilde{\mathcal{Z}}_{\partial;I}^\circ$. □

For $I\in\mathcal{P}^*(N)$, we define

$$\widetilde{\mathcal{Z}}_{I;i}^{>} = \big\{\big(x,(z_j)_{j\in I}\big)\in\widetilde{\mathcal{Z}}_I^\circ\colon\ z_i\in\mathbb{C}^*,\ z_j\in\mathbb{R}^+\ \forall\, j\in I-i\big\} \subset U_I\times\mathbb{C}^I. \tag{5.10}$$

By (4.12), $\widetilde{\mathcal{Z}}_{I;i}^{>}$ is a slice for the $\widetilde{\phi}_I$-action on $\widetilde{\mathcal{Z}}_I^\circ - \widetilde{\mathcal{Z}}_{0;I}^\circ$. By Lemma 4.6, $\widetilde{\mathcal{Z}}_{I;i}^{>}$ is a smooth submanifold of $U_I\times\mathbb{C}^I$ of $\dim_{\mathbb{R}}X+2$.

Lemma 5.4. *Let I, $\widetilde{f}_I$, and $\widetilde{\pi}_{\mathcal{Z};I}$ be as in Lemma 5.3. Then there exists an open subset $\widetilde{W}_I\subset\widetilde{\mathcal{Z}}_I^\circ$ containing $\widetilde{\mathcal{Z}}_{0;I}^\circ$ such that for every $i\in I$ the restriction of the smooth map*

$$\pi_1\times\widetilde{\pi}_{\mathcal{Z};I}\colon\widetilde{\mathcal{Z}}_{I;i}^{>}\longrightarrow U_I\times\mathbb{C} \tag{5.11}$$

to $\widetilde{\mathcal{Z}}_{I;i}^{>}\cap\widetilde{W}_I$ is a diffeomorphism onto the intersection of $U_I\times\mathbb{C}^*$ with an open neighborhood of $U_I\times 0$ in $U_I\times\mathbb{C}$.

Proof. Let $\widetilde{W}_I\subset\widetilde{\mathcal{Z}}_I^\circ$ be an open subspace provided by Lemma 5.3. By Lemma 5.3 and its proof, the homomorphisms

$$\mathrm{d}_{\widetilde{x}}\widetilde{\pi}_{\mathcal{Z};I}\colon\ker\mathrm{d}_{\widetilde{x}}\pi_1\big|_{T_{\widetilde{x}}\widetilde{\mathcal{Z}}_I^\circ}\longrightarrow\mathbb{C}\qquad\text{and}\qquad\mathrm{d}_{\widetilde{x}}\pi_1\colon T_{\widetilde{x}}\widetilde{\mathcal{Z}}_I^\circ\longrightarrow T_{\pi_1(\widetilde{x})}X$$

are surjective for all $\widetilde{x}\in\widetilde{W}_I-\widetilde{\mathcal{Z}}_{\partial;I}^\circ$. For dimensional reasons, this implies that the differential of (5.11) is an isomorphism for all $\widetilde{x}\in\widetilde{\mathcal{Z}}_{I;i}^{>}\cap\widetilde{W}_I$. From the Inverse Function Theorem [19, Theorem 1.30], we then conclude that (5.11) is a local diffeomorphism from $\widetilde{\mathcal{Z}}_{i;I}^{>}\cap\widetilde{W}_I$ onto an open subset of $U_I\times\mathbb{C}^*$.

We next show that (5.11) is injective on $\widetilde{\mathcal{Z}}_{I;i}^{>}\cap\widetilde{W}_I$, after possibly shrinking the neighborhood $\widetilde{W}_I$ of $\widetilde{\mathcal{Z}}_{0;I}^\circ$. For each $x\in U_I$, let

$$\varrho_{I;i}(x)=\max\bigl\{-2(\mu_I(x))_{ij}\colon j\in I\bigr\}\in\mathbb{R}^{\geqslant 0},$$

with $(\mu_I(x))_{ij}$ given by (4.3). For $\lambda\in\mathbb{C}^*$ and $j\in I$, define

$$g_{\lambda;I;i;j}\colon\bigl\{(x,\varrho)\in U_I\times\mathbb{R}\colon\ \varrho>\varrho_{I;i}(x)\bigr\}\longrightarrow\mathbb{C}^*,\quad g_{\lambda;I;i;j}(x,\varrho)=\begin{cases}\frac{\lambda}{|\lambda|}\sqrt{\varrho}, & \text{if } i=j;\\ \sqrt{2(\mu_I(x))_{ij}+\varrho}, & \text{if } j\in I-i.\end{cases}$$

By (4.14), (5.10), and (5.7), every preimage $\widetilde{x}\equiv(x,(z_j)_{j\in I})$ of $(x,\lambda)\in U_I\times\mathbb{C}^*$ under (5.11) is of the form

$$\bigl(x,(z_j)_{j\in I}\bigr)=\bigl(x,\bigl(g_{\lambda;I;i;j}(x,\varrho)\bigr)_{j\in I}\bigr)$$

with $\varrho=\varrho_i(\lambda,x)$ being a solution of the equation

$$\widetilde{f}_I\bigl(x,\bigl(g_{\lambda;I;i;j}(x,\varrho)\bigr)_{j\in I}\bigr)^2\varrho\prod_{j\in I-i}g_{\lambda;I;i;j}(x,\varrho)^2=|\lambda|^2,\quad\varrho\in\bigl(\varrho_{I;i}(x),\infty\bigr). \tag{5.12}$$

At $\varrho=\varrho_{I;i}(x)$, the first non-vanishing derivative of the left-hand side of (5.12) is strictly positive. Thus, there exist continuous functions

$$\delta_{I;i},\varepsilon_{I;i}\colon U_I\longrightarrow\mathbb{R}^+$$

such that the left-hand side of (5.12) as a function of ϱ is injective on $[\varrho_{I;i}(x),\varrho_{I;i}(x)+\delta_{I;i}(x)]$ and the image of this interval contains the interval $[0,\varepsilon_{I;i}(x)]$.

By the previous paragraph, the map (5.11) is injective on the intersection of $\widetilde{\mathcal{Z}}_{I;i}^{>}$ with

$$\widetilde{W}'_{i;I}\equiv\bigl\{(x,(z_j)_{j\in I})\in\widetilde{W}_I\colon\ \min_{j\in I}|z_j|^2<\delta_{I;i}(x)\bigr\}$$

and the image of this intersection under (5.11) contains

$$\big\{(x,z)\in U_I\times\mathbb{C}^*:\ |z|<\varepsilon_{I;i}(x)\big\}\subset U_I\times\mathbb{C}\,.$$

Since $\widetilde{W}'_{I;i}\subset\widetilde{W}_I$ is an open neighborhood of $\widetilde{\mathcal{Z}}^\circ_{0;I}$, the proof is completed by taking $\widetilde{W}_I$ in the statement of Lemma 5.4 to be the intersection of the sets $\widetilde{W}'_{I;i}$ over $i\in I$. $\square$

Corollary 5.5. *Let I, $\widetilde{f}_I$, and $\widetilde{\pi}_{\mathcal{Z};I}$ be as in Lemma 5.3. Then there exists an open subset $\widetilde{W}_I\subset\widetilde{\mathcal{Z}}^\circ_I$ containing $\widetilde{\mathcal{Z}}^\circ_{0;I}$ such that*

$$\widetilde{\omega}_I^n\big|_{T_{\widetilde{x}}(\widetilde{\pi}^{-1}_{\mathcal{Z};I}(\lambda))}\neq 0\quad\forall\ \widetilde{x}\in\widetilde{\pi}^{-1}_{\mathcal{Z};I}(\lambda),\ \lambda\in\mathbb{C}^*\,,\qquad\textit{where}\quad 2n=\dim_{\mathbb{R}}X.$$

Proof. Let $\lambda\in\mathbb{C}^*$, $\widetilde{W}_I\subset\widetilde{\mathcal{Z}}^\circ_I$ be an open subspace provided by Lemma 5.4, $i\in I$, and

$$\widetilde{\mathcal{Z}}^>_{\lambda;I;i}=\widetilde{\pi}^{-1}_{\mathcal{Z};I}(\lambda)\cap\widetilde{\mathcal{Z}}^>_{I;i}.$$

In particular, $\widetilde{\mathcal{Z}}^>_{\lambda;I;i}$ is a smooth submanifold of $U_I\times\mathbb{C}^I$ and the map

$$\pi_1\colon\widetilde{\mathcal{Z}}^>_{\lambda;I;i}\longrightarrow U_I$$

is a diffeomorphism onto an open subset $U_{\lambda;I;i}$. Thus, there exists a smooth function

$$g_{\lambda;I;i}\colon U_{\lambda;I;i}\longrightarrow\big\{(z_j)_{j\in I}\in\mathbb{C}^I:\ z_i\in\frac{\lambda}{|\lambda|}\mathbb{R}^+,\ z_j\in\mathbb{R}^+\ \forall\, j\in I-i\big\}$$
$$\text{s.t.}\qquad\widetilde{\mathcal{Z}}^>_{\lambda;I;i}=\big\{\big(x,g_{\lambda;I;i}(x)\big):\ x\in U_{\lambda;I;i}\big\}.$$

Since the argument of every component of $g_{\lambda;I;i}\colon U_{\lambda;I;i}\longrightarrow\mathbb{C}^I$ is fixed,

$$\big\{\mathrm{id}\times g_{\lambda;I;i}\big\}^*\widetilde{\omega}_I=\big\{\mathrm{id}\times g_{\lambda;I;i}\big\}^*\pi_1^*\omega+\big\{\mathrm{id}\times g_{\lambda;I;i}\big\}^*\pi_2^*\omega_{\mathbb{C}^I}=\omega|_{U_{\lambda;I;i}}+g^*_{\lambda;I;i}\omega_{\mathbb{C}^I}=\omega|_{U_{\lambda;I;i}}.\tag{5.13}$$

This implies that

$$\big\{\mathrm{id}\times g_{\lambda;I;i}\big\}^*\big(\widetilde{\omega}_I^n\big|_{T_{\widetilde{x}}(\widetilde{\pi}^{-1}_{\mathcal{Z};I}(\lambda))}\big)=\big\{\mathrm{id}\times g_{\lambda;I;i}\big\}^*\big(\widetilde{\omega}_I^n\big|_{\widetilde{x}}\big)=\omega^n\big|_{\pi_1(\widetilde{x})}\neq 0\qquad\forall\ \widetilde{x}\in\widetilde{\mathcal{Z}}^>_{\lambda;I;i},$$

since ω is a symplectic form. Since the $\widetilde{\phi}_I$-action consists of a family of symplectomorphisms on $U_I\times\mathbb{C}^I$, this establishes the claim. $\square$

5.2 One-parameter family of smoothings

By Lemma 5.1, there exists a $\mathscr{C}$-invariant open cover $(U'_I)_{I\in\mathcal{P}^*(N)}$ of X properly refining $(U_I)_{I\in\mathcal{P}^*(N)}$. By Lemma 5.2, there exists a tuple $(f_I\colon\mathcal{Z}'^\circ_I\longrightarrow\mathbb{R}^+)_{I\in\mathcal{P}^*(N)}$ of smooth functions such that each f_I is $\phi^\circ_{\mathcal{Z};I}$-invariant and (5.4) holds. Define

$$\pi\colon\mathcal{Z}\longrightarrow\mathbb{C}\quad\text{by}\quad\pi\big(\big[[x,(z_i)_{i\in I}]\big]\big)=f_I\big([x,(z_i)_{i\in I}]\big)\prod_{i\in I}z_i\quad\forall\,\big[x,(z_i)_{i\in I}\big]\in\mathcal{Z}'^\circ_I,\ I\in\mathcal{P}^*(N).$$

By (4.12), π is independent of the choice of representative for an element of $\mathcal{Z}'^\circ_I$ for $I\in\mathcal{P}^*(N)$ fixed. By (4.36) and (5.4), π is also independent of the choice of $I\in\mathcal{P}^*(N)$ and so is well-defined. Since all f_I take values in $\mathbb{R}^+$,

$$\mathcal{Z}_0\equiv\pi^{-1}(0)=\mathcal{Z}_{\{1\}}\cup\ldots\cup\mathcal{Z}_{\{N\}}=X_1\cup\ldots\cup X_N\equiv X_\emptyset\,;$$

the second equality above holds by (4.41).

Lemma 5.6. *There exists a neighborhood $\mathcal{Z}'$ of $\mathcal{Z}_0 = X_\emptyset$ in $\mathcal{Z}$ such that $(\mathcal{Z}', \omega_{\mathcal{Z}}|_{\mathcal{Z}'}, \pi|_{\mathcal{Z}'})$ is a one-parameter family of smoothings of the SC symplectic variety $X_\emptyset$ associated with the SC symplectic configuration $\mathbf{X}(\mathscr{C})$ of Corollary 4.13.*

Proof. For each $I \in \mathcal{P}^*(N)$, define

$$\begin{aligned} \widetilde{f}_I\colon \widetilde{\mathcal{Z}}_I'^{\circ} &\longrightarrow \mathbb{R}^+, & \widetilde{f}_I\big(x,(z_i)_{i\in I}\big) &= f_I\big([x,(z_i)_{i\in I}]\big), \\ \widetilde{\pi}_{\mathcal{Z};I}\colon \widetilde{\mathcal{Z}}_I'^{\circ} &\longrightarrow \mathbb{C}, & \widetilde{\pi}_{\mathcal{Z};I}\big(x,(z_i)_{i\in I}\big) &= \pi\big([[x,(z_i)_{i\in I}]]\big). \end{aligned} \tag{5.14}$$

Let $\widetilde{W}_I \subset \widetilde{\mathcal{Z}}_I'^{\circ}$ be an open subset as in Lemmas 5.3 and 5.4 with $(U_I)_{I\in\mathcal{P}^*(N)}$ replaced by $(U_I')_{I\in\mathcal{P}^*(N)}$. By Lemma 4.8, the image $W_I \subset \mathcal{Z}$ of $\widetilde{W}_I$ under the quotient map

$$q_{\mathcal{Z}} \circ q_{\mathcal{Z};I}\colon \widetilde{\mathcal{Z}}_I'^{\circ} \longrightarrow \mathcal{Z} \tag{5.15}$$

is an open subset. The union $\mathcal{Z}'$ of the open subsets W_I taken over all $I \in \mathcal{P}^*(N)$ is an open neighborhood of $\mathcal{Z}_0 = X_\emptyset$ in $\mathcal{Z}$.

Since $\widetilde{\pi}_{\mathcal{Z};I}$ factors through π, the surjectivity of the homomorphism (5.8) implies that π is a submersion on W_I outside of

$$X_\partial = \bigcup_{\substack{I\subset[N]\\ |I|=2}} \mathcal{Z}_I \subset \mathcal{Z}\,.$$

For any $\lambda \in \mathbb{C}^*$, the pullback of $\omega_{\mathcal{Z}}|_{\pi^{-1}(\lambda)}$ by (5.15) is $\widetilde{\omega}_I|_{T(\widetilde{\pi}_{\mathcal{Z};I}^{-1}(\lambda))}$. By Corollary 5.5, $\omega_{\mathcal{Z}}^n|_{\pi^{-1}(\lambda)}$ thus does not vanish over W_I and so the restriction of $\omega_{\mathcal{Z}}$ to $\pi^{-1}(\lambda) \cap W_I$ is nondegenerate. Thus, $(\mathcal{Z}', \omega_{\mathcal{Z}}|_{\mathcal{Z}'}, \pi|_{\mathcal{Z}'})$ is a nearly regular symplectic fibration in the sense of Definition 2.6 with $\{\pi|_{\mathcal{Z}'}\}^{-1}(0) = X_\emptyset$. $\square$

For each $I \in \mathcal{P}^*(N)$, the subspace $\widetilde{\mathcal{Z}}_{0;I}'^{\circ} \subset \widetilde{\mathcal{Z}}_I'^{\circ}$ is preserved by the $\widetilde{\phi}_{\mathcal{Z};I}$-action in (4.53). By replacing $\widetilde{W}_I$ in the proof of Lemma 5.6 with

$$\bigcap_{g\in(S^1)^I_\bullet} \widetilde{\phi}_{\mathcal{Z};I}\big(\{g\}\times\widetilde{W}_I\big) \subset U_I' \times \mathbb{C}^I\,,$$

we can thus assume that $\widetilde{W}_I$ is $\widetilde{\phi}_{\mathcal{Z};I}$-invariant. This implies that the subspace $W_I \subset U_{\mathcal{Z};I}$ is preserved by the $\phi_{\mathcal{Z};I}$-action in (4.58). The collection $(W_I)_{I\in\mathcal{P}^*(N)}$ is then a $\mathscr{C}_{\mathcal{Z}}$-invariant open cover of the subspace $\mathcal{Z}'$ of $\mathcal{Z}$ (corresponding to $(U'_{\mathscr{X};t;I})_{I\in\mathcal{P}^*(N)}$ in Theorem 2). We denote by

$$\mathscr{C}'_{\mathcal{Z}} \equiv \big(W_I, \phi'_{\mathcal{Z};I}, \mu'_{\mathcal{Z};I}\big)_{I\in\mathcal{P}^*(N)}$$

the restriction of $\mathscr{C}_{\mathcal{Z}}$ to $(W_I)_{I\in\mathcal{P}^*(N)}$.

Lemma 5.7. *For every $\lambda \in \mathbb{C}$, the restriction $\mathscr{C}'_{\mathcal{Z}}|_{\mathcal{Z}'\cap\pi^{-1}(\lambda)}$ of $\mathscr{C}'_{\mathcal{Z}}$ to $\mathcal{Z}' \cap \pi^{-1}(\lambda)$ is an N-fold Hamiltonian configuration for $(\mathcal{Z}'\cap\pi^{-1}(\lambda), \omega_{\mathcal{Z}}|_{\mathcal{Z}'\cap\pi^{-1}(\lambda)})$. It is a cutting configuration if $\lambda \in \mathbb{C}^*$.*

Proof. By Lemma 5.6, $\omega_{\mathcal{Z}}|_{\mathcal{Z}'\cap\pi^{-1}(\lambda)}$ is a symplectic form on $\mathcal{Z}'\cap\pi^{-1}(\lambda)$. Since the function f_I is $\phi^{\circ}_{\mathcal{Z};I}$-invariant for each $I \in \mathcal{P}^*(N)$, the restriction of π to W_I is $\phi'_{\mathcal{Z};I}$-invariant. Thus, $\mathcal{Z}'\cap\pi^{-1}(\lambda)$ is $\mathscr{C}'_{\mathcal{Z}}$-invariant. This establishes the first claim. If $\lambda \in \mathbb{C}^*$, then

$$\pi^{-1}(\lambda) \subset \mathcal{Z} - X_\emptyset \subset \mathcal{Z} - X_\partial$$

and so $\mathscr{C}'_{\mathcal{Z}}|_{\mathcal{Z}'\cap\pi^{-1}(\lambda)}$ is a restriction of the cutting configuration of Corollary 4.16. This establishes the second claim. $\square$

Lemma 5.8. *Suppose X is compact, $(U''_I)_{I\in\mathcal{P}^*(N)}$ is a $\mathscr{C}$-invariant open cover properly refining $(U'_I)_{I\in\mathcal{P}^*(N)}$, $\mathscr{C}''$ is the restriction of $\mathscr{C}$ to $(U''_I)_{I\in\mathcal{P}^*(N)}$, and $\mathscr{C}''_{\mathcal{Z}}$ is the analogue of the N-fold Hamiltonian configuration $\mathscr{C}_{\mathcal{Z}}$ in (4.52) for $\mathscr{C}''$. Then, there exist neighborhoods $\mathcal{Z}'\subset\mathcal{Z}$ of $X_\emptyset$ and $\Delta\subset\mathbb{C}$ of 0 such that the symplectic manifold $(\pi^{-1}(\lambda),\omega_{\mathcal{Z}}|_{\pi^{-1}(\lambda)})$ with the cutting configuration $\mathscr{C}''_{\mathcal{Z}}|_{\pi^{-1}(\lambda)}$ is canonically isomorphic to the symplectic manifold (X,ω) with the cutting configuration $\mathscr{C}''$ for every $\lambda\in\Delta\cap\mathbb{R}^+$.*

Proof. For $i\in I\subset[N]$ and $\lambda\in\mathbb{R}^+$, let

$$\varepsilon_{I;i}\colon U'_I\longrightarrow\mathbb{R}^+\qquad\text{and}\qquad g_{\lambda;I;i}\colon U'_{\lambda;I;i}\longrightarrow(\mathbb{R}^+)^I$$

be as in the proofs of Lemma 5.4 and Corollary 5.5, respectively, with $(U_I)_{I\in\mathcal{P}^*(N)}$ replaced by $(U'_I)_{I\in\mathcal{P}^*(N)}$. Since the subsets $\overline{U''_I}\subset U'_I$ are compact,

$$\epsilon\equiv\min_{i\in I\subset[N]}\big\{\varepsilon_{I;i}(x)\colon\, x\in\overline{U''_I}\big\}\in\mathbb{R}^+\,.$$

Let $\Delta\subset\mathbb{C}$ be the disk of radius ϵ around the origin and $\mathcal{Z}'=\pi^{-1}(\Delta)$. Since each function f_I is $\phi^\circ_{\mathcal{Z};I}$-invariant, $\mathcal{Z}'\subset\mathcal{Z}$ is a $\mathscr{C}_{\mathcal{Z}}$-invariant subspace. By the proofs of Corollary 5.5 and Lemma 5.6, it satisfies the condition of the latter.

By (5.14) and (5.4),

$$\widetilde{f}_I\big(x,(z_j)_{j\in I}\big)=\widetilde{f}_J\big(\widetilde{\Theta}_{J,I}\big(x,(z_j)_{j\in I}\big)\big)\prod_{j\in J-I}\sqrt{2(\mu_J(x))_{ij}+|z_i|^2}$$
$$\forall\ \big(x,(z_j)_{j\in I}\big)\in\widetilde{\mathcal{Z}}'^{\circ}_{I,J},\ i\in I\subset J\subset[N].$$

By the uniqueness of $g_{\lambda;i;I}$, this implies that

$$\widetilde{\Theta}_{J,I}\big(x,g_{\lambda;I;i}(x)\big)=\big(x,g_{\lambda;J;i}(x)\big)\qquad\forall\ x\in U'_{\lambda;I;i}\cap U'_{\lambda;J;i},\ i\in I,J\subset[N].\tag{5.16}$$

Furthermore,

$$g_{\lambda;I;i}(x)=g_{\lambda;I;j}(x)\qquad\forall\ x\in U'_{\lambda;I;i}\cap U'_{\lambda;I;j},\ i,j\in I\subset[N].\tag{5.17}$$

In contrast to (5.16), (5.17) does not hold for $\lambda\in\mathbb{C}^*-\mathbb{R}^+$ (even after passing to the quotient $\mathcal{Z}'^{\circ}_I$).

By the definition of ϵ, $U''_{\lambda;I;i}=U''_I$ for all $\lambda\in\Delta-0$ and $i\in I\subset[N]$. Define

$$f_\lambda\colon X\longrightarrow\pi^{-1}(\lambda)\subset\mathcal{Z}',\qquad f_\lambda(x)=q_{\mathcal{Z}}\big(q_{\mathcal{Z};I}\big(x,g_{\lambda;I;i}(x)\big)\big)\quad\forall\ x\in U''_I,\ i\in I\subset[N].$$

By (5.16) and (4.36), f_λ is independent of the choice of $I\in\mathcal{P}_i(N)$ for $i\in[N]$ fixed. By (5.17), f_λ is also independent of the choice of $i\in[N]$ and so is well-defined.

Since the graph of $g_{\lambda;I;i}|_{U''_I}$ is a slice for the $\widetilde{\phi}_I$-action on $\widetilde{\pi}^{-1}_{\mathcal{Z};I}(\lambda)\cap\widetilde{\mathcal{Z}}''^{\circ}_I$,

$$f_\lambda(U''_I)=\pi^{-1}(\lambda)\cap U''_{\mathcal{Z};I}\qquad\forall\ I\in\mathcal{P}^*(N).$$

Since the sets $U''_{\mathcal{Z};I}=q_{\mathcal{Z}}(q_{\mathcal{Z};I}(\widetilde{\mathcal{Z}}''^{\circ}_I))$ cover $\mathcal{Z}$, the map f_λ is thus surjective. By (4.36) and (5.16), it is injective. Since the restriction of f_λ to each U''_I is a composition of smooth maps, f_λ is a smooth map. By (4.17) and (5.13),

$$f_\lambda^*\omega_{\mathcal{Z}}\big|_{U''_I}=\big\{\mathrm{id}\times g_{\lambda;I;i}\big\}^*q^*_{\mathcal{Z};I}q^*_{\mathcal{Z}}\omega_{\mathcal{Z}}\big|_{U''_I}=\big\{\mathrm{id}\times g_{\lambda;I;i}\big\}^*q^*_{\mathcal{Z};I}\varpi_I\big|_{U''_I}$$
$$=\big\{\mathrm{id}\times g_{\lambda;I;i}\big\}^*\widetilde{\omega}_I\big|_{U''_I}=\omega\big|_{U''_I}\,.$$

Thus, f_λ is a symplectomorphism from (X,ω) to $(\pi^{-1}(\lambda),\omega_{\mathcal{Z}}|_{\pi^{-1}(\lambda)})$.

By the construction of $\mu_{\mathcal{Z};I}$ in Section 4.5,

$$\mu_{\mathcal{Z};I}\circ f_\lambda\big|_{U''_I}=\widetilde{\mu}_{\mathcal{Z};I}\circ\big\{\mathrm{id}\times g_{\lambda;I;i}\big\}\big|_{U''_I}=\mu_I\big|_{U''_I}\qquad\forall\ I\in\mathcal{P}^*(N).$$

Since $\widetilde{f}_I$ is a $\widetilde{\phi}_{\mathcal{Z};I}$-invariant map, the uniqueness of $g_{\lambda;I;i}$ implies that $g_{\lambda;I;i}|_{U''_I}$ is a ϕ_I-invariant map. Thus, the maps $\{\mathrm{id}\times g_{\lambda;I;i}\}|_{U''_I}$ and $f_\lambda|_{U''_I}$ are $(\widetilde{\phi}_{\mathcal{Z};I},\phi_I)$-equivariant and $(\phi_{\mathcal{Z};I},\phi_I)$-equivariant, respectively. We conclude that f_λ is an isomorphism from the cutting configuration $\mathscr{C}''$ for (X,ω) to the cutting configuration $\mathscr{C}''_{\mathcal{Z}}|_{\pi^{-1}(\lambda)}$ for $(\pi^{-1}(\lambda),\omega_{\mathcal{Z}}|_{\pi^{-1}(\lambda)})$. $\square$

The projection $\pi:\mathcal{Z}\longrightarrow\Delta$ depends on the choice of the functions f_I. By the same inductive reasoning as in the proof of Lemma 5.2, any two such collections of functions are homotopic after shrinking their domains. The latter does not effect the space $\mathcal{Z}$. If $U_{[N]}=X$, there is a natural choice of π given by $f_{[N]}=1$; see the proof of Proposition 6.2.

6 NC degenerations of Hamiltonian manifolds

The tori $(S^1)^I_\bullet$ whose actions ϕ_I appear in a Hamiltonian configuration (1.3) are subtori of

$$(S^1)^N_\bullet\equiv(S^1)^{[N]}_\bullet=\rho_\bullet^{-1}(\mathrm{id}),\qquad\text{where}\quad\rho_\bullet\colon(S^1)^N\longrightarrow S^1,\ \ \rho_\bullet\big((\mathrm{e}^{\mathrm{i}\theta_i})_{i\in[N]}\big)=\prod_{i\in[N]}\mathrm{e}^{\mathrm{i}\theta_i}.\tag{6.1}$$

If the domain $U_{[N]}$ of the $\phi_{[N]}$-action is the entire manifold X, the constructions of Sections 4 and 5 greatly simplify and result in a richer structure. By Proposition 6.2, there is then a one-parameter family π of smoothings of the associated SC symplectic variety $X_\emptyset$ defined over the entire symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$. Furthermore, a Hamiltonian pair $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ for (X,ω) compatible with $(\phi_{[N]},\mu_{[N]})$ induces a Hamiltonian $\mathbb{T}$-action on $(\mathcal{Z},\omega_{\mathcal{Z}})$ preserving the associated cut symplectic manifolds (X_i,ω_i) and the projection π. We illustrate this situation on the simple local example of Section 6.2 when all relevant objects can be readily described explicitly.

If the $\phi_{[N]}$-action is in addition the restriction of a Hamiltonian action of the entire torus $(S^1)^N$ on (X,ω), then there is a natural Hamiltonian S^1-pair $(\phi_{\mathcal{Z};S^1},\mu_{\mathcal{Z};S^1})$ for $(\mathcal{Z},\omega_{\mathcal{Z}})$ so that the projection π is S^1-equivariant; see Lemma 6.5. If X is also compact, then π can be "cut" to a one-parameter family

$$\widehat{\pi}\colon\widehat{\mathcal{Z}}_a\longrightarrow\mathbb{P}^1\tag{6.2}$$

of smoothings of $X_\emptyset$ with all smooth fibers (i.e. those not over $[1,0]$) canonically isomorphic to (X,ω); see Corollary 6.6.

As shown in Section 6.4, a Hamiltonian action of an abstract torus $\mathbb{T}$ on (X,ω) gives rise to a plethora of Hamiltonian $(S^1)^N$-pairs (ϕ,μ) for (X,ω) and often to cutting configurations. Theorem 3 describes the effect of the constructions of Sections 4 and 5 with cutting configurations arising in this way on the moment polytope of the original torus action. Proposition 6.10 provides a combinatorial criterion for a Hamiltonian configuration obtained as in Section 6.4 to be a cutting configuration; it is especially effective in the case of toric symplectic manifolds.

Let $\mathbb{T}\approx(S^1)^k$ be a k-torus. A Hamiltonian $\mathbb{T}$-manifold (Hamiltonian $\mathbb{T}$-space of [3, Definition 22.1]) is a tuple (X,ω,ϕ,μ) such that (X,ω) is a symplectic manifold and (ϕ,μ) is a Hamiltonian $\mathbb{T}$-pair for (X,ω). We call such a Hamiltonian $\mathbb{T}$-manifold compact (resp. connected) if X is compact (resp. connected). We call Hamiltonian $\mathbb{T}$-pair (ϕ,μ) and $\mathbb{T}'$-pair (ϕ',μ') for (X,ω) compatible if the actions of ϕ and ϕ' commute, μ' is ϕ-invariant, and μ is ϕ'-invariant. This means that the two actions and the two moment maps form a Hamiltonian $(\mathbb{T}\times\mathbb{T}')$-pair for (X,ω). If (ϕ,μ) is a Hamiltonian $\mathbb{T}$-pair for (X,ω), then the pair obtained by restricting the action ϕ to a subtorus $\mathbb{T}'\subset\mathbb{T}$ and composing μ with the restriction $\mathfrak{t}^*\longrightarrow\mathfrak{t}'^*$ is compatible with (ϕ,μ). Compatible pairs of a different kind play an important role in Theorem 3.

Suppose X and X' are topological spaces with $\mathbb{T}$-actions ϕ and ϕ', respectively, and μ and μ' are $\mathfrak{t}^*$-valued maps on X and X', respectively. A map $f\colon X\longrightarrow X'$ intertwines (ϕ,μ) and (ϕ',μ') if

$$f\big(\phi(g;x)\big)=\phi'\big(g;f(x)\big)\quad\text{and}\quad\mu(x)=\mu'\big(f(x)\big)\qquad\forall\,g\in\mathbb{T},\,x\in X.$$

In such a case, we call

$$f\colon(X,\phi,\mu)\longrightarrow(X',\phi',\mu')$$

a morphism.

6.1 Basic setup and output

Let $N\in\mathbb{Z}^+$ and (X,ω,ϕ,μ) be a Hamiltonian $(S^1)^N_\bullet$-manifold. For each $I\in\mathcal{P}^*(N)$, define

$$\begin{gathered}U_I=\big\{x\in X\colon(\mu(x))_i<(\mu(x))_j\ \forall\,i\in I,\,j\in[N]-I\big\},\\ \phi_I=\phi|_{(S^1)^I_\bullet\times U_I}\colon(S^1)^I_\bullet\times U_I\longrightarrow U_I,\qquad\mu_I=r_I\circ\mu|_{U_I}\colon U_I\longrightarrow\mathfrak{t}^*_{I;\bullet},\end{gathered}\tag{6.3}$$

where $r_I\colon\mathfrak{t}^*_{N;\bullet}\longrightarrow\mathfrak{t}^*_{I;\bullet}$ is the restriction homomorphism. We call (X,ω,ϕ,μ) regular if the ϕ_I-action of $(S^1)^I_\bullet$ on $\mu_I^{-1}(0)$ is free for every $I\in\mathcal{P}^*(N)$.

Lemma 6.1. *Let $N\in\mathbb{Z}^+$ and (X,ω,ϕ,μ) be a Hamiltonian $(S^1)^N_\bullet$-manifold. The tuple*

$$\mathscr{C}_{\phi,\mu}\equiv\big(U_I,\phi_I,\mu_I\big)_{I\in\mathcal{P}^*(N)}\tag{6.4}$$

defined by (6.3) is a maximal N-fold Hamiltonian configuration. It is a cutting configuration if and only if (X,ω,ϕ,μ) is regular.

Proof. Since $U_{[N]}=X$ and μ is continuous, $\{U_I\}_{I\in\mathcal{P}^*(N)}$ is an open cover of X. Since μ is ϕ-invariant, each U_I is preserved by the ϕ-action and thus by its restriction to $(S^1)^I_\bullet$. It follows that ϕ_I is a Hamiltonian $(S^1)^I_\bullet$-action on U_I with moment map μ_I. It is immediate from (6.3) that the tuple (6.4) satisfies (a)-(c) in Definition 1.1 and (3.7). This establishes the first claim. The regularity conditions on (X,ω,ϕ,μ) are the $I_0=I$ cases of the conditions of Definition 1.2 for a Hamiltonian configuration to be a cutting configuration. On the other hand, (6.4) is a maximal Hamiltonian configuration and so the domain of $\mu_{I_0;I}$ in (1.4) is an open subspace of U_{I_0}. Furthermore, $\mu_{I_0;I_0}=\mu_{I_0}$. By (6.3), $\mu_{I_0;I}$ and the restriction of the ϕ_I-action to $(S^1)^{I_0}_\bullet$ are the restrictions of μ_{I_0} and the ϕ_{I_0}-action to $\mathrm{Dom}(\mu_{I_0;I})$. This establishes the second claim. $\square$

Proposition 6.2. *Suppose $N\in\mathbb{Z}^+$, (X,ω,ϕ,μ) is a regular Hamiltonian $(S^1)^N_\bullet$-manifold, $(\mathcal{Z},\omega_{\mathcal{Z}})$ is the symplectic manifold determined by (6.4) via the construction of Section 4, and $q_\emptyset\colon X\longrightarrow X_\emptyset$ is the corresponding surjection. There are then natural continuous maps*

$$\pi\colon \mathcal{Z}\longrightarrow\mathbb{C} \qquad \textit{and} \qquad F\colon \mathbb{R}^{\geqslant 0}\times(S^1)^N\times X\longrightarrow\mathcal{Z} \tag{6.5}$$

so that π is a one-parameter family of smoothings of $X_\emptyset$ representing the germ of deformation equivalence classes of Theorem 2 determined by (6.4), F is smooth outside of $F^{-1}(X_\partial)$,

$$F(0,\mathrm{id},x)=q_\emptyset(x)\quad\forall\,x\in X,\quad \pi\big(F(r,g,x)\big)=r\,\rho_\bullet(g)\quad\forall\ (r,g,x)\in\mathbb{R}^{\geqslant 0}\times(S^1)^N\times X, \tag{6.6}$$

with $\rho_\bullet$ given by (6.1), and

$$F_{r,g}\colon (X,\omega)\longrightarrow\big(\pi^{-1}\big(r\rho_\bullet(g)\big),\omega_{\mathcal{Z}}\big|_{\pi^{-1}(r\rho_\bullet(g))}\big),\quad x\longrightarrow F(r,g,x), \tag{6.7}$$

is a symplectomorphism for every $(r,g)\in\mathbb{R}^+\times(S^1)^N$. Every Hamiltonian $\mathbb{T}$-pair $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ for (X,ω) compatible with (ϕ,μ) determines a Hamiltonian $\mathbb{T}$-pair $(\phi_{\mathcal{Z};\mathbb{T}},\mu_{\mathcal{Z};\mathbb{T}})$ for $(\mathcal{Z},\omega_{\mathcal{Z}})$ so that $(\phi_{\mathcal{Z};\mathbb{T}},\mu_{\mathcal{Z};\mathbb{T}})$ is compatible with the Hamiltonian $(S^1)^N_\bullet$-pair $(\phi_{\mathcal{Z}},\mu_{\mathcal{Z}})$ for $(\mathcal{Z},\omega_{\mathcal{Z}})$ determined by (ϕ,μ) and intertwined with $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ by F. The Hamiltonian configuration (4.52) for $(\mathcal{Z},\omega_{\mathcal{Z}})$ determined by (6.4) is $\mathscr{C}_{\phi_{\mathcal{Z}},\mu_{\mathcal{Z}}}$.

Proof. Since $U_{[N]}=X$ in this case,

$$\begin{gathered}\big(\mathcal{Z},\omega_{\mathcal{Z}}\big)=\big(\mathcal{Z}^\circ_{[N]},\varpi_{[N]}\big),\qquad (X_i,\omega_i)=\big(\mathcal{Z}^\circ_{\{i\};[N]},\varpi_{\{i\};[N]}\big)\quad\forall\,i\in[N],\\ (X_I,\omega_I)=\big(\mathcal{Z}^\circ_{I;[N]},\varpi_{I;[N]}\big)\quad\forall\,I\in\mathcal{P}^*(N);\end{gathered} \tag{6.8}$$

see (4.36), (4.41), and (4.39). The $I=[N]$ cases of (4.12) and (4.14) become

$$\widetilde\phi\equiv\widetilde\phi_{[N]}:(S^1)^N_\bullet\times\big(X\times\mathbb{C}^N\big)\longrightarrow X\times\mathbb{C}^N,\quad \widetilde\phi(g;x,z)=\big(\phi(g;x),\phi_{\mathbb{C}^N}(g^{-1};z)\big), \tag{6.9}$$

$$\widetilde{\mathcal{Z}}\equiv\widetilde{\mathcal{Z}}^\circ_{[N]}=\big\{\big(x,(z_j)_{j\in[N]}\big)\in X\times\mathbb{C}^N\colon (\mu(x))_i-\frac12|z_i|^2=(\mu(x))_j-\frac12|z_j|^2\ \forall\,i,j\in[N]\big\}. \tag{6.10}$$

By (6.8), (4.16), and (4.17), the symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$ of Corollary 4.10 is given by

$$\mathcal{Z}=\widetilde{\mathcal{Z}}/\widetilde\phi,\qquad q^*\omega_{\mathcal{Z}}=\big(\pi_1^*\omega+\pi_2^*\omega_{\mathbb{C}^N}\big)\big|_{\widetilde{\mathcal{Z}}}, \tag{6.11}$$

where $q\colon\widetilde{\mathcal{Z}}\longrightarrow\mathcal{Z}$ is the quotient map. The manifolds (X_i,ω_i) and their submanifolds (X_I,ω_I) are the symplectic submanifolds of $\mathcal{Z}$ with $z_i=0$ and $z_j=0$ for all $j\in I$, respectively. They are the images under q of the subspaces $\widetilde X_i$ and $\widetilde X_I$ of $\widetilde{\mathcal{Z}}$ described in the same way. The map $q_\emptyset$ in (4.49) is now given by

$$\begin{gathered}q_\emptyset\colon U_i^{\leqslant}=\big\{x\in X:(\mu(x))_i\leqslant(\mu(x))_j\ \forall\,j\in[N]\big\}\longrightarrow X_i\subset X_\emptyset,\\ q_\emptyset(x)=q\Big(x,\Big(\sqrt{2(\mu(x))_{ij}}\Big)_{j\in[N]}\Big),\end{gathered} \tag{6.12}$$

for each $i\in[N]$, with $(\mu(x))_{ij}$ as in (4.3).

A smooth map $\pi : \mathcal{Z} \longrightarrow \mathbb{C}$ as in Section 5.2 is determined by a tuple $(f_I)_{I\in\mathcal{P}^*(N)}$ of smooth functions as in Lemma 5.2. Since $U_{\mathcal{Z};[N]} = \mathcal{Z}$ in this case, such a tuple can be constructed without shrinking the original cover. For $I\in\mathcal{P}^*(N)$, $i\in I$, and $[x,(z_j)_{j\in I}]\in\mathcal{Z}_I^\circ$, define

$$f_I\big([x,(z_j)_{j\in I}]\big) = \prod_{j\in[N]-I}\sqrt{2(\mu(x))_{ij}+|z_i|^2} > 0\,;$$

the inequality holds by (6.3). By (6.10), the function f_I does not depend on $i\in I$. The associated projection map is then given by

$$\pi\big([x,(z_i)_{i\in[N]}]\big) = f_{[N]}\big([x,(z_i)_{i\in[N]}]\big)\prod_{i\in[N]}z_i = \prod_{i\in[N]}z_i\,. \tag{6.13}$$

The proofs of Lemmas 5.3 and 5.4 with $I=[N]$ and $\widetilde{f}_{[N]}=1$ show that $\widetilde{W}_{[N]}=\widetilde{\mathcal{Z}}_{[N]}^\circ$ satisfies the conditions stated in these lemmas. Thus, $\widetilde{W}_{[N]}=\widetilde{\mathcal{Z}}_{[N]}^\circ$ also satisfies the condition of Corollary 5.5. In the proof of Lemma 5.6, $W_{[N]}$ then becomes $q_{\mathcal{Z}}(\mathcal{Z}_{[N]}^\circ)$. From (6.8), we conclude that

$$\mathcal{Z}' = q_{\mathcal{Z}}(\mathcal{Z}_{[N]}^\circ) = \mathcal{Z}\,,$$

i.e. (6.13) defines a one-parameter family π of smoothings of $X_\emptyset = \pi^{-1}(0)$ without shrinking $\mathcal{Z}$. This conclusion also follows from (6.6) and (6.7).

Fix $i\in[N]$ and define

$$\varrho_i\colon X \longrightarrow \mathbb{R}^{\geqslant 0}, \qquad \varrho_i(x) = \max\big\{-2(\mu(x))_{ij}\colon j\in[N]\big\}. \tag{6.14}$$

By the same reasoning as in the proof of Lemma 5.4, the equation

$$\prod_{j\in[N]}\big(2(\mu(x))_{ij}+\varrho\big) = r^2 \tag{6.15}$$

has a unique solution $\varrho=\varrho_i(r,x)$ in $[\varrho_i(x),\infty)$ for each $r\in\mathbb{R}^{\geqslant 0}$. For $r\in\mathbb{R}^+$, it lies in $(\varrho_i(x),\infty)$ and depends smoothly on (r,x). For $x\in X$ such that the maximum in (6.14) is achieved at a unique value $j\in[N]$, the function

$$(r,x') \longrightarrow \sqrt{2(\mu(x'))_{ij}+\varrho_i(r,x')}$$

is smooth around $(x,0)$. Define F in (6.5) by

$$F\big(r,(\mathrm{e}^{\mathrm{i}\theta_j})_{j\in[N]},x\big) = q\Big(x,\big(\mathrm{e}^{\mathrm{i}\theta_j}\sqrt{2(\mu(x))_{ij}+\varrho_i(r,x)}\big)_{j\in[N]}\Big). \tag{6.16}$$

This function is independent of the choice of $i\in[N]$. It is continuous everywhere and smooth outside of the points $(0,g,x)$ such that

$$\big|\big\{j\in[N]\colon \varrho_i(x)=-2(\mu(x))_{ij}\big\}\big| \geqslant 2,$$

i.e. on the complement of $F^{-1}(X_\partial)$. It restricts to (6.12) over $\{(0,\mathrm{id})\}\times X$. By (6.13), (6.1), and (6.15), F satisfies the second property in (6.6) as well. By the same reasoning as in the proof of Corollary 4.13, each map (6.7) is a symplectomorphism.

Let $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ be a Hamiltonian $\mathbb{T}$-pair for (X,ω) compatible with (ϕ,μ). We define a Hamiltonian $\mathbb{T}$-pair for $(X\times\mathbb{C}^N,\pi_1^*\omega+\pi_2^*\omega_{\mathbb{C}^N})$ by

$$\widetilde{\phi}_{\mathcal{Z};\mathbb{T}}\colon \mathbb{T}\times\big(X\times\mathbb{C}^N\big)\longrightarrow X\times\mathbb{C}^N, \qquad \widetilde{\phi}_{\mathcal{Z};\mathbb{T}}(g;x,z)=\big(\phi_{\mathbb{T}}(g;x),z\big), \tag{6.17}$$

$$\widetilde{\mu}_{\mathcal{Z};\mathbb{T}}\colon X\times\mathbb{C}^N\longrightarrow \mathfrak{t}^*, \qquad \widetilde{\mu}_{\mathcal{Z};\mathbb{T}}(x,z)=\mu_{\mathbb{T}}(x)\,. \tag{6.18}$$

Since the $\phi_{\mathbb{T}}$-action preserves μ and commutes with the ϕ-action, the action (6.17) restricts to an action on (6.10) and descends to a $\mathbb{T}$-action

$$\phi_{\mathcal{Z};\mathbb{T}}\colon \mathbb{T}\times\mathcal{Z}\longrightarrow\mathcal{Z}$$

on $\mathcal{Z}$. Since the moment map $\mu_{\mathbb{T}}$ is ϕ-invariant, (6.18) descends to a smooth map

$$\mu_{\mathcal{Z};\mathbb{T}}\colon \mathcal{Z}\longrightarrow\mathfrak{t}^*.$$

By (6.11), $(\phi_{\mathcal{Z};\mathbb{T}},\mu_{\mathcal{Z};\mathbb{T}})$ is a Hamiltonian $\mathbb{T}$-pair for $(\mathcal{Z},\omega_{\mathcal{Z}})$. Since the actions of ϕ and $\phi_{\mathbb{T}}$ commute, $\mu_{\mathbb{T}}$ is ϕ-invariant, and μ is $\phi_{\mathbb{T}}$-invariant, (6.17) and (6.18) imply that $(\phi_{\mathcal{Z};\mathbb{T}},\mu_{\mathcal{Z};\mathbb{T}})$ is compatible $(\phi_{\mathcal{Z}},\mu_{\mathcal{Z}})$. By (6.16), (6.17), and (6.18),

$$\phi_{\mathcal{Z};\mathbb{T}}\big(g';F(r,g,x)\big)=F\big(r,g,\phi_{\mathbb{T}}(g';x)\big), \qquad \mu_{\mathcal{Z};\mathbb{T}}\big(F(r,g,x)\big)=\mu_{\mathbb{T}}(x), \tag{6.19}$$

i.e. F intertwines $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ and $(\phi_{\mathcal{Z};\mathbb{T}},\mu_{\mathcal{Z};\mathbb{T}})$.

It remains to establish the last claim. Since $U_{\mathcal{Z};I}=q_{\mathcal{Z}}(q_{\mathcal{Z};I}(\widetilde{\mathcal{Z}}_I^{\circ}))$, (6.8) and (4.36) give

$$U_{\mathcal{Z};I}=q\big(\widetilde{\mathcal{Z}}_{[N],I}^{\circ}\big)=q\big(\widetilde{\mathcal{Z}}\cap(U_I\times\mathbb{C}^N)\big).$$

Combining this with the first equation in (6.3) and (6.18) for $\mu_{\mathbb{T}}=\mu$, we obtain

$$\begin{aligned} U_{\mathcal{Z};I}&=\big\{[x,z]\in\mathcal{Z}\colon (\mu(x))_i<(\mu(x))_j\ \forall\,i\in I,\ j\in[N]-I\big\}\\ &=\big\{[x,z]\in\mathcal{Z}\colon \big(\mu_{\mathcal{Z}}([x,z])\big)_i<\big(\mu_{\mathcal{Z}}([x,z])\big)_j\ \forall\,i\in I,\ j\in[N]-I\big\}. \end{aligned} \tag{6.20}$$

By (4.53), the second equation in (6.3), and (6.17) for $\phi_{\mathbb{T}}=\phi$,

$$\phi_{\mathcal{Z};I}=\phi_{\mathcal{Z}}\big|_{(S^1)^I_{\bullet}\times U_{\mathcal{Z};I}}. \tag{6.21}$$

By (4.54), the third equation in (6.3), and (6.18) for $\mu_{\mathbb{T}}=\mu$,

$$\mu_{\mathcal{Z};I}=r_I\circ\mu_{\mathcal{Z}}\big|_{U_{\mathcal{Z};I}}. \tag{6.22}$$

By (6.20)-(6.22), $\mathscr{C}_{\mathcal{Z}}=\mathscr{C}_{\phi_{\mathcal{Z}},\mu_{\mathcal{Z}}}$. □

By (6.6) and the first property in (6.19),

$$\phi_{\mathcal{Z};\mathbb{T}}(\mathbb{T}\times X_i)=X_i\quad\forall\,i\in[N], \qquad \pi\big(\phi_{\mathcal{Z};\mathbb{T}}(g;y)\big)=\pi(y).$$

By (6.7) and (6.19), the restriction (6.7) of F induces an isomorphism

$$F_{r,g}\colon \big(X,\omega,\phi_{\mathbb{T}},\mu_{\mathbb{T}}\big)\longrightarrow\big(\pi^{-1}(\lambda),\omega_{\mathcal{Z}}|_{\pi^{-1}(\lambda)},\phi_{\mathcal{Z};\mathbb{T}}|_{\mathbb{T}\times\pi^{-1}(\lambda)},\mu_{\mathcal{Z};\mathbb{T}}|_{\mathbb{T}\times\pi^{-1}(\lambda)}\big)$$

whenever $\lambda\equiv r\rho_{\bullet}(g)\neq0$. Since $\mathscr{C}_{\mathcal{Z}}=\mathscr{C}_{\phi_{\mathcal{Z}},\mu_{\mathcal{Z}}}$, the $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})=(\phi,\mu)$ case of this statement implies that $F_{r,g}$ identifies the cutting configuration (6.4) for (X,ω) with the restriction of the induced Hamiltonian configuration $\mathscr{C}_{\mathcal{Z}}$ to $\pi^{-1}(\lambda)$.

6.2 A local example

For $N\in\mathbb{Z}^+$, let

$$\mathbb{C}^N_0=\bigcup_{i\in[N]}\mathbb{C}^N_i \qquad\text{and}\qquad \mathbb{C}^N_\partial=\bigcup_{\substack{I\in\mathcal{P}^*(N)\\ |I|=2}}\mathbb{C}^N_I$$

be the union of the coordinate hyperplanes and the union of the codimension 2 coordinate subspaces, respectively. Thus, $\mathbb{C}^N_0$ and $\mathbb{C}^N_\partial$ are the SC symplectic variety and its singular locus associated to the SC symplectic configuration

$$\mathbf{X}_{\mathbb{C}^N}\equiv\big((\mathbb{C}^N_I)_{I\in\mathcal{P}^*(N)},(\omega_{\mathbb{C}^N}|_{\mathbb{C}^N_i})_{i\in[N]}\big) \tag{6.23}$$

as in (2.2) and (2.3). Let

$$\pi_{\mathbb{C}^N}\colon\mathbb{C}^N\longrightarrow\mathbb{C},\qquad \pi_{\mathbb{C}^N}(x_1,\ldots,x_N)=x_1\ldots x_N\,. \tag{6.24}$$

The tuple $(\mathbb{C}^N,\omega_{\mathbb{C}^N},\pi_{\mathbb{C}^N})$ is then a one-parameter family of smoothings of the SC symplectic variety $\mathbb{C}^N_0$ in the sense of the sentence after Definition 2.6.

Under the identifications (1.2), the restriction homomorphism

$$r_\bullet\colon\mathfrak{t}^*_N\longrightarrow\mathfrak{t}^*_{N;\bullet}=\mathbb{R}^N/\big\{(a,\ldots,a)\in\mathbb{R}^N\colon a\in\mathbb{R}\big\},\qquad \eta\longrightarrow\eta|_{\mathfrak{t}_{N;\bullet}},$$

is the quotient map. Let $(\phi_{\mathbb{C}^N},\mu_{\mathbb{C}^N})$ be the standard Hamiltonian pair for $(\mathbb{C}^N,\omega_{\mathbb{C}^N})$ given by (4.11). The maximal N-fold Hamiltonian configuration

$$\mathscr{C}_{\mathbb{C}^N}\equiv\mathscr{C}_{\phi_{\mathbb{C}^N}|_{(S^1)^N_\bullet\times\mathbb{C}^N},r_\bullet\circ\mu_{\mathbb{C}^N}}\equiv\big(U_{\mathbb{C}^N;I},\phi_{\mathbb{C}^N;I},\mu_{\mathbb{C}^N;I}\big)_{I\in\mathcal{P}^*(N)} \tag{6.25}$$

determined by this pair via (6.3) is given by

$$U_{\mathbb{C}^N;I}=\big\{(x_1,\ldots,x_N)\in\mathbb{C}^N\colon |x_i|<|x_j|\ \forall\,i\in I,\ j\in[N]-I\big\},$$
$$\phi_{\mathbb{C}^N;I}\colon(S^1)^I_\bullet\times U_I\longrightarrow U_I,\quad \mu_{\mathbb{C}^N;I}\big((x_i)_{i\in[N]}\big)=\frac{1}{2}\big[\big(|x_i|^2\big)_{i\in I}\big]\in\mathfrak{t}^*_{I;\bullet}\,.$$

The associated (S^1)-actions and Hamiltonians in (3.5) are

$$\phi_{\mathbb{C}^N;ij}\colon S^1\times\mathbb{C}^N\longrightarrow\mathbb{C}^N,\quad \big(\phi_{\mathbb{C}^N;ij}(\mathrm{e}^{\mathrm{i}\theta};x_1,\ldots,x_N)\big)_k=\begin{cases}\mathrm{e}^{-\mathfrak{s}_{ij}\mathrm{i}\theta}x_k, & \text{if } k=i;\\ \mathrm{e}^{\mathfrak{s}_{ij}\mathrm{i}\theta}x_k, & \text{if } k=j;\\ x_k, & \text{otherwise};\end{cases}$$
$$h_{\mathbb{C}^N;ij}\colon\mathbb{C}^N\longrightarrow\mathbb{R},\qquad h_{\mathbb{C}^N;ij}(x_1,\ldots,x_N)=\frac{\mathfrak{s}_{ij}}{2}\big(|x_j|^2-|x_i|^2\big).$$

The Hamiltonian configuration (6.25) is not a cutting configuration, as it does not satisfy the additional conditions of Definition 1.2 if $N\geqslant 2$. On the other hand, the restriction of $\mathscr{C}_{\mathbb{C}^N}$ to $\mathbb{C}^N-\mathbb{C}^N_\partial$ satisfies these conditions and thus is an N-fold cutting configuration.

Fix $\lambda\in\mathbb{C}^*$. Let

$$(S^1)^N_\lambda=\rho_\bullet^{-1}\big(\lambda/|\lambda|\big),\quad X=\big\{(x_1,\ldots,x_N)\in\mathbb{C}^N\colon x_1\ldots x_N=\lambda\big\},\quad \omega=\omega_{\mathbb{C}^N}\big|_X. \tag{6.26}$$

Since X is preserved by the restriction of the $\phi_{\mathbb{C}^N}$-action to $(S^1)^N_\bullet$,

$$\mathscr{C} \equiv \mathscr{C}_{\mathbb{C}^N}\big|_X \equiv \mathscr{C}_{\phi_{\mathbb{C}^N}|_{(S^1)^N_\bullet\times X}, r_\bullet\circ\mu_{\mathbb{C}^N}|_X} \equiv \big(U_I, \phi_I, \mu_I\big)_{I\in\mathcal{P}^*(N)} \tag{6.27}$$

is an N-fold Hamiltonian configuration. Since the full $\phi_{\mathbb{C}^N}$-action does not preserve X, the present situation is a special case of the setting of Lemma 6.1 and not of Lemma 6.5.

Lemma 6.3. *The tuple $\mathscr{C}$ in (6.27) is a maximal N-fold cutting configuration for (X,ω).*

Proof. By the first statement of Lemma 6.1, (6.27) is a maximal N-fold Hamiltonian configuration for (X,ω). Since the $\phi_{\mathbb{C}^N}$-action is free on $\mathbb{C}^N-\mathbb{C}^N_0$, the ϕ_I-action is free on X for each $I\in\mathcal{P}^*(N)$. By the second statement of Lemma 6.1, (6.27) is thus a cutting configuration. □

Proposition 6.4. *Suppose $(\mathcal{Z},\omega_{\mathcal{Z}})$ is the symplectic manifold determined by (6.27) via the construction of Section 4 and π, F, $\phi_{\mathcal{Z}}$, and $\mu_{\mathcal{Z}}$ are the associated maps provided by Proposition 6.2. There is a natural smooth map*

$$\breve{F}\colon (S^1)^N_\lambda\times\mathbb{C}^N \longrightarrow \mathcal{Z} \tag{6.28}$$

such that

$$\breve{F}_g\colon \big(\mathbb{C}^N,\omega_{\mathbb{C}^N}\big) \longrightarrow \big(\mathcal{Z},\omega_{\mathcal{Z}}\big), \qquad x\longrightarrow \breve{F}(g,x), \tag{6.29}$$

is a symplectomorphism for every $g\in(S^1)^N_\lambda$,

$$\breve{F}\big((S^1)^N_\lambda\times\mathbb{C}^N_i\big) = X_i \;\forall\, i\in[N],\quad \pi\big(\breve{F}(g,z)\big) = \pi_{\mathbb{C}^N}(z),\quad \breve{F}|_{(S^1)^N_\lambda\times X} = F|_{\{|\lambda|\}\times(S^1)^N_\lambda\times X}, \tag{6.30}$$

$$\phi_{\mathcal{Z}}\big(g';\breve{F}(g,z)\big) = \breve{F}\big(g,\phi_{\mathbb{C}^N}(g';z)\big), \qquad \mu_{\mathcal{Z}}\big(\breve{F}(g,z)\big) = r_\bullet\big(\mu_{\mathbb{C}^N}(z)\big). \tag{6.31}$$

Proof. The action (6.9) in this case reduces to

$$\widetilde{\phi}\colon (S^1)^N_\bullet\times\big(X\times\mathbb{C}^N\big) \longrightarrow X\times\mathbb{C}^N, \quad \widetilde{\phi}(g;x,z) = \big(\phi_{\mathbb{C}^N}(g;x),\phi_{\mathbb{C}^N}(g^{-1};z)\big). \tag{6.32}$$

The symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$ of Corollary 4.10 is described by (6.10) and (6.11), which become

$$\widetilde{\mathcal{Z}} = \Big\{\big((x_j)_{j\in[N]},(z_j)_{j\in[N]}\big)\in\mathbb{C}^N\times\mathbb{C}^N\colon \prod_{i\in[N]}x_i=\lambda,\ |x_i|^2-|z_i|^2=|x_j|^2-|z_j|^2\ \forall\, i,j\in[N]\Big\},$$

$$\mathcal{Z} = \widetilde{\mathcal{Z}}/\widetilde{\phi}, \qquad q^*\omega_{\mathcal{Z}} = \big(\pi_1^*\omega_{\mathbb{C}^N}+\pi_2^*\omega_{\mathbb{C}^N}\big)\big|_{\widetilde{\mathcal{Z}}}, \tag{6.33}$$

where $q\colon\widetilde{\mathcal{Z}}\longrightarrow\mathcal{Z}$ is the quotient map. The manifolds (X_i,ω_i) and their submanifolds (X_I,ω_I) are again the symplectic submanifolds of $\mathcal{Z}$ with $z_i=0$ and $z_j=0$ for all $j\in I$, respectively. They are the images under q of the subspaces $\widetilde{X}_i$ and $\widetilde{X}_I$ of $\widetilde{\mathcal{Z}}$ described in the same way.

Fix $i\in[N]$ and define

$$\varrho_i\colon\mathbb{C}^N\longrightarrow\mathbb{R}^{\geqslant 0}, \qquad \varrho_i(z) = \max\big\{-2\big(\mu_{\mathbb{C}^N}(z)\big)_{ij}\colon j\in[N]\big\},$$
$$\text{where} \qquad \big(\mu_{\mathbb{C}^N}(z)\big)_{ij} = \frac{1}{2}\big(|z_j|^2-|z_i|^2\big)\ \ \forall\ z\equiv(z_j)_{j\in[N]}\in\mathbb{C}^N.$$

For each $z\in\mathbb{C}^N$, let $\varrho_{\lambda;i}(z)\in(\varrho_i(z),\infty)$ be the unique solution of the equation

$$\prod_{j\in[N]}\big(2(\mu_{\mathbb{C}^N}(z))_{ij}+\varrho\big) = |\lambda|^2 \tag{6.34}$$

in ϱ; this is a special case of the equation (6.15). Define $\breve{F}$ in (6.28) by

$$\breve{F}\big((\mathrm{e}^{\mathrm{i}\theta_j})_{j\in[N]},(z_j)_{j\in[N]}\big) = q\Big(\big(\mathrm{e}^{\mathrm{i}\theta_j}\sqrt{|z_j|^2-|z_i|^2+\varrho_{\lambda;i}((z_k)_{k\in[N]})}\big)_{j\in[N]},(z_j)_{j\in[N]}\Big). \tag{6.35}$$

This function is independent of the choice of $i\in[N]$.

By the reasoning in the proof of Corollary 4.13 with the x and z components of $\mathcal{Z}$ interchanged, each map (6.29) is a symplectomorphism. Since $X_i=q(\widetilde{X}_i)$, $\breve{F}$ satisfies the first property in (6.30). By (6.13) and (6.24), it also satisfies the second property in (6.30). By (6.18) for $\mu_{\mathbb{T}}=r_\bullet\circ\mu_{\mathbb{C}^N}$, (6.35), and (4.11),

$$\begin{aligned}\mu_{\mathcal{Z}}\big(\breve{F}((\mathrm{e}^{\mathrm{i}\theta_j})_{j\in[N]},(z_j)_{j\in[N]})\big) &= \{r_\bullet\circ\mu_{\mathbb{C}^N}\}\Big(\big(\mathrm{e}^{\mathrm{i}\theta_j}\sqrt{|z_j|^2-|z_i|^2+\varrho_{\lambda;i}((z_k)_{k\in[N]})}\big)_{j\in[N]}\Big)\\ &= \frac{1}{2}r_\bullet\Big(\big(|z_j|^2-|z_i|^2+\varrho_{\lambda;i}((z_k)_{k\in[N]})\big)_{j\in[N]}\Big) = \frac{1}{2}r_\bullet\big((|z_j|^2)_{j\in[N]}\big) = r_\bullet\big(\mu_{\mathbb{C}^N}((z_j)_{j\in[N]})\big).\end{aligned}$$

This establishes the second claim in (6.31). Since each map (6.29) is a symplectomorphism, the second claim in (6.31) implies the first one.

Suppose $(\mathrm{e}^{\mathrm{i}\theta_j})_{j\in[N]}\in(S^1)^N_\lambda$ and $(z_j)_{j\in[N]}\in X$. Let

$$\big(\mathrm{e}^{\mathrm{i}\theta'_j}\big)_{j\in[N]} = \big(\mathrm{e}^{-\mathrm{i}\theta_j}z_j/|z_j|\big)_{j\in[N]} \in (S^1)^N_\bullet\,.$$

By the uniqueness of the solution of (6.34),

$$|z_j|^2-|z_i|^2+\varrho_{\lambda;i}\big((z_j)_{j\in[N]}\big) = |z_j|^2 \qquad \forall\, j\in[N]. \tag{6.36}$$

Combining this with (6.35), (6.33), and (6.32), we obtain

$$\breve{F}\big((\mathrm{e}^{\mathrm{i}\theta_j})_{j\in[N]},(z_j)_{j\in[N]}\big) = q\Big(\big(\mathrm{e}^{\mathrm{i}\theta'_j}\mathrm{e}^{\mathrm{i}\theta_j}|z_j|\big)_{j\in[N]},\big(\mathrm{e}^{-\mathrm{i}\theta'_j}z_j\big)_{j\in[N]}\Big) = q\Big((z_j)_{j\in[N]},\big(\mathrm{e}^{\mathrm{i}\theta_j}|z_j|\big)_{j\in[N]}\Big).$$

The last property in (6.30) now follows from (6.36) and (6.16). □

By Proposition 6.4, the restriction (6.29) of $\breve{F}$ induces an isomorphism

$$\breve{F}_g\colon \big(\mathbb{C}^N,\omega_{\mathbb{C}^N},\mathbf{X}(\mathscr{C}),X_\emptyset,\pi_{\mathbb{C}^N},\phi_{\mathbb{C}^N}|_{(S^1)^N_\bullet\times\mathbb{C}^N},r_\bullet\circ\mu_{\mathbb{C}^N}\big) \longrightarrow \big(\mathcal{Z},\omega_{\mathcal{Z}},\mathbf{X}_{\mathbb{C}^N},\mathbb{C}^N_0,\pi,\phi_{\mathcal{Z}},\mu_{\mathcal{Z}}\big),$$

where $\mathbf{X}(\mathscr{C})$ is the SC symplectic configuration determined by (6.27) and $X_\emptyset$ is the associated SC symplectic divisor. By (6.25) and the last statement of Proposition 6.2, this implies that $\breve{F}_g$ identifies the Hamiltonian configuration (6.25) for $(\mathbb{C}^N,\omega_{\mathbb{C}^N})$ and the induced Hamiltonian configuration $\mathscr{C}_{\mathcal{Z}}$ for $(\mathcal{Z},\omega_{\mathcal{Z}})$.

6.3 Further refinements

For a Hamiltonian $(S^1)^N$-manifold (X,ω,ϕ,μ), the global quotient description of the main constructions of this paper provided by the proof of Proposition 6.2 gives rise to a Hamiltonian S^1-action on $(\mathcal{Z},\omega_{\mathcal{Z}})$ which is free on the complement of the SC symplectic divisor $X_\emptyset\subset\mathcal{Z}$; see Lemma 6.5. If in addition X is compact, this action can be used to "cut" off a precompact neighborhood of $X_\emptyset$ in $\mathcal{Z}$ to form a one-parameter family of smoothings of $X_\emptyset$ with compact total space $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$ and projection map (6.2) which is equivariant with respect to the induced S^1-action on $\widehat{\widetilde{\mathcal{Z}}}_a$ and the standard S^1-action on $\mathbb{P}^1$ given by

$$\phi_{\mathbb{P}^1}: S^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \quad \phi_{\mathbb{P}^1}\big(\mathrm{e}^{\mathrm{i}\theta}; [w,z]\big) = \big[w, \mathrm{e}^{\mathrm{i}\theta} z\big]. \tag{6.37}$$

For the purposes of Corollary 6.6 below, let

$$\widehat{\mathbb{P}}^1_0 = \Big(\big[0,1)\times S^1\big) \sqcup \big(\mathbb{P}^1 - \{[1,0]\}\big)\Big)\Big/\!\sim, \quad (0,1)\times S^1 \ni \big(r, \mathrm{e}^{\mathrm{i}\theta}\big) \sim \big[1, r\mathrm{e}^{\mathrm{i}\theta}\big] \in \mathbb{P}^1 - \big\{[1,0]\big\}\,; \tag{6.38}$$

this is the disk obtained from $\mathbb{P}^1$ by replacing $[1,0]$ with S^1. Let

$$q_{\mathbb{P}^1}: \widehat{\mathbb{P}}^1_0 \longrightarrow \mathbb{P}^1$$

be the natural projection map restricting to the inclusion of $\mathbb{P}^1 - \{[1,0]\}$.

Lemma 6.5. *Suppose $N \in \mathbb{Z}^+$, (X,ω,ϕ,μ) is a regular Hamiltonian $(S^1)^N$-manifold, $(\mathcal{Z},\omega_{\mathcal{Z}})$ is the symplectic manifold determined by (6.4) via the construction of Section 4, and π is the associated map provided by Proposition 6.2. There is then a natural Hamiltonian S^1-pair $(\phi_{\mathcal{Z};S^1}, \mu_{\mathcal{Z};S^1})$ for $(\mathcal{Z},\omega_{\mathcal{Z}})$ such that*

$$\phi_{\mathcal{Z};S^1}(S^1 \times X_i) = X_i \quad \forall\, i \in [N], \qquad \pi\big(\phi_{\mathcal{Z};S^1}(\mathrm{e}^{\mathrm{i}\theta}; y)\big) = \mathrm{e}^{\mathrm{i}\theta}\pi(y). \tag{6.39}$$

If $(\phi_{\mathbb{T}}, \mu_{\mathbb{T}})$ is a Hamiltonian $\mathbb{T}$-pair for (X,ω) compatible with (ϕ,μ), then the associated Hamiltonian $\mathbb{T}$-pair $(\phi_{\mathcal{Z};\mathbb{T}}, \mu_{\mathcal{Z};\mathbb{T}})$ for $(\mathcal{Z},\omega_{\mathcal{Z}})$ provided by Proposition 6.2 is compatible with $(\phi_{\mathcal{Z};S^1}, \mu_{\mathcal{Z};S^1})$.

Proof. We continue with the notation and setup of Proposition 6.2. In this case, the components $(\mu(x))_i \in \mathbb{R}$ of $\mu(x) \in \mathfrak{t}^*$ are well-defined. For $i \in [N]$, let

$$\iota_i: S^1 \longrightarrow (S^1)^N$$

be the inclusion as the i-th component. Define a Hamiltonian S^1-pair for $(X \times \mathbb{C}^N, \pi_1^*\omega + \pi_2^*\omega_{\mathbb{C}^N})$ by

$$\widetilde{\phi}_{\mathcal{Z};S^1;i}: S^1 \times \big(X \times \mathbb{C}^N\big) \longrightarrow X \times \mathbb{C}^N, \quad \widetilde{\phi}_{\mathcal{Z};S^1;i}\big(\mathrm{e}^{\mathrm{i}\theta}; x, z\big) = \big(\phi\big(\iota_i(\mathrm{e}^{-\mathrm{i}\theta}); x\big), \phi_{\mathbb{C}^N}\big(\iota_i(\mathrm{e}^{\mathrm{i}\theta}); z\big)\big), \tag{6.40}$$

$$\widetilde{\mu}_{\mathcal{Z};S^1;i}: X \times \mathbb{C}^N \longrightarrow \mathbb{R}, \qquad \widetilde{\mu}_{\mathcal{Z};S^1;i}(x,z) = -\big(\mu(x)\big)_i + \frac{1}{2}|z_i|^2\,. \tag{6.41}$$

By the same reasoning as below (6.18), this pair descends to a Hamiltonian S^1-pair

$$\phi_{\mathcal{Z};S^1}: S^1 \times \mathcal{Z} \longrightarrow \mathcal{Z}, \qquad \mu_{\mathcal{Z};S^1}: \mathcal{Z} \longrightarrow \mathbb{R}, \tag{6.42}$$

for $(\mathcal{Z},\omega_{\mathcal{Z}})$. By (6.10), the restriction of (6.41) to $\widetilde{\mathcal{Z}}$ is independent of the choice of $i \in [N]$. Thus, so is the pair (6.42). Since (6.40) preserves the subspace $\widetilde{X}_i$ of $\widetilde{\mathcal{Z}}$, $\phi_{\mathcal{Z};S^1}$ preserves the subspace X_i of $\mathcal{Z}$. By (6.13) and (6.40), (6.42) satisfies the second property in (6.39) as well. If $(\phi_{\mathbb{T}}, \mu_{\mathbb{T}})$ is a Hamiltonian $\mathbb{T}$-pair for (X,ω) compatible with (ϕ,μ), then the actions of $\widetilde{\phi}_{\mathcal{Z};S^1;i}$ and $\widetilde{\phi}_{\mathcal{Z};\mathbb{T}}$ commute, $\widetilde{\mu}_{\mathcal{Z};\mathbb{T}}$ is $\widetilde{\phi}_{\mathcal{Z};S^1;i}$-invariant, and $\widetilde{\mu}_{\mathcal{Z};S^1;i}$ is $\widetilde{\phi}_{\mathbb{T};\mathcal{Z}}$-invariant. This implies the last claim. □

In the setting of Proposition 6.2, an extra Hamiltonian S^1-action on $(\mathcal{Z},\omega_{\mathcal{Z}})$ can be obtained via the diagonal action of S^1 on the $\mathbb{C}^N$-component in (6.10). However, the action of $(S^1)^{N+1}$ on $\mathcal{Z}$ obtained by combining this action with $\phi_{\mathcal{Z}}$ is not effective even if the $(S^1)^N$-action ϕ is effective. In contrast, the action of $(S^1)^{N+1}$ on $\mathcal{Z}$ obtained by combining the S^1-action of Lemma 6.5 with with $\phi_{\mathcal{Z}}$ is effective if the ϕ-action is effective; see Theorem 3.

Corollary 6.6. *Suppose $N\in\mathbb{Z}^+$, (X,ω,ϕ,μ) is a compact regular Hamiltonian $(S^1)^N$-manifold, and $(\mathcal{Z},\omega_{\mathcal{Z}})$, $X_\emptyset$, $q_\emptyset$, and $(\phi_{\mathcal{Z};S^1},\mu_{\mathcal{Z};S^1})$ are as in Proposition 6.2 and Lemma 6.5. For every $a\in\mathbb{R}$ sufficiently large, there exist a natural compact symplectic manifold $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$, an open neighborhood $\mathcal{Z}'_a$ of $X_\emptyset$ in $\mathcal{Z}$, continuous surjections*

$$\widehat{\pi}\colon \widehat{\mathcal{Z}}_a\longrightarrow\mathbb{P}^1,\qquad f_a\colon \overline{\mathcal{Z}}'_a\longrightarrow\widehat{\mathcal{Z}}_a,\qquad\text{and}\quad \widehat{F}_a\colon \widehat{\mathbb{P}}^1_0\times X\longrightarrow\widehat{\mathcal{Z}}_a,\tag{6.43}$$

and a Hamiltonian S^1-pair $(\phi_{\widehat{\mathcal{Z}};S^1;a},\mu_{\widehat{\mathcal{Z}};S^1;a})$ for $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$ so that $\widehat{\pi}$ is a one-parameter family of smoothings of the SC symplectic variety $\widehat{\mathcal{Z}}_{a;0}\equiv\widehat{\pi}^{-1}([1,0])$, $\widehat{F}_a$ is smooth outside of $\widehat{F}_a^{-1}(f_a(X_\partial))$,

$$\widehat{F}_a(0,1,x)=f_a\big(q_\emptyset(x)\big),\quad \widehat{\pi}\big(\widehat{F}_a(w,x)\big)=q_{\mathbb{P}^1}(w),\quad \widehat{\pi}\big(\phi_{\widehat{\mathcal{Z}};S^1;a}(\mathrm{e}^{\mathrm{i}\theta};y)\big)=\mathrm{e}^{\mathrm{i}\theta}\widehat{\pi}(y),\tag{6.44}$$

and the maps

$$f_a\colon \big(\mathcal{Z}'_a,\omega_{\mathcal{Z}},X_\emptyset,\phi_{\mathcal{Z};S^1},\mu_{\mathcal{Z};S^1}\big)\longrightarrow\big(\widehat{\mathcal{Z}}_a-\widehat{\pi}^{-1}([0,1]),\omega_{\widehat{\mathcal{Z}};a},\widehat{\mathcal{Z}}_{a;0},\phi_{\widehat{\mathcal{Z}};S^1;a},\mu_{\widehat{\mathcal{Z}};S^1;a}\big),\tag{6.45}$$

$$\widehat{F}_{a;w}\colon (X,\omega)\longrightarrow\big(\widehat{\pi}^{-1}(w),\omega_{\widehat{\mathcal{Z}};a}\big|_{\widehat{\pi}^{-1}(w)}\big),\quad x\longrightarrow\widehat{F}_a(w,x),\quad \forall\,w\in\mathbb{P}^1-\{[1,0]\},\tag{6.46}$$

are isomorphisms. Every Hamiltonian $\mathbb{T}$-pair $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ for (X,ω) compatible with (ϕ,μ) determines a Hamiltonian $\mathbb{T}$-pair $(\phi_{\widehat{\mathcal{Z}};\mathbb{T}},\mu_{\widehat{\mathcal{Z}};\mathbb{T}})$ for $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$ so that $(\phi_{\widehat{\mathcal{Z}};\mathbb{T}},\mu_{\widehat{\mathcal{Z}};\mathbb{T}})$ is compatible with the pair $(\phi_{\widehat{\mathcal{Z}};S^1;a},\mu_{\widehat{\mathcal{Z}};S^1;a})$ and with the Hamiltonian $(S^1)^N_\bullet$-pair $(\phi_{\widehat{\mathcal{Z}}},\mu_{\widehat{\mathcal{Z}}})$ for $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$ determined by (ϕ,μ) and is intertwined with $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ by $\widehat{F}_a$ and with the pair $(\phi_{\mathcal{Z};\mathbb{T}},\mu_{\mathcal{Z};\mathbb{T}})$ of Proposition 6.2 by f_a.

Proof. We split $(\mathcal{Z},\omega_{\mathcal{Z}})$ into a compact piece $(\widehat{\mathcal{Z}},\omega_{\widehat{\mathcal{Z}}})$ containing a neighborhood $\mathcal{Z}'_a$ of $X_\emptyset$ and the "infinite remainder". We use the symplectic cut construction of [11] with the S^1-action $\phi_{\mathcal{Z};S^1}$, moment map $\mu_{\mathcal{Z};S^1}$, and its value

$$a>-\min_{i\in[N]}\min_{x\in X}\big(\mu(x)\big)_i\,.\tag{6.47}$$

This corresponds to the construction of Proposition 6.2 with $N=2$,

$$\begin{aligned}\phi=\phi_{S^1;\bullet}\colon (S^1)^2_\bullet\times\mathcal{Z}\longrightarrow\mathcal{Z},&\qquad \phi_{S^1;\bullet}\big((\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{-\mathrm{i}\theta});y\big)=\phi_{\mathcal{Z};S^1}\big(\mathrm{e}^{\mathrm{i}\theta};y\big),\\ \mu=\mu_{S^1;\bullet}\colon \mathcal{Z}\longrightarrow\mathfrak{t}^*_{2;\bullet},&\qquad \mu_{S^1;\bullet}(y)=r_\bullet\big(\mu_{\mathcal{Z};S^1}(y),a\big).\end{aligned}$$

By (6.39), the restriction of the $\phi_{S^1;\bullet}$-action to $\mathcal{Z}-X_\emptyset$ is free. By (6.41) and (6.47), $\mu_{S^1;\bullet}^{-1}(0)$ is disjoint from $X_\emptyset$ and so the restriction of the $\phi_{S^1;\bullet}$-action to $\mu_{S^1;\bullet}^{-1}(0)$ is free. Thus, $(\mathcal{Z},\omega,\phi_{S^1;\bullet},\mu_{S^1;\bullet})$ is a regular Hamiltonian $(S^1)^2_\bullet$-manifold. By Lemma 6.1 and Corollary 4.13, $(\mathcal{Z},\omega,\phi_{S^1;\bullet},\mu_{S^1;\bullet})$ thus determines symplectic manifolds $(\widehat{\mathcal{Z}}_1,\omega_{\widehat{\mathcal{Z}};1})$ and $(\widehat{\mathcal{Z}}_2,\omega_{\widehat{\mathcal{Z}};2})$ with a common smooth symplectic divisor $\widehat{\mathcal{Z}}_{12}$. We denote $(\widehat{\mathcal{Z}}_1,\omega_{\widehat{\mathcal{Z}};1})$ and $\widehat{\mathcal{Z}}_{12}$ by $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$ and $\widehat{\mathcal{Z}}_{a;\infty}$, respectively.

By the proof of Proposition 6.2,

$$\begin{gathered}\widehat{\mathcal{Z}}_a=\check{\mathcal{Z}}_a/\!\sim,\quad \widehat{\mathcal{Z}}_{a;\infty}=\check{\mathcal{Z}}_{a;\infty}/\!\sim\;\subset\widehat{\mathcal{Z}}_a,\qquad\text{where}\\ \check{\mathcal{Z}}_a=\big\{(y,w)\in\mathcal{Z}\times\mathbb{C}\colon \mu_{\mathcal{Z};S^1}(y)=a-\frac{1}{2}|w|^2\big\},\qquad \check{\mathcal{Z}}_{a;\infty}=\check{\mathcal{Z}}_a\cap\big(\mathcal{Z}\times\{0\}\big),\\ (y,w)\sim\big(\phi_{\mathcal{Z};S^1}(\mathrm{e}^{\mathrm{i}\theta};y),\mathrm{e}^{\mathrm{i}\theta}w\big)\,.\end{gathered}\tag{6.48}$$

Let $\widehat{q}\colon \breve{\mathcal{Z}}_a \longrightarrow \widehat{\mathcal{Z}}_a$ be the quotient map. The map f_a in (6.43) is the restriction of the collapsing map $q_\emptyset$ for this symplectic cut to the preimage of $\widehat{\mathcal{Z}}_a$,

$$f_a\colon \overline{\mathcal{Z}}'_a \equiv \big\{y\in\mathcal{Z}\colon \mu_{\mathcal{Z};S^1}(y)\leqslant a\big\} \longrightarrow \widehat{\mathcal{Z}}_a, \qquad f_a(y) = \widehat{q}\Big(y, \sqrt{2\big(a-\mu_{\mathcal{Z};S^1}(y)\big)}\Big). \tag{6.49}$$

By Corollary 4.13, its restriction

$$f_a\colon \mathcal{Z}'_a \equiv \big\{y\in\mathcal{Z}\colon \mu_{\mathcal{Z};S^1}(y)<a\big\} \longrightarrow \widehat{\mathcal{Z}}_a - \widehat{\mathcal{Z}}_{a;\infty}$$

is a symplectomorphism with respect to the symplectic forms $\omega_{\mathcal{Z}}$ and $\omega_{\widehat{\mathcal{Z}};a}$.

Let $\pi\colon \mathcal{Z}\longrightarrow\mathbb{C}$ be as in Proposition 6.2 and define

$$\widehat{\pi}\colon \widehat{\mathcal{Z}}_a \longrightarrow \mathbb{P}^1, \qquad \widehat{\pi}\big(\widehat{q}(y,w)\big) = \big[w,\pi(y)\big]. \tag{6.50}$$

Since $\widehat{\mathcal{Z}}_{a;\infty}$ is disjoint from $f_a(X_\emptyset)=f_a(\pi^{-1}(0))$, this map is well-defined. Furthermore,

$$\widehat{\mathcal{Z}}_{a;0} \equiv \widehat{\pi}^{-1}\big([1,0]\big) = f_a\big(\pi^{-1}(0)\big) = f_a(X_\emptyset). \tag{6.51}$$

Thus, f_a identifies $X_\emptyset$ with $\widehat{\mathcal{Z}}_{a;0}$. The facts that $\widehat{\pi}$ is a submersion outside of $\widehat{\mathcal{Z}}_{a;0}$ and $\omega_{\widehat{\mathcal{Z}};a}|_{\widehat{\pi}^{-1}(\lambda)}$ is a symplectic form whenever $\lambda\neq[1,0]$ follow from the first statement in (6.44) and (6.46).

With the notation as in the proof of Lemma 6.5, define an $(S^1)^N$-action on $X\times\mathbb{C}^{N+1}$ by

$$\widetilde{\phi}_{N+1}\colon (S^1)^N\times\big(X\times\mathbb{C}^{N+1}\big) \longrightarrow X\times\mathbb{C}^{N+1}, \quad \widetilde{\phi}_{N+1}\big(\iota_i(\mathrm{e}^{\mathrm{i}\theta});x,z,w\big) = \big(\widetilde{\phi}_{\mathcal{Z};S^1;i}(\mathrm{e}^{\mathrm{i}\theta};x,z),\mathrm{e}^{\mathrm{i}\theta}w\big).$$

By (6.9), (6.10), and (6.48),

$$\begin{gathered}\widehat{\mathcal{Z}}_a = \widetilde{\mathcal{Z}}_{N+1;a}/\widetilde{\phi}_{N+1}, \qquad \text{where}\\ \widetilde{\mathcal{Z}}_{N+1;a} = \Big\{\big(x,(z_i)_{i\in[N]},w\big)\in X\times\mathbb{C}^{N+1}\colon -\big(\mu(x)\big)_i+\frac{1}{2}|z_i|^2 = a-\frac{1}{2}|w|^2\ \forall\, i\in[N]\Big\}.\end{gathered} \tag{6.52}$$

The symplectic form $\omega_{\widehat{\mathcal{Z}};a}$ on $\widehat{\mathcal{Z}}_a$ is determined by the condition

$$\widehat{q}_{N+1;a}^*\omega_{\widehat{\mathcal{Z}};a} = \big(\pi_1^*\omega_X+\pi_2^*\omega_{\mathbb{C}^{N+1}}\big)\big|_{\widetilde{\mathcal{Z}}_{N+1;a}}, \tag{6.53}$$

where $\widehat{q}_{N+1;a}\colon \widetilde{\mathcal{Z}}_{N+1;a}\longrightarrow\widehat{\mathcal{Z}}_a$ is the quotient map. By (6.51),

$$\widehat{\mathcal{Z}}_{a;0} = \widetilde{\mathcal{Z}}_{N+1;a;0}/\widetilde{\phi}_{N+1}, \quad \text{where}\ \ \widetilde{\mathcal{Z}}_{N+1;a;0} = \bigcup_{i\in[N]}\big\{\big(x,(z_j)_{j\in[N]},w\big)\in\widetilde{\mathcal{Z}}_{N+1;a}\colon z_i=0\big\}.$$

In particular,

$$\widetilde{\mathcal{Z}}^{>}_{N+1;a} \equiv \widetilde{\mathcal{Z}}_{N+1;a}\cap\big(X\times(\mathbb{R}^+)^N\times\mathbb{C}\big)$$

is a slice for the $\widetilde{\phi}_{N+1}$-action on $\widetilde{\mathcal{Z}}_{N+1;a}-\widetilde{\mathcal{Z}}_{N+1;a;0}$ identified by $\widehat{q}_{N+1;a}$ with $\widehat{\mathcal{Z}}_a-\widehat{\mathcal{Z}}_{a;0}$.

For each $x\in\mathbb{C}\times X$, let

$$\varrho(x) = 2\min\big\{a+(\mu(x))_i\colon i\in[N]\big\}\in\mathbb{R}^+; \tag{6.54}$$

see (6.47). For each $(w,x)\in\mathbb{C}\times X$, let $\varrho(w,x)\in[0,\varrho(x)]$ be the unique solution of the equation

$$|w|^2\prod_{i\in[N]}\big(2\big(a+(\mu(x))_i\big)-\varrho\big)=\varrho \tag{6.55}$$

in ϱ. As ϱ increases from 0 to $\varrho(x)$, the left-hand side of (6.55) decreases from a positive value to 0 if $w\neq0$. Thus, $\varrho(w,x)$ is well-defined, depends smoothly on (w,x), and

$$\sqrt{\varrho(w,x)}=|w|\,g(w,x) \tag{6.56}$$

for some smooth $\mathbb{R}^+$-valued function g on $\mathbb{C}\times X$. Define $\widehat{F}_a$ in (6.43) by

$$\widehat{F}_a([w,1],x)=\widehat{q}_{N+1;a}\Big(x,\big(\sqrt{2\big(a+(\mu(x))_i\big)-\varrho(w,x)}\big)_{i\in[N]},wg(w,x)\Big). \tag{6.57}$$

By (6.50), (6.55), and (6.56), $\widehat{F}_a$ satisfies the second property in (6.44). By the same reasoning as in the proof of Corollary 4.13 and (6.53), each map (6.46) is a symplectomorphism.

Under the identification in (6.38), (6.57) becomes

$$\widehat{F}_a(r,\mathrm{e}^{\mathrm{i}\theta},x)=\widehat{q}_{N+1;a}\Big(x,\big(\sqrt{2\big(a+(\mu(x))_i\big)-\rho(r,x)}\big)_{i\in[N]},\mathrm{e}^{-\mathrm{i}\theta}\sqrt{\rho(r,x)}\Big), \tag{6.58}$$

where $\rho=\rho(r,x)$ is the unique solution of

$$\prod_{i\in[N]}\big(2\big(a+(\mu(x))_i\big)-\rho\big)=r^2\rho$$

in $[0,\varrho(x)]$. It extends continuously over $r=0$ as $\rho(0,x)=\varrho(x)$. Thus, (6.58) extends continuously over $\{0\}\times S^1\times X$. By (6.12) and (6.49), this extension satisfies the first property in (6.44). By the continuity of both sides, it also satisfies the second property in (6.44). The functions

$$(r,x)\longrightarrow\sqrt{2\big(a+(\mu(x))_i\big)-\rho(r,x)},\qquad i\in[N],$$

are smooth at $(0,x)$ if the minimum in (6.54) is reached at a unique $i\in[N]$. Thus, the function (6.58) is smooth outside of the preimage of $f_a(X_\partial)$.

By Proposition 6.2, the Hamiltonian S^1-pair $(\phi_{\mathcal{Z};S^1},\mu_{\mathcal{Z};S^1})$ for $(\mathcal{Z},\omega_{\mathcal{Z}})$ determines a Hamiltonian S^1-pair $(\phi_{\widehat{\mathcal{Z}};S^1;a},\mu_{\widehat{\mathcal{Z}};S^1;a})$ for $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$ such that

$$\phi_{\widehat{\mathcal{Z}};S^1;a}\big(\mathrm{e}^{\mathrm{i}\theta};f_a(y)\big)=f_a\big(\phi_{\mathcal{Z};S^1}(\mathrm{e}^{\mathrm{i}\theta};y)\big),\quad \mu_{\widehat{\mathcal{Z}};S^1;a}\big(f_a(y)\big)=\mu_{\mathcal{Z};S^1}(y)\qquad\forall\,\mathrm{e}^{\mathrm{i}\theta}\in S^1,\ y\in\mathcal{Z}_a';$$

this establishes (6.45). By (6.50), the second property in (6.39), and (6.37), $\phi_{\widehat{\mathcal{Z}};S^1;a}$ also satisfies the third property in (6.44).

Let $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ be a Hamiltonian $\mathbb{T}$-pair for (X,ω) compatible with (ϕ,μ). By Proposition 6.2 applied to $(\phi_{\mathcal{Z};\mathbb{T}},\mu_{\mathcal{Z};\mathbb{T}})$, $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ determines a Hamiltonian pair $(\phi_{\widehat{\mathcal{Z}};\mathbb{T}},\mu_{\widehat{\mathcal{Z}};\mathbb{T}})$ for $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$. By this proposition and Lemma 6.5, $(\phi_{\widehat{\mathcal{Z}};\mathbb{T}},\mu_{\widehat{\mathcal{Z}};\mathbb{T}})$ is compatible with $(\phi_{\widehat{\mathcal{Z}};S^1;a},\mu_{\widehat{\mathcal{Z}};S^1;a})$ and $(\phi_{\widehat{\mathcal{Z}}},\mu_{\widehat{\mathcal{Z}}})$ and is intertwined with $(\phi_{\widehat{\mathcal{Z}};\mathbb{T}},\mu_{\widehat{\mathcal{Z}};\mathbb{T}})$ by f_a. Since $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ is compatible with (ϕ,μ), the solution $\varrho(w,x)$ of (6.55) is $\phi_{\mathbb{T}}$-invariant. By (6.57) and the construction of the induced Hamiltonian pair,

$$\phi_{\widehat{\mathcal{Z}};\mathbb{T}}\big(g;\widehat{F}_a(w,x)\big)=\widehat{F}_a\big(w,\phi_{\mathbb{T}}(g;x)\big),\qquad \mu_{\widehat{\mathcal{Z}};\mathbb{T}}\big(\widehat{F}_a(w,x)\big)=\mu_{\mathbb{T}}(x), \tag{6.59}$$

i.e. $\widehat{F}_a$ thus intertwines $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ and $(\phi_{\widehat{\mathcal{Z}};\mathbb{T}},\mu_{\widehat{\mathcal{Z}};\mathbb{T}})$. $\square$

By the second property in (6.44) and the first property in (6.59),

$$\widehat{\pi}\big(\phi_{\widehat{\mathcal{Z}};\mathbb{T}}(g;y)\big) = \widehat{\pi}(y).$$

An analogue $\mathscr{C}_{\widehat{\mathcal{Z}}}$ of the cutting configuration (4.52) for the symplectic manifold $(\widehat{\mathcal{Z}}_a, \omega_{\widehat{\mathcal{Z}};a})$ of Corollary 6.6 can be constructed from the initial cutting configuration (6.4) via (6.48) as in Section 4.5. By the same reasoning as in the proof of Proposition 6.2, $\mathscr{C}_{\widehat{\mathcal{Z}}}$ is then the cutting configuration determined by the Hamiltonian pair $(\phi_{\widehat{\mathcal{Z}}}, \mu_{\widehat{\mathcal{Z}}})$ for $(\widehat{\mathcal{Z}}_a, \omega_{\widehat{\mathcal{Z}};a})$. By (6.46) and (6.59), the restriction (6.46) of $\widehat{F}_a$ induces an isomorphism

$$\widehat{F}_{a;w}\colon \big(X, \omega, \phi_{\mathbb{T}}, \mu_{\mathbb{T}}\big) \longrightarrow \big(\widehat{\pi}^{-1}(\lambda), \omega_{\widehat{\mathcal{Z}};a}|_{\widehat{\pi}^{-1}(\lambda)}, \phi_{\widehat{\mathcal{Z}};\mathbb{T}}|_{\mathbb{T}\times\widehat{\pi}^{-1}(\lambda)}, \mu_{\widehat{\mathcal{Z}};\mathbb{T}}|_{\widehat{\pi}^{-1}(\lambda)}\big)$$

whenever $\lambda \equiv q_{\mathbb{P}^1}(w) \neq [1,0]$. Since $\mathscr{C}_{\widehat{\mathcal{Z}}} = \mathscr{C}_{\phi_{\widehat{\mathcal{Z}}}, \mu_{\widehat{\mathcal{Z}}}}$, the $(\phi_{\mathbb{T}}, \mu_{\mathbb{T}}) = (\phi, \mu)$ case of this statement implies that $\widehat{F}_{a;w}$ identifies the cutting configuration (6.4) for (X, ω) with the restriction of $\mathscr{C}_{\widehat{\mathcal{Z}}}$ to $\widehat{\pi}^{-1}(\lambda)$.

6.4 Degenerations and moment polytopes

Suppose $\mathbb{T} \approx (S^1)^k$ is a k-torus and $(X, \omega, \phi_{\mathbb{T}}, \mu_{\mathbb{T}})$ is a compact connected Hamiltonian $\mathbb{T}$-manifold. By the Atiyah-Guillemin-Sternberg Convexity Theorem [3, Theorem 27.1],

$$\Delta \equiv \mu_{\mathbb{T}}(X) \subset \mathfrak{t}^*$$

is then a convex polytope. The Hamiltonian $\mathbb{T}$-pair $(\phi_{\mathbb{T}}, \mu_{\mathbb{T}})$ for (X, ω) gives rise to Hamiltonian $(S^1)^N$-pairs for (X, ω); the latter in turn determine Hamiltonian configurations for (X, ω) as in (6.3). In this section, we describe the effect of the constructions of this paper with cutting configurations arising in this way on the moment polytope Δ.

For $\xi \in \mathfrak{t}$, define

$$L_\xi\colon \mathfrak{t}^* \longrightarrow \mathbb{R}, \quad L_\xi(\eta) = \langle \eta, \xi \rangle, \qquad h_\xi \equiv L_\xi \circ \mu_{\mathbb{T}}\colon X \longrightarrow \mathbb{R}, \quad h_\xi(x) = L_\xi\big(\mu_{\mathbb{T}}(x)\big).$$

A vector $\xi \in \mathfrak{t}$ is called integral if its time 2π-flow in $\mathbb{T}$ generates a circle subgroup $S^1_\xi \subset \mathbb{T}$. An integral vector ξ determines a homomorphism and an S^1-action,

$$\varrho_\xi\colon S^1 \longrightarrow S^1_\xi \qquad \text{and} \qquad \phi_\xi\colon S^1 \times X \longrightarrow X,$$

respectively; the latter is the composition of the $\mathbb{T}$-action $\phi_{\mathbb{T}}$ with the former. The action ϕ_ξ commutes with the $\mathbb{T}$-action and has Hamiltonian h_ξ. We denote by $\Lambda_{\mathfrak{t}} \subset \mathfrak{t}$ the lattice of integral vectors and by $\Lambda^*_{\mathfrak{t}} \subset \mathfrak{t}^*$ the dual lattice. An integral vector $\xi \in \mathfrak{t}$ is called primitive if ξ/m is not integral for any integer $m > 1$. The homomorphism ϱ_ξ is injective if and only if ξ is primitive.

Suppose $\xi \in \Lambda_{\mathfrak{t}}$ is primitive, $\epsilon \in \mathbb{R}$ is a regular value of h_ξ, and the S^1_ξ-action on the hypersurface

$$\widetilde{V}_\epsilon \equiv \big\{x \in X\colon\, h_\xi(x) = \epsilon\big\} \subset X$$

is free. By [11, Remark 1.5], the 2-fold symplectic cut construction with the S^1_ξ-action and Hamiltonian $h_\xi - \epsilon$ then cuts X along $\widetilde{V}_\epsilon$ into two symplectic manifolds, (X_-, ω_-) and (X_+, ω_+), with Hamiltonian $\mathbb{T}$-actions and moment polytopes

$$\Delta_- = \big\{\eta \in \Delta\colon L_\xi(\eta) \leqslant \epsilon\big\} \qquad \text{and} \qquad \Delta_+ = \big\{\eta \in \Delta\colon L_\xi(\eta) \geqslant \epsilon\big\},$$

respectively.

Our N-fold symplectic cut construction produces more complicated subdivisions of the moment polytope Δ. Fix a tuple $(\xi_i,\epsilon_i)_{i\in[N]}$ in $(\Lambda_{\mathfrak t}\times\mathbb R)^N$. For $i\in[N]$ and $a\in\mathbb R$, define

$$\begin{split} L_i = L_{\xi_i}-\epsilon_i\colon \mathfrak t^*\longrightarrow\mathbb R, \quad \Delta_i\equiv\Delta_i\big((\xi_j,\epsilon_j)_{j\in[N]}\big)=\big\{\eta\in\Delta\colon\ L_i(\eta)\leqslant L_j(\eta)\ \forall\,j\in[N]\big\},\\ \widehat\Delta_a\equiv\widehat\Delta_a\big((\xi_j,\epsilon_j)_{j\in[N]}\big)=\big\{(\eta,u)\in\Delta\times\mathbb R\colon -\min_{i\in[N]}L_i(\eta)\leqslant u\leqslant a\big\}. \end{split} \tag{6.60}$$

For $I\in\mathcal P^*(N)$, let

$$\Delta_I\equiv\Delta_I\big((\xi_i,\epsilon_i)_{i\in[N]}\big)=\big\{\eta\in\Delta\colon L_i(\eta)\leqslant L_j(\eta)\ \forall\,i\in I,\ j\in[N]\big\}.$$

For our purposes, a pair (ξ_i,ϵ_i) such that the polytope Δ_i is empty can be dropped from the consideration (thus reducing the value of N).

The polytopes Δ_i with $i\in[N]$ subdivide Δ; the first diagram in Figure 2 shows such a subdivision for the data of Example 6.9. For generic choices of $\epsilon_i\in\mathbb R$, the intersection of each Δ_I with a facet of Δ is a polytope of codimension $|I|-1$ in the facet. By Proposition 6.10 in the next section, this property needs to hold for the N-fold Hamiltonian configuration (6.63) for (X,ω) determined by the tuple $(\xi_i,\epsilon_i)_{i\in[N]}$ to be a cutting configuration. By Theorem 3 below, the polytopes Δ_i are then the moment polytopes of the Hamiltonian $\mathbb T$-manifolds $(X_i,\omega_i,\phi_{\mathbb T;i},\mu_{\mathbb T;i})$ determined by the construction of Section 4. If

$$a>-\min_{i\in[N]}\min_{\eta\in\Delta}L_i(\eta)\,, \tag{6.61}$$

then $\widehat\Delta_a$ is the polytope rising from the graph of the function

$$\Delta\longrightarrow\mathbb R,\qquad \eta\longrightarrow-\min_{i\in[N]}\big(L_i(\eta)\big)\,,$$

to the "horizontal" hyperplane $\mathfrak t^*\times\{a\}$ in $\mathfrak t^*\times\mathbb R$. This graph consists of N polytopes Δ_i' with $\pi_{\mathfrak t^*}(\Delta_i')=\Delta_i$, where

$$\pi_{\mathfrak t^*}\colon\mathfrak t^*\oplus\mathbb R\longrightarrow\mathfrak t^*$$

is the projection map. The second diagram in Figure 2 shows such a polytope $\widehat\Delta_a$ for the data of Example 6.9. If (6.63) is a cutting configuration and (6.61) holds, then $\widehat\Delta_a$ is the moment polytope of the symplectic manifold $(\widehat{\mathcal Z}_a,\omega_{\widehat{\mathcal Z};a})$ of Corollary 6.6 with the Hamiltonian $(\mathbb T\times S^1)$-pair obtained by combing the pairs $(\phi_{\widehat{\mathcal Z};\mathbb T},\mu_{\widehat{\mathcal Z};\mathbb T})$ and $(\phi_{\widehat{\mathcal Z};S^1;a},\mu_{\widehat{\mathcal Z};S^1;a})$.

With $(\xi_i,\epsilon_i)_{i\in[N]}$ as above and ι_i as in the proof of Lemma 6.5, we define a Hamiltonian $(S^1)^N$-pair for (X,ω) by

$$\begin{split} \phi\colon(S^1)^N\times X\longrightarrow X, \qquad \phi\big(\iota_i(\mathrm e^{\mathrm i\theta});x\big)=\phi_{\xi_i}\big(\mathrm e^{\mathrm i\theta};x\big),\\ \mu\colon X\longrightarrow\mathbb R^N, \qquad \mu(x)=\big(L_i(\mu_{\mathbb T}(x))\big)_{i\in[N]}. \end{split} \tag{6.62}$$

Let

$$\mathscr C_{\phi,\mu}\equiv\mathscr C_{\phi_{\mathbb T},\mu_{\mathbb T}}\big((\xi_i,\epsilon_i)_{i\in[N]}\big)\equiv\big(U_I,\phi_I,\mu_I\big)_{I\in\mathcal P^*(N)} \tag{6.63}$$

be the associated maximal N-fold Hamiltonian configuration for (X,ω) as in (6.4). In particular,

$$U_I^{\leqslant}\equiv\big\{x\in X\colon L_i(\mu_{\mathbb T}(x))\leqslant L_j(\mu_{\mathbb T}(x))\ \forall\,i\in I,\ j\in[N]\big\}=\mu_{\mathbb T}^{-1}\big(\Delta_I\big)\quad\forall\,I\in\mathcal P^*(N). \tag{6.64}$$

For an arbitrary Hamiltonian $(S^1)^N$-pair for (X,ω), the configuration (6.4) is the case of (6.63) with $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})=(\phi,\mu)$, $\xi_i\in\mathbb{Z}^N$ being the standard i-th coordinate vector, and $\epsilon_i=0$.

We call a tuple $(\xi_i,\epsilon_i)_{i\in[N]}$ as above **regular** if the associated $(S^1)^N$-Hamiltonian manifold (X,ω,ϕ,μ) is regular as defined above Lemma 6.1. Proposition 6.10 in Section 6.5 provides a geometric interpretation of this criterion in terms of the Delzant condition of [3, Definition 28.1].

Theorem 3. *Suppose $\mathbb{T}$ is a k-torus, $(X,\omega,\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ is a compact connected Hamiltonian $\mathbb{T}$-space, and $(\xi_i,\epsilon_i)_{i\in[N]}$ is a regular tuple in $(\Lambda_{\mathfrak{t}}\times\mathbb{R})^N$. The tuple (6.63) is then a maximal N-fold cutting configuration for (X,ω). For every $a\in\mathbb{R}$ sufficiently large, this tuple determines*

(1) a compact symplectic manifold $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$ containing the tuple $(X_i,\omega_i)_{i\in[N]}$ of the cut symplectic manifolds of Corollary 4.13 as an SC symplectic divisor $X_\emptyset$ and

(2) a Hamiltonian $(\mathbb{T}\times S^1)$-pair $(\phi_{\widehat{\mathcal{Z}};a},\mu_{\widehat{\mathcal{Z}};a})$ such that

$$\phi_{\widehat{\mathcal{Z}};a}\big(\mathbb{T}\times S^1\times X_i\big)=X_i\quad\forall\,i\in[N],\qquad \mu_{\widehat{\mathcal{Z}};a}\big(\widehat{\mathcal{Z}}_a\big)=\widehat{\Delta}_a\,. \tag{6.65}$$

For all $i\in I\subset[N]$, the preimage of Δ_I under $\pi_{\mathfrak{t}^*}\circ\mu_{\widehat{\mathcal{Z}};a}|_{X_i}$ is the symplectic submanifold (X_I,ω_I) of (X_i,ω_i) determined by (6.63). If the $\phi_{\mathbb{T}}$-action is effective, then so is the $\phi_{\widehat{\mathcal{Z}};a}$-action. The deformation equivalence class of smoothings of $X_\emptyset$ determined by (6.63) is represented by a nearly regular symplectic fibration $\widehat{\pi}\colon\widehat{\mathcal{Z}}_a\longrightarrow\mathbb{P}^1$ equivariant with respect to the projection $\mathbb{T}\times S^1\longrightarrow S^1$.

Proof. Since the tuple $(\xi_i,\epsilon_i)_{i\in[N]}$ is regular, (6.63) is a maximal N-fold cutting configuration for (X,ω) by Lemma 6.1. By Theorem 1, it thus determines a symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$ and a tuple $(X_i,\omega_i)_{i\in[N]}$ of symplectic manifolds contained in $(\mathcal{Z},\omega_{\mathcal{Z}})$ as an SC symplectic divisor $X_\emptyset$. For $a\in\mathbb{R}$ sufficiently large, let $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$,

$$\widehat{\pi}\colon\widehat{\mathcal{Z}}_a\longrightarrow\mathbb{P}^1,\qquad f_a\colon\mathcal{Z}'_a\longrightarrow\widehat{\mathcal{Z}}_a,$$

$(\phi_{\widehat{\mathcal{Z}};S^1;a},\mu_{\widehat{\mathcal{Z}};S^1;a})$, and $(\phi_{\widehat{\mathcal{Z}};\mathbb{T}},\mu_{\widehat{\mathcal{Z}};\mathbb{T}})$ be the associated objects provided by Corollary 6.6. In particular, f_a embeds $X_\emptyset$ as an SC symplectic divisor into the compact symplectic manifold $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$. The required condition (6.47) on $a\in\mathbb{R}$ in this case becomes (6.61).

Since the Hamiltonian $\mathbb{T}$-pair $(\phi_{\widehat{\mathcal{Z}};\mathbb{T}},\mu_{\widehat{\mathcal{Z}};\mathbb{T}})$ and the Hamiltonian S^1-pair $(\phi_{\widehat{\mathcal{Z}};S^1;a},\mu_{\widehat{\mathcal{Z}};S^1;a})$ are compatible, they determine a Hamiltonian $(\mathbb{T}\times S^1)$-pair $(\phi_{\widehat{\mathcal{Z}};a},\mu_{\widehat{\mathcal{Z}};a})$ with

$$\mu_{\widehat{\mathcal{Z}};a}=\big(\mu_{\widehat{\mathcal{Z}};\mathbb{T}},\mu_{\widehat{\mathcal{Z}};S^1;a}\big)\colon\widehat{\mathcal{Z}}_a\longrightarrow\mathfrak{t}^*\oplus\mathbb{R}.$$

Since $\widehat{\pi}$ is S^1-equivariant, it is equivariant with respect to the projection $\mathbb{T}\times S^1\longrightarrow S^1$. Since the $\phi_{\widehat{\mathcal{Z}};\mathbb{T}}$- and $\phi_{\widehat{\mathcal{Z}};S^1;a}$-actions preserve $X_i\subset\widehat{\mathcal{Z}}_a$, so does the $\phi_{\widehat{\mathcal{Z}};a}$-action. By Corollary 6.6 and Proposition 6.2,

$$\mu_{\mathcal{Z};\mathbb{T}}=\mu_{\widehat{\mathcal{Z}};\mathbb{T}}\circ f_a\qquad\text{and}\qquad \mu_{\mathbb{T}}=\mu_{\mathcal{Z};\mathbb{T}}\circ q_\emptyset\,, \tag{6.66}$$

respectively. Combining these statements with (6.64) and Corollary 4.13, we obtain

$$\begin{aligned}\big\{\pi_{\mathfrak{t}^*}\circ\mu_{\widehat{\mathcal{Z}};a}\big\}^{-1}(\Delta_I)\cap X_\emptyset&\equiv\mu_{\widehat{\mathcal{Z}};\mathbb{T}}^{-1}(\Delta_I)\cap f_a(X_\emptyset)=f_a\big(\mu_{\mathcal{Z};\mathbb{T}}^{-1}(\Delta_I)\cap X_\emptyset\big)=f_a\big(\mu_{\mathcal{Z};\mathbb{T}}^{-1}(\Delta_I)\cap q_\emptyset(X)\big)\\&=f_a\big(q_\emptyset\big(\mu_{\mathbb{T}}^{-1}(\Delta_I)\big)\big)=f_a\big(q_\emptyset(U_I^{\leqslant})\big)=f_a(X_I)\equiv X_I\end{aligned}$$

for every $I\in\mathcal{P}^*(N)$.

With the notation as in (6.52), define

$$\widetilde{\phi}_{\widehat{\mathcal{Z}};a}\colon \mathbb{T}\times S^1\times\widetilde{\mathcal{Z}}_{N+1;a}\longrightarrow\widetilde{\mathcal{Z}}_{N+1;a},\quad \widetilde{\phi}_{\widehat{\mathcal{Z}};a}\big(g,\mathrm{e}^{\mathrm{i}\theta};x,z,w\big)=\big(\phi_{\mathbb{T}}(g;x),z,\mathrm{e}^{-\mathrm{i}\theta}w\big), \tag{6.67}$$

$$\widetilde{\mu}_{\widehat{\mathcal{Z}};a}\colon\widetilde{\mathcal{Z}}_{N+1;a}\longrightarrow\mathfrak{t}^*\oplus\mathbb{R},\qquad \widetilde{\mu}_{\widehat{\mathcal{Z}};a}(x,z,w)=\Big(\mu_{\mathbb{T}}(x),a-\frac{1}{2}|w|^2\Big). \tag{6.68}$$

By construction, the pair $(\widetilde{\phi}_{\widehat{\mathcal{Z}};a},\widetilde{\mu}_{\widehat{\mathcal{Z}};a})$ descends to the pair $(\phi_{\widehat{\mathcal{Z}};a},\mu_{\widehat{\mathcal{Z}};a})$ on the quotient in (6.52). By (6.68) and (6.66), $\pi_{\mathfrak{t}^*}(\mu_{\widehat{\mathcal{Z}};a}(\widehat{\mathcal{Z}}_a))=\Delta$. Since $(\mu(x))_i=L_i(\mu_{\mathbb{T}}(x))$, (6.52) and (6.68) give

$$\big\{u\in\mathbb{R}\colon(\eta,u)\in\mu_{\widehat{\mathcal{Z}};a}(\widehat{\mathcal{Z}}_a)\big\}=\big\{u\in\mathbb{R}\colon -L_i(\eta)\leqslant u\leqslant a\ \forall\,i\in[N]\big\}\quad\forall\,\eta\in\Delta.$$

This establishes the second statement in (6.65).

Suppose the $\phi_{\widehat{\mathcal{Z}};a}$-action of $(g,\mathrm{e}^{\mathrm{i}\theta})\in\mathbb{T}\times S^1$ on $\widehat{\mathcal{Z}}_a$ is trivial. Let

$$z=(z_i)_{i\in[N]}\in\mathbb{C}^N,\qquad \widetilde{y}\equiv\big(x,z,w\big)\in\widetilde{\mathcal{Z}}_{N+1;a},\qquad\text{and}\quad\big(\mathrm{e}^{\mathrm{i}\theta_i}\big)_{i\in[N]}\in(S^1)^N$$

be such that

$$z_i\neq0\ \ \forall\,i\in[N],\quad w\neq0,\quad \widetilde{\phi}_{\widehat{\mathcal{Z}};a}\big(g,\mathrm{e}^{\mathrm{i}\theta};x,z,w\big)=\widetilde{\phi}_{N+1}\big((\mathrm{e}^{\mathrm{i}\theta_i})_{i\in[N]};x,z,w\big). \tag{6.69}$$

By (6.67) and the first and last assumptions in (6.69), $\mathrm{e}^{\mathrm{i}\theta_i}=1$ for every $i\in[N]$. By the second assumption in (6.69), this in turn implies that $\mathrm{e}^{\mathrm{i}\theta}=1$. Since the projection

$$\widetilde{\mathcal{Z}}_{N+1;a}\cap\big(X\times(\mathbb{C}^{N+1}-\mathbb{C}^{N+1}_0)\big)\longrightarrow X$$

is surjective, it follows that the $\phi_{\mathbb{T}}$-action of g on X is trivial. If the $\phi_{\mathbb{T}}$-action is effective, the $\phi_{\widehat{\mathcal{Z}};a}$-action is thus effective as well. □

The compatible Hamiltonian pairs $(\phi_{\mathcal{Z};\mathbb{T}},\mu_{\mathcal{Z};\mathbb{T}})$ and $(\phi_{\mathcal{Z};S^1},\mu_{\mathcal{Z};S^1})$ for $(\mathcal{Z},\omega_{\mathcal{Z}})$ provided by Proposition 6.2 and Lemma 6.5 give rise to a Hamiltonian $(\mathbb{T}\times S^1)$-pair $(\phi_{\mathcal{Z};\infty},\mu_{\mathcal{Z};\infty})$ for the symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$ determined by (6.63) if the tuple $(\xi_i,\epsilon_i)_{i\in[N]}$ is regular. The corresponding moment "polytope" is

$$\mu_{\mathcal{Z};\infty}(\mathcal{Z})=\widehat{\Delta}_\infty\equiv\big\{(\eta,u)\in\Delta\times\mathbb{R}\colon -\min_{i\in[N]}L_i(\eta)\leqslant u\big\}.$$

The moment polytope of $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a},\phi_{\widehat{\mathcal{Z}};a},\mu_{\widehat{\mathcal{Z}};a})$ is obtained by cutting off this infinite "polytope" at the level $u=a$ of the moment map for the S^1-action $\phi_{\mathcal{Z};S^1}$, as expected from the proof of Corollary 6.6 and [11, Remark 1.5].

By Corollary 6.6, the fibers $(\widehat{\mathcal{Z}}_{a;\lambda},\omega_{\widehat{\mathcal{Z}};a;\lambda})$ of the map $\widehat{\pi}$ in Theorem 3 over $\mathbb{P}^1-\{[1,0]\}$ with the Hamiltonian $\mathbb{T}$-pair $(\phi_{\widehat{\mathcal{Z}};\mathbb{T}},\mu_{\widehat{\mathcal{Z}};\mathbb{T}})$ are canonically isomorphic to (X,ω) with the pair $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$. The fibers over $[1,0]$ and $[0,1]$ are preserved by the full $\mathbb{T}\times S^1$-action $\phi_{\widehat{\mathcal{Z}};a}$ on $\widehat{\mathcal{Z}}_a$. The restriction of the S^1-action to the latter is in fact trivial; this is reflected in the "top" face of the polytope $\widehat{\Delta}_a$ in (6.60) being "horizontal". The fibers over $\mathbb{C}^*\subset\mathbb{P}^1$ are not preserved by the S^1-action. Their images $\mu_{\widehat{\mathcal{Z}};a}(\widehat{\mathcal{Z}}_{a;\lambda})\subset\widehat{\Delta}_a$ do not depend the angular component of λ. The restriction of $\pi_{\mathfrak{t}*}$ to $\mu_{\widehat{\mathcal{Z}};a}(\widehat{\mathcal{Z}}_{a;\lambda})$ is surjective onto Δ and has one-dimensional fibers.

A compact connected Hamiltonian $\mathbb{T}$-manifold $(X,\omega,\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ is a symplectic toric manifold if the $\mathbb{T}$-action $\phi_{\mathbb{T}}$ is effective and $\dim_{\mathbb{R}}X=2\dim_{\mathbb{R}}\mathbb{T}$. By Theorem 3, $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a},\phi_{\widehat{\mathcal{Z}};a},\mu_{\widehat{\mathcal{Z}};a})$ is a symplectic toric manifold if $(X,\omega,\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ is. The projection $\mathfrak{t}\oplus\mathbb{R}\longrightarrow\mathbb{R}$ then induces a projection from the toric fan of $\widehat{\mathcal{Z}}_a$ to the toric fan of $\mathbb{P}^1$. The projection $\widehat{\pi}$ in Theorem 3 is the projective morphism induced by $\pi_{\mathbb{R}}$; see [2, Proposition VII.1.16]. We can think of $\widehat{\pi}$ as a one-parameter family of Kähler manifolds smoothing $\widehat{\mathcal{Z}}_{a;0}=X_{\emptyset}$ into $\widehat{\mathcal{Z}}_{a;\infty}\approx X$. The vertical edges of the polytope $\widehat{\Delta}_a$ in (6.60) correspond to holomorphic sections of $\widehat{\pi}$.

The configuration (6.63), which determines the output of Theorems 1 and 2, depends only on the restriction of the Hamiltonian $(S^1)^N$-pair (ϕ,μ) defined by (6.62) to the subtorus $(S^1)^N_{\bullet}\subset(S^1)^N$. The latter is determined by a tuple $(\xi_{ij},\epsilon_{ij})_{i,j\in[N]}$ in $(\Lambda_{\mathfrak{t}}\times\mathbb{R})^{N\times N}$ satisfying the one-cocycle condition

$$\big(\xi_{ij},\epsilon_{ij}\big)+\big(\xi_{jk},\epsilon_{jk}\big)=\big(\xi_{ik},\epsilon_{ik}\big)\qquad\forall\ i,j,k\in[N].$$

A tuple $(\xi_i,\epsilon_i)_{i\in[N]}$ in $(\Lambda_{\mathfrak{t}}\times\mathbb{R})^N$ cobounding $(\xi_{ij},\epsilon_{ij})_{i,j\in[N]}$, i.e.

$$\big(\xi_{ij},\epsilon_{ij}\big)=\big(\xi_j,\epsilon_j\big)-\big(\xi_i,\epsilon_i\big)\qquad\forall\,i,j\in[N],\tag{6.70}$$

determines an extension of the Hamiltonian $(S^1)^N_{\bullet}$-pair determined by $(\xi_{ij},\epsilon_{ij})_{i,j\in[N]}$ to an $(S^1)^N$-pair. If $(\xi_i',\epsilon_i')_{i\in[N]}$ is another tuple satisfying (6.70), then there exists (ξ,ϵ) in $\Lambda_{\mathfrak{t}}\times\mathbb{R}$ such that

$$\big(\xi_i',\epsilon_i'\big)=\big(\xi_i,\epsilon_i\big)+(\xi,\epsilon)\qquad\forall\,i\in[N].$$

Along with (6.40) and (6.41), this implies that

$$\phi'_{\mathcal{Z};S^1}\big(\mathrm{e}^{\mathrm{i}\theta};y\big)=\phi_{\mathcal{Z};\mathbb{T}}\big(\rho_{\xi}(\mathrm{e}^{-\mathrm{i}\theta});\phi_{\mathcal{Z};S^1}(\mathrm{e}^{\mathrm{i}\theta};y)\big),\quad \mu'_{\mathcal{Z};S^1}(y)=\mu_{\mathcal{Z};S^1}(y)-L_{\xi}\big(\mu_{\mathcal{Z};\mathbb{T}}(y)\big)+\epsilon\,.$$

As indicated by Example 6.7, the deformation equivalence class of the symplectic manifold $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$ of Theorem 3 in general depends on the choice of such a coboundary.

Example 6.7. Let $\omega_{\mathbb{P}^1}$ denote the doubled Fubini-Study symplectic form on $\mathbb{P}^1$, i.e.

$$\omega_{\mathbb{P}^1}\big|_z=\frac{2\,\omega_{\mathbb{C}}}{(1+|z|^2)^2}\qquad\forall\ z\in\mathbb{C}\subset\mathbb{P}^1.$$

We take $(X,\omega)=(\mathbb{P}^1,\omega_{\mathbb{P}^1})$, $\mathbb{T}=S^1$, $\phi_{\mathbb{T}}=\phi_{\mathbb{P}^1}$ as in (6.37),

$$\mu_{\mathbb{T}}\equiv\mu_{\mathbb{P}^1}:\mathbb{P}^1\longrightarrow\mathbb{R},\quad \mu_{\mathbb{P}^1}\big([w,z]\big)=\frac{|z|^2}{|w|^2+|z|^2}\,,\quad N=1,\quad \xi_1=m\in\mathbb{Z}=\Lambda_{\mathfrak{t}},\quad \epsilon_1=0.$$

Thus, $\mathcal{Z}=\mathbb{P}^1\times\mathbb{C}$, $\widehat{\Delta}_a$ is the polytope in the first diagram of Figure 1, and

$$\widehat{\pi}\colon\widehat{\mathcal{Z}}_a=\mathbb{F}_m\longrightarrow\mathbb{P}^1$$

is the m-th Hirzebruch surface with the canonical projection; see [3, Homework 22.3]. We conclude that $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$ depends on the parity of m in this case.

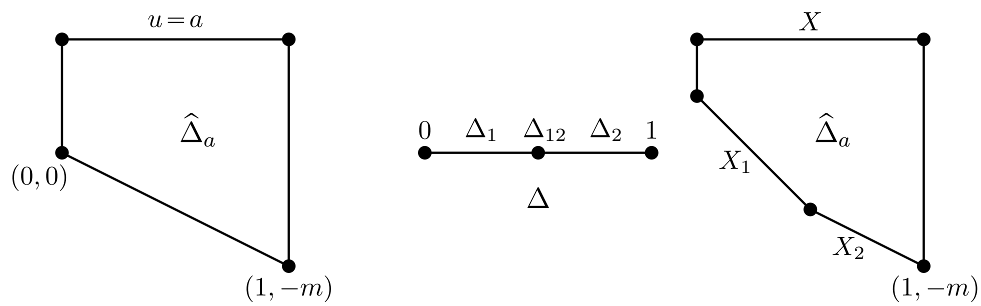

Figure 1: The polytope $\widehat{\Delta}_a$ for the one-parameter "degeneration" of symplectic manifolds for the data of Example 6.7, the subdivision of Δ into two polytopes for the data of Example 6.8, and the polytope $\widehat{\Delta}_a$ for the associated one-parameter family of symplectic manifolds degenerating $X=\mathbb{P}^1$ into $X_\emptyset=\mathbb{P}^1\vee\mathbb{P}^1$.

Example 6.8. With $(X,\omega,\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ as in Example 6.7 and $m\in\mathbb{Z}$, we now take

$$N=2,\quad \xi_1=m+1,\quad \xi_2=m,\quad \epsilon_1=\frac{1}{2},\quad \epsilon_2=0.$$

The tuple $(\xi_i,\epsilon_i)_{i\in[2]}$ is then regular. The polytopes Δ for $(\phi_{\mathbb{T}},\mu_{\mathbb{T}})$, Δ_i for the cut symplectic manifolds (X_i,ω_i) with $i=1,2$, and $\widehat{\Delta}_a$ for the symplectic manifold $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$ containing $X_1\cup X_2$ as an SC symplectic divisor are shown in the second and third diagrams of Figure 1. By [3, Homework 22.2], $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a},\phi_{\widehat{\mathcal{Z}};a},\mu_{\widehat{\mathcal{Z}};a})$ is a symplectic manifold that can be constructed as either a toric blowup of $\mathbb{F}_m$ or a toric blowup of $\mathbb{F}_{m+1}$. By Delzant's Classification Theorem [3, Theorem 28.2], this quadruple depends on the choice of m. However, the deformation equivalence class of the symplectic manifold $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a})$ is independent of this choice.

Example 6.9. With the notation as in Example 6.7, we now take

$$X=\mathbb{P}^1\times\mathbb{P}^1,\qquad \omega=3\pi_1^*\omega_{\mathbb{P}^1}+2\pi_2^*\omega_{\mathbb{P}^1},\qquad \mathbb{T}=(S^1)^2,$$
$$\phi_{\mathbb{T}}=\phi_{\mathbb{P}^1}\times\phi_{\mathbb{P}^1}\colon\mathbb{T}\times X\longrightarrow X,\qquad \mu_{\mathbb{T}}=\big(3\mu_{\mathbb{P}^1}\circ\pi_1-2,2\mu_{\mathbb{P}^1}\circ\pi_2-1\big)\colon X\longrightarrow\mathbb{R}^2,$$
$$\xi_1=(0,0),\qquad \xi_2=(1,0),\qquad \xi_3=(0,1),\qquad \epsilon_1,\epsilon_2,\epsilon_3=0.$$

The associated subdivision of $\Delta=[-2,1]\times[-1,1]$ is shown in the first diagram of Figure 2. It corresponds to a 3-fold symplectic cut of $\mathbb{P}^1\times\mathbb{P}^1$ into $X_1=\mathbb{P}^1\times\mathbb{P}^1$, a one-point blowup X_2 of $\mathbb{P}^1\times\mathbb{P}^1$, and $X_3=\mathbb{F}_1$. The smooth divisors $X_{12},X_{13}\subset X_1$ are one of the horizontal lines and one of the vertical lines and thus have normal bundles of degree 0. The smooth divisors $X_{12},X_{23}\subset X_2$ are the proper transform of a ruling of $\mathbb{P}^1\times\mathbb{P}^1$ through the blowup point and the exceptional divisor, respectively, and thus have normal bundles of degree -1. The smooth divisors $X_{13},X_{23}\subset X_3$ are the exceptional divisor and a fiber and thus have normal bundles of degrees -1 and 0, respectively. Since X_{123} is a single point in this case, the restrictions of the line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ in (2.9) to X_{12},X_{13},X_{23} are thus of degree 0. Since $X_{12},X_{13},X_{23}\approx\mathbb{P}^1$ are simply connected, this line bundle has a unique homotopy class of trivializations. The second diagram in Figure 2 shows the polytope (6.60) for the associated symplectic toric $\mathbb{T}\times S^1$-manifold $(\widehat{\mathcal{Z}}_a,\omega_{\widehat{\mathcal{Z}};a},\phi_{\widehat{\mathcal{Z}};a},\mu_{\widehat{\mathcal{Z}};a})$.

6.5 Admissible decompositions of polytopes

Suppose $\mathbb{T}$ is a k-torus and $\Delta\subset\mathfrak{t}^*$ is a convex polytope as before. For each $v\in\Delta$, let $\mathfrak{d}_v(\Delta)\in\mathbb{Z}^{\geqslant 0}$ be the dimension of the minimal facet Δ_v of Δ containing v. If $v\in\Delta$ is a vertex, let $\mathrm{E}_v(\Delta)$ be

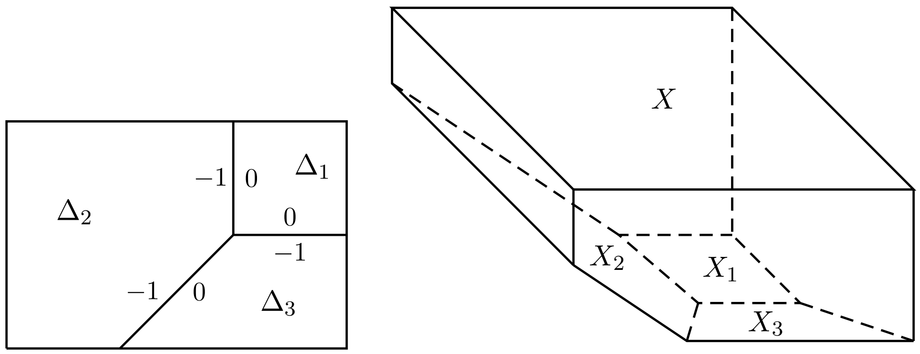

Figure 2: The subdivision of Δ into three polytopes for the data of Example 6.9 and the polytope $\widehat{\Delta}_a$ for the associated one-parameter family of symplectic manifolds degenerating X (the top face) into the SC symplectic variety $X_1 \cup X_2 \cup X_3$. The numbers next to each edge $\Delta_{ij} = \Delta_i \cap \Delta_j$ in the first diagram are the degrees of the normal bundles of the symplectic submanifold (X_{ij}, ω_{ij}) in the symplectic manifolds (X_i, ω_i).

the set of edges of Δ containing v. For each $e \in \mathrm{E}_v(\Delta)$, we denote by $e/v \in \Delta$ the vertex of e other than v. A vertex v of a polytope $\Delta \subset \mathfrak{t}^*$ is called smooth if there exist a $\mathbb{Z}$-basis $\{u_e\}_{e \in \mathrm{E}_v(\Delta)}$ for $\Lambda_{\mathfrak{t}}^*$ and a tuple $(t_{e,v})_{e \in \mathrm{E}_v(\Delta)}$ in $\mathbb{R}^{\mathrm{E}_v(\Delta)}$ so that

$$e/v = v + t_{e,v} u_e \in \mathfrak{t}^* \qquad \forall\, e \in \mathrm{E}_v(\Delta).$$

The smoothness of a vertex $v \in \Delta$ implies that $|\mathrm{E}_v(\Delta)| = k$ and the slope of each edge $e \in \mathrm{E}_v(\Delta)$ is rational with respect to the lattice $\Lambda_{\mathfrak{t}}^*$. A polytope $\Delta \subset \mathfrak{t}^*$ is Delzant if every vertex $v \in \Delta$ is smooth.

Let $(X, \omega, \phi_{\mathbb{T}}, \mu_{\mathbb{T}})$ be a compact connected Hamiltonian $\mathbb{T}$-manifold with moment polytope Δ. For a tuple $(\xi_i, \epsilon_i)_{i \in [N]}$ in $(\Lambda_{\mathfrak{t}} \times \mathbb{R})^N$, let $\Delta_i \subset \Delta$ be as in (6.60). For each $v \in \Delta$, define

$$I_v = \big\{ i \in [N] \colon v \in \Delta_i \big\}.$$

We call $(\xi_i, \epsilon_i)_{i \in [N]}$ a quasi-regular tuple if every polytope $\Delta_i \subset \Delta$ is Delzant, $\xi_i - \xi_j$ is a primitive element of $\Lambda_{\mathfrak{t}}$ for all $i, j \in [N]$ distinct such that $\Delta_{ij} \neq \emptyset$, and

$$|I_v| \leqslant \mathfrak{d}_v(\Delta) + 1 \qquad \forall\, v \in \Delta. \tag{6.71}$$

The last condition is equivalent to the same condition for all vertices v of the polytopes Δ_i with $i \in [N]$. By dimensional considerations, the inequality in (6.71) for any given vertex $v \in \Delta_i$ is equivalent to the equality for the same vertex. An example of a subdivision for a quasi-regular tuple is shown in the first diagram in Figure 2.

By (6.71), the combinatorics of the subdivision of Δ determined by a quasi-regular tuple $(\xi_i, \epsilon_i)_{i \in [N]}$ does not change under sufficiently small changes in the values of ϵ_i. The next proposition relates the combinatorial notion of quasi-regularity for $(\xi_i, \epsilon_i)_{i \in [N]}$ to the geometric notion of regularity for the induced Hamiltonian configuration (6.63).

Proposition 6.10. *Suppose $N \in \mathbb{Z}^+$, $(X, \omega, \phi_{\mathbb{T}}, \mu_{\mathbb{T}})$ is a compact connected Hamiltonian $\mathbb{T}$-manifold, and $(\xi_i, \epsilon_i)_{i \in [N]}$ is a tuple in $(\Lambda_{\mathfrak{t}} \times \mathbb{R})^N$. If this tuple is regular, then it is quasi-regular. If it satis-*

fies (6.71) and $(\epsilon_i)_{i\in[N]}$ *is generic, then the* ϕ_I*-action of* $(S^1)^I_\bullet$ *on* $\mu_I^{-1}(0)$ *has at most finite stabilizers for every* $I\in\mathcal{P}^*(N)$*. If* $(X,\omega,\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ *is a toric symplectic manifold and the tuple* $(\xi_i,\epsilon_i)_{i\in[N]}$ *is quasi-regular, then it is regular.*

For $i,j\in[N]$, let $\xi_{ij}=\xi_i-\xi_j$. For each $v\in\mathfrak{t}^*$, let

$$\mathfrak{t}_v \equiv \big\{\xi\in\mathfrak{t}\colon\ L_\xi(\eta-v)=0\ \forall\,\eta\in\Delta_v\big\}$$

be the annihilator of the vectors contained in Δ_v. By [2, Corollary IV.4.13], $\mathfrak{t}_v$ is the $\mathbb{R}$-span of a sublattice of $\Lambda_{\mathfrak{t}}$ and thus generates a subtorus $\mathbb{T}_v\subset\mathbb{T}$. For $I\in\mathcal{P}^*(N)$, define

$$L_{\mathfrak{t}^*;I}\colon\mathfrak{t}^*\longrightarrow\mathfrak{t}^*_{I;\bullet}=\mathbb{R}^I/\big\{(a,\ldots,a)\in\mathbb{R}^I\colon\ a\in\mathbb{R}\big\},\qquad L_{\mathfrak{t}^*;I}(\eta)=\big[\big(L_i(\eta)\big)_{i\in I}\big],$$
$$\mathcal{K}_{\mathfrak{t}^*;I}=\big\{\eta\in\mathfrak{t}^*\colon\ L_{\xi_i}(\eta)=L_{\xi_j}(\eta)\ \forall\,i,j\in I\big\}.$$

Thus,

$$\begin{aligned}\mathrm{Ann}\big(\mathcal{K}_{\mathfrak{t}^*;I}\big)\equiv\big\{\xi\in\mathfrak{t}\colon\ L_\xi(\eta)=0\ \forall\,\eta\in\mathcal{K}_{\mathfrak{t}^*;I}\big\}&=\mathrm{Span}_{\mathbb{R}}\big\{\xi_{ij}\colon i,j\in I\big\},\\ \ker\big(\mathrm{d}_vL_{\mathfrak{t}^*;I}\colon T_v\mathfrak{t}^*\longrightarrow\mathfrak{t}^*_{I;\bullet}\big)=\mathcal{K}_{\mathfrak{t}^*;I}\subset\mathfrak{t}^*&\qquad\forall\ v\in\mathfrak{t}^*.\end{aligned}\tag{6.72}$$

Since the dimension of the space on the right-hand side of (6.72) is at most $|I|-1$,

$$\dim\big(T_v\Delta_v\cap\mathcal{K}_{\mathfrak{t}^*;I_v}\big)\geqslant\mathfrak{d}_v(\Delta)-\dim\big(\mathrm{Span}_{\mathbb{R}}\big\{\xi_{ij}\colon i,j\in I_v\big\}\big)\geqslant\mathfrak{d}_v(\Delta)+1-|I_v|\quad\forall\ v\in\mathfrak{t}^*.\tag{6.73}$$

For $x\in X$, let

$$\mathbb{T}_x=\big\{g\in\mathbb{T}\colon\phi_{\mathbb{T}}(g;x)=x\big\}\quad\text{and}\quad\mathfrak{t}_x=\mathrm{Ann}\big(\mathrm{Im}\big(\mathrm{d}_x\mu_{\mathbb{T}}\colon T_xX\longrightarrow T_v\mathfrak{t}^*\big)\big)\tag{6.74}$$

be the stabilizer of x and the Lie algebra of $\mathbb{T}_x$, respectively.

Lemma 6.11. *Let* $N\in\mathbb{Z}^+$ *and* $(\xi_j,\epsilon_j)_{j\in[N]}\in(\Lambda_{\mathfrak{t}}\times\mathbb{R})^N$*. The conditions (6.71), the homomorphism*

$$\mathrm{d}_vL_{\mathfrak{t}^*;I_v}\colon T_v\Delta_v\longrightarrow\mathfrak{t}^*_{I;\bullet}\tag{6.75}$$

is surjective for all $v\in\Delta$*, and the kernel of the homomorphism*

$$\mathbb{T}_v\times(S^1)^{I_v}_\bullet\longrightarrow\mathbb{T},\qquad\big(g,(\mathrm{e}^{\mathrm{i}\theta_i})_{i\in I_v}\big)\longrightarrow g\prod_{i\in I_v}\varrho_{\xi_i}\big(\mathrm{e}^{\mathrm{i}\theta_i}\big),\tag{6.76}$$

is finite for all $v\in\Delta$ *are equivalent.*

Proof. We show that each of the three conditions of Lemma 6.11 is equivalent to

$$\dim\big(\mathrm{Span}_{\mathbb{R}}\big\{\xi_{ij}\colon i,j\in I_v\big\}\big)=|I_v|-1,\quad\mathfrak{t}_v\cap\mathrm{Span}_{\mathbb{R}}\big\{\xi_{ij}\colon i,j\in I_v\big\}=\big\{0\big\}\quad\forall\,v\in\Delta.\tag{6.77}$$

Suppose $v\in\Delta$. The first equality in (6.77) is equivalent to the second inequality in (6.73) being an equality; the second equality in (6.77) is equivalent to the first inequality in (6.73) being an equality. Thus, the two equalities in (6.77) for a fixed $v\in\Delta$ are equivalent to

$$\dim\big(T_v\Delta_v\cap\mathcal{K}_{\mathfrak{t}^*;I_v}\big)=\mathfrak{d}_v(\Delta)+1-|I_v|\tag{6.78}$$

and thus to the surjectivity of the homomorphism (6.75).

The kernel of the homomorphism (6.76) is finite if and only if the homomorphism

$$\mathfrak{t}_v \oplus \mathfrak{t}_{I_v;\bullet} \longrightarrow \mathfrak{t}, \qquad \big(\xi, (r_i)_{i\in I_v}\big) \longrightarrow \xi + \sum_{i\in I_v} r_i\xi_i\,,$$

is injective. By (1.2), the latter is the case if and only if (6.77) holds.

If v is a vertex of Δ_i for some $i\in[N]$, then

$$T_v\Delta_v \cap \mathcal{K}_{\mathfrak{t}^*;I_v} = \{0\} \qquad \text{and} \qquad \mathfrak{d}_v(\Delta) \leqslant |I_v|-1.$$

The inequality (6.71) for a given vertex $v \in \Delta_i$ is thus equivalent to (6.78) and so to the two equalities in (6.77) for v. If $v\in\Delta_i$ is arbitrary and $v'\in(\Delta_i)_v$ is a vertex of the minimal facet of Δ_i containing v, then

$$I_{v'} \supset I_v, \quad \mathrm{Span}_{\mathbb{R}}\big\{\xi_{ij} : i, j\in I_{v'}\big\} \supset \mathrm{Span}_{\mathbb{R}}\big\{\xi_{ij} : i, j\in I_v\big\}, \quad \mathfrak{t}_{v'} \supset \mathfrak{t}_v.$$

The two equalities in (6.77) with v replaced by v' thus imply the two equalities in (6.77) themselves. Since the same is also the case for the inequality in (6.71), we conclude that the conditions (6.71) and (6.77) are equivalent. □

Lemma 6.12. *Let $N\in\mathbb{Z}^+$ and $(\xi_j, \epsilon_j)_{j\in[N]} \in (\Lambda_{\mathfrak{t}}\times\mathbb{R})^N$. The tuple $(\xi_j, \epsilon_j)_{j\in[N]}$ is quasi-regular if and only if the homomorphism (6.76) is injective for all $v\in\Delta$.*

Proof. Let $v\in\Delta$ and $i\in I_v$. By (1.2), the injectivity of the homomorphism (6.76) is equivalent to

$$\Lambda_{\mathfrak{t}} \cap \big(\mathfrak{t}_v + \mathrm{Span}_{\mathbb{R}}\big\{\xi_{ij} : j\in I_v\big\}\big) = \Lambda_{\mathfrak{t}_v} \oplus \bigoplus_{j\in I_v - i} \mathbb{Z}\xi_{ij}\,. \tag{6.79}$$

If (6.79) holds, then $\xi_{ij}\in\Lambda_t$ is a primitive element for every $j\in I_v - i$. If in addition v is a vertex of Δ_i, then (6.79) implies that it is smooth. Along with Lemma 6.11, this implies that the tuple $(\xi_j, \epsilon_j)_{j\in[N]}$ is quasi-regular.

Suppose the tuple $(\xi_j, \epsilon_j)_{j\in[N]}$ is quasi-regular. Let $i\in[N]$ and $v\in\Delta_i$. By the proof of Lemma 6.11, the two equalities in (6.77) hold. Thus, (6.79) holds when tensored with $\mathbb{R}$. Since $\xi_{ij}\in\Lambda_t$ is primitive for every $j\in I_v - i$ and a vertex $v'\in(\Delta_i)_v$ is smooth, (6.79) itself holds as well. □

Lemma 6.13. *Suppose $N\in\mathbb{Z}^+$, $(\xi_j, \epsilon_j)_{j\in[N]} \in (\Lambda_{\mathfrak{t}}\times\mathbb{R})^N$ satisfies (6.71), $v\in\Delta$, and $x\in\mu^{-1}(v)$. The homomorphism*

$$\mathrm{d}_x\mu_{I_v} : T_xX \longrightarrow \mathfrak{t}^*_{I_v;\bullet} \tag{6.80}$$

is surjective if and only if the kernel of the homomorphism

$$\mathbb{T}_x \times (S^1)^{I_v}_{\bullet} \longrightarrow \mathbb{T}, \qquad \big(g, (\mathrm{e}^{\mathrm{i}\theta_i})_{i\in I_v}\big) \longrightarrow g \prod_{i\in I_v} \varrho_{\xi_i}\big(\mathrm{e}^{\mathrm{i}\theta_i}\big), \tag{6.81}$$

is finite.

Proof. By the definitions of I_v and Δ_i, $(\mu(x))_i \leqslant (\mu(x))_j$ for all $i \in I_v$ and $j \in [N]$ and the equality holds if and only if $j \in I_v$. Thus, $x \in U_{I_v}$ and the homomorphism (6.80) is well-defined. We show that each condition of Lemma 6.13 is equivalent to

$$\mathfrak{t}_x \cap \mathrm{Span}_{\mathbb{R}}\{\xi_{ij} : i, j \in I_v\} = \{0\}. \tag{6.82}$$

By Lemma 6.11 and its proof, the homomorphism (6.75) is surjective and both equalities in (6.77) hold in the present case.

Since $\mu_{I_v} = L_{\mathfrak{t}^*;I_v} \circ \mu_{\mathbb{T}}$,

$$\mathrm{d}_x\mu_{I_v} = \mathrm{d}_v L_{\mathfrak{t}^*;I_v} \circ \mathrm{d}_x\mu_{\mathbb{T}} : T_xX \longrightarrow T_v\mathfrak{t}^* \longrightarrow \mathfrak{t}^*_{I_v;\bullet}. \tag{6.83}$$

By the definition of Δ_v,

$$\mathrm{Im}\big(\mathrm{d}_x\mu_{\mathbb{T}} : T_xX \longrightarrow T_v\mathfrak{t}^*\big) \subset T_v\Delta_v. \tag{6.84}$$

By the surjectivity of the homomorphism (6.75), the surjectivity of the homomorphism (6.80) is thus equivalent to

$$T_v\Delta_v \subset \mathrm{Im}\big(\mathrm{d}_x\mu_{\mathbb{T}}\big) + \mathcal{K}_{\mathfrak{t}^*;I_v}.$$

By (6.72) and (6.74), this condition is in turn equivalent to

$$\mathfrak{t}_v \supset \mathfrak{t}_x \cap \mathrm{Span}_{\mathbb{R}}\{\xi_{ij} : i, j \in I_v\}.$$

In light of the second equality in (6.77), the last condition is equivalent to (6.82).

The finiteness of the kernel of (6.81) is equivalent to the injectivity of the homomorphism

$$\mathfrak{t}_x \oplus \mathfrak{t}_{I_v;\bullet} \longrightarrow \mathfrak{t}, \qquad \big(\xi, (r_i)_{i\in I_v}\big) \longrightarrow \xi + \sum_{i\in I_v} r_i\xi_i.$$

By (1.2) and the first equality in (6.77), the latter is equivalent to (6.82). □

Proof of Proposition 6.10. Let $v \in \Delta$ and $x \in \mu^{-1}(v)$. Thus, $x \in \mu_{I_v}^{-1}(0)$ and $\mathbb{T}_x \supset \mathbb{T}_v$. Furthermore, the projection

$$\mathbb{T}_x \times (S^1)^{I_v}_\bullet \longrightarrow (S^1)^{I_v}_\bullet$$

restricts to an isomorphism from the kernel of the homomorphism (6.81) to the stabilizer of the ϕ_{I_v}-action on x. If this stabilizer is trivial, then the homomorphisms (6.81) and (6.76) are injective. Since $\mu^{-1}(v) \neq \emptyset$ for every $v \in \Delta$, Lemma 6.12 thus implies the first claim of Proposition 6.10.

Suppose the tuple $(\xi_j, \epsilon_j)_{j\in[N]}$ satisfies (6.71) and the homomorphism (6.80) is surjective. By Lemma 6.13, the homomorphism (6.81) then has finite kernel and so the stabilizer of the ϕ_{I_v}-action on x is finite. The homomorphism (6.80) is surjective if $[(\varepsilon_i)_{i\in I}] \in \mathfrak{t}^*_{I;\bullet}$ is a regular value of the smooth map μ_I on U_I for all $I \in \mathcal{P}^*(N)$. By Sard's Theorem [14, p10], this is the case if the tuple $(\varepsilon_i)_{i\in[N]}$ is generic. This establishes the second claim of Proposition 6.10.

By (6.83) and Lemma 6.11, the homomorphism (6.80) is surjective if the tuple $(\xi_i, \epsilon_i)_{i\in[N]}$ satisfies (6.71) and the inclusion (6.84) is an equality. Suppose $(X, \omega, \phi_{\mathbb{T}}, \mu_{\mathbb{T}})$ is a toric symplectic manifold. By [2, Corollary IV.4.14], the inclusion (6.84) is then an equality. Furthermore, $\mathbb{T}_x = \mathbb{T}_v$.

Since the inclusion (6.84) is an equality, this statement is equivalent to the connectedness of $\mathbb{T}_x$. The latter is implied by each toric symplectic manifold being the quotient of a subset $\widetilde{X}_M^\tau$ of $\mathbb{C}^{k'}$ by the restriction of the standard $(S^1)^{k'}$-action to an action of a $(k'-k)$-subtorus $\mathbb{T}'\subset(S^1)^{k'}$ with $\mathbb{T}=(S^1)^{k'}/\mathbb{T}'$ and $\phi_{\mathbb{T}}$ being the induced action; see [18, Section 2.1] for the relevant details. Thus, the stabilizer $\mathbb{T}_x$ of the point $x\in X$ determined by a point $\widetilde{x}\in\widetilde{X}_M^\tau$ is the quotient of the stabilizer $(S^1)^{k'}_{\widetilde{x}}\subset(S^1)^{k'}$ of $\widetilde{x}$ by its intersection with $\mathbb{T}'$. Since $(S^1)^{k'}_{\widetilde{x}}$ is connected, so is $\mathbb{T}_x$.

Suppose $(X,\omega,\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ is a toric symplectic manifold and the tuple $(\xi_i,\epsilon_i)_{i\in[N]}$ is quasi-regular. By the previous paragraph, the homomorphism (6.80) is then surjective. Since $\mathbb{T}_x=\mathbb{T}_v$, the triviality of the stabilizer of the ϕ_{I_v}-action on x is equivalent to the injectivity of the homomorphism (6.76). The latter is the case by Lemma 6.12. This establishes the last claim of Proposition 6.10. □

The example below provides decompositions of moment polytopes for Hamiltonian S^1-manifolds $(X,\omega,\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ with effective actions. These decompositions arise from quasi-regular tuples $(\xi_i,\epsilon_i)_{i\in[2]}$ that are not regular. Example 6.14 illustrates that assuming that the action is Hamiltonian and the tuple $(\xi_i,\epsilon_i)_{i\in[N]}$ is quasi-regular is not sufficient to overcome either of the two deficiencies of the second statement of Proposition 6.10 as compared to the third. On the other hand, our degeneration and symplectic cut construction can be applied whenever the ϕ_I-action on $\mu_I^{-1}(0)$ has at most finite stabilizers for all $I\in\mathcal{P}^*(N)$. It would then produce symplectic orbifolds. Thus, the combinatorics of polytope decompositions in the context of Theorem 3 fit most naturally with the category of symplectic orbifolds (rather than just manifolds).

Example 6.14. Let $\omega_{\mathbb{P}^1}$ and $(\phi_{\mathbb{P}^1},\mu_{\mathbb{P}^1})$ be as in Example 6.7. Let

$$(X,\omega)=\big(\mathbb{P}^1\times\mathbb{P}^1,\pi_1^*\omega_{\mathbb{P}^1}+\pi_2^*\omega_{\mathbb{P}^1}\big)\qquad\text{and}\qquad\mathbb{T}=S^1.$$

Fix $m_1,m_2\in\mathbb{Z}^+$ and define

$$\begin{aligned}\phi_{\mathbb{T}}\colon S^1\times X\longrightarrow X,&\qquad \phi_{\mathbb{T}}\big(\mathrm{e}^{\mathrm{i}\theta};z_1,z_2\big)=\big(\phi_{\mathbb{P}^1}(\mathrm{e}^{\mathrm{i}m_1\theta};z_1),\phi_{\mathbb{P}^1}(\mathrm{e}^{\mathrm{i}m_2\theta};z_2)\big),\\ \mu_{\mathbb{T}}\colon X\longrightarrow\mathbb{R},&\qquad \mu_{\mathbb{T}}(z_1,z_2)=m_1\mu_{\mathbb{P}^1}(z_1)+m_2\mu_{\mathbb{P}^1}(z_2)\,.\end{aligned}$$

The tuple $(X,\omega,\phi_{\mathbb{T}},\mu_{\mathbb{T}})$ is then a Hamiltonian S^1-manifold. The associated moment polytope Δ is the interval $[0,m_1{+}m_2]$. Two of the four $\phi_{\mathbb{T}}$-fixed points, $P_{\infty0}$ and $P_{0\infty}$, are mapped to the interior points m_1 and m_2. A setting for the usual $N=2$ symplectic cut configuration of [11] in this case is obtained by taking

$$\xi\equiv\xi_2-\xi_1=1\in\mathbb{Z}=\Lambda_{\mathbb{R}}\qquad\text{and}\qquad\epsilon=\epsilon_1-\epsilon_2\in\big(0,m_1+m_2\big).$$

The associated decomposition breaks Δ into the intervals $\Delta_1=[0,\epsilon]$ and $\Delta_2=[\epsilon,m_1{+}m_2]$. In this case, $\phi_{12}=\phi_{\mathbb{T}}$ under a suitable identification $(S^1)^2_\bullet=S^1$. Since $\mathrm{d}_x\mu_{\mathbb{T}}=0$ if and only if x is a $\phi_{\mathbb{T}}$-fixed point, the ϕ_{12}-action on $\mu_{12}^{-1}(0)=\mu_{\mathbb{T}}^{-1}(\epsilon)$ is non-trivial if and only if $\epsilon\neq m_1,m_2$. This illustrates the necessity of the "generic" assumption in Proposition 6.10. Since the subgroups $\mathbb{Z}_{m_1},\mathbb{Z}_{m_2}\subset S^1$ act trivially on $\mathbb{P}^1\times\{0,\infty\}$ and $\{0,\infty\}\times\mathbb{P}^1$, respectively, the ϕ_{12}-action on $\mu_{12}^{-1}(0)$ is not free for any $\epsilon\in(0,m_1{+}m_2)$ unless $m_1,m_2=1$. However, the $\phi_{\mathbb{T}}$-action on X is effective if m_1 and m_2 are relatively prime.

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