Basic Riemannian Geometry and Sobolev Estimates
used in Symplectic Topology

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Abstract
This note collects a number of standard statements in Riemannian geometry and in Sobolev-space theory that play a prominent role in analytic approaches to symplectic topology. These include relations between connections and complex structures, estimates on exponential-like maps, and dependence of constants in Sobolev and elliptic estimates.

Contents
1 Connections in real vector bundles 2
  1.1 Connections and splittings 2
  1.2 Metric-compatible connections 5
  1.3 Torsion-free connections 5

2 Complex structures 6
  2.1 Complex linear connections 6
  2.2 Generalized $\bar{\partial}$-operators 8
  2.3 Connections and $\bar{\partial}$-operators 10
  2.4 Holomorphic vector bundles 11
  2.5 Deformations of almost complex submanifolds 12

3 Riemannian geometry estimates 15
  3.1 Parallel transport 15
  3.2 Poincare lemmas 19
  3.3 Exponential-like maps and differentiation 21
  3.4 Expansion of the $\bar{\partial}$-operator 23

4 Sobolev and elliptic inequalities 24
  4.1 Euclidean case 25
  4.2 Bundle sections along smooth maps 29
  4.3 Elliptic estimates 31

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1 Connections in real vector bundles

1.1 Connections and splittings

Suppose $M$ is a smooth manifold and $\pi_E: E \rightarrow M$ is a vector bundle. We identify $M$ with the zero section of $E$. Denote by $a: E \oplus E \rightarrow E$ and $\pi_{E \oplus E}: E \oplus E \rightarrow M$ the associated addition map and the induced projection map, respectively. For $f \in C^\infty(M; \mathbb{R})$, define

$$m_f: E \rightarrow E \quad \text{by} \quad m_f(v) = f(\pi_E(v)) \cdot v \quad \forall v \in E.$$  

(1.1)

In particular,

$$\pi_{E \oplus E} = \pi_E \circ a, \quad \pi_E = \pi_E \circ m_f \quad \forall f \in C^\infty(M; \mathbb{R}).$$

The total spaces of the vector bundles $\pi_{E \oplus E}: E \oplus E \rightarrow M$ and $\pi^*_E E \rightarrow E$ consist of the pairs $(v, w)$ in $E \times E$ such that $\pi_E(v) = \pi_E(w)$.

Define a smooth bundle homomorphism

$$\iota_E: \pi^*_E E \rightarrow TE, \quad \iota_E(v, w) = \frac{d}{dt}(v + tw)|_{t=0}.$$  

(1.2)

Since the restriction of $\iota_E$ to the fiber over $v \in E$ is the composition of the isomorphism

$$E_{\pi_E(v)} \rightarrow T_v E_{\pi_E(v)}, \quad w \rightarrow \frac{d}{dt}(v + tw)|_{t=0},$$

with the differential of the embedding of the fiber $E_{\pi_E(v)}$ into $E$, $\iota_E$ is an injective bundle homomorphism. Furthermore,

$$d\pi_E \circ \iota_E = 0, \quad m^*_f \iota_E \circ \pi^*_E m_f = dm_f \circ \iota_E, \quad a^* \iota_E \circ \pi^*_E a = da \circ \iota_{E \oplus E};$$

$$TE|_M \approx TM \oplus \text{Im} \iota_E.$$  

(1.3)

(1.4)

By the first statement in (1.3), the injectivity of $\iota_E$, and surjectivity of $d\pi_E$,

$$0 \rightarrow \pi^*_E E \xrightarrow{\iota_E} TE \xrightarrow{d\pi_E} \pi^*_E TM \rightarrow 0$$  

(1.5)

is an exact sequence of vector bundles over $E$. By the second statement in (1.3), the diagram

$$0 \rightarrow \pi^*_E E \xrightarrow{\iota_E} TE \xrightarrow{d\pi_E} \pi^*_E TM \xrightarrow{\pi^*_E \text{id}} 0$$  

is exact.
of vector bundle homomorphisms over $E$ commutes. By the third statement in (1.3), the diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & \pi^*_E E(E \oplus E) & \longrightarrow & T(E \oplus E) & \longrightarrow & \pi^*_E TM & \longrightarrow & 0 \\
\downarrow & & \pi^*_E E \circ a & \downarrow da & & \pi^*_E E \circ id & \downarrow 0 \\
0 & \longrightarrow & \pi^*_E E E \oplus E(E \oplus E) & \longrightarrow & \pi^*_E E \oplus (E \oplus E) & \longrightarrow & \pi^*_E TM & \longrightarrow & 0
\end{array}
$$

(1.7)

of vector bundle homomorphisms over $E \oplus E$ commutes.

A connection in $E$ is an $\mathbb{R}$-linear map

$$
\nabla: \Gamma(M; E) \longrightarrow \Gamma(M; T^* M \otimes \mathbb{R} E) \quad \text{s.t.} \quad \nabla(f\xi) = df \otimes \xi + f \nabla \xi \quad \forall f \in C^\infty(M), \xi \in \Gamma(M; E).
$$

(1.8)

The Leibnitz property (1.8) implies that any two connections in $E$ differ by a 1-form on $M$. In other words, if $\nabla$ and $\tilde{\nabla}$ are connections in $E$ there exists $\theta \in \Gamma(M; T^* M \otimes \mathbb{R} \text{Hom}_\mathbb{R}(E, E))$ s.t.

$$
\tilde{\nabla}_v \xi = \nabla_v \xi + \{\theta(v)\} \xi \quad \forall \xi \in \Gamma(M; E), v \in T_x M, x \in M.
$$

(1.9)

If $U$ is a neighborhood of $x \in M$ and $f$ is a smooth function on $M$ supported in $U$ such that $f(x) = 1$, then

$$
\nabla \xi \big|_x = \nabla(f \xi) \big|_x - dx f \otimes \xi(x)
$$

by (1.8). The right-hand side of (1.10) depends only on $\xi|_U$. Thus, a connection $\nabla$ in $E$ is a local operator, i.e. the value of $\nabla \xi$ at a point $x \in M$ depends only on the restriction of $\xi$ to any neighborhood $U$ of $x$.

Suppose $U$ is an open subset of $M$ and $\xi_1, \ldots, \xi_n \in \Gamma(U; E)$ is a frame for $E$ on $U$, i.e.

$$
\xi_1(x), \ldots, \xi_n(x) \in E_x
$$

is a basis for $E_x$ for all $x \in U$. By definition of $\nabla$, there exist

$$
\theta^k_l \in \Gamma(U; T^* U) \quad \text{s.t.} \quad \nabla \xi_l = \sum_{k=1}^{k=n} \xi_k \theta^k_l \equiv \sum_{k=1}^{k=n} \theta^k_l \otimes \xi_k \quad \forall l = 1, \ldots, n.
$$

We call

$$
\theta \equiv (\theta^k_l)_{k,l=1,\ldots,n} \in \Gamma(U; T^* U \otimes \mathbb{R} \text{Mat}_n(\mathbb{R}))
$$

the connection 1-form of $\nabla$ with respect to the frame $(\xi_k)_k$.

For an arbitrary section

$$
\xi = \sum_{l=1}^{l=n} f^l \xi_l \in \Gamma(U; E),
$$

It is called the connection 1-form of $\nabla$ with respect to the frame $(\xi_k)_k$. 

3
by (1.8) we have

\[ \nabla \xi = \sum_{k=1}^{k=n} \xi_k \left( df^k + \sum_{l=1}^{l=n} \theta_l^k f^l \right), \quad \text{i.e.} \quad \nabla(\xi \cdot f^t) = \xi \cdot \{ d + \theta \} f^t, \quad (1.11) \]

where \( \xi = (\xi_1, \ldots, \xi_n) \), \( f = (f^1, \ldots, f^n) \). \quad (1.12)

This implies that

\[ \nabla \xi \big|_x = \pi_2 \circ d_x \xi : T_x M \to E_x \quad \forall \xi \in \Gamma(U; E) \text{ s.t. } \xi(x) = 0, \quad (1.13) \]

where \( \pi_2 : T_x E \to E_x \) is the projection to the second component in (1.4).

By (1.11), \( \nabla \) is a first-order differential operator. By (1.8), its symbol is given by

\[ \sigma_{\nabla} : T^* M \to \text{Hom}(E, T^* M \otimes \mathbb{R}^n), \quad \{ \sigma_{\nabla}(\eta) \} (f) = \eta \otimes f. \]

**Lemma 1.1.** Suppose \( M \) is a smooth manifold and \( \pi_E : E \to M \) is a vector bundle. A connection \( \nabla \) in \( E \) induces a splitting

\[ TE \approx \pi_E^* TM \oplus \pi_E^* E \quad (1.14) \]

of the exact sequence (1.5) extending the splitting (1.4) such that

\[ \nabla \xi \big|_x = \pi_2 \circ d_x \xi : T_x M \to E_x \quad \forall \xi \in \Gamma(M; E), \ x \in M, \quad (1.15) \]

where \( \pi_2 : T_x E \to E_x \) is the projection onto the second component in (1.14). Furthermore,

\[ d_m t \approx \pi_E^* \text{id} \oplus \pi_E^* m_t \quad \forall \ t \in \mathbb{R} \quad \text{and} \quad a \approx \pi_E^* \text{id} \oplus \pi_E^* a, \quad (1.16) \]

with respect to the splitting (1.14), i.e. it is consistent with the commutative diagrams (1.6) and (1.7).

**Proof.** Given \( x \in M \) and \( v \in E_x \), choose \( \xi \in \Gamma(M; E) \) such that \( \xi(x) = v \) and let

\[ T_v E^h = \text{Im} \{ d\xi - \nabla \xi \} \big|_x \subset T_v E. \]

Since \( \pi_E \circ \xi = \text{id}_M \),

\[ d_v \pi_E \circ \{ d\xi - \nabla \xi \} \big|_x = \text{id}_{T_x M} \quad \implies \quad T_v E \approx T_v E^h \oplus E_x \approx T_x M \oplus E_x. \]

This splitting of \( T_v E \) satisfies (1.15) at \( v \).

With the notation as in (1.11),

\[ \{ d\xi - \nabla \xi \} \big|_x = \left( d_x \text{id}_M, - \sum_{l=1}^{l=n} f^l(x) \theta_l^1 |_x, \ldots, - \sum_{l=1}^{l=n} f^l(x) \theta_l^n |_x \right) : T_x M \to T_x M \oplus \mathbb{R}^n \]

with respect to the identification \( E \big|_U \approx U \times \mathbb{R}^k \) determined by the frame \( (\xi_k)_k \). Thus, \( T_v E^h \) is independent of the choice of \( \xi \). Furthermore, the resulting splitting (1.14) of (1.5) extends (1.4) and satisfies (1.16).
1.2 Metric-compatible connections

Suppose $E \rightarrow M$ is a smooth vector bundle. Let $g$ be a metric on $E$, i.e.

$$g \in \Gamma(M; E^* \otimes \mathbb{R}) \quad \text{s.t.} \quad g(v, w) = g(w, v), \quad g(v, v) > 0 \quad \forall \, v, w \in E_x, \, v \neq 0, \, x \in M.$$ 

A connection $\nabla$ in $E$ is $g$-compatible if

$$d(g(\xi, \zeta)) = g(\nabla \xi, \zeta) + g(\xi, \nabla \zeta) \in \Gamma(M; T^*M) \quad \forall \, \xi, \zeta \in \Gamma(M; E).$$

Suppose $U$ is an open subset of $M$ and $\xi_1, \ldots, \xi_n \in \Gamma(U; E)$ is a frame for $E$ on $U$. For $i, j = 1, \ldots, n$, let

$$g_{ij} = g(\xi_i, \xi_j) \in C^\infty(U).$$

If $\nabla$ is a connection in $E$ and $\theta_{kl}$ is the connection 1-form for $\nabla$ with respect to the frame $\{\xi_k\}_k$, then $\nabla$ is $g$-compatible on $U$ if and only if

$$\sum_{k=1}^{n} (g_{ik} \theta_{j}^k + g_{jk} \theta_{i}^k) = dg_{ij} \quad \forall \, i, j = 1, 2, \ldots, n. \quad (1.17)$$

1.3 Torsion-free connections

If $M$ is a smooth manifold, a connection $\nabla$ in $TM$ is torsion-free if

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

If $(x_1, \ldots, x_n): U \rightarrow \mathbb{R}^n$ is a coordinate chart on $M$, let

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \in \Gamma(U; TM)$$

be the corresponding frame for $TM$ on $U$. If $\nabla$ is a connection in $TM$, the corresponding connection 1-form $\theta$ can be written as

$$\theta^k_j = \sum_{i=1}^{n} \Gamma^k_{ij} dx^i, \quad \text{where} \quad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma^k_{ij} \frac{\partial}{\partial x_k}.$$ 

The connection $\nabla$ is torsion-free on $TM|_U$ if and only if

$$\Gamma^k_{ij} = \Gamma^k_{ji} \quad \forall \, i, j, k = 1, \ldots, n. \quad (1.18)$$

Lemma 1.2. If $(M, g)$ is a Riemannian manifold, there exists a unique torsion-free $g$-compatible connection $\nabla$ in $TM$. 

5
Proof. (1) Suppose \( \nabla \) and \( \tilde{\nabla} \) are torsion-free \( g \)-compatible connections in \( TM \). By (1.9), there exists

\[
\theta \in \Gamma(M; T^*M \otimes_{\mathbb{R}} \text{Hom}(TM, TM)) \quad \text{s.t.}
\]

\[
\tilde{\nabla}_X Y - \nabla_X Y = \{ \theta(X) \} Y \ \forall \ Y \in \Gamma(M; TM), \ X \in \mathcal{T}_x M, \ x \in M.
\]

Since \( \nabla \) and \( \tilde{\nabla} \) are torsion-free,

\[
\{ \theta(X) \} Y = \{ \theta(Y) \} X \ \forall \ X, Y \in \mathcal{T}_x M, \ x \in M.
\]

(1.19)

Since \( \nabla \) and \( \tilde{\nabla} \) are \( g \)-compatible,

\[
\begin{aligned}
&g(\{ \theta(X) \} Y, Z) + g(Y, \{ \theta(X) \} Z) = 0 \\
g(\{ \theta(Y) \} X, Z) + g(X, \{ \theta(Y) \} Z) = 0 \\
g(\{ \theta(Z) \} X, Y) + g(X, \{ \theta(Z) \} Y) = 0
\end{aligned}
\]

(1.20)

Adding the first two equations in (1.20), subtracting the third, and using (1.19) and the symmetry of \( g \), we obtain

\[
2g(\{ \theta(X) \} Y, Z) = 0 \ \forall \ X, Y, Z \in \mathcal{T}_x M, \ x \in M \quad \Rightarrow \quad \theta \equiv 0.
\]

Thus, \( \tilde{\nabla} = \nabla \).

(2) Let \((x_1, \ldots, x_n) : U \rightarrow \mathbb{R}^n\) be a coordinate chart on \( M \). With notation as in the paragraph preceding Lemma 1.2, \( \nabla \) is \( g \)-compatible on \( TM|_U \) if and only if

\[
\sum_{l=1}^{l=n} (g_{il} \Gamma^l_k + g_{jl} \Gamma^l_i) = \partial_{x_k} g_{ij};
\]

(1.21)

see (1.17). Define a connection \( \nabla \) in \( TM|_U \) by

\[
\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{l=n} g^{kl}(\partial_{x_l} g_{jl} + \partial_{x_j} g_{il} - \partial_{x_i} g_{lj}) \quad \forall \ i, j, k = 1, \ldots, n,
\]

where \( g^{ij} \) is the \((i, j)\)-entry of the inverse of the matrix \((g_{ij})_{i,j=1,\ldots,n}\). Since \( g_{ij} = g_{ji} \), \( \Gamma^k_{ij} \) satisfies (1.18); a direct computation shows that \( \Gamma^k_{ij} \) also satisfies (1.21). Therefore, \( \nabla \) is a torsion-free \( g \)-compatible connection on \( TM|_U \). In this way, we can define a torsion-free \( g \)-compatible connection on every coordinate chart. By the uniqueness property, these connections agree on the overlaps. \( \square \)

2 Complex structures

2.1 Complex linear connections

Suppose \( M \) is a smooth manifold and \( \pi : (E, \mathfrak{i}) \rightarrow M \) is a complex vector bundle. Similarly to Section 1.1, there is an exact sequence

\[
0 \longrightarrow \pi_E^* E \xrightarrow{\mathfrak{i}_E} TE \xrightarrow{d\pi_E} \pi_E^* TM \longrightarrow 0
\]

(2.1)
of vector bundles over $E$. The homomorphism $\iota_E$ is now $\mathbb{C}$-linear. If $f \in C^\infty(M; \mathbb{C})$ and $m_f : E \to E$ is defined as in (1.1), there is a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & \pi_E^* E & \xrightarrow{\iota_E} & TE & \xrightarrow{d\pi_E} & \pi_E^* TM & \to & 0 \\
\downarrow{\pi_E^* m_f} & & \downarrow{dm_f} & & \downarrow{\pi_E^* \text{id}} & & & \\
0 & \to & \pi_E^* E & \xrightarrow{m_f^* \iota_E} & m_f^* TE & \xrightarrow{m_f^* d\pi_E} & \pi_E^* TM & \to & 0 \\
\end{array} \tag{2.2}
\]

of bundle maps over $E$.

Suppose

\[\nabla : \Gamma(M; E) \to \Gamma(M; T^* M \otimes \mathbb{R} E)\]

is a $\mathbb{C}$-linear connection, i.e.

\[\nabla_v (i\xi) = i(\nabla_v \xi) \quad \forall \xi \in \Gamma(M; E), \; v \in TM.\]

If $U$ is an open subset of $M$ and $\xi_1, \ldots, \xi_n \in \Gamma(U; E)$ is a $\mathbb{C}$-frame for $E$ on $U$, then there exist

\[\theta^k_l \in \Gamma(M; T^* M) \quad \text{s.t.} \quad \nabla \xi_l = \sum_{k=1}^{n} \xi_k \theta^k_l \equiv \sum_{k=1}^{n} \theta^k_l \otimes \xi_k \quad \forall \; l = 1, \ldots, n.\]

We will call

\[\theta \equiv (\theta^k_l)_{k,l=1,\ldots,n} \in \Gamma(\Sigma; T^* M \otimes \mathbb{R} \text{Mat}_n \mathbb{C})\]

the complex connection 1-form of $\nabla$ with respect to the frame $(\xi_k)_k$. For an arbitrary section

\[\xi = \sum_{l=1}^{n} f^{l} \xi_l \in \Gamma(U; E),\]

by (1.8) and $\mathbb{C}$-linearity of $\nabla$ we have

\[\nabla \xi = \sum_{k=1}^{n} \xi_k \left( df^k + \sum_{l=1}^{n} \theta^l_k f^l \right), \quad \text{i.e.} \quad \nabla (\xi \cdot f^l) = \xi \cdot \{d + \theta\} f^l, \tag{2.3}\]

where $\xi$ and $f$ are as (1.12).

Let $g$ be a hermitian metric on $E$, i.e.

\[g \in \Gamma\left(M; \text{Hom}_\mathbb{C}(\bar{E} \otimes \mathbb{C}, E)\right) \quad \text{s.t.} \quad g(v, w) = \overline{g(w, v)}, \quad g(v, v) > 0 \quad \forall \; v, w \in E_x, \; v \neq 0, \; x \in M.\]

A $\mathbb{C}$-linear connection $\nabla$ in $E$ is $g$-compatible if

\[d(g(\xi, \zeta)) = g(\nabla \xi, \zeta) + g(\xi, \nabla \zeta) \in \Gamma(M; T^* M \otimes \mathbb{R} \mathbb{C}) \quad \forall \; \xi, \zeta \in \Gamma(M; E).\]

With notation as in the previous paragraph, let

\[g_{ij} = g(\xi_i, \xi_j) \in C^\infty(U; \mathbb{C}) \quad \forall \; i, j = 1, \ldots, n.\]

Then $\nabla$ is $g$-compatible on $U$ if and only if

\[\sum_{k=1}^{n} (g_{ik} \theta^k_j + \bar{g}_{jk} \bar{\theta}^k_i) = dg_{ij} \quad \forall \; i, j = 1, 2, \ldots, n. \tag{2.4}\]
2.2 Generalized $\bar{\partial}$-operators

If $(\Sigma, j)$ is an almost complex manifold, let

$$T^*\Sigma^{1,0} \equiv \{ \eta \in T^*\Sigma \otimes \mathbb{C}: \eta \circ j = i \eta \} \quad \text{and} \quad T^*\Sigma^{0,1} \equiv \{ \eta \in T^*\Sigma \otimes \mathbb{C}: \eta \circ j = -i \eta \}$$

be the bundles of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear 1-forms on $\Sigma$. If $(\Sigma, j)$ and $(M, J)$ are smooth almost complex manifolds and $u: \Sigma \rightarrow M$ is a smooth function, define

$$\bar{\partial}_{J,j}u \in \Gamma(\Sigma; T^*\Sigma^{0,0} \otimes_{\mathbb{C}} u^*TM) \quad \text{by} \quad \bar{\partial}_{J,j}u = \frac{1}{2}(du + J \circ du \circ j). \quad (2.5)$$

A smooth map $u: (\Sigma, j) \rightarrow (M, J)$ will be called $(J, j)$-holomorphic if $\bar{\partial}_{J,j}u = 0$.

**Definition 2.1.** Suppose $(\Sigma, j)$ is an almost complex manifold and $\pi: (E, i) \rightarrow \Sigma$ is a complex vector bundle. A $\bar{\partial}$-operator on $(E, i)$ is a $\mathbb{C}$-linear map $\bar{\partial}: \Gamma(\Sigma; E) \rightarrow \Gamma(\Sigma; T^*\Sigma^{0,0} \otimes_{\mathbb{C}} E)$ such that

$$\bar{\partial}(f\xi) = (\bar{\partial}f) \otimes \xi + f(\bar{\partial}\xi) \quad \forall \ f \in C^\infty(\Sigma), \ \xi \in \Gamma(\Sigma; E), \quad (2.6)$$

where $\bar{\partial}f = \bar{\partial}_i f$ is the usual $\bar{\partial}$-operator on complex-valued functions.

Similarly to Section 1.1, a $\bar{\partial}$-operator on $(E, i)$ is a first-order differential operator. If $U$ is an open subset of $M$ and $\xi_1, \ldots, \xi_n \in \Gamma(U; E)$ is a $\mathbb{C}$-frame for $E$ on $U$, then there exist

$$\theta^k_l \in \Gamma(U; T^*U^{0,1}) \quad \text{s.t.} \quad \bar{\partial}\xi_l = \sum_{k=1}^n \xi_k \theta^k_l \equiv \sum_{k=1}^n \theta^k_l \otimes \xi_k \quad \forall \ l = 1, \ldots, n. \quad (2.7)$$

We call

$$\theta \equiv (\theta^k_l)_{k,l=1,\ldots,n} \in \Gamma(U; T^*U^{0,1} \otimes_{\mathbb{C}} \text{Mat}_n \mathbb{C})$$

the connection 1-form of $\bar{\partial}$ with respect to the frame $(\xi_k)_k$. For an arbitrary section

$$\xi = \sum_{l=1}^n f^l \xi_l \in \Gamma(U; E),$$

by (2.6) we have

$$\bar{\partial}\xi = \sum_{k=1}^n \xi_k \left( \bar{\partial}f^k + \sum_{l=1}^n \theta^k_l f^l \right), \quad \text{i.e.} \quad \bar{\partial}(\xi \cdot f^l) = \xi \cdot \left( \bar{\partial} + \theta \right) f^l, \quad (2.7)$$

where $\xi$ and $f$ are as in (1.12). It is immediate from (2.6) that the symbol of $\bar{\partial}$ is given by

$$\sigma_{\bar{\partial}}: T^*\Sigma \rightarrow \text{Hom}_{\mathbb{C}}(E, T^*\Sigma^{0,1} \otimes_{\mathbb{C}} E), \quad \{ \sigma_{\bar{\partial}}(\eta) \}(f) = (\eta + i \eta \circ j) \otimes f.$$  

In particular, $\bar{\partial}$ is an elliptic operator (i.e. $\sigma_{\bar{\partial}}(\eta)$ is an isomorphism for $\eta \neq 0$) if $(\Sigma, j)$ is a Riemann surface.
Lemma 2.2. Suppose $(\Sigma, j)$ is an almost complex manifold and $\pi: (E, i) \to \Sigma$ is a complex vector bundle. If
\[
\bar{\partial}: \Gamma(\Sigma; E) \to \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes_\mathbb{C} E)
\]
is a $\bar{\partial}$-operator on $(E, i)$, there exists a unique almost complex structure $J = J_\partial$ on (the total space of) $E$ such that $\pi$ is a $(i, J)$-holomorphic map, the restriction of $J$ to the vertical tangent bundle $TE^v \cong \pi^*E$ agrees with $i$, and
\[
\bar{\partial}_J \xi = 0 \in \Gamma(U; T^*\Sigma^{0,1} \otimes_\mathbb{C} \xi^*TE) \iff \bar{\partial} \xi = 0 \in \Gamma(U; T^*\Sigma^{0,1} \otimes_\mathbb{C} E) \quad (2.8)
\]
for every open subset $U$ of $\Sigma$ and $\xi \in \Gamma(U; E)$.

Proof. (1) With notation as above, define
\[
\varphi: U \times \mathbb{C}^n \to E|_U \quad \text{by} \quad \varphi(x, c^1, \ldots, c^n) = \xi(x) \cdot c^t = \sum_{k=1}^{k=n} c^k \xi_k(x) \in E_x.
\]
The map $\varphi$ is a trivialization of $E$ over $U$. If $J \equiv J_\partial$ is an almost complex structure on $E$, let $\tilde{J}$ be the almost complex structure on $U \times \mathbb{C}^n$ given by
\[
\tilde{J}_{(x, c)} = \left\{ d_{(x, c)} \varphi \right\}^{-1} \circ J_{(x, c)} \circ d_{(x, c)} \varphi \quad \forall (x, c) \in U \times \mathbb{C}^n. \quad (2.9)
\]
The almost complex structure $J$ restricts to $i$ on $TE^v$ if and only if
\[
\tilde{J}_{(x, c)} w = iw \in T_{\overline{\pi}} \mathbb{C}^n \subset T_{(x, c)}(U \times \mathbb{C}^n) \quad \forall w \in T_x \mathbb{C}^n. \quad (2.10)
\]
If $J$ restricts to $i$ on $TE^v$, the projection $\pi$ is $(i, J)$-holomorphic on $E|_U$ if and only if there exists
\[
\tilde{J}^v \in \Gamma(U \times \mathbb{C}^n, \text{Hom}_\mathbb{R}(\pi^*_U TU, \pi^*_U \mathbb{C}^n)) \quad \text{s.t.}
\]
\[
\tilde{J}_{(x, c)} w = J_{(x, c)} w + \tilde{J}^v_{(x, c)} w \quad \forall w \in T_x U \subset T_{(x, c)}(U \times \mathbb{C}^n). \quad (2.11)
\]
If $\xi \in \Gamma(U; E)$, let
\[
\tilde{\xi} \equiv \varphi^{-1} \circ \xi \equiv (\text{id}_U, \tilde{f}), \quad \text{where} \quad \tilde{f} \in C^\infty(U; \mathbb{C}^n).
\]
By (2.9)-(2.11),
\[
2 \bar{\partial}_J \xi |_x = d\tilde{\xi}(x) \varphi \circ 2\bar{\partial}_J \tilde{\xi} |_x = d\tilde{\xi}(x) \varphi \circ \left\{ (\text{Id}_{T_x U}, d_x \tilde{f}) + \tilde{J}^v_{\tilde{\xi}(x)} \circ (\text{Id}_{T_x U}, d_x f) \circ i_x \right\} \quad \text{and}
\]
\[
= d\tilde{\xi}(x) \varphi \circ \left\{ 0, 2\bar{\partial} f |_x + \tilde{J}^v_{\tilde{\xi}(x)} \circ i_x \right\}. \quad (2.12)
\]
On the other hand, by (2.7),
\[
\bar{\partial} \xi |_x = \bar{\partial}(\xi \cdot f^t) |_x = \xi(x) \cdot \left\{ \bar{\partial} + \theta \right\} f^t |_x = \varphi(\bar{\partial} f |_x + \theta_x \cdot f(x)^t).
\]
(2.13)
By (2.12) and (2.13), the property (2.8) is satisfied for all $\xi \in \Gamma(U; E)$ if and only if
\[
\tilde{J}^v_{(x, c)} = 2(\theta_x \cdot c^t) \circ (-i_x) = 2i \theta_x \cdot c^t \quad \forall (x, c) \in U \times \mathbb{C}^n.
\]
In summary, the almost complex structure \( J = J_\bar{\partial} \) on \( E \) has the three desired properties if and only if for every trivialization of \( E \) over an open subset \( U \) of \( \Sigma \)

\[
\bar{J}_{(x,\zeta)}(w_1, w_2) = (j_{x}w_1, iw_2 + 2i\theta_x(w_1) \cdot \zeta) \quad (2.14)
\]
\[
\forall (x, \zeta) \in U \times \mathbb{C}^n, \quad (w_1, w_2) \in T_x U \oplus T_x \mathbb{C}^n \cong T_{(x,\zeta)}(U \times \mathbb{C}^n),
\]

where \( \bar{J} \) is the almost complex structure on \( U \times \mathbb{C}^n \) induced by \( J \) via the trivialization and \( \theta \) is the connection 1-form corresponding to \( \partial \) with respect to the frame inducing the trivialization.

(2) By (2.14), there exists at most one almost complex structure \( J \) satisfying the three properties. Conversely, (2.14) determines such an almost complex structure on \( E \). Since

\[
\bar{J}^2_{(x,\zeta)}(w_1, w_2) = \bar{J}_{(x,\zeta)}(iw_1, iw_2 + 2i\theta_x(w_1) \cdot \zeta) = (j^2w_1, i(iw_2 + 2i\theta_x(w_1) \cdot \zeta) + 2i\theta_x(iw_1) \cdot \zeta)
\]
\[
= -(w_1, w_2),
\]

\( \bar{J} \) is indeed an almost complex structure on \( E \). The almost complex structure induced by \( \bar{J} \) on \( E|_U \) satisfies the three properties by part (a). By the uniqueness property, the almost complex structures on \( E \) induced by the different trivializations agree on the overlaps. Therefore, they define an almost complex structure \( J = J_\bar{\partial} \) on the total space of \( E \) with the desired properties. \( \square \)

### 2.3 Connections and \( \bar{\partial} \)-operators

Suppose \((\Sigma, j)\) is an almost complex manifold, \( \pi: (E, i) \longrightarrow \Sigma \) is a complex vector bundle, and

\[
\bar{\partial}: \Gamma(\Sigma; E) \longrightarrow \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes \mathbb{C}E)
\]

is a \( \bar{\partial} \)-operator on \((E, i)\). A \( \mathbb{C} \)-linear connection \( \nabla \) in \((E, i)\) is \( \bar{\partial} \)-compatible if

\[
\bar{\partial}\xi = \bar{\partial}_\nabla \xi \equiv \frac{1}{2}(\nabla \xi + i\nabla \xi \circ j) \quad \forall \xi \in \Gamma(M; \Sigma). \quad (2.15)
\]

**Lemma 2.3.** Suppose \((\Sigma, j)\) is an almost complex manifold, \( \pi: (E, i) \longrightarrow \Sigma \) is a complex vector bundle,

\[
\bar{\partial}: \Gamma(\Sigma; E) \longrightarrow \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes \mathbb{C}E)
\]

is a \( \bar{\partial} \)-operator on \((E, i)\), and \( J_\bar{\partial} \) is the complex structure in the vector bundle \( TE \longrightarrow E \) provided by Lemma 2.2. A \( \mathbb{C} \)-linear connection \( \nabla \) in \((E, i)\) is \( \bar{\partial} \)-compatible if and only if the splitting (1.14) determined by \( \nabla \) respects the complex structures.

**Proof.** Since \( J_\bar{\partial} = \pi^*i \) on \( \pi^*E \subset TE \), the splitting (1.14) determined by \( \nabla \) respects the complex structures if and only if

\[
J_{\bar{\partial}|_v} \circ (d\xi - \nabla \xi) \big|_x = (d\xi - \nabla \xi) \circ j_x: T_x \Sigma \longrightarrow T_v E
\]

for all \( x \in \Sigma, \ v \in E_x, \) and \( \xi \in \Gamma(\Sigma; E) \) such that \( \xi(x) = 0 \); see the proof of Lemma 1.1. This identity is equivalent to

\[
\bar{\partial}_{J_{\bar{\partial}|_v}} \xi = \bar{\partial}_{\nabla} \xi \quad \forall \xi \in \Gamma(\Sigma; E). \quad (2.16)
\]

On the other hand, by the proof of Lemma 2.2,

\[
\bar{\partial}_{J_{\bar{\partial}|_v}} \xi = \bar{\partial} \xi \quad \forall \xi \in \Gamma(\Sigma; E); \quad (2.17)
\]

see (2.12)-(2.14). The lemma follows immediately from (2.16) and (2.17). \( \square \)
2.4 Holomorphic vector bundles

Let \((\Sigma, j)\) be a complex manifold. A holomorphic vector bundle \((E, i)\) on \((\Sigma, j)\) is a complex vector bundle with a collection of trivializations that overlap holomorphically.

A collection of holomorphically overlapping trivializations of \((E, i)\) determines a holomorphic structure \(J\) on the total space of \(E\) and a \(\bar{\partial}\)-operator

\[
\bar{\partial} : \Gamma(\Sigma; E) \rightarrow \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes_C E).
\]

The latter is defined as follows. If \(\xi_1, \ldots, \xi_n\) is a holomorphic complex frame for \(E\) over an open subset \(U\) of \(M\), then

\[
\bar{\partial} \sum_{k=1}^{k=n} f^k \xi_k = \sum_{k=1}^{k=n} \bar{\partial} f^k \otimes \xi_k \quad \forall f^1, \ldots, f^k \in C^\infty(U; \mathbb{C}).
\]

In particular, for all \(\xi \in \Gamma(M; E)\)

\[
\bar{\partial}_{J^i} \xi = 0 \iff \bar{\partial} \xi = 0.
\]

Thus, \(J = J_\beta\); see Lemma 2.2.

**Lemma 2.4.** Suppose \((\Sigma, j)\) is a Riemann surface and \(\pi: (E, i) \rightarrow \Sigma\) is a complex vector bundle. If

\[
\bar{\partial} : \Gamma(\Sigma; E) \rightarrow \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes_C E)
\]

is a \(\bar{\partial}\)-operator on \((E, i)\), the almost complex structure \(J = J_\beta\) on \(E\) is integrable. With this complex structure, \(\pi: E \rightarrow \Sigma\) is a holomorphic vector bundle and \(\bar{\partial}\) is the corresponding \(\bar{\partial}\)-operator.

**Proof.** By (2.8), it is sufficient to show that there exists a \((J, j)\)-holomorphic local section through every point \(v \in E\), i.e. there exist a neighborhood \(U\) of \(x \equiv \pi(v)\) in \(\Sigma\) and \(\xi \in \Gamma(U; E)\) such that

\[
\xi(x) = v \quad \text{and} \quad \bar{\partial}_{J^i} \xi = 0.
\]

By Lemma 2.2 and (2.13), this is equivalent to showing that the equation

\[
\left\{ \bar{\partial} + \theta \right\} f^t = 0, \quad f(x) = v, \quad f \in C^\infty(U; \mathbb{C}^n),
\]

has a solution for every \(v \in \mathbb{C}^n\). We can assume that \(U\) is a small disk contained in \(S^2\). Let

\[
\eta: S^2 \rightarrow [0, 1]
\]

be a smooth function supported in \(U\) and such that \(\eta \equiv 1\) on a neighborhood of \(x\). Then,

\[
\eta \in \Gamma(S^2; (T^*S^2)^{0,1} \otimes_C \text{Mat}_n\mathbb{C}).
\]

Choose \(p > 2\). The operator

\[
\Theta : L^p_1(S^2; \mathbb{C}^n) \rightarrow L^p(S^2; (T^*S^2)^{0,1} \otimes_C \mathbb{C}^n) \oplus \mathbb{C}^n, \quad \Theta(f) = (\bar{\partial}_{J^i} f, f(x)),
\]

is surjective. If \(\eta\) has sufficiently small support, so is the operator

\[
\Theta_\eta : L^p_1(S^2; \mathbb{C}^n) \rightarrow L^p(S^2; (T^*S^2)^{0,1} \otimes_C \mathbb{C}^n) \oplus \mathbb{C}^n, \quad \Theta_\eta(f) = (\left\{ \bar{\partial}_{J^i} + \eta \theta \right\} f, f(x)).
\]

Then, the restriction of \(\Theta_\eta^{-1}(0, v)\) to a neighborhood of \(x\) on which \(\eta \equiv 1\) is a solution of (2.18). By elliptic regularity, \(\Theta_\eta^{-1}(0, v) \in C^\infty(S^2; \mathbb{C}^n)\). 

\(\square\)
2.5 Deformations of almost complex submanifolds

If \((M, J)\) is a complex manifold, holomorphic coordinate charts on \((M, J)\) determine a holomorphic structure in the vector bundle \((TM, i) \rightarrow M\). If \((\Sigma, i) \subset (M, J)\) is a complex submanifold, holomorphic coordinate charts on \(\Sigma\) can be extended to holomorphic coordinate charts on \(M\). Thus, the holomorphic structure in \(T\Sigma \rightarrow \Sigma\) induced from \((\Sigma, i)\) is the restriction of the holomorphic structure in \(TM|_\Sigma\). It follows that

\[
\bar{\partial}_M = \bar{\partial}_\Sigma: \Gamma(\Sigma; T\Sigma) \rightarrow \Gamma(\Sigma; T^*\Sigma^0.1 \otimes_C T\Sigma) \subset \Gamma(\Sigma; T^*\Sigma^0.1 \otimes_C TM|_\Sigma),
\]

where \(\bar{\partial}_M\) and \(\bar{\partial}_\Sigma\) are the \(\bar{\partial}\)-operators in \(TM|_\Sigma\) and \(T\Sigma\) induced from the holomorphic structures in \(\Sigma\) and \(M\). Therefore, \(\bar{\partial}_M\) descends to a \(\bar{\partial}\)-operator on the quotient

\[
\bar{\partial}: \Gamma(\Sigma; N_M\Sigma) = \Gamma(\Sigma; TM|_\Sigma)/\Gamma(\Sigma; T\Sigma) \rightarrow \Gamma(\Sigma; T^*\Sigma^0.1 \otimes_C N_M\Sigma),
\]

where

\[
N_M\Sigma \equiv TM|_\Sigma/T\Sigma \rightarrow \Sigma
\]

is the normal bundle of \(\Sigma\) in \(M\). This vector bundle inherits a holomorphic structure from that of \(TM|_\Sigma\) and \(\Sigma\). The above \(\bar{\partial}\)-operator on \(N_M\Sigma\) is the \(\bar{\partial}\)-operator corresponding to this induced holomorphic structure on \(N_M\Sigma\).

Suppose \((M, J)\) is an almost complex manifold and \((\Sigma, i) \subset (M, J)\) is an almost complex submanifold. Let \(\nabla\) be a torsion-free connection in \(TM\). Define

\[
D_{J,\Sigma}: \Gamma(\Sigma; TM|_\Sigma) \rightarrow \Gamma(\Sigma; T^*\Sigma^0.1 \otimes_C TM|_\Sigma)
\]

by

\[
D_{J,\Sigma} \xi = \frac{1}{2}(\nabla \xi + J \circ \nabla \circ i j) - \frac{1}{2} J \circ \nabla \circ i j: T\Sigma \rightarrow TM|_\Sigma.
\]

(2.19)

If \(\nabla\) is the Levi-Civita connection (the connection of Lemma 1.2) for a \(J\)-compatible metric on \(M\) (and \(\Sigma\) is a Riemann surface), then \(D_{J,\Sigma}\) is the linearization of the \(\bar{\partial}_J\)-operator at the inclusion map \(\iota: \Sigma \rightarrow M\); see [4, Proposition 3.1.1].

In fact, \(D_{J,\Sigma}\) is independent of the choice of a torsion-free connection in \(TM\). Let

\[
\tilde{\nabla} = \nabla + \theta, \quad \theta \in \Gamma\bigl(M; T^*M \otimes \text{Hom}_R(TM, TM)\bigr),
\]

(2.20)

be another torsion-free connection; see (1.9). Since \(\tilde{\nabla}\) and \(\nabla\) are torsion-free connections,

\[
\{\theta(X)\}Y = \{\theta(Y)\}X \quad \forall X, Y \in T_xM, \ x \in M.
\]

(2.21)

If \(x \in M\) and \(X, Y \in \Gamma(M; TM)\),

\[
\{\nabla_Y J\}X = \nabla_Y (JX) - J\nabla_Y X, \quad \{\tilde{\nabla}_Y J\}X = \tilde{\nabla}_Y (JX) - J\tilde{\nabla}_Y X \quad \Rightarrow
\]

\[
\{\nabla_Y J\}X - \{\nabla_Y J\}X = \{\theta(Y)\}(JX) - J\{\theta(Y)\}X = \{\theta(JX)\}Y - J\{\theta(X)\}Y
\]

by (2.20) and (2.21). On the other hand, by (2.20) for all \(X \in T\Sigma\) and \(\xi \in \Gamma(\Sigma; TM|_\Sigma)\),

\[
\{\tilde{\nabla}_Y J \circ \tilde{\nabla} \circ i j\}(X) - \{\nabla_Y J \circ \nabla \circ i j\}(X) = \{\theta(X)\}\xi + J\{\theta(JX)\}\xi = J\{\theta(JX)\}\xi - J\{\theta(X)\}\xi
\]

(2.23)
Since \( j = J|_{T\Sigma} \) and \( J^2 = -\text{Id} \). By (2.22) and (2.23), \( D_{J,\Sigma} \) is independent of the choice of torsion-free connection \( \nabla \).

Since any torsion-free connection on \( \Sigma \) extends to a torsion-free connection on \( M \), the above observation implies that

\[
D_{J,\Sigma} : \Gamma(\Sigma; T\Sigma) \to \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes_{\mathbb{C}} TM|_{\Sigma}).
\]

(2.24)

Thus, an almost complex submanifold \((\Sigma, j)\) of an almost complex manifold \((M, J)\) induces a well-defined generalized Cauchy-Riemann operator\(^1\) on the normal bundle of \( \Sigma \) in \( M \),

\[
D^N_{J,\Sigma} : \Gamma(\Sigma; N_M\Sigma) \to \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes_{\mathbb{C}} N_M\Sigma), \quad D^N_{J,\Sigma}(\pi(\xi)) = \pi(D_{J,\Sigma}(\xi)) \quad \forall \xi \in \Gamma(\Sigma; TM|_{\Sigma}),
\]

where \( \pi : TM|_{\Sigma} \to N_M\Sigma \) is the quotient projection map. The \( \mathbb{C} \)-linear part of \( D^N_{J,\Sigma} \) determines a \( \bar{\partial} \)-operator on the normal bundle of \( \Sigma \) in \( M \):

\[
\bar{\partial}^N_{J,\Sigma} : \Gamma(\Sigma; N_M\Sigma) \to \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes_{\mathbb{C}} N_M\Sigma), \quad \bar{\partial}^N_{J,\Sigma}(\xi) = \frac{1}{2}(D^N_{J,\Sigma}(\xi) - JD^N_{J,\Sigma}(\xi)) \quad \forall \xi \in \Gamma(\Sigma; N_M\Sigma).
\]

Both operators are determined by the almost complex submanifold \((\Sigma, j)\) of the almost complex manifold \((M, J)\) only and are independent of the choice of torsion-free connection \( \nabla \) in (2.19).

Any connection \( \nabla \) in \( TM \) induces a \( J \)-linear connection in \( TM \) by

\[
\nabla^J_X \xi = \nabla_X \xi - \frac{1}{2} J(\nabla_X J)\xi \quad \forall X \in TM, \xi \in \Gamma(M; TM).
\]

(2.25)

If \( \nabla \) is as in (2.19),

\[
\{ D_{J,\Sigma} \xi \} (X) = \{ \bar{\partial}^J \nabla \xi \} (X) + A_J(X, \xi) - \frac{1}{4} \{ (\nabla J J^2) + J(\nabla J J^2) \} (X)
\]

(2.26)

for all \( \xi \in \Gamma(\Sigma; TM|_{\Sigma}) \) and \( X \in T\Sigma \), where \( A_J \) is the Nijenhuis tensor of \( J \):

\[
A_J(\xi_1, \xi_2) = \frac{1}{4} \left( [\xi_1, \xi_2] + J[\xi_1, J\xi_2] + J[J\xi_1, \xi_2] - [J\xi_1, J\xi_2] \right) \quad \forall \xi_1, \xi_2 \in \Gamma(M; TM).
\]

(2.27)

Since the sum of the terms in the curly brackets in (2.26) is \( \mathbb{C} \)-linear in \( \xi \), while the Nijenhuis tensor is \( \mathbb{C} \)-antilinear, the \( \mathbb{C} \)-linear operator

\[
\Gamma(\Sigma; TM|_{\Sigma}) \to \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes_{\mathbb{C}} TM|_{\Sigma}), \quad \xi \to \bar{\partial}^J \nabla \xi \xi - \frac{1}{4} \{ (\nabla J J^2) + J(\nabla J J^2) \},
\]

(2.28)

takes \( \Gamma(\Sigma; T\Sigma) \) to \( \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes_{\mathbb{C}} T\Sigma) \) by (2.24). Thus, it induces a \( \bar{\partial} \)-operator on \( N_M\Sigma \) and this induced operator is \( \bar{\partial}^N_{J,\Sigma} \). If the image of the homomorphism

\[
TM \to T^*\Sigma^{0,1} \otimes_{\mathbb{C}} TM|_{\Sigma}, \quad \xi \to \nabla J J^2 - J(\nabla J J^2),
\]

\(^1\)see Section 4.3
is contained in $T^*\Sigma \otimes \mathbb{C} T\Sigma$, then $\bar{\partial}_{\nabla J}$ preserves $T\Sigma$ and induces a $\bar{\partial}$-operator $\bar{\partial}_{\nabla J}^N$ on $N_M\Sigma$ with $\bar{\partial}_{\nabla J}^N = \bar{\partial}_{\nabla J}^N$. In this case,

$$D_{J;\Sigma}^N(\pi(\xi)) = \pi(\bar{\partial}_{\nabla J}^N \xi + A_J(\cdot, \xi)) : T\Sigma \to N_M\Sigma \quad \forall \xi \in \Gamma(\Sigma; TM|\Sigma).$$

This is the case in particular if $J$ is compatible with a symplectic form $\omega$ on $M$ and $\nabla$ is the Levi-Civita connection for the metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$, as the sum in the curly brackets in (2.26) then vanishes by [4, (C.7.5)].

It is immediate that $A_J$ takes $T\Sigma \otimes \mathbb{R} T\Sigma$ to $T\Sigma$ and thus induces a bundle homomorphism

$$A_J^N : T\Sigma \otimes \mathbb{R} N_M\Sigma \to N_M\Sigma.$$

If $\zeta$ is any vector field on $M$ such that $\zeta(x) = X \in T_x\Sigma$ for some $x \in \Sigma$, then

$$\{D_{J;\Sigma}\zeta\}(X) = \frac{1}{2} ([\zeta, \xi] + J[J\zeta, \xi])|_x,$$

$$\{\bar{\partial}_{\nabla J}(\zeta) - \frac{1}{4} ((\nabla J\zeta) + J(\nabla J\zeta))\}(X) = \frac{1}{4} ([\zeta, \xi] + J[J\zeta, \xi] - J[\zeta, J\xi] + [J\zeta, J\xi])|_x,$$

since $\nabla$ is torsion-free. These two identities immediately imply that the operators (2.19) and (2.28) preserve $T\Sigma \subset TM|\Sigma$ and thus induce operators

$$\Gamma(\Sigma; N_M\Sigma) \to \Gamma(\Sigma; T^*\Sigma \otimes \mathbb{C} N_M\Sigma)$$

as claimed above.

If $g$ is a $J$-compatible metric on $TM|\Sigma$ and $\pi^\perp : TM|\Sigma \to T\Sigma^\perp$ is the projection to the $g$-orthogonal complement of $T\Sigma$ in $TM|\Sigma$, the composition $\nabla^\perp$

$$\Gamma(\Sigma; T\Sigma^\perp) \to \Gamma(\Sigma; TM|\Sigma) \xrightarrow{\nabla^J} \Gamma(\Sigma; T^*\Sigma \otimes \mathbb{R} TM|\Sigma) \xrightarrow{\pi^\perp} \Gamma(\Sigma; T^*\Sigma \otimes \mathbb{R} T\Sigma^\perp),$$

with $\nabla^J$ as in (2.25), is a $g$-compatible $J$-linear connection in $T\Sigma^\perp$. Via the isomorphism $\pi : T\Sigma^\perp \to N_M\Sigma$, it induces a $J$-linear connection $\nabla^N$ in $N_M\Sigma$ which is compatible with the metric $g^N$ induced via this isomorphism from $g|_{T\Sigma^\perp}$. If the image of the homomorphism

$$T\Sigma^\perp \to T^*\Sigma \otimes \mathbb{C} T\Sigma|\Sigma, \quad \xi \mapsto \nabla_\xi J - J\nabla J\xi,$$  

(30)

is contained in $T^*\Sigma \otimes \mathbb{C} T\Sigma$, then $\bar{\partial}_{\nabla J}^N = \bar{\partial}_{\nabla J}^N$ and so

$$D_{J;\Sigma}^N(\pi(\xi)) = \pi(\bar{\partial}_{\nabla J}^N \xi + A_J(\cdot, \xi)) : T\Sigma \to N_M\Sigma \quad \forall \xi \in \Gamma(\Sigma; T\Sigma^\perp).$$

This is the case if $\Sigma$ is a divisor in $M$, i.e. $\text{rk}_\mathbb{C} N = 1$, since $(\nabla_\xi J)\xi$ is $g$-orthogonal to $\xi$ and $J\xi$ for all $\xi, \zeta \in T_xM$ and $x \in M$ by [4, (C.7.1)]. This is also the case if $J$ is compatible with a symplectic form $\omega$ on $M$ and $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$, as the homomorphism (2.30) is then trivial by [4, (C.7.5)].

\footnote{Since LHS and RHS of these identities depend only $\xi$ and $X = \zeta(x)$, and not on $\zeta$, it is sufficient to verify them under the assumption that $\nabla_\xi|_x = 0$.}
3 Riemannian geometry estimates

This section is based on [1, Chapter 1] and [2, Section 3] and culminates in a Poincare lemma for closed curves in Proposition 3.6 and an expansion for the $\partial$-operator in Proposition 3.13. If $u: \Sigma \rightarrow M$ is a smooth map between smooth manifolds and $E \rightarrow M$ is a smooth vector bundle, let

$$\Gamma(u; E) = \Gamma(\Sigma; u^* E), \quad \Gamma^1(u; E) = \Gamma(\Sigma; T^* \Sigma \otimes_{\mathbb{R}} u^* E).$$

We denote the subspace of compactly supported sections in $\Gamma(u; E)$ by $\Gamma_c(u; E)$.

An exponential-like map on a smooth manifold $M$ is a smooth map $\exp: T M \rightarrow M$ such that $\exp|_M = \text{id}_M$ and

$$d_x \exp = (\text{id}_{T_x M} \text{id}_{T_x M}) : T_x (T M) = T_x M \oplus T_x M \rightarrow T_x M \quad \forall x \in M,$$

where the second equality is the canonical splitting of $T_x (T M)$ into the horizontal and vertical tangent space along the zero section. Any connection $\nabla$ in $T M$ gives rise to a smooth map $\exp^\nabla: W \rightarrow M$ from some neighborhood $W$ of the zero section $M$ in $T M$; see [1, Section 1.3]. If $\eta: T M \rightarrow \mathbb{R}$ is a smooth function which equals 1 on a neighborhood of $M$ in $T M$ and 0 outside of $W$, then

$$\exp: T M \rightarrow M, \quad v \mapsto \exp^\nabla (\eta(v) v),$$

is an exponential-like map. If $M$ is compact, then $W$ can be taken to be all of $T M$ and $\exp = \exp^\nabla$.

If $(M, g, \exp)$ is a Riemannian manifold with an exponential-like map and $x \in M$, let $r_{\exp}(x) \in \mathbb{R}^+$ be the supremum of the numbers $r \in \mathbb{R}$ such that the restriction

$$\exp: \{ v \in T_x M : |v| < r \} \rightarrow M$$

is a diffeomorphism onto an open subset of $M$. Set

$$r^g_{\exp}(x) = \inf \{ d_g(x, \exp(v)) : v \in T_x M, |v| = r_{\exp}(x) \} \in \mathbb{R}^+,$$

where $d_g$ is the metric on $M$ induced by $g$. If $K \subset M$, let

$$r^g_{\exp}(K) = \inf_{x \in K} r^g_{\exp}(x);$$

this number is positive if $\overline{K} \subset M$ is compact.

3.1 Parallel transport

Let $(E, \langle \cdot, \cdot \rangle, \nabla) \rightarrow M$ be a vector bundle, real or complex, with an inner-product $\langle \cdot, \cdot \rangle$ and a metric-compatible connection $\nabla$. If $\alpha: (a, b) \rightarrow M$ is a piecewise smooth curve, denote by

$$\Pi_\alpha: E_{\alpha(a)} \rightarrow E_{\alpha(b)}$$

the parallel-transport map along $\alpha$ with respect to the connection $\nabla$. If $\exp: T M \rightarrow M$ is an exponential-like map, $x \in M$, and $v \in T_x M$, let

$$\Pi_v: E_x \rightarrow E_{\exp(v)}$$

15
be the parallel transport along the curve
\[ \gamma_v: [0, 1] \rightarrow M, \quad \gamma_v(t) = \exp(tv). \]

If \( u: [a, b] \times [c, d] \rightarrow M \) is a smooth map, let
\[ \Pi_{\partial u}: E_{u(a,c)} \rightarrow E_{u(a,c)} \]
be the parallel transport along \( u \) restricted to the boundary of the rectangle traversed in the positive direction. If \( u: \Sigma \rightarrow M \) is any smooth map, \( \nabla \) induces a connection
\[ \nabla^u: \Gamma(u; E) \rightarrow \Gamma^1(u; E) \]
in the vector bundle \( u^*E \rightarrow \Sigma \). If \( \alpha \) is a smooth curve as above and \( \zeta \in \Gamma(\alpha; E) \), let
\[ \frac{D}{dt}\zeta = \nabla^\partial_t \zeta \in \Gamma(\alpha; E), \]
where \( \partial_t \) is the standard unit vector field on \( \mathbb{R} \).

**Lemma 3.1.** If \( (M, g) \) is a Riemannian manifold and \( (E, \langle \cdot, \cdot \rangle, \nabla) \) is a normed vector bundle with connection over \( M \), for every compact subset \( K \subset M \) there exists \( C_K \in \mathbb{R}^+ \) such that for every smooth map \( u: [a, b] \times [c, d] \rightarrow M \) with \( \text{Im} \ u \subset K \)
\[ |\Pi_{\partial u} - I| \leq C_K \int_a^b \int_c^d |u_s||u_t|dsdt, \]
where the norm of \( (\Pi_{\partial u} - I) \in \text{End}(E_{u(a,c)}) \) is computed with respect to the inner-product in \( E_{u(a,c)} \).

**Proof.** (1) Choose an orthonormal frame \( \{v_i\} \) for \( E_{u(a,c)} \). Extend each \( v_i \) to
\[ \xi_i \in \Gamma(u|_{a \times [c, d]}; E) \]
by parallel-transporting along the curve \( t \rightarrow u(a, t) \) and then to \( \zeta_i \in \Gamma(u; E) \) by parallel-transporting \( \xi_i(a, t) \) along the curve \( s \rightarrow u(s, t) \); see Figure 1. By construction,
\[ \frac{D}{ds}\zeta_i = 0 \in \Gamma(u; E). \]

Let \( A \) be the matrix-valued function on \([a, b] \times [c, d]\) such that
\[ \frac{D}{dt}\zeta_i|_{(s,t)} = \sum_{l=1}^{k} A_{il}(s,t)\zeta_l(s,t), \] (3.1)
where \( k \) is the rank of \( E \). Note that \( A_{ij}(a, t) = 0 \) and
\[ \langle R_{\nabla} (u_s, u_t)\zeta_i, \zeta_j \rangle = \left\langle \frac{D}{ds}\frac{D}{dt}\zeta_i - \frac{D}{dt}\frac{D}{ds}\zeta_i, \zeta_j \right\rangle = \sum_{l=1}^{k} \left\langle \left( \frac{\partial}{\partial s} A_{il} \right) \zeta_l, \zeta_j \right\rangle = \frac{\partial}{\partial s} A_{ij}, \] (3.2)
where \( R_{\nabla} \) is the curvature tensor of the connection of \( \nabla \). Since \( K \) is compact and the image of \( u \) is contained in \( K \), it follows that
\[ |A_{ij}(b, t)| \leq C_K \int_a^b |u_s|_{(s,t)}|u_t|_{(s,t)} ds. \] (3.3)
Figure 1: Extending a basis \( \{v_i\} \) for \( E_{u(a,c)} \) to a frame \( \{\zeta_i\} \) over \([a,b] \times [c,d]\)

(2) The parallel transport of \( \zeta_i \) along the curves

\[
\tau \rightarrow u(\tau, c), \quad \tau \rightarrow u(\tau, d), \quad \tau \rightarrow u(a, \tau)
\]

is \( \zeta_i \) itself. Thus, it remains to estimate the parallel transport of each \( \zeta_i \) along the curve \( \tau \rightarrow u(b, \tau) \).

Let \( h_{ij} \) be the \( \text{SO}_k \)-valued function (\( \text{U}_k \)-valued function if \( E \) is complex) on \([c,d]\) such that

\[
h(c) = I, \quad \sum_{j=1}^{j=k} \frac{D}{dt} (h_{ij})_{(b,t)} = 0 \quad \forall \ i, t.
\]

The second equation is equivalent to

\[
\sum_{j=1}^{j=k} h_{ij}^j(t) \zeta_j(b, t) + \sum_{j=1}^{j=k} \sum_{l=1}^{l=k} h_{ij}(t) A_{jl}(b, t) \zeta_l(b, t) = 0 \quad \iff \quad h' = -hA(b, \cdot). \tag{3.4}
\]

Since (the real part of) the trace of \( (A_{ij}) \) is zero by (3.2), equation (3.4) has a unique solution in \( \text{SO}_k \) (or \( \text{U}_k \)) such that \( h(c) = I \). Furthermore, by (3.3)

\[
|h(d) - I| \leq \int_c^d |h'(t)| dt \leq \int_c^d \|A\| dt \leq k \int_c^d \int_a^b C_K |u_s| |u_t| ds dt. \tag{3.5}
\]

Since \( \Pi_{\partial \alpha} v_i = \sum_{j=1}^{j=k} h^j_{ij}(d) v_j \) by the above, the claim follows from equation (3.5).

Corollary 3.2. If \( (M, g) \) is a Riemannian manifold and \( (E, \langle \cdot, \cdot \rangle, \nabla) \) is a normed vector bundle with connection over \( M \), for every compact subset \( K \subset M \) there exists \( C_K \in \mathbb{R}^+ \) such that for every smooth closed curve \( \alpha: [a,b] \to M \) with \( \text{Im} \alpha \subset K \)

\[
|\Pi_{\alpha} - I| \leq C_K \min \left( \|\alpha\|_1, (b-a) \|\alpha\|_2^2 \right).
\]

Proof. Let \( \exp : TM \to M \) be an exponential-like map. Since the group \( \text{SO}_k \) (or \( \text{U}_k \) if \( E \) is complex) is compact and

\[
\|\alpha\|_1^2 \leq (b-a) \|\alpha\|_2^2
\]

by Hölder’s inequality, it is enough to assume that

\[
\|\alpha\|_1 \leq \min(\exp_0^a(K)/2, 1).
\]
Thus, there exists
\[ \tilde{\alpha} \in C^\infty([a, b]; T_{\alpha(a)}M) \quad \text{s.t.} \quad \alpha(t) = \exp(\tilde{\alpha}(t)), \quad |\tilde{\alpha}(t)|_{\alpha(a)} < r_{\exp}(\alpha(a)). \]
Define
\[ u: [0, 1] \times [a, b] \to K \subset M \quad \text{by} \quad u(s, t) = \exp \left( s \tilde{\alpha}(t) \right). \]
Using
\[ |\tilde{\alpha}(t)| \leq C_K d_g(\alpha(a), \alpha(t)) \leq C_K \|d\alpha\|_1, \]
\[ |\tilde{\alpha}'(t)| = \|d_{\tilde{\alpha}(t)} \exp^{-1}(\alpha'(t))\| \leq C_K |d\alpha|, \]
we find that
\[ u_s(s, t) = \{ d_{s\tilde{\alpha}(t)} \exp \} (\tilde{\alpha}(t)) \implies |u_s|_{s,t} \leq C_K' \|d\alpha\|_1; \]
\[ u_t(s, t) = s \{ d_{s\tilde{\alpha}(t)} \exp \} (\tilde{\alpha}'(t)) \implies |u_t|_{s,t} \leq C_K' |d_t\alpha|. \]
Thus, by Lemma 3.1,
\[ |\Pi_{\alpha} - \mathbb{I}| = |\Pi_{\partial u} - \mathbb{I}| \leq C_K \int_a^b \|u_s\|_{s,t} |u_t|_{s,t} dsdt \leq C_K' \|d\alpha\|_1^2 \leq C_K'(b-a) \|d\alpha\|_2^2. \]

Since \( \|d\alpha\|_1 \leq r_{\exp}(K) \), it follows that \( |\Pi_{\alpha} - \mathbb{I}| \leq C_K \|d\alpha\|_1. \)

**Corollary 3.3.** If \((M, g, \exp)\) is a Riemannian manifold with an exponential-like map and \((E, \langle \cdot, \cdot \rangle, \nabla)\) is a normed vector bundle with connection over \(M\), for every compact subset \(K \subset M\) there exists \(C_K \in C^\infty(\mathbb{R}; \mathbb{R})\) such that for all \(x \in K\) and smooth maps \(\tilde{\alpha}: (-\epsilon, \epsilon) \to T_x M\) and \(\xi: (-\epsilon, \epsilon) \to E_x\)
\[ \left| \frac{D}{dt} \left( \Pi_{\tilde{\alpha}(t)} \xi(t) \right) \right|_{t=0} - \Pi_{\tilde{\alpha}(0)} \xi'(0) \leq C_K \|\tilde{\alpha}(0)\| |\tilde{\alpha}(0)| \|\tilde{\alpha}'(0)\| \|\xi(0)\|. \] (3.8)

**Proof.** Define
\[ u: [0, 1] \times [0, \epsilon/2] \to K \subset M \quad \text{by} \quad u(s, t) = \exp \left( s \tilde{\alpha}(t) \right). \]
Let \(\{v_i\}\) be an orthonormal basis for \(E_x\). Extend each \(v_i\) to \(\zeta_i \in \Gamma(u|_{[0,1] \times t}; E)\) by parallel-transporting along the curves \(s \to u(s, t)\). If
\[ \xi(t) = \sum_{i=1}^{i=k} f_i(t) v_i, \]
where \(k\) is the rank of \(E\), then
\[ \Pi_{\tilde{\alpha}(t)} \xi(t) = \sum_{i=1}^{i=k} f_i(t) \zeta_i(1, t) \implies \]
\[ \left| \frac{D}{dt} \left( \Pi_{\tilde{\alpha}(t)} \xi(t) \right) \right|_{t=0} = \sum_{i=1}^{i=k} f_i'(0) \zeta_i(1, 0) + \sum_{i=1}^{i=k} f_i(0) \frac{D}{dt} \zeta_i(1, t) \bigg|_{t=0} \]
\[ = \Pi_{\tilde{\alpha}(0)} \xi'(0) + \sum_{i=1}^{i=k} f_i(0) \frac{D}{dt} \zeta_i(1, t) \bigg|_{t=0}. \] (3.9)
On the other hand, by (3.1), (3.3), and the first identities in (3.6) and (3.7),
\[
\left| \frac{d}{dt} \zeta(1,t) \right|_{t=0} = \sum_{j=1}^{j=k} |A_{ij}(1,0)| \leq kC_K'(\|\tilde{\alpha}(0)\|) \int_{0}^{1} |u_i|_{t=0} |u_j|_{t=0} ds \\
\leq C_K(\|\tilde{\alpha}(0)\|)\|\tilde{\alpha}(0)\|. \tag{3.10}
\]
The claim follows from (3.9) and (3.10).

**Remark 3.4.** Note that (3.3) is applied above with $K$ replaced by the compact set
\[
\exp \left\{ \{ v \in T_x M : x \in K, |v| \leq |\tilde{\alpha}(0)| \} \right\}.
\]
Thus, the constants $C_K'(\|\tilde{\alpha}(0)\|)$ and $C_K(\|\tilde{\alpha}(0)\|)$ may depend on $|\tilde{\alpha}(0)|$. If $M$ is compact, then the first constant does not depend on $|\tilde{\alpha}(0)|$, since (3.3) can then be applied with $K=M$. The second constant is then also independent of $K$ and $|\tilde{\alpha}(0)|$ if $\exp = \exp^\nabla$ for some connection $\nabla$ in $TM$. So, in this case, the function $C_K$ in (3.8) can be taken to be a constant independent of $K$.

### 3.2 Poincaré lemmas

**Lemma 3.5.** If $\zeta : S^1 \to \mathbb{R}^k$ is a smooth function such that $\int_0^{2\pi} \zeta(\theta) d\theta = 0$,
\[
\int_0^{2\pi} |\zeta(\theta)|^2 d\theta \leq \int_0^{2\pi} |\zeta'(\theta)|^2 d\theta.
\]

**Proof.** Write
\[
\zeta(\theta) = \sum_{n < \infty} c_n e^{in\theta};
\]
see [6, Section 6.16]. Since $\zeta$ integrates to 0, $\zeta_0 = 0$. Thus,
\[
\int_0^{2\pi} |\zeta(\theta)|^2 d\theta = 2\pi \sum_{n < \infty} |c_n|^2 \leq 2\pi \sum_{n > -\infty} |n c_n|^2 = \int_0^{2\pi} |\zeta'(\theta)|^2 d\theta,
\]
as claimed. \qed

**Proposition 3.6.** If $(M, g)$ is a Riemannian manifold and $(E, \langle \cdot, \cdot \rangle, \nabla)$ is a normed vector bundle with connection over $M$, for every compact subset $K \subset M$ there exists $C_K \in \mathbb{R}^+$ with the following property. If $\alpha \in C^\infty(S^1; M)$ is such that $\text{Im} \alpha \subset K$ and $\xi, \zeta \in \Gamma(\alpha; E)$, then
\[
|\langle \nabla_\theta \xi, \zeta \rangle| \leq \|\nabla_\theta \xi\|_2 \|\nabla_\theta \zeta\|_2 + C_K \min \left( \|d\alpha\|_1, \|d\alpha\|_2 \right) \|\xi\|_2 \|\zeta\|_2,
\]
where $\nabla_\theta \equiv \nabla_{\partial_\theta}$ is the covariant derivative with respect to the oriented unit field on $S^1$ and all the norms are computed with respect to the standard metric on $S^1$.

**Proof.** Identify $E_{\alpha(0)}$ with $\mathbb{R}^k$ (or $\mathbb{C}^k$), preserving the metric. Denote by $\text{so}(E_{\alpha(0)}) \approx \text{so}_k$ (or $\text{u}(E_{\alpha(0)}) \approx \text{u}_k$) the Lie algebra of the Lie group $\text{SO}(E_{\alpha(0)}) \approx \text{SO}_k$ (or of $U(E_{\alpha(0)}) \approx \text{U}_k$). For each $\chi \in \text{so}(E_{\alpha(0)})$ (or $\chi \in \text{u}(E_{\alpha(0)})$), let $e^\chi \in \text{SO}(E_{\alpha(0)})$ (or $e^\chi \in \text{U}(E_{\alpha(0)})$) be the exponential of $\chi$. Let $\Pi_\theta : E_{\alpha(0)} \to E_{\alpha(\theta)}$
be the parallel transport along the curve \( t \to \alpha(t) \) with \( t \in [0, \theta] \). By Corollary 3.2, there exists \( \chi \in so(E_\alpha(0)) \) (or \( \chi \in u(E_\alpha(0)) \)) such that

\[
\Pi_{2\pi} = e^\chi \quad \text{and} \quad |\chi| \leq C_K \min \left( \|d\alpha\|_1, \|d\alpha\|_2^2 \right).
\]  

(3.11)

By the first statement in (3.11),

\[
\Psi: S^1 \times E_\alpha(0) \to \alpha^*E, \quad (\theta, v) \to e^{-\theta\chi/2\pi} \Pi_\theta(v),
\]

is a smooth isometry. Let \( \Phi_2 = \pi_2 \circ \Psi^{-1}: \alpha^*E \to E_\alpha(0) \) and

\[
\tilde{\zeta} = \frac{1}{2\pi} \int_0^{2\pi} \{\Phi_2 \zeta\}(\theta)d\theta \in E_\alpha(0).
\]

By Hölder’s inequality and Lemma 3.5,

\[
|\langle \nabla_\theta \xi, \zeta - \Psi \tilde{\zeta} \rangle| \leq \|\nabla_\theta \xi\|_2 \|\zeta - \Psi \tilde{\zeta}\|_2 = \|\nabla_\theta \xi\|_2 \|\Phi_2 \zeta - \tilde{\zeta}\|_2 \leq \|\nabla_\theta \xi\|_2 \|d(\Phi_2 \zeta)\|_2.
\]

(3.12)

By the product rule,

\[
\|d(\Phi_2 \zeta)\|_2 \leq \|d(\Pi^{-1}_\theta \zeta)\|_2 + |\chi/2\pi| \|\Pi^{-1}_\theta \zeta\|_2 = \|\nabla_\theta \zeta\|_2 + |\chi/2\pi| \|\Pi^{-1}_\theta \zeta\|_2 \leq \|\nabla_\theta \zeta\|_2 + C_K \min \left( \|d\alpha\|_1, \|d\alpha\|_2^2 \right) \|\zeta\|_2.
\]

(3.13)

On the other hand, by integration by parts, we obtain

\[
\langle \nabla_\theta \xi, \zeta - \Psi \tilde{\zeta} \rangle = \langle \nabla_\theta \xi, \zeta \rangle + \langle \xi, \nabla_\theta (\Psi \tilde{\zeta}) \rangle.
\]

(3.14)

Since \( \Psi \tilde{\zeta} \) is the parallel transport of \( e^{\theta\chi/2\pi} \tilde{\zeta} \),

\[
|\langle \xi, \nabla_\theta (\Psi \tilde{\zeta}) \rangle| \leq \|\xi\|_2 \|\nabla_\theta (\Psi \tilde{\zeta})\|_2 = \|\xi\|_2 \|\chi/2\pi\| \|\Psi \tilde{\zeta}\|_2 \leq C_K \min \left( \|d\alpha\|_1, \|d\alpha\|_2^2 \right) \|\zeta\|_2 \|\xi\|_2.
\]

(3.15)

The claim follows from equations (3.12)-(3.15). \( \square \)

Let \( B_{R_\rho} \subset \mathbb{R}^2 \) denote the open annulus with radii \( r < R \) centered at the origin.

**Corollary 3.7** (of Lemma 3.5). There exists \( C \in C^\infty(\mathbb{R}; \mathbb{R}) \) such that for all \( R \in \mathbb{R}^+ \)

\[
r \in (0, R], \quad \zeta \in C^\infty(B_{R_\rho}; \mathbb{R}^k), \quad \int_{B_{R_\rho}} \zeta = 0 \quad \implies \quad \|\zeta\|_1 \leq C(R/r)R^2 \|d\zeta\|_2.
\]

Proof. It is sufficient to assume that \( k = 1 \). Define

\[
\xi: S^1 \to \mathbb{R} \quad \text{by} \quad \xi(\theta) = \int_r^R \zeta(\rho, \theta) \rho d\rho.
\]

By Hölder’s inequality and Lemma 3.5,

\[
\left( \int_0^{2\pi} \left| \int_r^R \zeta(\rho, \theta) \rho d\rho \right| d\theta \right)^2 \leq 2\pi \int_0^{2\pi} |\xi(\theta)|^2 d\theta \leq 2\pi \int_0^{2\pi} |\xi'\theta(\theta)|^2 d\theta \leq 2\pi \int_0^{2\pi} \left( \int_r^R |d(\rho, \theta)\zeta(\rho^2 d\rho) \right)^2 d\theta \leq \frac{\pi R^4}{2} \int_0^{2\pi} |d(\rho, \theta)\zeta(\rho^2 d\rho) \|d\zeta\|_2.
\]

(3.16)
If the function \( \rho \rightarrow \zeta(\rho, \theta) \) does not change sign on \((r, R)\), then
\[
\int_r^R |\zeta(\rho, \theta)| \rho d\rho = \int_r^R \zeta(\rho, \theta) \rho d\rho.
\]
On the other hand, if this function vanishes somewhere on \((r, R)\), then
\[
|\zeta(\rho, \theta)| \leq \int_r^R |d_{(t, \theta)} \zeta| dt \quad \forall \rho \implies \int_r^R |\zeta(\rho, \theta)| \rho d\rho \leq \frac{R^2}{2} \int_r^R |d_{(t, \theta)} \zeta| dt.
\]
Combining these two cases and using (3.16) and Hölder’s inequality, we obtain
\[
\int_0^{2\pi} \int_r^R |\zeta(\rho, \theta)| \rho d\rho d\theta \leq \int_0^{2\pi} \int_r^R \zeta(\rho, \theta) \rho d\rho d\theta + \frac{R^2}{2} \int_0^{2\pi} \int_r^R |d_{(\rho, \theta)} \zeta| \rho d\rho d\theta
\leq \frac{\sqrt{\pi} R^2}{\sqrt{2}} \|d\zeta\|_2 + \frac{R^2}{2} \|d\zeta\|_2 \left( \int_0^{2\pi} \int_r^R \rho^{-1} d\rho d\theta \right)^{1/2} \tag{3.17}
= \sqrt{\frac{\pi}{2}} \left( 1 + \sqrt{\ln(R/r)} \right) R^2 \|d\zeta\|_2,
\]
as claimed. \(\square\)

**Remark 3.8.** By Corollary 4.7 below, \(C\) can in fact be chosen to be a constant function. Corollary 3.7 suffices for gluing \(J\)-holomorphic maps in symplectic topology, but Corollary 4.7 leads to a sharper version of Proposition 4.14; see Remark 4.13.

### 3.3 Exponential-like maps and differentiation

Let \((M, g, \text{exp}, \nabla)\) be a smooth Riemannian manifold with an exponential-like map \(\text{exp}\) and connection \(\nabla\) in \(TM\), which is \(g\)-compatible, but not necessarily torsion-free. Let
\[
T_{\nabla}(\xi(x), \zeta(x)) \equiv (\nabla \xi \zeta - \nabla \zeta \xi - [\xi, \zeta])_{|x} \quad \forall x \in M, \xi, \zeta \in \Gamma(M; TM),
\]
be the torsion tensor of \(\nabla\). If \(\alpha: (-\epsilon, \epsilon) \rightarrow M\) is a smooth curve and \(\xi \in \Gamma(\alpha; TM)\), put
\[
\Phi_{\alpha(0)}(\alpha'(0); \xi(0); \frac{D}{ds} \xi_{|s=0}) = \Pi^{-1}_{\xi(0)} \left( \frac{d}{ds} \exp (\xi(s))_{|s=0} \right) = \Pi^{-1}_{\xi(0)} (\{d\xi(0) \exp\}(\xi'(0))),
\]
where \(\xi'(0) \in T_{\xi(0)}(TM)\) is the tangent vector to the curve \(\xi: (-\epsilon, \epsilon) \rightarrow TM\) at \(s = 0\).

**Lemma 3.9.** If \((M, g, \text{exp}, \nabla)\) is a smooth Riemannian manifold with an exponential-like map and a \(g\)-compatible connection, there exists \(C \in C^\infty(TM; \mathbb{R})\) such that
\[
|\Phi_x(v; w_0, w_1) - (v + w_1 - T_{\nabla}(v, w_0))| \leq C(w_0)(|v||w_0|^2 + |w_0||w_1|)
\]
for all \(x \in M\) and \(v, w_0, w_1 \in T_x M\).

**Proof.** Let \(\alpha: (-\epsilon, \epsilon) \rightarrow M\) be a smooth curve and \(\xi \in \Gamma(\alpha; TM)\) such that
\[
\alpha(0) = x, \quad \alpha'(0) = v, \quad \xi(0) = w_0, \quad \frac{D}{ds} \xi(s)_{|s=0} = w_1.
\]
Put
\[ F_{v,w_0,w_1}(t) = \frac{d}{ds} \exp (t\xi(s)) \bigg|_{s=0} = \{dtw_0,\exp\} \{d_{w_0}m_t(\xi'(0))\}, \]
\[ H_{v,w_0,w_1}(t) = \Pi_{tw_0}(v + tw_1 - tT_v(v, w_0)), \]
where \( m_t : TM \rightarrow TM \) is the scalar multiplication by \( t \). Then,
\[ F_{v,w_0,w_1}(0) = \frac{d}{ds}\alpha(s) \bigg|_{s=0} = v = H_{v,w_0,w_1}(0), \]
\[ \frac{D}{dt}F_{v,w_0,w_1}(t) \bigg|_{t=0} = \frac{D}{ds}\frac{d}{dt} \exp (t\xi(s)) \bigg|_{t=0} = -T_v(v, w_0) = w_1 - T_v(v, w_0) = \frac{D}{dt}H_{v,w_0,w_1}(t) \bigg|_{t=0}; \]
see Corollary 3.3. Since
\[ F_{v,w_0,(t)} - H_{v,w_0,(t)} \in \text{Hom}(T_xM \oplus T_xM, T_{\exp(tw_0)}M), \]
combining the last two equations, we obtain
\[ |F_{v,w_0,w_1}(t) - H_{v,w_0,w_1}(t)| \leq C(w_0, t^2(|v| + |w_1|)) \quad \forall \, v, w_0, w_1 \in T_xM, \, x \in M, \, t \in \mathbb{R}, \]
where \( C \) is a smooth function on \( TM \times \mathbb{R} \). Since
\[ F_{v,w_0,w_1}(t) - H_{v,w_0,w_1}(t) = F_{v,tw_0,tw_1}(1) - H_{v,tw_0,tw_1}(1), \]
we conclude that there exists \( C \in C^\infty(TM) \) such that
\[ |F_{v,w_0,w_1}(1) - H_{v,w_0,w_1}(1)| \leq C(w_0)(|w_0|^2|v| + |w_0||w_1|) \quad \forall \, v, w_0, w_1 \in T_xM, \, x \in M, \, (3.18) \]
as claimed.

For any \( v, w_0, w_1 \in T_xM, \) let \( \Phi_x(v; w_0, w_1) = \Phi_x(v; w_0, w_1) - (v + w_1 - T_v(v, w_0)). \)

**Corollary 3.10.** If \( (M, g, \exp, \nabla) \) is a smooth Riemannian manifold with an exponential-like map and a \( g \)-compatible connection, there exists \( C \in C^\infty(TM \times_M TM; \mathbb{R}) \) such that
\[ |\Phi_x(v; w_0, w_1) - \Phi_x(v; w_0', w_1')| \leq C(w_0, w_0') \left( ((|w_0| + |w_0'|)|v| + |w_1| + |w_1'|)|w_0 - w_0'| + (|w_0| + |w_0'|)|w_1 - w_1'| \right) \]
for all \( x \in M \) and \( v, w_0, w_1, w_0', w_1' \in T_xM. \)

**Proof.** By the proof of Lemma 3.9,
\[ \Phi(v; w_0, w_1) = \Phi_1(w_0; v) + \Phi_2(w_0; w_1) \]
for some smooth bundle sections \( \Phi_1, \Phi_2 : TM \rightarrow \pi_TM \text{Hom}(TM, TM) \) such that
\[ |\Phi_1(w_0; \cdot)| \leq C_1(w_0)|w_0|^2, \quad |\Phi_2(w_0; \cdot)| \leq C_2(w_0)|w_0| \quad \forall \, w_0 \in TM. \]
Thus,
\[ |\Phi_1(w_0; \cdot) - \Phi_1(w_0'; \cdot)| \leq C_1(w_0, w_0')(|w_0| + |w_0'|)|w_0 - w_0'| \quad \forall \, w_0, w_0' \in T_xM. \]
\[ |\Phi_2(w_0; \cdot) - \Phi_2(w_0'; \cdot)| \leq C_2(w_0, w_0')|w_0 - w_0'| \quad \forall \, w_0, w_0' \in T_xM. \]
From the linearity of \( \Phi_1(w_0; \cdot) \) and \( \Phi_2(w_0; \cdot) \) in the second input, we conclude that
\[ |\Phi_1(w_0; v) - \Phi_1(w_0'; v)| \leq C_1(w_0, w_0')(|w_0| + |w_0'|)|w_0 - w_0'||v|, \]
\[ |\Phi_2(w_0; w_1) - \Phi_2(w_0'; w_1)| \leq C_2(w_0, w_0')(|w_0 - w_0'||w_1| + C_2(w_0)|w_0'||w_1 - w_1'|. \]
This establishes the claim. \( \square \)
3.4 Expansion of the $\overline{\partial}$-operator

Let $(M, J)$ and $(\Sigma, i)$ be almost-complex manifolds. If $u : \Sigma \to M$ is a smooth map, let
\[
\Gamma(u) = \Gamma(\Sigma; u^*TM), \quad \Gamma_{J,i}^{0,1}(u) = \Gamma(\Sigma; T^*\Sigma^{0,1} \otimes \mathbb{C} u^*TM),
\]
\[
\overline{\partial}_{J,i}u = \frac{1}{2}(du + J \circ du \circ j) \in \Gamma_{J,i}^{0,1}(u),
\]
as in (2.5). If $\nabla$ is a connection in $TM$, define
\[
D_{J,i}^\nabla : \Gamma(u) \to \Gamma_{J,i}^{0,1}(u) \quad \text{by} \quad D_{J,i}^\nabla u = \frac{1}{2}(\nabla u^* \xi + J \nabla_i^u \xi) - \frac{1}{2}(\nabla (du, \xi) + J \nabla (du \circ i, \xi)).
\]
If in addition $\exp : TM \to M$ is an exponential-like map and $\nabla J = 0$, define
\[
\exp, \Gamma(u) \to C^\infty(\Sigma; M), \quad \overline{\partial}_u, N_{\exp}^\nabla : \Gamma(u) \to \Gamma_{J,i}^{0,1}(u) \quad \text{by} \quad \{\exp_u(\xi)\}_z = \exp(\xi(z)) \quad \forall z \in \Sigma, \quad \{\overline{\partial}_u \xi\}_z(v) = \Pi^{-1}_z(\{\overline{\partial}_{J,i}(\exp_u(\xi))\}_z(v)) \quad \forall z \in \Sigma, \ v \in T_z \Sigma,
\]
\[
\overline{\partial}_u \xi = \overline{\partial}_{J,i}u + D_{J,i}^\nabla u \xi + N_{\exp}^\nabla(\xi).
\]

Lemma 3.11. If $(M, J, g, \exp, \nabla)$ is an almost-complex Riemannian manifold with an exponential-like map and a $g$-compatible connection in $(TM, J)$, there exists $C \in C^\infty(TM \times_M TM; \mathbb{R})$ with the following property. If $(\Sigma, i)$ is an almost complex manifold, $u : \Sigma \to M$ is a smooth map, and $\xi, \xi' \in \Gamma(\Sigma)$, then
\[
\left| \left\{ N_{\exp}^\nabla(\xi) \right\}_z(v) - \left\{ N_{\exp}^\nabla(\xi') \right\}_z(v) \right| \leq C(\xi(z), \xi'(z)) \left( |\xi(z)| + |\xi'(z)| \right) \left\{ |\nabla u^{\xi}(\xi - \xi')| + |\nabla v^{\xi}(\xi - \xi')| \right\}
+ \left( |d_z u(v)| + |d_z u(v')| \right) \left( |\xi(z)| + |\xi'(z)| \right) + \left( |\nabla u^{\xi}(\xi) + |\nabla v^{\xi}(\xi)| \right) \left( \xi(z) - \xi'(z) \right)
\]
for all $z \in \Sigma, v \in T_z \Sigma$. Furthermore, $N_{\exp}^\nabla(0) = 0$.

Proof. Since the connection $\nabla$ commutes with $J$, so does the parallel transport $\Pi$. Thus, with notation as in Section 3.3,
\[
\left\{ N_{\exp}^\nabla(\xi) \right\}_z(v) = \frac{1}{2} \left( \Phi(d_z u(v); \xi(z), \nabla u \xi) + J(u(z)) \tilde{\Phi}(d_z u(v); \xi(z), \nabla_u \xi) \right).
\]
The claim now follows from Corollary 3.10. \hfill \Box

Definition 3.12. Let $M$ be a smooth manifold and $(\mathcal{E}, \langle \cdot , \cdot \rangle, \nabla)$ a normed vector bundle with connection over $M$. If $C_0 \in \mathbb{R}^+$, $(\Sigma, i)$ is an almost complex manifold, and $u : \Sigma \to M$ is a smooth map, norms $\| \cdot \|_p$ and $\| \cdot \|_p$ on $\Gamma(\Sigma; \mathcal{E})$ and $\Gamma^1(\Sigma; \mathcal{E})$, respectively, are $C_0$-admissible if for all $\xi \in \Gamma(\Sigma; \mathcal{E})$, $\eta \in \Gamma^1(\Sigma; \mathcal{E})$, and every continuous function $f : \Sigma \to \mathbb{R}$,
\[
\| f \eta \|_p \leq \| f \|_{C^0} \| \eta \|_p, \quad \| \eta \circ i \|_p = \| \eta \|_p, \quad \| \nabla u^\xi \eta \|_p \leq \| \xi \|_{p,1} + \| \xi \|_{C^0} \leq C_0 \| \xi \|_{p,1}.
\]

Proposition 3.13. If $(M, J, g, \exp, \nabla)$ is an almost-complex Riemannian manifold with an exponential-like map and a $g$-compatible connection in $(TM, J)$, for every compact subset $K \subset M$ there exists $C_K \in C^\infty(\mathbb{R}^+; \mathbb{R})$ with the following property. If $(\Sigma, i)$ is an almost complex manifold, $u : \Sigma \to K$ is a smooth map, and $\| \cdot \|_{p,1}$ and $\| \cdot \|_p$ are $C_0$-admissible norms on $\Gamma(\Sigma; TM)$ and $\Gamma^1(\Sigma; TM)$, respectively, then
\[
\| N_{\exp}^\nabla(\xi) - N_{\exp}^\nabla(\xi') \|_{p,1} \leq C_K(\| du \|_p + \| \xi \|_{p,1} + \| \xi' \|_{p,1}) \| \xi - \xi' \|_{p,1}
\]
for all $\xi, \xi' \in \Gamma(\Sigma)$. Furthermore, $N_{\exp}^\nabla(0) = 0$. If the $g$-ball $B_{g,\delta}(u(z))$ of radius $\delta$ around $f(z)$ for some $z \in \Sigma$ is isomorphic to an open subset of $\mathbb{C}^n$ and $|\xi(z)| < \delta$, then $\{ N_{\exp}^\nabla \}_{z} = 0$. 

23
Remark 3.14. As the notation suggests, one possibility for the norms $\| \cdot \|_{p,1}$ and $\| \cdot \|_p$ is the usual Sobolev $L^p$ and $L^p$-norms with respect to some Riemannian metric on $\Sigma$, where $p > \dim \Sigma$. Another natural possibility in the $\dim \Sigma = 2$ case is the modified Sobolev norms introduced in [3, Section 3]; these are particularly suited for gluing pseudo-holomorphic curves. By Proposition 4.10 below, in the $\dim \Sigma = 2$ case the constant $C_0$ itself is a function of $\|du\|_p$ only for either of these two choices of norms.

Remark 3.15. By Proposition 3.13, the operator $D_{\bar{\partial}u}^{\nabla}$ defined above is a linearization of the $\bar{\partial}$-operator on the space of smooth maps to $M$ at $u$. If $\nabla'$ is any connection in $TM$, the connection

$$\nabla : \Gamma(M; TM) \to \Gamma(M; T^*M \otimes \mathbb{R}TM), \quad \nabla_v \xi = \frac{1}{2} \left( \nabla'_v \xi - J \nabla'_v (J \xi) \right) \quad \forall v \in TM, \xi \in \Gamma(M; TM),$$

is $J$-compatible. If in addition $\nabla'$ and $J$ are compatible with a Riemannian metric $g$ on $M$, then so is $\nabla$. If $\nabla'$ is also the Levi-Civita connection of the metric $g$ (i.e. $T_{\nabla'} = 0$),

$$T_{\nabla} (v, w) = \frac{1}{2} (J(\nabla'_v J)v - J(\nabla'_v J)w) \quad \forall v, w \in T_x M, x \in M.$$

If the 2-form $\omega(\cdot, \cdot) \equiv g(J \cdot, \cdot)$ is closed as well, then

$$\nabla_{\nabla'v} J = -J \nabla_v J \quad \forall v \in TM$$

by [4, (C.7.5)] and thus

$$T_{\nabla} (v, w) = -\frac{1}{4} \left( J(\nabla'_v J)w - J(\nabla'_v J)v - (\nabla'_{Jv} J)w + (\nabla'_{Jw} J)v \right) = -A_J(v, w) \quad \forall v, w \in T_x M, x \in M,$$

where $A_J$ is the Nijenhuis tensor of $J$ as in (2.27). The operator $D_{\bar{\partial}u}^{\nabla}$ then becomes

$$D_{\bar{\partial}u}^{\nabla} : \Gamma(u) \to \Gamma_{0,1}^{\partial,\bar{\partial}u} (u), \quad D_{\bar{\partial}u}^{\nabla} \xi = \bar{\partial} \nabla_u \xi + A_J(\partial_{\bar{\partial}u}, \xi),$$

where

$$\bar{\partial} \nabla_u \xi = \frac{1}{2} \left( \nabla_u \xi + J \nabla_u \xi \right) \in \Gamma_{0,1}^{\partial,\bar{\partial}u} (u),$$

$$\partial_{\bar{\partial}u} = \frac{1}{2} (du - J \circ du \circ j) \in \Gamma(\Sigma; T^*\Sigma \otimes C u^* TM).$$

This agrees with [4, (3.1.5)], since the Nijenhuis tensor of $J$ is defined to be $-4A_J$ in [4, p18].

4 Sobolev and elliptic inequalities

This appendix refines, in the $n = 2$ case, the proofs of Sobolev Embedding Theorems given in [5] to obtain a $C^0$-estimate in Proposition 4.10 and elliptic estimates for the $\bar{\partial}$-operator in Propositions 4.14 and 4.16. If $R, r \in \mathbb{R}$, let

$$B_R = \{ x \in \mathbb{R}^2 : |x| < R \}, \quad B_R = B_R - \bar{B}_r, \quad \bar{B}_{R,r} = B_R - B_r.$$
4.1 Euclidean case

If \( \xi \) is an \( \mathbb{R}^k \)-valued function defined on a subset \( B \) of \( \mathbb{R}^2 \), let \( \text{supp}_{\mathbb{R}^2}(\xi) \) be the closure of \( \text{supp}(\xi) \subset B \) in \( \mathbb{R}^2 \). If \( U \) is an open subset of \( \mathbb{R}^2 \), \( \xi \in C^\infty(U; \mathbb{R}^k) \), and \( p \geq 1 \), let

\[
\| \xi \|_p \equiv \left( \int_U |\xi|^p \right)^{1/p}, \quad \| \xi \|_{p,1} \equiv \| \xi \|_p + \| d\xi \|_p,
\]

be the usual Sobolev norms of \( \xi \).

Lemma 4.1. For every bounded convex domain \( D \subset \mathbb{R}^2 \), \( \xi \in C^\infty(D; \mathbb{R}^k) \), and \( x \in D \),

\[
|\xi_D - \xi(x)| \leq 2r_0 \frac{\| \xi \|_p}{|D|} \int_D |d\xi||y-x|^{-1}dy,
\]

where \( 2r_0 \) is the diameter of \( D \), \( |D| \) is the area of \( D \), and

\[
\xi_D = \frac{1}{|D|} \left( \int_D \xi(y)dy \right)
\]

is the average value of \( \xi \) on \( D \).

Proof. For any \( y \in D \),

\[
\xi(y) - \xi(x) = \int_0^1 \frac{d}{dt} \xi(x + t(y-x))dt = \int_0^1 d_{x + t(y-x)} \xi(y-x)dt.
\]

Putting \( g(z) = |d\xi| \) if \( \xi \in D \) and \( g(z) = 0 \) otherwise, we obtain

\[
|\xi_D - \xi(x)| \leq \frac{1}{|D|} \int_{y \in D} |\xi(y) - \xi(x)|dy \leq \frac{1}{|D|} \int_{y \in D} \int_0^\infty g(x + t(y-x))|y-x|dtdy.
\]

Rewriting the last integral in polar coordinates \((r, \theta)\) centered at \( x \), we obtain

\[
|\xi_D - \xi(x)| \leq \frac{1}{|D|} \int_0^{2\pi} \int_0^{2r_0} \int_0^\infty g(tr, \theta)r^2dtdr \theta = \frac{1}{|D|} \int_0^{2\pi} \int_0^{2r_0} g(t, \theta)rdr \theta
\]

\[
= \frac{2r_0^2}{|D|} \int_D |d\xi||y-x|^{-1}dy.
\]

This establishes the claim.

Corollary 4.2. For every \( p > 2 \), there exists \( C_p > 0 \) such that

\[
\forall x, y \in B_{R,r}, \quad |\xi(x) - \xi(y)| \leq C_p R^{\frac{n-2}{p}} \| d\xi \|_p
\]

Proof. For any \( x \in B_{R,r} \), put

\[
D_x = \{ y \in B_{R,r} : \langle x, |x+y-rx| \rangle > 0 \}.
\]
If \( x \neq 0 \), \( D_x \) is the part of the annulus on the same side of the line \( \langle x, y - rx/|x| \rangle = 0 \) as \( x \); see Figure 2. In particular,

\[
\text{diam}(D_x) \leq 2R, \quad |D_x| \geq \left( \frac{\pi}{3} - \frac{\sqrt{3}}{4} \right) R^2.
\]

Thus, by Lemma 4.1 and Hölder’s inequality,

\[
|\xi(x) - \xi_{D_x}| \leq 12 \int_{y \in D_x} |d_y \xi||y - x|^{-1} dy \\
\leq 12 \left( \int_{y \in B_{2R}(x)} |y - x|^{-\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \|d\xi\|_p \leq C_p R^{\frac{p-2}{p}} \|d\xi\|_p,
\]

since \( \frac{p}{p-1} < 2 \). Let

\[
x_{\pm} = \left( \pm \frac{(R-r)}{2}, 0 \right), \quad y_{\pm} = (0, \pm \frac{(R-r)}{2}).
\]

Since each of the convex regions \( D_{x_{\pm}} \) intersects \( D_{y_{\pm}} \) and \( D_{y_{-}} \) and \( D_x \) intersects at least one (in fact precisely two if \( r \neq 0 \)) of these four convex regions for every \( x \in B_{R,r} \),

\[
|\xi(x) - \xi(y)| \leq 8C_p R^{-\frac{p-2}{p}} \|\xi\|_p \quad \forall x, y \in B_{R,r}
\]

by (4.1) and triangle inequality.

\( \square \)

**Corollary 4.3.** For every \( p > 2 \), there exists \( C_p \in \mathcal{C}^\infty(\mathbb{R}^+; \mathbb{R}) \) such that

\[
r \in [0, R/2], \quad \xi \in \mathcal{C}^\infty(B_{R,r}; \mathbb{R}^k) \quad \Rightarrow \quad \|\xi\|_{C^0} \leq C_p(R)\|\xi\|_{p,1}.
\]

**Proof.** By Corollary 4.2 and Hölder’s inequality, for every \( x \in B_{R,r} \)

\[
|\xi(x)| \leq |\xi_{B_{R,r}}| + C_p R^{-\frac{p-2}{p}} \|d\xi\|_p \leq \frac{1}{|B_{R,r}|} \|\xi\|_1 + C_p R^{-\frac{p-2}{p}} \|d\xi\|_p \\
\leq |B_{R,r}|^{-\frac{1}{p}} \|\xi\|_p + C_p R^{-\frac{p-2}{p}} \|d\xi\|_p \leq (1+C_p)R^{-\frac{2}{p}} \left( \|\xi\|_p + R \|d\xi\|_p \right).
\]

This implies the claim.

\( \square \)
Lemma 4.4. For all $R > 0$ and $r \in [0, R)$,
\[ \zeta \in C^\infty(B_{R,r}; \mathbb{R}^k), \quad \text{supp}_{\mathbb{R}^2}(\zeta) \subset \overline{B}_{R,r} \quad \implies \quad \|\zeta\|_2 \leq \|d\zeta\|_1. \]

Proof. Such a function $\zeta$ can be viewed as a function on the complement of the ball $B_r$ in $\mathbb{R}^2$. Since $\zeta$ vanishes at infinity, for any $(x, y) \in B_{R,r}$,
\[ \zeta(x, y) = \begin{cases} \int_{-\infty}^{x} \zeta(s, y) \, ds, & \text{if } x \leq 0; \\ -\int_{x}^{\infty} \zeta(s, y) \, ds, & \text{if } x \geq 0; \end{cases} \quad \zeta(x, y) = \begin{cases} \int_{-\infty}^{y} \zeta(x, t) \, dt, & \text{if } y \leq 0; \\ -\int_{y}^{\infty} \zeta(x, t) \, dt, & \text{if } y \geq 0. \end{cases} \]

Taking the absolute value in these equations, we obtain
\[ |\zeta(x, y)| \leq \int_{-\infty}^{\infty} |d_{(s,y)}\zeta| \, ds \quad \text{and} \quad |\zeta(x, y)| \leq \int_{-\infty}^{\infty} |d_{(x,t)}\zeta| \, dt, \quad (4.3) \]
where we formally set $\zeta$ and $d\zeta$ to be zero on the smaller disk. Multiplying the two inequalities in (4.3) and integrating with respect to $x$ and $y$, we conclude
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\zeta(x, y)|^2 \, dx \, dy \leq \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |d_{(x,y)}\zeta| \, dx \, dy \right)^2, \]
as claimed. \qed

Corollary 4.5. For all $p, q \geq 1$ with $1 - 2/p \geq -2/q$, there exists $C_{p,q} \in \mathbb{R}^+$ such that
\[ r \in [0, R), \quad \xi \in C^\infty(B_{R,r}; \mathbb{R}^k), \quad \text{supp}_{\mathbb{R}^2}(\xi) \subset \overline{B}_{R,r} \quad \implies \quad \|\xi\|_q \leq C_{p,q} R^{1 - \frac{2}{p} + \frac{2}{q}} \|d\xi\|_p. \]

Proof. We can assume that $k = 1$. For $\epsilon > 0$, let $\zeta_\epsilon = (\xi^2 + \epsilon)^{\frac{q}{2}} - \epsilon^{\frac{q}{2}}$. By Lemma 4.4 and Hölder’s inequality,
\[ \|\xi\|_q^2 \leq \|\zeta_\epsilon + \epsilon^{\frac{q}{2}}\|_2^2 \leq 2\|d\zeta_\epsilon\|_1^2 + 2\epsilon^{\frac{q}{2}} \pi R^2 = 2\left( \left( \xi^2 + \epsilon \right)^{\frac{q}{2}} - 1 \xi \, d\xi \right)_1^2 + 2\epsilon^{\frac{q}{2}} \pi R^2 \leq q^2 \left( \xi^2 + \epsilon \right)^{\frac{q}{2}} \, d\xi_1^2 + 2\epsilon^{\frac{q}{2}} \pi R^2 \leq q^2 \|d\xi\|_p^2 \left( \xi^2 + \epsilon \right)^{\frac{q}{2}} \|d\xi\|_p^2 + 2\epsilon^{\frac{q}{2}} \pi R^2. \quad (4.4) \]

Note that
\[ 1 - \frac{2}{p} = -\frac{2}{q} \quad \implies \quad q - \frac{2}{p} = \frac{q - 2}{q - 2} = \frac{q}{2}. \]
Thus, letting $\epsilon$ go to zero in (4.4), we obtain
\[ \|\xi\|_q^2 \leq q^2 \|d\xi\|_p^2 \|\xi\|_q^{q-2} \implies \|\xi\|_q \leq q \|d\xi\|_p. \]
The case $1 - \frac{2}{p} > -\frac{2}{q}$ follows by Hölder’s inequality. \qed

Remark 4.6. By Hölder’s inequality, the constant $C_{p,q}$ can be taken to be
\[ C_{p,q} = \max(2, q) \pi^{\frac{1}{2}} \left( 1 - \frac{2}{p} + \frac{2}{q} \right). \]

Corollary 4.7 (of Lemmas 4.1, 4.4). There exists $C > 0$ such that for all $R \in \mathbb{R}^+$
\[ r \in [0, R), \quad \zeta \in C^\infty(B_{R,r}; \mathbb{R}^k), \quad \int_{B_{R,r}} \zeta = 0 \quad \implies \quad \|\zeta\|_1 \leq C R^2 \|d\zeta\|_2. \]
Proof. (1) If $\zeta \in C^\infty(B_{R,r}; \mathbb{R}^k)$ integrates to 0 over its domain, then so does the function

$$\tilde{\zeta} \in C^\infty(B_{1,r}/R; \mathbb{R}^k), \quad \tilde{\zeta}(z) = \zeta(Rz).$$

Furthermore, $\|\tilde{\zeta}\|_1 = \|\zeta\|_1/R^2$ and $\|d\tilde{\zeta}\|_2 = \|d\zeta\|_2$. Thus, it is sufficient to prove the claim for $R = 1$.

(2) If $r = 0$, for some open half-disk $D \subset B_{1,0}$

$$\int_D \zeta = 0, \quad \|\zeta\|_1 \geq \frac{1}{2} \|\zeta\|_1.$$  \hspace{1cm} (4.5)

By the first condition, Lemma 4.1, and H"older’s inequality

$$\|\zeta\|_1 \leq \frac{4}{\pi} \int_D \int_D |y-x|^{-1} dy dx \leq 16 \int_D |d_y \zeta| dy \leq 8\sqrt{2\pi} \|d\zeta\|_2.$$  \hspace{1cm} (4.6)

Along with the second assumption in (4.5), this implies the claim for $r = 0$ with $C = 16\sqrt{2\pi}$.

(3) Let $\beta: \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that

$$\beta(t) = \begin{cases} 1, & \text{if } t \leq 1/2; \\ 0, & \text{if } t \geq 1. \end{cases}$$

It remains to prove the claim for all $r > 0$ and $R = 1$. By (3.17), we can assume that

$$r \leq \frac{1}{48\sqrt{3\pi}} \|\beta\|_{C^0} < \frac{1}{96\sqrt{3\pi}}.$$  \hspace{1cm} (4.7)

We first consider the case

$$\|\zeta\|_{B_{2r}} \geq \frac{1}{25} \|\zeta\|_1.$$  \hspace{1cm} (4.8)

By H"older’s inequality and Lemma 4.4,

$$\|\zeta\|_{B_{2r}} \leq \sqrt{3\pi r} \|\zeta\|_2 \leq \sqrt{3\pi r} \|d\zeta\|_1 \leq \sqrt{3\pi r} \left(\|d\zeta\|_1 + \|\beta\|_{C^0} \|\zeta\|_{B_{1,1/2}}\right).$$

Along with the assumptions (4.6) and (4.7), this implies the bound with

$$C = \frac{25}{1 - 24\sqrt{3\pi}} \left(\frac{\sqrt{3\pi r}}{1 - 24\sqrt{3\pi}}\right) \leq \frac{25}{48}.$$  \hspace{1cm} (4.9)

Finally, suppose

$$\|\zeta\|_{B_{2r}} \leq \frac{1}{25} \|\zeta\|_1.$$  \hspace{1cm} (4.10)

Split the annulus $B_{1,r}$ into 3 wedges of equal area; split each wedge into a large convex outer portion and a small inner portion by drawing the line segment tangent to the circle of radius $r$ and with the end points on the sides of the wedges $2r$ from the center as in Figure 3. By (4.8),

$$A \equiv \|\zeta\|_{D_1} \geq \frac{8}{25} \|\zeta\|_1.$$  \hspace{1cm} (4.11)
Figure 3: A large convex region $D_+$ of an annulus $D$

for the outer piece $D_+$ of some wedge $D$. If

$$\left| \int_{D_+} \zeta \right| \leq \frac{3}{10} A,$$

then by Lemma 4.1, (4.6), and Hölder’s inequality,

$$A \leq \frac{3}{10} A + \frac{2}{\pi} \left( \frac{\sqrt{3}}{2} \right)^2 \int_{D_+} \int_{D_+} |d_y \zeta| |y-x|^{-1} dy dx$$

$$\leq \frac{3}{10} A + \frac{9}{2\pi} \cdot \frac{7\sqrt{2}}{9} \cdot 2\pi \sqrt{3} \int_D |d_y \zeta| dy \leq \frac{3}{10} A + 7\sqrt{2\pi} \|d\zeta\|_2.$$

Along with the assumption (4.9), this implies the bound with $C = 125\sqrt{2\pi}/4$. If

$$\left| \int_{D_+} \zeta \right| \geq \frac{3}{10} A,$$

then by (4.8), (4.9), and (3.16),

$$A \leq \|\xi\|_1 \leq \|\zeta\|_1 - \left| \int_{D} \zeta \right| + \int_0^{2\pi} \int_0^1 \zeta(\rho, \theta) \rho d\rho d\theta$$

$$\leq \left( A + \frac{1}{8} A \right) - \left( \frac{3}{10} A - \frac{1}{8} A \right) + \sqrt{\frac{\pi}{2}} \|d\zeta\|_2 = \frac{19}{20} A + \sqrt{\frac{\pi}{2}} \|d\zeta\|_2.$$

Along with the assumption (4.9), this implies the bound with $C = 125\sqrt{2\pi}/4$. Since $\beta$ can be chosen so that $\|\beta\|_{C^0} < 3$ (actually arbitrarily close to 2), comparing with (3.17) for $R/r = 144\sqrt{3\pi}$ we conclude that the claim holds with $C = 125\sqrt{2\pi}/4$ for all $r$. \hfill $\square$

### 4.2 Bundle sections along smooth maps

Let $(M, g)$ be a Riemannian manifold and $(E, \langle \cdot, \cdot \rangle, \nabla)$ a normed vector bundle with connection over $M$. If $u \in C^\infty(B_{R,r}; M)$, $\xi \in \Gamma(u; E)$, and $p \geq 1$, let

$$\|\xi\|_p \equiv \left( \int_{B_{R,r}} |\xi|^p \right)^{1/p}, \quad \|\xi\|_{p,1} \equiv \|\xi\|_p + \|u\xi\|_p.$$
Lemma 4.8. If $(M,g)$ is a Riemannian manifold, $(E, ⟨⟩, ∇)$ is a normed vector bundle with connection over $M$, and $p,q ≥ 1$ are such that $1−2/p ≥ −2/q$, for every compact subset $K ⊂ M$ there exists $C_{K,p,q} ∈ ℝ^+$ with the following property. If $R ∈ ℝ^+$, $r ∈ [0,R)$, $u ∈ C^∞(B_{R,r}; M)$ is such that $\text{Im} u ⊂ K$, and $ξ ∈ Γ_c(u; E)$, then

$$||ξ||_q ≤ C_{K,p,q} R^{1−\frac{2}{p}+\frac{2}{q}} (||∇^u ξ||_p + ||ξ ◦ du||_p).$$

Proof. Let $\text{exp}: TM → M$ be an exponential-like map and $\{U_i: i ∈ [N]\}$ a finite open cover of $K$ such that the $g$-diameter of each set $U_i$ is at most $r_{\exp}(K)/2$. Let $\{W_i: i ∈ [N]\}$ be an open cover of $K$ such that $\overline{W}_i ⊂ U_i$. Choose smooth functions $η_i: M → [0,1]$ such that $η_i = 1$ on $W_i$ and $η_i = 0$ outside of $U_i$. For each $i ∈ [N]$, pick $x_i ∈ W_i$. For each $z ∈ u^{-1}(U_i) ⊂ B_{R,r}$, define $\tilde{u}_i(z) ∈ T_{x_i} M$ and $ξ_i(z) ∈ E_{x_i}$ by

$$\exp_{x_i} \tilde{u}_i(z) = u(z), \quad |\tilde{u}_i(z)| < r_{\exp}(x_i); \quad Π_{\tilde{u}_i(z)} ξ_i(z) = ξ(z).$$

For any $z ∈ B_{R,r}$, put $\tilde{ξ}_i(z) = η_i(u(z)) ξ_i(z)$. Since $\tilde{ξ}_i ∈ C^∞(\overline{B}_{R,r}; E_{x_i})$, by Corollary 4.5 there exists $C_{i,p,q} > 0$ such that

$$||ξ||_{u^{-1}(W_i)}||_q = ||\tilde{ξ}_i||_{u^{-1}(W_i)}||_q ≤ ||\tilde{ξ}_i||_q ≤ C_{i,p,q} R^{1−\frac{2}{p}+\frac{2}{q}} ||d\tilde{ξ}_i||_p. \quad (4.10)$$

Since $d\tilde{ξ}_i = (dη_i ◦ du/ξ_i + (η ◦ du/ξ_i$ on $u^{-1}(U_i)$ and vanishes outside of $u^{-1}(U_i)$,

$$||d\tilde{ξ}_i||_p ≤ ||dξ_i||_{u^{-1}(U_i)}||_p + C_i ||ξ ◦ du||_p. \quad (4.11)$$

On the other hand, by Corollary 3.3, if $u(z) ∈ U_i$

$$|\nabla^u ξ|_z − Π_{\tilde{u}_i(z)} ◦ d_z ξ_i| ≤ C_K |d_z u||ξ(z)|. \quad (4.12)$$

Combining equations (4.10)-(4.12), we obtain

$$||ξ||_{u^{-1}(W_i)}||_q ≤ \tilde{C}_{i,p,q} R^{1−\frac{2}{p}+\frac{2}{q}} (||ξ||_{p,1} + ||ξ ◦ du||_p).$$

The claim follows by summing the last inequality over all $i$. □

Lemma 4.9. If $(M,g)$ is a Riemannian manifold, $(E, ⟨⟩, ∇)$ is a normed vector bundle with connection over $M$, and $p > 2$, for every compact subset $K ⊂ M$ there exists $C_{K,p} ∈ C^∞(ℝ^+; ℝ)$ with the following property. If $R ∈ ℝ^+$, $r ∈ [0,R/2)$, $u ∈ C^∞(B_{R,r}; M)$ is such that $\text{Im} u ⊂ K$, and $ξ ∈ Γ(u; E)$, then

$$||ξ||_{C^0} ≤ C_{K,p}(R) (||ξ||_{p,1} + ||ξ ◦ du||_p).$$

Proof. We continue with the setup in the proof of Lemma 4.8. By Corollary 4.3,

$$||ξ||_{u^{-1}(W_i)}||_{C^0} ≤ ||\tilde{ξ}_i||_{C^0} ≤ C_{i,p}(R) ||\tilde{ξ}_i||_{p,1} ≤ C_{i,p}(R) (||ξ||_{u^{-1}(U_i)}||_p + ||d\tilde{ξ}_i||_p).$$

As above, we obtain

$$||d\tilde{ξ}_i||_p ≤ C_i (||∇^u ξ||_p + ||ξ ◦ du||_p),$$

and the claim follows. □
**Proposition 4.10.** If \((M, g)\) is a Riemannian manifold, \((E, \langle , \rangle, \nabla)\) is a normed vector bundle with connection over \(M\), and \(p > 2\), for every compact subset \(K \subset M\) there exists \(C_{K, p} \in C^\infty(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})\) with the following property. If \(R \in \mathbb{R}^+, \ r \in [0, R/2], \ u \in C^\infty(B_{R,r}; M)\) is such that \(\text{Im} u \subset K\), and \(\xi \in \Gamma_c(u; E)\), then
\[
\|\xi\|_{C^0} \leq C_{K, p}(R, \|du\|_p)\|\xi\|_{p,1}.
\]
The same statement holds if \(B_{R,r}\) is replaced by a fixed compact Riemann surface \((\Sigma, g_{\Sigma})\).

**Proof.** By Lemma 4.9 applied with \(\tilde{p} = (p+2)/2\) and Hölder’s inequality,
\[
\|\xi\|_{C^0} \leq C_{K, \tilde{p}}(R)(\|\xi\|_{\tilde{p}, 1} + \|\xi \otimes \partial u\|_{\tilde{p}}) \leq \tilde{C}_{K, \tilde{p}}(R)(\|\xi\|_{p,1} + \|du\|_p\|\xi\|_{q_1}),
\]
where \(q_1 = p(p+2)/(p-2)\). If \(q_1 \leq p\), then the proof is complete. Otherwise, apply Lemma 4.8 with \(p_1 = 2q_1/(q_1+2)\) and Hölder’s inequality:
\[
\|\xi\|_{q_1} \leq C_{K, p_1 q_1}(R)(\|\xi\|_{p_1, 1} + \|\xi \otimes \partial u\|_{p_1}) \leq C_{K, 1}(R)(\|\xi\|_{p,1} + \|du\|_p\|\xi\|_{q_2}),
\]
where \(q_2 = pp_1/(p-p_1)\). If \(q_2 \leq p\), then the claim follows from equations (4.13) and (4.14). Otherwise, we can continue and construct sequences \(\{p_i\}, \{q_i\}, \{C_{K, i}\}\) such that
\[
p_i = \frac{2q_i}{q_i + 2}, \quad q_{i+1} = \frac{pp_i}{p-p_i};
\]
\[
\|\xi\|_{q_i} \leq C_{K, i}(R)(\|\xi\|_{p,1} + \|du\|_p\|\xi\|_{q_{i+1}}).
\]
The recursion (4.15) implies that
\[
q_{i+1} = \frac{2p}{2p + (p-2)q_i}q_i \quad \implies \quad \text{if } q_i > 0, \text{ then } 0 < q_{i+1} < q_i.
\]
Thus, if \(q_i > 2\) for all \(i\), then the sequence \(\{q_i\}\) must have a limit \(q \geq 2\) with
\[
q = \frac{2p}{2p + (p-2)q} \quad \implies \quad (p-2)q = 0 \quad \implies \quad q = 0,
\]
since \(p > 2\) by assumption. Thus, \(q_N \leq p\) for \(N\) sufficiently large and the first claim follows from (4.13) and the equations (4.16) with \(i\) running from 1 to \(N\), where \(N\) is the smallest integer such that \(q_{N+1} \leq p\). The second claim follows immediately from the first. \(\square\)

### 4.3 Elliptic estimates
If \(A_1 = B_{R_1,r_1}\) and \(A_2 = \bar{B}_{R_2,r_2}\) are two annuli in \(\mathbb{R}^2\), we write \(A_2 \subseteq_{\delta} A_1\) if \(R_1 - R_2 > \delta\) and \(r_2 - r_1 \geq \delta\).

**Lemma 4.11.** For any \(\delta > 0, \ p \geq 1,\) and open annulus \(A_1\), there exists \(C_{\delta, p}(A_1) > 0\) such that for any annulus \(A_2 \subseteq_{\delta} A_1\) and \(\xi \in C^\infty(A_1; \mathbb{C}^k)\),
\[
\|\xi|_{A_2}\|_{p,1} \leq C_{\delta, p}(A_1)(\|\bar{\partial} \xi\|_p + \|d\xi\|_2 + \|\xi\|_1),
\]
where the norms are taken with respect to the standard metric on \(\mathbb{R}^2\).
Proof. We can assume that $A_2$ is the maximal annulus such that $A_2 \Subset A_1$. Let $\eta: A_1 \to [0,1]$ be a compactly supported smooth function such that $\eta|_{A_2} = 1$. By the fundamental elliptic inequality for the $\overline{\partial}$-operator on $S^2$ [4, Lemma C.2.1],
\[
\|\xi|_{A_2}\|_p, p, 1 \leq \|\eta\xi\|_{p, 1} \leq C_p(A_1)(\|\overline{\partial}(\eta\xi)\|_p + \|\eta\xi\|_p) \\
\leq C_p(A_1)(\|\overline{\partial}\xi\|_p + \|d\eta\xi\|_p + \|\eta\xi\|_p).
\]
(4.17)
By Corollary 4.5 with $(p, q) = (2, p)$ and $(p, q) = (1, 2)$ and Hölder’s inequality,
\[
\|\eta\xi\|_p \leq C_p(A_1)\|d(\eta\xi)\|_2 \leq C_p(A_1)(\|d\xi\|_2 + \|d(\eta\xi)\|_2) \\
\leq \tilde{C}_p(A_1)(\|d\xi\|_2 + \|d((\eta, \xi))\|_1) \leq \tilde{C}_{p, \delta}(A_1)(\|d\xi\|_2 + \|d\xi\|_1 + \|\xi\|_1) \\
\leq C_{\delta, p}(A_1)(\|d\xi\|_2 + \|\xi\|_1).
\]
(4.18)
Similarly,
\[
\|d(\eta)\xi\|_p \leq C_{\delta, p}(A_1)(\|d\xi\|_2 + \|\xi\|_1).
\]
(4.19)
The claim follows by plugging (4.18) and (4.19) into (4.17).

Corollary 4.12. For any $\delta > 0$, $p \geq 1$, and open annulus $A_1$, there exists $C_{\delta, p}(A_1) > 0$ such that for any annulus $A_2 \Subset A_1$, and $\xi \in C^\infty(A_1; \mathbb{C}^n),$
\[
|d\xi|_{A_2}\|_p \leq C_{\delta, p}(A_1)(\|\overline{\partial}\xi\|_p + |d\xi|_2).
\]

Proof. With $|A_1|$ denoting the area of $A_1$, let
\[
\bar{\xi} = \frac{1}{|A_1|} \int_{A_1} \xi
\]
be the average value of $\xi$. By Lemma 4.11,
\[
|d\xi|_{A_2}\|_p = |d(\xi - \bar{\xi})|_{A_2}\|_p \leq C_{\delta, p}(A_1)(|\overline{\partial}(\xi - \bar{\xi})|_p + |d(\xi - \bar{\xi})|_2 + |\xi - \bar{\xi}|_1) \\
= C_{\delta, p}(A_1)(|\overline{\partial}\xi|_p + |d\xi|_2 + |\xi - \bar{\xi}|_1).
\]
(4.20)
The claim follows by applying Corollary 4.7 with $\zeta = \xi - \bar{\xi}$.

Remark 4.13. The case $r_1 > 0$ (which is the case needed for gluing pseudo-holomorphic maps in symplectic topology) follows from Corollary 3.7; Corollary 4.7 can be used to obtain a sharper statement in this case (that $C_{\delta, p}(A_1)$ does not depend on $r_1$). The $r_1 = 0$ case requires only the first two steps in the proof of Corollary 4.7.

A smooth generalized CR-operator in a smooth complex vector bundle $(E, \nabla)$ with connection over an almost complex manifold $(M, J)$ is an operator of the form
\[
D = \overline{\partial} + A: \Gamma(M; E) \to \Gamma(M; T^*M^{0,1} \otimes \mathbb{C} E),
\]
where
\[
\overline{\partial}\xi = \frac{1}{2}(\nabla\xi + i\nabla J\xi) \quad \forall \xi \in \Gamma(M; TM), \quad A \in \Gamma(M; \text{Hom}(E; T^*M^{0,1} \otimes \mathbb{C} E)).
\]
If in addition $u: \Sigma \to M$ is a smooth map from an almost complex manifold $(\Sigma, j)$, the pull-back CR-operator is given by
\[
D_u = \overline{\partial}_{\bar{u}} + A \circ \overline{\partial} u: \Gamma(u; E) \to \Gamma^{0,1}(u; E).
\]
Proposition 4.14. If \((M, g)\) is a Riemannian manifold with an almost complex structure \(J\), \((E, \langle \cdot, \cdot \rangle, \nabla)\) is a normed complex vector bundle with connection over \(M\) and a smooth generalized CR-operator \(D\), and \(p \geq 1\), then for every compact subset \(K \subset M\), \(\delta > 0\), and open annulus \(A_1 \subset \mathbb{R}^2\), there exists \(C_{K; \delta, p}(A_1) \in \mathbb{R}^+\) with the following property. If \(u \in C^\infty(A_1; M)\) is such that \(\text{Im} u \subset K\), \(\xi \in \Gamma(u; E)\), and \(A_2 \subset_\delta A_1\) is an annulus, then

\[
\|\nabla^u \xi|_{A_2}\|_p \leq C_{K; \delta, p}(A_1)(\|Du \xi\|_p + \|\nabla^u \xi\|_2 + \|\xi \otimes du\|_p),
\]

where the norms are taken with respect to the standard metric on \(\mathbb{R}^2\).

Proof. We continue with the setup in the proof of Lemma 4.8. By Corollary 4.12,

\[
\|d\tilde{\xi}_i|_{A_2}\|_p \leq C_{i; \delta, p}(A_1)(\|\tilde{\partial}_i \xi\|_p + \|d\tilde{\xi}_i\|_2) \leq C'_{i; \delta, p}(A_1)(\|\tilde{\partial}_i \xi|_{u^{-1}(U_i)}\|_p + \|d\xi|_{u^{-1}(U_i)}\|_2 + \|\xi \otimes du\|_p). \tag{4.21}
\]

Since \(\nabla\) commutes with the complex structure in \(E\) and \(\tilde{\xi}_i = \xi_i\) on \(u^{-1}(W_i)\), it follows from (4.12) and (4.21) that

\[
\|\nabla^u \xi|_{A_2 \cap u^{-1}(W_i)}\|_p \leq \|d\tilde{\xi}_i|_{A_2}\|_p + C_K\|\xi \otimes du\|_p
\]

\[\leq \tilde{C}_{i; \delta, p}(A_1)(\|\tilde{\partial}_i \xi\|_p + \|\nabla^u \xi\|_2 + \|\xi \otimes du\|_p) \tag{4.22}
\]

\[\leq \tilde{C}'_{i; \delta, p}(A_1)(\|Du \xi\|_p + \|\nabla^u \xi\|_2 + \|\xi \otimes du\|_p).
\]

The claim is obtained by summing the last equation over all \(i\).

Lemma 4.15. If \((M, g)\) is a Riemannian manifold with an almost complex structure \(J\), \((E, \langle \cdot, \cdot \rangle, \nabla)\) is a normed complex vector bundle with connection over \(M\) and a smooth generalized CR-operator \(D\), and \(p > 2\), then for every compact subset \(K \subset M\) and open ball \(B \subset \mathbb{R}^2\), there exists \(C_{K; B, p} \in C^\infty(\mathbb{R}, \mathbb{R})\) with the following property. If \(u \in C^\infty(B; M)\) is such that \(\text{Im} u \subset K\) and \(\xi \in \Gamma_c(u; E)\), then

\[
\|\xi\|_{p, 1} \leq C_{K; B, p}(\|du\|_p)(\|Du \xi\|_p + \|\xi\|_p),
\]

where the norms are taken with respect to the standard metric on \(\mathbb{R}^2\).

Proof. By an argument nearly identical to the proof of Proposition 4.14,

\[
\|\xi\|_{p', 1} \leq C_{K; p'}(B)(\|Du \xi\|_{p'} + \|\xi\|_{p'} + \|\xi \otimes du\|_{p'})
\]

for any \(p' \geq 1\). On the other hand, by Proposition 4.10,

\[
\|\xi\|_{C^\alpha} \leq C_{K; \bar{p}, \bar{p}}(\|du\|_{\bar{p}})(\|\xi\|_{\bar{p}, 1}),
\]

where \(\bar{p} = (p + 2)/2\). Proceeding as in the proof of Proposition 4.10, we then obtain

\[
\|\xi\|_{p, 1} \leq C_{K; B, p}(\|du\|_{\bar{p}})(\|Du \xi\|_{p} + \|\xi\|_{p} + \|du\|_{p}(\|\xi\|_{\bar{p}, 1})\),
\]

\[
\|\xi\|_{\bar{p}, 1} \leq C_{K; \bar{p}, \bar{p}}(B)(\|Du \xi\|_{\bar{p}} + \|\xi\|_{\bar{p}} + \|du\|_{\bar{p}}(\|\xi\|_{q_1}),
\]

\[
\|\xi\|_{q_1} \leq C_{K; p, q_1}(B)(\|\xi\|_{p, 1} + \|\xi \otimes du\|_{p}),
\]

\[
\leq C_{K; B, \bar{p}}(\|du\|_{\bar{p}})(\|Du \xi\|_{p} + \|\xi\|_{p} + \|du\|_{p}(\|\xi\|_{q_1})\);
\]

we stop the recursion at the same value of \(i = N\) as in the proof of Proposition 4.10. □
Proposition 4.16. If \((M, g)\) is a Riemannian manifold with an almost complex structure \(J\), 
\((E, \langle \cdot, \cdot \rangle, \nabla)\) is a normed complex vector bundle with connection over \(M\) and a smooth generalized CR-operator \(D\), and \(p > 2\), then for every compact subset \(K \subset M\) and compact Riemann surface \((\Sigma, g_\Sigma)\), there exists \(C_{K, \Sigma, p} \in C^\infty(\mathbb{R}; \mathbb{R})\) with the following property. If \(u \in C^\infty(\Sigma; M)\) is such that \(\text{Im } u \subset K\) and \(\xi \in \Gamma(u; E)\), then

\[
\|\xi\|_{p, 1} \leq C_{K, \Sigma, p}(\|du\|_p)(\|Du\xi\|_p + \|\xi\|_p).
\]

Proof. This statement is immediate from Lemma 4.15. \(\square\)

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References