Orientability in Real Gromov-Witten Theory

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Abstract

The orientability problem in real Gromov-Witten theory is one of the fundamental hurdles to enumerating real curves. In this paper, we describe topological conditions on the target manifold which ensure that the uncompactified moduli spaces of real maps are orientable for *all* genera of and for *all* types of involutions on the domain. In contrast to the typical approaches to this problem, we do not compute the signs of any diffeomorphisms, but instead compare them. Many projective complete intersections, including the renowned quintic threefold, satisfy our topological conditions. Our main result yields real Gromov-Witten invariants of arbitrary genus for real symplectic manifolds that satisfy these conditions and have empty real locus and illustrates the significance of previously introduced moduli spaces of maps with crosscaps. We also apply it to study the orientability of the moduli spaces of real Hurwitz covers.

Contents

1 Introduction

The theory of J-holomorphic maps plays a prominent role in symplectic topology, algebraic geometry, and string theory. The foundational work of [14, 28, 18, 23, 7, 16] has established the theory of (closed) Gromov-Witten invariants, i.e. counts of J-holomorphic maps from closed Riemann surfaces to symplectic manifolds. In contrast, the theory of real Gromov-Witten invariants, i.e. counts of J-holomorphic maps from symmetric Riemann surfaces commuting with the involutions on the domain and the target, is still in early stages of development, especially in positive genera. The two main obstacles to defining real Gromov-Witten invariants are the potential non-orientability of

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the moduli space of real J-holomorphic maps and the existence of real codimension-one boundary strata. In this paper, we address the former, obtaining sufficient topological conditions on the target manifold for these moduli spaces to be orientable for *all* genera of and for *all* types of involutions on the domain; see Definition 1.1 and Theorem 1.2. Theorem 1.2 yields real Gromov-Witten invariants of arbitrary genus for real symplectic manifolds that satisfy these conditions and have empty real locus; see Theorem 1.3. Many projective complete intersections, including the quintic threefold which plays a central role in Gromov-Witten theory, satisfy our topological conditions; see Corollary 1.4.

The orientability question in real Gromov-Witten theory is studied in [27, 22, 25, 8, 10, 5, 6, 11, 12]. Real maps can be naturally divided into two groups, depending on whether the involution σ on the domain Σ has separating fixed locus Σ^σ or not. In the first case, one can use bordered surfaces to obtain a good understanding of the orientability of the moduli spaces of such maps; see [25, 8, 10, 5, 11, 12]. In the second case, however, understanding the orientability in the bordered case is not sufficient beyond genus 1, due to the presence of real diffeomorphisms of (Σ,σ) not preserving any half of Σ; see Example 2.4. The subtle effect of such diffeomorphisms on the orientability is hard to determine. In [5], this problem is studied for the diffeomorphisms of (Σ,σ) preserving some additional structure determined by a distinguished component of Σ^σ and a polarizing divisor on the target manifold X, obtaining orientability results for certain hypersurfaces in projective spaces. In this paper, we adopt a fundamentally different approach: instead of computing the signs of the actions induced by such diffeomorphisms, we directly compare the orientation systems of the moduli spaces with the orientation systems of certain bundles over them naturally suggested by our previous study [12]; see the end of this section for more details. The orientable cases we discover include the orientable cases described in [5] and go far beyond them.

An involution on a smooth manifold X is a diffeomorphism $\phi\colon X\longrightarrow X$ such that $\phi\circ\phi=\mathrm{id}_X$. Let

$$X^\phi=\{x\in X:\ \phi(x)=x\}$$

denote the fixed locus. An anti-symplectic involution ϕ on a symplectic manifold (X,ω) is an involution $\phi\colon X\longrightarrow X$ such that $\phi^*\omega=-\omega$. A real symplectic manifold is a triple (X,ω,ϕ) consisting of a symplectic manifold (X,ω) and an anti-symplectic involution ϕ. For example, the maps

$$\tau_n\colon \mathbb{P}^{n-1}\longrightarrow\mathbb{P}^{n-1},\qquad [Z_1,\ldots,Z_n]\longrightarrow[\bar Z_1,\ldots,\bar Z_n],$$
$$\eta_{2m}\colon \mathbb{P}^{2m-1}\longrightarrow\mathbb{P}^{2m-1},\qquad [Z_1,Z_2,\ldots,Z_{2m-1},Z_{2m}]\longrightarrow\big[-\bar Z_2,\bar Z_1,\ldots,-\bar Z_{2m},\bar Z_{2m-1}\big],$$

are anti-symplectic involutions with respect to the standard Fubini-Study symplectic forms ω_n on $\mathbb{P}^{n-1}$ and ω_{2m} on $\mathbb{P}^{2m-1}$, respectively. If

$$k\ge 0,\qquad \mathbf{a}\equiv(a_1,\ldots,a_k)\in(\mathbb{Z}^+)^k,$$

and $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ is a complete intersection of multi-degree $\mathbf{a}$ preserved by τ_n, $\tau_{n;\mathbf{a}}\equiv\tau_n|_{X_{n;\mathbf{a}}}$ is an anti-symplectic involution on $X_{n;\mathbf{a}}$ with respect to the symplectic form $\omega_{n;\mathbf{a}}=\omega_n|_{X_{n;\mathbf{a}}}$. Similarly, if $X_{2m;\mathbf{a}}\subset\mathbb{P}^{2m-1}$ is preserved by η_{2m}, $\eta_{2m;\mathbf{a}}\equiv\eta_{2m}|_{X_{2m;\mathbf{a}}}$ is an anti-symplectic involution on $X_{2m;\mathbf{a}}$ with respect to the symplectic form $\omega_{2m;\mathbf{a}}=\omega_{2m}|_{X_{2m;\mathbf{a}}}$.

Let (X,ϕ) be a manifold with an involution. A conjugation on a complex vector bundle $V\longrightarrow X$ lifting an involution ϕ is a vector bundle homomorphism $\tilde\phi\colon V\longrightarrow V$ covering ϕ (or equivalently a vector bundle homomorphism $\tilde{\phi}\colon V\longrightarrow\phi^*V$ covering id_X) such that the restriction of $\tilde{\phi}$ to each fiber is anti-complex linear and $\tilde{\phi}\circ\tilde{\phi}=\mathrm{id}_V$. A real bundle pair $(V,\tilde{\phi})\longrightarrow(X,\phi)$ consists of a complex vector bundle $V\longrightarrow X$ and a conjugation $\tilde{\phi}$ on V lifting ϕ. For example,

$$(TX,\mathrm{d}\phi)\longrightarrow(X,\phi) \qquad\text{and}\qquad (X\times\mathbb{C},\phi\times\mathfrak{c}_{\mathbb{C}})\longrightarrow(X,\phi),$$

where $\mathfrak{c}_{\mathbb{C}}\colon\mathbb{C}\longrightarrow\mathbb{C}$ is the standard conjugation on $\mathbb{C}$, are real bundle pairs. For any real bundle pair $(V,\tilde{\phi})\longrightarrow(X,\phi)$, we denote by

$$\Lambda^{\mathrm{top}}_{\mathbb{C}}(V,\tilde{\phi})=(\Lambda^{\mathrm{top}}_{\mathbb{C}}V,\Lambda^{\mathrm{top}}_{\mathbb{C}}\tilde{\phi})$$

the top exterior power of V over $\mathbb{C}$ with the induced conjugation. Direct sums, duals, and tensor products over $\mathbb{C}$ of real bundle pairs over (X,ϕ) are again real bundle pairs over (X,ϕ).

A symmetric surface (Σ,σ) is a closed connected oriented smooth surface Σ (manifold of real dimension 2) with an orientation-reversing involution σ. The fixed locus of σ is a disjoint union of circles. If in addition (X,ϕ) is a manifold with an involution, a real map

$$u\colon(\Sigma,\sigma)\longrightarrow(X,\phi)$$

is a smooth map $u\colon\Sigma\longrightarrow X$ such that $u\circ\sigma=\phi\circ u$. We denote the space of such maps by $\mathfrak{B}_g(X)^{\phi,\sigma}$.

For a symplectic manifold (X,ω), we denote by $\mathcal{J}_\omega$ the space of ω-compatible almost complex structures on X. If ϕ is an anti-symplectic involution on (X,ω), let

$$\mathcal{J}_\phi=\big\{J\in\mathcal{J}_\omega\colon\ \phi^*J=-J\big\}.$$

For a genus g symmetric surface (Σ,σ), we similarly denote by $\mathcal{J}_\sigma$ the space of complex structures on Σ compatible with the orientation such that $\sigma^*\mathfrak{j}=-\mathfrak{j}$. For $J\in\mathcal{J}_\phi$, $\mathfrak{j}\in\mathcal{J}_\sigma$, and $u\in\mathfrak{B}_g(X)^{\phi,\sigma}$, let

$$\bar{\partial}_{J,\mathfrak{j}}u=\frac{1}{2}\big(\mathrm{d}u+J\circ\mathrm{d}u\circ\mathfrak{j}\big).$$

If $l\in\mathbb{Z}^{\ge0}$, $J\in\mathcal{J}_\phi$, and $B\in H_2(X;\mathbb{Z})$, let

$$\begin{split}\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}=\big\{(u,(z_1,\sigma(z_1)),\ldots,(z_l,\sigma(z_l)),\mathfrak{j})\in\mathfrak{B}_g(X)^{\phi,\sigma}\times\Sigma^{2l}\times\mathcal{J}_\sigma\colon\\ u_*[\Sigma]_{\mathbb{Z}}=B,\ \bar{\partial}_{J,\mathfrak{j}}u=0\big\}/\!\sim\end{split}$$

be the moduli space of equivalence classes of degree B real J-holomorphic maps from (Σ,σ) to (X,ϕ) with l pairs of non-real conjugate distinct points; two J-holomorphic maps are equivalent in this space if they differ by an orientation-preserving diffeomorphism of Σ commuting with σ. Let

$$\mathfrak{M}_{g,l}(X,B;J)^{\phi}=\bigcup_{\sigma}\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$$

denote the union over all topological types of orientation-reversing involutions on a genus g surface Σ. Using the geometric perturbations of [23] adapted to the real case as in [10, Section 2], we can perturb $\bar{\partial}_{J,\mathfrak{j}}$ to achieve transversality; thus, we may assume that $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ is an orbifold if the domain is stable. Under the assumptions of Remark 2.3, $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ is a manifold outside codimension 2 strata and has a first Stiefel-Whitney class.

Definition 1.1. A real symplectic manifold (X,ω,ϕ) is real-orientable if there exists a rank 1 real bundle pair $(L,\tilde{\phi})\longrightarrow(X,\phi)$ such that

$$w_2(TX^\phi)=w_1(L^{\tilde{\phi}})^2 \qquad \text{and} \qquad \Lambda_{\mathbb{C}}^{\mathrm{top}}(TX,\mathrm{d}\phi)=(L,\tilde{\phi})^{\otimes 2}\,. \tag{1.1}$$

Theorem 1.2. *Let (X,ω,ϕ) be a real-orientable $2n$-manifold, $B\in H_2(X,\mathbb{Z})$, $J\in\mathcal{J}_\phi$, $l\in\mathbb{Z}^{\ge0}$, and (Σ,σ) be a symmetric surface of genus $g\ge2$.*

(1) If n is odd, then the moduli space $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ is orientable.

(2) If $g+2l\ge4$, then

$$w_1(\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma})=(n+1)\,\mathfrak{f}^*w_1(\mathcal{M}_{g,l}^\sigma),$$

where $\mathfrak{f}\colon\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}\longrightarrow\mathcal{M}_{g,l}^\sigma$ is the forgetful morphism to the Deligne-Mumford moduli space of σ-compatible complex structures on Σ.

The genus 0 and 1 analogues of Theorem 1.2 are essentially [11, Theorems 1.1, 1.2], respectively; less general versions of [11, Theorem 1.1] are contained in [8, Theorem 1.1] and [6, Theorem 1.3]. The second requirement in (1.1) is not necessary for the conclusion of Theorem 1.2 if $\Sigma-\Sigma^\sigma$ is disconnected; see [12, Theorem 1.4]. We note that Definition 1.1 forces the fixed locus X^ϕ to be orientable; so Theorem 1.2 does not consider any situations with unorientable Lagrangians.

Under the assumptions on (X,ω,ϕ) in Theorem 1.2(1), an orientation on $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ can be specified as follows. Fix $\mathfrak{j}\in\mathcal{J}_\sigma$ and choose an orientation of the tangent space of the Deligne-Mumford moduli space $\mathcal{M}_{g,0}^\sigma$ at $\mathfrak{j}$. Let $[\Sigma,X]_B^{\phi,\sigma}$ denote the set of homotopy classes of real maps from (Σ,σ) to (X,ϕ) so that $u_*[\Sigma]=B$. The group $\mathcal{D}^\sigma$ of orientation-preserving diffeomorphisms of Σ commuting with σ acts on $[\Sigma,X]_B^{\phi,\sigma}$ by composition on the right. For each coset of this group action, choose a representative u_i and an orientation of the index of a linearization of a real Cauchy-Riemann operator in u_i^*TX compatible with $\mathfrak{j}$. The latter can be obtained by choosing a spin structure on the real vector bundle

$$TX^\phi\oplus L^{\tilde{\phi}}\oplus L^{\tilde{\phi}}\longrightarrow X^\phi\,,$$

fixing the second identification in (1.1), and choosing a symmetric half-surface Σ^b, or a surface with crosscaps, doubling to Σ; see Section 2 and [11, Section 4]. The resulting orientation on $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ does not depend on J. However, if $g\ge2$, it does depend on the choice of Σ^b for most topological types of orientation-reversing involutions σ on Σ. This dependence is suggested by Example 2.4 and illustrates the significance of maps with crosscaps in real Gromov-Witten theory in positive genera; the mathematical construction of moduli spaces of such maps in [11] is motivated by their role in the description of localization data for real GW-invariants of the quintic threefold in [26].

Theorem 1.3. *Let (X,ω,ϕ) be a real symplectic $2n$-manifold, $g,l\in\mathbb{Z}^{\ge0}$, $B\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}_\phi$. If $n\notin2\mathbb{Z}$, $X^\phi=\emptyset$, and*

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}(TX,\mathrm{d}\phi)=(L,\tilde{\phi})^{\otimes2}$$

for some rank 1 real bundle pair $(L,\tilde{\phi})\longrightarrow(X,\phi)$, then the moduli space $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ carries a virtual fundamental class and thus gives rise to real genus g Gromov-Witten invariants of (X,ω,ϕ).

If $X^\phi=\emptyset$, $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}=\emptyset$ for every topological type of orientation-reversing involutions σ on a genus g surface Σ except for the involutions σ_g with $\Sigma^{\sigma_g}=\emptyset$. Thus,

$$\mathfrak{M}_{g,l}(X,B;J)^\phi = \mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma_g}\,.$$

By Theorem 1.2 and [11, Theorems 1.1, 1.2], this moduli space is orientable under the assumptions of Theorem 1.3. The (virtual) codimension-one boundary of $\mathfrak{M}_{g,l}(X,B;J)^\phi$ consists of maps to X from two-component domains sending the node to X^ϕ. If $X^\phi=\emptyset$, the boundary is empty and the moduli space $\mathfrak{M}_{g,l}(X,B;J)^\phi$ with a choice of orientation determines a virtual fundamental class, obtained by a suitable adaptation of the usual VFC constructions of [7, 16], as in [25, Section 7] and [10, Remark 3.3].

Corollary 1.4. *Let $n\in\mathbb{Z}^+$, $g,k,l\in\mathbb{Z}^{\ge0}$, $\mathbf{a}\equiv(a_1,\ldots,a_k)\in(\mathbb{Z}^+)^k$, and (Σ,σ) be a genus g symmetric surface.*

(1) If $n-k\in 2\mathbb{Z}$, $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ is a complete intersection of multi-degree $\mathbf{a}$ preserved by τ_n,

$$\sum_{i=1}^k a_i\equiv n \mod 2, \qquad \textit{and} \qquad \sum_{i=1}^k a_i^2\equiv\sum_{i=1}^k a_i \mod 4,$$

the moduli space $\mathfrak{M}_{g,l}(X_{n;\mathbf{a}},B;J)^{\tau_{n;\mathbf{a}},\sigma}$ is orientable for every $B\in H_2(X_{n;\mathbf{a}};\mathbb{Z})$ and $J\in\mathcal{J}_{\tau_{n;\mathbf{a}}}$.

(2) If $k\in2\mathbb{Z}$, $X_{2n;\mathbf{a}}\subset\mathbb{P}^{2n-1}$ is a complete intersection of multi-degree $\mathbf{a}$ preserved by η_{2n} and

$$a_1+\ldots+a_k\equiv 2n \mod 4,$$

the moduli space $\mathfrak{M}_{g,l}(X_{2n;\mathbf{a}},B;J)^{\eta_{2n;\mathbf{a}},\sigma}$ carries a virtual fundamental class for every $B\in H_2(X_{2n;\mathbf{a}};\mathbb{Z})$ and $J\in\mathcal{J}_{\eta_{2n;\mathbf{a}}}$ and thus gives rise to real genus g Gromov-Witten invariants of $(X_{2n;\mathbf{a}},\omega_{2n;\mathbf{a}},\eta_{2n;\mathbf{a}})$.

In the genus 0 case, the conclusions of Corollary 1.4 apply without the parity assumptions on k (which correspond to the dimension of $X_{n;\mathbf{a}}$ being odd).

Let $X_{n;\delta}\subset\mathbb{P}^{n-1}$ denote a hypersurface of degree $\delta\in\mathbb{Z}^+$ preserved by τ_n. Theorem 1.2 applies to $X_{n;\delta}$ if

$$\delta=0,1 \mod 4 \qquad \text{and} \qquad \delta\equiv n \mod 2.$$

With the second condition strengthened to $\delta\equiv n \mod 4$, the conclusion of Theorem 1.2 is obtained in [5, Corollaire 2.4] under the additional assumption that Σ^σ is a single circle; if Σ^σ consists of more than one circle, [5, Corollaire 2.4] shows that the conclusion of Theorem 1.2 holds after pulling back to a cover of $\mathfrak{M}_g(X,B;J)^{\phi,\sigma}$.

By [6, Lemma 2.5], the canonical line bundle K_X of a Kahler Calabi-Yau manifold X, i.e. K_X is trivial as a holomorphic line bundle, is trivial as a rank 1 real bundle pair with respect to any involution ϕ which is anti-holomorphic with respect to the Kahler complex structure. By [6, Lemma 2.6], the canonical line bundle K_X of a simply connected symplectic Calabi-Yau manifold (X,ω), i.e. $c_1(X,\omega)=0$, is trivial with respect to any anti-symplectic involution ϕ on X. Thus, Theorems 1.2 and 1.3 also apply to Kahler Calabi-Yau manifolds X with an anti-holomorphic involution and to simply connected symplectic Calabi-Yau manifolds with any anti-symplectic involution, provided the fixed locus is spin, i.e. $w_2(X^\phi)=0$, in the case of Theorem 1.2 and empty in the case of Theorem 1.3.

Theorem 1.2 and Corollary 1.4 can be extended to the moduli spaces $\mathfrak{M}_{g,k,l}(X,B;J)^{\phi,\sigma}$ of real maps with k boundary and l interior marked points, as the effect of adding marked points on the sign of the relevant automorphisms can be easily determined. The moduli spaces $\mathfrak{M}_{g,k,l}(X,B;J)^{\phi,\sigma}$ typically have codimension-one boundary and often of more than one type. The codimension-one boundary stratum consisting of maps from Σ with a bubble attached at a real point of the domain can be eliminated by the gluing procedure of [3, 25], which is adapted to maps with decorated marked points in [10, Section 3]. By [10, Theorems 1.3], the proof of [10, Corollary 6.1], and [12, Propositions 4.1, 4.2], Theorem 1.2 can be extended to the glued moduli space $\widetilde{\mathfrak{M}}_{g,0,l}(X,B;J)^{\phi,\sigma}$. The remaining types of codimension-one boundary strata of $\mathfrak{M}_{g,0,l}(X,B;J)^{\phi,\sigma}$ correspond to one-nodal degenerations of Σ passing between involutions on Σ of different topological types, as described in detail in [15, Section 4], [24, Section 5], and [17, Sections 3,4]. As suggested in [21, Section 1.5] and carried out in [6, Section 3] in the case $\Sigma=\mathbb{P}^1$, the moduli spaces $\overline{\mathfrak{M}}_{g,0,l}(X,B;J)^{\phi,\sigma}$ with different types of involutions σ on Σ should in general be combined to get well-defined invariants by gluing along codimension-one boundaries. We intend to consider this question in a future paper.

In Section 5, we apply Theorem 1.2 to show that some moduli spaces of real Hurwitz covers are orientable. We also establish the orientability of such moduli spaces directly in some cases with the target $(X,\phi)=(\mathbb{P}^1,\eta)$; Theorem 1.2 does not apply to these cases as the rank 1 real bundle pair $(T\mathbb{P}^1,\mathrm{d}\eta)$ does not admit a real square root. This suggests that perhaps the second requirement in (1.1) can be replaced by the requirement that the equivariant w_2 of $\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX,\mathrm{d}\phi)$ be a square class; by [11, Corollary 2.4], the latter is the case if the second condition in (1.1) holds or if $\pi_1(X)=0$ and $w_2(TX)=0$. By [11, Corollary 1.6], the w_2 requirement suffices whenever the domain of the maps is $\mathbb{P}^1$. On the other hand, the orientability problem for the moduli spaces of real Hurwitz covers appears to be a purely combinatorial question about the topology of various Hurwitz covers and can perhaps be addressed by more classical methods.

The typical approaches to the orientability problem in real Gromov-Witten theory involve computing the signs of the actions of appropriate diffeomorphisms on determinant lines of real Cauchy-Riemann operators over some coverings of $\mathfrak{M}_g(X,B;J)^{\phi,\sigma}$, as done in [25, 8, 5, 12]. These approaches work as long as the relevant diffeomorphisms are homotopically fairly simple and in particular preserve a bordered surface in Σ that doubles to Σ or map it to its conjugate half; more general diffeomorphisms are considered in [5]. In contrast to [25, 8], in [12] we allowed the complex structure on a bordered domain to vary and considered diffeomorphisms that interchange the boundary components. We discovered that

(1) these diffeomorphisms act with the same signs on a natural cover of $\mathcal{M}_g^\sigma$ and on the determinant line bundle for the trivial rank 1 real bundle pair over it;

(2) the signs for the square of a rank 1 real bundle pair are often the same as for the rank 1 trivial real bundle pair;

(3) the signs for a real bundle pair and its top exterior power are often related just by the parity of the rank of the bundle pair;

see Corollary 2.2 and Propositions 4.1 and 4.2 in [12]. In this paper, we show that suitable interpretations of these statements apply to arbitrary real diffeomorphisms of the closed surface. Proposition 4.1, which appears to be of its own interest, establishes (1) in the general case. Proposition 3.3 captures the phenomena (2) and (3) for arbitrary diffeomorphisms and is used in the proof of Proposition 4.1. In contrast to the typical approaches, we compare the signs directly, instead of computing each sign separately. In Section 4, we combine Proposition 3.3 and Proposition 4.1 in order to confirm Theorem 1.2 and then deduce Corollary 1.4.

Remark 1.5. After completing this paper, we discovered that [5, Proposition 1.2] contains what can be viewed as a more general version of one of the main stepping stones in this paper, Proposition 3.2. The former applies in broader range of cases; while the conclusion of our Proposition 3.2 need not hold in some of these cases, the conclusion of [5, Proposition 1.2] suffices for the purposes of [5] and of this paper. In the appendix, we provide an alternative, less technical, argument for this broader result and extend our Theorem 1.2 further; unlike the main part of this paper, this argument relies on an actual sign computation. As explained in the appendix, the benefit of this extension is unclear to us at this point.

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2 Setup

Let (Σ,σ) be a genus g symmetric surface. We denote by $|\sigma|_0\in\mathbb{Z}^{\ge0}$ the number of connected components of Σ^σ; each of them is a circle. Let $\langle\sigma\rangle\!=\!0$ if the quotient Σ/σ is orientable, i.e. $\Sigma\!-\!\Sigma^\sigma$ is disconnected, and $\langle\sigma\rangle\!=\!1$ otherwise. There are $\left\lfloor\frac{3g+4}{2}\right\rfloor$ different topological types of orientation-reversing involutions σ on Σ classified by the triples $(g,|\sigma|_0,\langle\sigma\rangle)$; see [20, Colollary 1.1]. The two equivalence classes of orientation-reversing involutions on $S^2=\mathbb{P}^1$ are described in Example 5.2, while the three equivalence classes of orientation-reversing involutions on $\mathbb{T}\!=\!S^1\!\times\!S^1$ are described in (2) of the proof of Theorem 5.3.

An oriented symmetric half-surface (or simply oriented sh-surface) is a pair (Σ^b,c) consisting of an oriented bordered smooth surface Σ^b and an involution $c\colon\partial\Sigma^b\longrightarrow\partial\Sigma^b$ preserving each component and the orientation of $\partial\Sigma^b$. The restriction of c to a boundary component is either the identity or the antipodal map

$$\mathfrak{a}\colon S^1\longrightarrow S^1,\qquad z\longrightarrow -z, \tag{2.1}$$

for a suitable identification $(\partial\Sigma^b)_i$ with $S^1\subset\mathbb{C}$; the latter type of boundary structure is called crosscap in the string theory literature. We define

$$c_i=c|_{(\partial\Sigma^b)_i},\qquad |c_i|=\begin{cases}0, & \text{if } c_i=\text{id};\\ 1, & \text{otherwise};\end{cases}\qquad |c|_k=\big|\{(\partial\Sigma^b)_i\!\subset\!\Sigma^b\colon\,|c_i|\!=\!k\}\big|\quad k=0,1.$$

Thus, $|c|_0$ is the number of standard boundary components of $(\Sigma^b,\partial\Sigma^b)$ and $|c|_1$ is the number of crosscaps. Up to isomorphism, each oriented sh-surface (Σ^b,c) is determined by the genus g of Σ^b, the number $|c|_0$ of ordinary boundary components, and the number $|c|_1$ of crosscaps. We denote by $(\Sigma_{g,m_0,m_1}, c_{g,m_0,m_1})$ the genus g oriented sh-surface with $|c_{g,m_0,m_1}|_0 = m_0$ and $|c_{g,m_0,m_1}|_1 = m_1$.

An oriented sh-surface (Σ^b, c) of type (g, m_0, m_1) doubles to a symmetric surface (Σ, σ) of type

$$(g(\Sigma), |\sigma|_0, \langle\sigma\rangle) = \begin{cases} (2g+m_0+m_1-1, m_0, 0), & \text{if } m_1 = 0; \\ (2g+m_0+m_1-1, m_0, 1), & \text{if } m_1 \neq 0; \end{cases}$$

so that σ restricts to c on the cutting circles (the boundary of Σ^b); see [11, (1.6)].

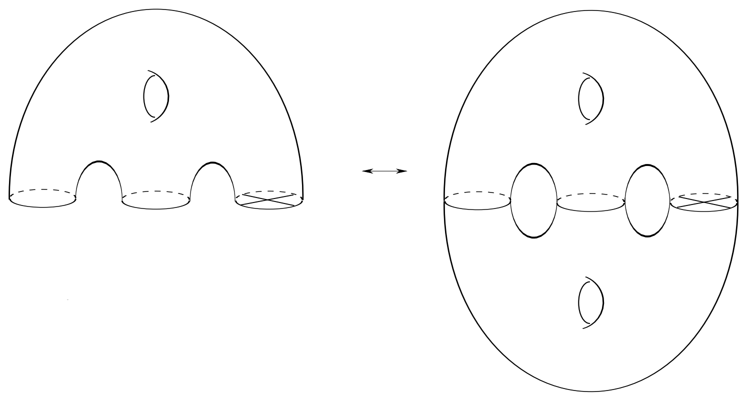

Figure 1: Doubling an oriented sh-surface

Since this doubling construction covers all topological types of orientation-reversing involutions σ on Σ, for every symmetric surface (Σ, σ) there is an oriented sh-surface (Σ^b, c) which doubles to (Σ, σ). In general, the topological type of such an sh-surface is not unique. There is a topologically unique oriented sh-surface (Σ^b, c) doubling to a symmetric surface (Σ, σ) if $\langle\sigma\rangle = 0$, in which case (Σ^b, c) has no crosscaps, or $|\sigma|_0 \geq g(\Sigma)-1$, in which case (Σ^b, c) is either of genus at most 1 and has no crosscaps or of genus 0 and has at most 2 crosscaps.

A real Cauchy-Riemann operator on a real bundle pair $(V, \tilde{\sigma}) \longrightarrow (\Sigma, \sigma)$, where (Σ, σ) is an oriented symmetric surface, is a linear map of the form

$$\begin{aligned} D = \bar{\partial}+A\colon \Gamma(\Sigma; V)^{\tilde{\sigma}} &\equiv \big\{\xi \in \Gamma(\Sigma; V)\colon\ \xi\circ\sigma = \tilde{\sigma}\circ\xi\big\} \\ &\longrightarrow \Gamma^{0,1}_{\mathfrak{j}}(\Sigma; V)^{\tilde{\sigma}} \equiv \big\{\zeta \in \Gamma(\Sigma; (T^*\Sigma, \mathfrak{j})^{0,1}\otimes_{\mathbb{C}} V)\colon\ \zeta\circ \mathrm{d}\sigma = \tilde{\sigma}\circ\zeta\big\}, \end{aligned} \tag{2.2}$$

where $\bar{\partial}$ is the holomorphic $\bar{\partial}$-operator for some $\mathfrak{j} \in \mathcal{J}_\sigma$ and a holomorphic structure in V and

$$A \in \Gamma\big(\Sigma; \mathrm{Hom}_{\mathbb{R}}(V, (T^*\Sigma, \mathfrak{j})^{0,1}\otimes_{\mathbb{C}} V)\big)^{\tilde{\sigma}}$$

is a zeroth-order deformation term. A real Cauchy-Riemann operator on a real bundle pair is Fredholm in the appropriate completions. A continuous family of such Fredholm operators D_t over a topological space $\mathcal{H}$ determines a line bundle over $\mathcal{H}$, called the determinant line bundle of $\{D_t\}$ and denoted $\det D$; see [19, Section A.2] and [29] for a construction. More specifically, if X, Y are Banach spaces and $D : X \to Y$ is a Fredholm operator, the determinant line of D is defined as

$$\det(D) \equiv \Lambda^{\mathrm{top}}_{\mathbb{R}} \ker(D) \otimes \big(\Lambda^{\mathrm{top}}_{\mathbb{R}} \mathrm{cok}(D)\big)^*.$$

A short exact sequence of Fredholm operators

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle D'} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D''} & & \\ 0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0 \end{array}$$

determines a canonical isomorphism

$$\det(D) \cong \det(D') \otimes \det(D''). \tag{2.3}$$

For a continuous family of Fredholm operators $D_t : X_t \to Y_t$ parametrized by a topological space $\mathcal{H}$, the determinant lines $\det(D_t)$ form a line bundle over $\mathcal{H}$. For a short exact sequence of such families, the isomorphisms (2.3) give rise to a canonical isomorphism between determinant line bundles.

Remark 2.1. Families of real Cauchy-Riemann operators often arise by pulling back data from a target manifold by smooth maps as follows. Suppose (X, ϕ, J) is an almost complex manifold with an anti-complex involution $\phi\colon X \longrightarrow X$ and $(V, \tilde{\phi}) \longrightarrow (X, \phi)$ is a real bundle pair. Let ∇ be a connection in V and

$$A \in \Gamma\big(X; \mathrm{Hom}_{\mathbb{R}}(V, (T^*X, J)^{0,1} \otimes_{\mathbb{C}} V)\big)^{\tilde{\phi}}.$$

For any real map $u\colon (\Sigma, \sigma) \longrightarrow (X, \phi)$ and $\mathfrak{j} \in \mathcal{J}_\sigma$, let ∇^u denote the induced connection in u^*V and

$$A_{\mathfrak{j};u} = A \circ \partial_{\mathfrak{j}} u \in \Gamma(\Sigma; \mathrm{Hom}_{\mathbb{R}}(u^*V, (T^*\Sigma, \mathfrak{j})^{0,1} \otimes_{\mathbb{C}} u^*V))^{u^*\tilde{\phi}}.$$

The homomorphisms

$$\bar{\partial}_u^{\nabla} = \frac{1}{2}(\nabla^u + \mathfrak{i} \circ \nabla^u \circ \mathfrak{j}), \;\; D_u \equiv \bar{\partial}_u^{\nabla} + A_{\mathfrak{j};u} \colon \Gamma(\Sigma; u^*V)^{u^*\tilde{\phi}} \longrightarrow \Gamma_{\mathfrak{j}}^{0,1}(\Sigma; u^*V)^{u^*\tilde{\phi}}$$

are real Cauchy-Riemann operators on $u^*(V, \tilde{\phi}) \longrightarrow (\Sigma, \sigma)$ that form families of real Cauchy-Riemann operators over families of maps. We denote the determinant line bundle of such a family by $\det D_{(V,\tilde{\phi})}$.

Denote by $\mathcal{D}_\sigma$ the group of orientation preserving diffeomorphisms of Σ commuting with the involution σ. If (X, ϕ) is a smooth manifold with an involution, $l \in \mathbb{Z}^{\geq 0}$, and $B \in H_2(X; \mathbb{Z})$, let

$$\mathfrak{B}_{g,l}(X, B)^{\phi,\sigma} \subset \mathfrak{B}_g(X)^{\phi,\sigma} \times \Sigma^{2l}$$

denote the space of real maps $u\colon (\Sigma, \sigma) \longrightarrow (X, \phi)$ with $u_*[\Sigma]_{\mathbb{Z}} = B$ and l pairs of conjugate non-real marked distinct points. We define

$$\mathcal{H}_{g,l}(X, B)^{\phi,\sigma} = \big(\mathfrak{B}_{g,l}(X, B)^{\phi,\sigma} \times \mathcal{J}_\sigma\big)/\mathcal{D}_\sigma.$$

The action of $\mathcal{D}_\sigma$ on $\mathcal{J}_\Sigma$ given by $h \cdot \mathfrak{j} = h^*\mathfrak{j}$ preserves $\mathcal{J}_\sigma$; thus, the above quotient is well-defined. If J is an almost complex structure on X such that $\phi^*J = -J$, the moduli space of marked real J-holomorphic maps in the class $B \in H_2(X; \mathbb{Z})$ is defined to be

$$\mathfrak{M}_{g,l}(X, J, B)^{\phi,\sigma} = \big\{[u, (z_1, \sigma(z_1)), \ldots, (z_l, \sigma(z_l)), \mathfrak{j}] \in \mathcal{H}_{g,l}(X, B)^{\phi,\sigma} \colon \; \bar{\partial}_{J,\mathfrak{j}} u = 0\big\},$$

where $\bar{\partial}_{J,\mathfrak{j}}$ is the usual Cauchy-Riemann operator with respect to the complex structures J on X and $\mathfrak{j}$ on Σ. If X is a point and B is zero, we denote by

$$\mathcal{M}_{g,l}^{\sigma} \equiv \mathfrak{M}_{g,l}(\mathrm{pt},0)^{\mathrm{id},\sigma} \equiv \mathcal{H}_{g,l}(\mathrm{pt},0)^{\mathrm{id},\sigma}$$

the moduli space of marked symmetric domains. There is a natural forgetful map

$$\mathfrak{f}: \mathcal{H}_{g,l}(X,B)^{\phi,\sigma} \longrightarrow \mathcal{M}_{g,l}^{\sigma}.$$

The determinant line bundle of a family of real Cauchy-Riemann operators $D_{(V,\tilde{\phi})}$ on

$$\mathfrak{B}_{g,l}(X,B)^{\phi,\sigma} \times \mathcal{J}_{\sigma}$$

induced by a real bundle pair $(V,\widetilde{\phi}) \to (X,\phi)$ as in Remark 2.1 descends to a line bundle over $\mathcal{H}_{g,l}(X,B)^{\phi,\sigma}$, which we still denote by $\det D_{(V,\tilde{\phi})}$.

Example 2.2. If $\mathfrak{c}_{\mathbb{C}}$ denotes the standard conjugation on $\mathbb{C}$ and $(V,\widetilde{\phi}) = (\mathbb{C},\mathfrak{c}_{\mathbb{C}}) \longrightarrow (\mathrm{pt},\mathrm{id})$, the induced family of operators $\bar{\partial}_{\mathbb{C}} \equiv D_{(\mathbb{C},\mathfrak{c}_{\mathbb{C}})}$ on $\mathcal{M}_{g,l}^{\sigma}$ defines a line bundle

$$\det \bar{\partial}_{\mathbb{C}} \longrightarrow \mathcal{M}_{g,l}^{\sigma}.$$

If (X,ϕ) is an almost complex manifold with anti-complex involution ϕ and

$$(V,\widetilde{\phi}) = (X \times \mathbb{C}, \phi \times \mathfrak{c}_{\mathbb{C}}) \longrightarrow (X,\phi),$$

then there is a canonical isomorphism

$$\det D_{(\mathbb{C},\mathfrak{c}_{\mathbb{C}})} \approx \mathfrak{f}^* \bar{\partial}_{\mathbb{C}}$$

of line bundles over $\mathcal{H}_{g,l}(X,B)^{\phi,\sigma}$.

Remark 2.3. For simplicity, we will assume that the action of $\mathcal{D}_{\sigma}$ has no fixed points on the relevant subspaces of $\mathfrak{B}_{g,l}(X,B)^{\phi,\sigma} \times \mathcal{J}_{\sigma}$. This happens for example if sufficiently many marked points are added to Σ. In more general cases, this issue can be avoided by working with Prym structures on Riemann surfaces; see [Loo]. This assumption ensures that $\mathfrak{M}_{g,l}(X,J,B)^{\phi,\sigma}$ is a manifold if cut out transversely and thus has a first Stiefel-Whitney class. Alternatively, if $g+2l \geq 4$, the subspace of $\mathcal{M}_{g,l}^{\sigma}$ consisting of $(\Sigma,\mathfrak{j})$ with non-trivial automorphisms is of codimension at least 2, and so $\mathfrak{M}_{g,l}(X,J,B)^{\phi,\sigma}$ is a manifold outside of subspaces of codimension 2 if cut out transversely and thus again has a first Stiefel-Whitney class.

The following example shows that the orientability of a moduli space of symmetric half-surfaces is not sufficient for the orientability of the corresponding component of the moduli space of symmetric doubles.

Example 2.4. The moduli space $\mathcal{M}_{\Sigma}^{\mathfrak{c}}$ of sh-surfaces Σ of genus 2 with one boundary component and non-trivial involution (Figure 2) is orientable by [11, Lemma 6.1] and [12, Lemma 2.1]. The natural automorphisms of $\mathcal{M}_{\Sigma}^{\mathfrak{c}}$ associated with real orientation-reversing diffeomorphisms of Σ are orientation-preserving by [11, Lemma 6.1] and [12, Corollary 2.3]. On the natural double of Σ (Figure 2), which is a symmetric surface of genus 4 with an involution σ without fixed locus, these diffeomorphisms correspond to flipping the surface across the crosscap. The real moduli space $\mathcal{M}_4^{\sigma}$,

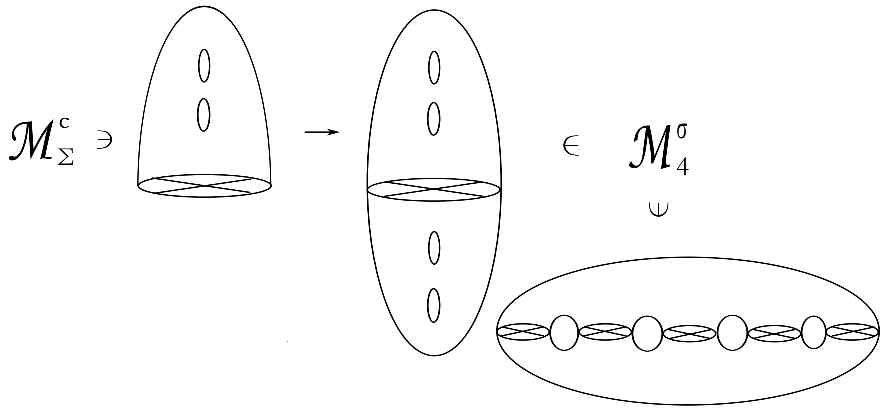

Figure 2: Orientability of crosscaps vs. real moduli spaces

parameterizing such symmetric surfaces, is not orientable. A particular loop supporting its first Stiefel-Whitney class can be described as follows. By [20, Theorem 1.2], every representative of a point in $\mathcal{M}_4^\sigma$ has 5 invariant circles which separate the surface. There is a real diffeomorphism h which fixes 3 of these circles and interchanges the other 2. By [12, Corollary 2.2], the mapping torus of h defines a loop in $\mathcal{M}_4^\sigma$ which pairs non-trivially with the first Stiefel-Whitney class of the moduli space.

3 Topological preliminaries

In this section, we establish Proposition 3.3, which relates the determinant lines of real Cauchy-Riemann operators on a real bundle pair and on its top exterior power. This is the key statement used in the proof of Theorem 1.2.

Lemma 3.1. *Let (Σ,σ) be a symmetric surface with fixed components $\Sigma_1^\sigma,\ldots,\Sigma_m^\sigma$ and $n\in\mathbb{Z}^+$ with $n\geq 3$. For every $a\in\pi_1(\mathrm{O}(n))=\mathbb{Z}_2\times\mathbb{Z}_2$ and $i=1,\ldots,m$, there is a map $\psi_i\colon\Sigma\longrightarrow\mathrm{U}(n)$ such that*

- *$\psi(z)=\overline{\psi(z)}$,*
- *ψ is the identity outside of a small neighborhood of Σ_i^σ, and*
- *$\psi_{|\Sigma_i^\sigma}=a\in\pi_1(\mathrm{O}(n))$.*

Proof. Let $S^1\times(-2,2)$ be a σ-equivariant parametrization of a neighborhood U of Σ_i^σ with $S^1\times 0$ corresponding to Σ_i^σ. Since the homomorphism $\pi_1(\mathrm{O}(n))\longrightarrow\pi_1(\mathrm{U}(n))$ induced by the inclusion is trivial, we can homotope a to the identity-valued constant map through maps $h_t\colon S^1\longrightarrow\mathrm{U}(n)$. We define ψ on U by

$$\psi(\theta,t)=\begin{cases}h_t(\theta), & \text{if } t\in[0,1];\\ I_n, & \text{if } t\in[1,2);\\ \overline{\psi(\theta,-t)}, & \text{if } t\in(-2,0];\end{cases}$$

and extend it as the identity-valued constant map over $\Sigma-U$. $\square$

If (Σ,σ) is a symmetric surface and $G\colon(\Sigma,\sigma)\longrightarrow(\Sigma,\sigma)$ is a real diffeomorphism, we define the mapping cylinder (M_G,σ_G) of G by

$$\begin{aligned} M_G&=\mathbb{I}\times\Sigma/\!\sim, && (1,z)\sim\big(0,G(z)\big)\ \ \forall\,z\in\Sigma,\\ \sigma_G\colon M_G&\longrightarrow M_G, && [s,z]\longrightarrow\big[s,\sigma(z)\big]\ \ \forall\,(s,z)\in\mathbb{I}\times\Sigma. \end{aligned}$$

Proposition 3.2. *Let (Σ,σ) be a symmetric surface, $G\colon(\Sigma,\sigma)\longrightarrow(\Sigma,\sigma)$ be a real orientation-preserving diffeomorphism, and $(W,\tilde\phi)$ be a rank n real bundle pair over (M_G,σ_G). If $n\ge3$, $W^{\tilde\phi}\longrightarrow M_G^{\sigma_G}$ is orientable, $w_2(W^{\tilde\phi})=0$, and $c_1(W)|_{\Sigma_s}=0$ for any $s\in\mathbb{I}$, there is an isomorphism*

$$(W,\tilde\phi)\approx\Lambda^{\mathrm{top}}_{\mathbb{C}}(W,\tilde\phi)\oplus(n-1)(M_G\times\mathbb{C},\sigma_G\times\mathfrak{c}_{\mathbb{C}})$$

of real bundle pairs.

Proof. By [2, Propositions 4.1, 4.2], there is an isomorphism of real bundle pairs

$$(W,\tilde\phi)\approx\big(\mathbb{I}\times\Sigma\times\mathbb{C}^n,\mathrm{id}_{\mathbb{I}}\times\sigma\times\mathfrak{c}_{\mathbb{C}^n}\big)/\!\sim_g,\quad\text{where}\quad(1,z,v)\sim_g\big(0,G(z),g(z)v\big)\quad\forall\,(z,v)\in\Sigma\times\mathbb{C}^n,$$

for some $g\colon\Sigma\longrightarrow\mathrm{U}(n)$ such that $g(\sigma(z))=\overline{g(z)}$ for all $z\in\Sigma$.

We first show that the above isomorphism can be chosen so that

$$g_{|\Sigma_i^\sigma}:\Sigma_i^\sigma\longrightarrow O(n)$$

is homotopic to the identity-valued constant map Id on each fixed component Σ_i^σ for $i=1,\ldots,m$. The map G defines a permutation on $\{\Sigma_i^\sigma\}$. Every cycle $(i_1,\ldots,i_k)$ in this permutation defines a connected component C of $M_G^{\sigma_G}$. Since $W^{\tilde\phi}$ is spin and $\mathrm{rk}(W^{\tilde\phi})\ge3$, the bundle $W^{\tilde\phi}|_C$ is trivial. Thus,

$$\sum_{l=1}^{k}\big[g_{|\Sigma_{i_l}^\sigma}\big]=1\in\pi_1(\mathrm{O}(n)).\tag{3.1}$$

For $j=2,\ldots,k$, let

$$\big[a_{i_j}\big]=\sum_{l=1}^{j-1}\big[g_{|\Sigma_{i_l}^\sigma}\big]\in\pi_1(\mathrm{O}(n))\tag{3.2}$$

and $\psi_{i_j}:\Sigma\longrightarrow\mathrm{U}(n)$ be the map constructed in Lemma 3.1 corresponding to $(i,[a])=(i_j,[a_{i_j}])$. Let

$$\Psi=\psi_{i_k}\cdot\ldots\cdot\psi_{i_2}\colon\Sigma\longrightarrow\mathrm{U}(n).$$

There is a real bundle pair isomorphism over (M_G,σ_G)

$$\begin{gathered}\big(\mathbb{I}\times\Sigma\times\mathbb{C}^n,\mathrm{id}_{\mathbb{I}}\times\sigma\times\mathfrak{c}_{\mathbb{C}^n}\big)/\!\sim_g\longrightarrow\big(\mathbb{I}\times\Sigma\times\mathbb{C}^n,\mathrm{id}_{\mathbb{I}}\times\sigma\times\mathfrak{c}_{\mathbb{C}^n}\big)/\!\sim_{\tilde g},\qquad(s,z,v)\longrightarrow(s,z,\Psi(z)v),\\ \text{where}\qquad\tilde g(z)=\Psi(G(z))g(z)\Psi^{-1}(z).\end{gathered}$$

Since $\Psi_{|\Sigma_{i_j}^\sigma}=\psi_{i_j|\Sigma_{i_j}^\sigma}$ for $j=2,\ldots,k$ and $G(\Sigma_{i_j}^\sigma)=\Sigma_{i_{j+1}}^\sigma$ for $j=1,\ldots,k-1$,

$$\big[\tilde g_{|\Sigma_{i_j}^\sigma}\big]=\big[a_{i_{j+1}}\big]+\big[g_{|\Sigma_{i_j}^\sigma}\big]+\big[a_{i_j}\big]=1\in\pi_1(O(n))\qquad\forall\ j=2,\ldots,k-1;$$

the second equality follows from (3.2). Furthermore, since $\Psi_{|\Sigma_{i_1}^\sigma} = \mathrm{Id}$ and $G(\Sigma_{i_k}^\sigma) = \Sigma_{i_1}^\sigma$,

$$\big[\tilde{g}_{|\Sigma_{i_1}^\sigma}\big] = \big[a_{i_2}\big] + \big[g_{|\Sigma_{i_1}^\sigma}\big] = 1 \in \pi_1(O(n)) \quad \text{and} \quad \big[\tilde{g}_{|\Sigma_{i_k}^\sigma}\big] = \big[g_{|\Sigma_{i_k}^\sigma}\big] + \big[a_{i_k}\big] = 1 \in \pi_1(O(n))$$

by (3.2) and by (3.1), respectively.

Let $e_1, \ldots, e_n$ be the standard coordinate basis for $\mathbb{C}^n$. We define a vector space isomorphism

$$\Xi\colon \mathbb{C}^n \longrightarrow \Lambda_{\mathbb{C}}^{\mathrm{top}}\mathbb{C}^n \oplus \mathbb{C}^{n-1} \qquad \text{by}$$
$$e_1 \longrightarrow (e_1\wedge\ldots\wedge e_n, 0), \qquad e_i \longrightarrow (0, e_{i-1}) \quad \forall\, i=2,\ldots,n.$$

In particular, the composition

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}\mathbb{C}^n \xrightarrow{\Lambda_{\mathbb{C}}^{\mathrm{top}}\Xi} \Lambda_{\mathbb{C}}^{\mathrm{top}}\big(\Lambda_{\mathbb{C}}^{\mathrm{top}}\mathbb{C}^n \oplus \mathbb{C}^{n-1}\big) = \Lambda_{\mathbb{C}}^{\mathrm{top}}\mathbb{C}^n \otimes \Lambda_{\mathbb{C}}^{\mathrm{top}}\mathbb{C}^{n-1} \longrightarrow \Lambda_{\mathbb{C}}^{\mathrm{top}}\mathbb{C}^n,$$

where the last map sends $w\otimes e_1\wedge\ldots\wedge e_{n-1}$ to w, is the identity. Define

$$f\colon \Sigma \longrightarrow \mathrm{U}(n) \qquad \text{by} \qquad f(z)\Xi\big(g(z)v\big) = \begin{pmatrix} \det g(z) & 0 \\ 0 & \mathbb{I}_{n-1} \end{pmatrix} \Xi(v) \quad \forall\,(z,v)\in\Sigma\times\mathbb{C}^n.$$

In particular, $f(z) \in \mathrm{SU}(n)$ and $f(\sigma(z)) = \overline{f(z)}$ for all $z \in \Sigma$. It is sufficient to show that f is homotopic to the constant map Id subject to these conditions.

Let (Σ^b, c) be an oriented sh-surface that doubles to (Σ, σ). Since the map g can be chosen to be homotopic to Id on each fixed component Σ_i^σ, the map $f\colon (\partial\Sigma^b)_i \longrightarrow \mathrm{O}(n)$ is homotopic to $\mathrm{Id}|_{(\partial\Sigma^b)_i}$ on each boundary component $(\partial\Sigma^b)_i$ with $|c_i| = 0$. For each boundary component $(\partial\Sigma^b)_i$ with $|c_i|=1$, the map $f_{|(\partial\Sigma^b)_i}\colon (\partial\Sigma^b)_i \longrightarrow \mathrm{SU}(n)$ is homotopic to Id; see [6, Lemma 2.3]. In both cases, the homotopies are through maps $f_t\colon (\partial\Sigma^b)_i \longrightarrow \mathrm{SU}(n)$ such that $f_t(\sigma(z))=\overline{f_t(z)}$ for all $z\in(\partial\Sigma^b)_i$. They extend over Σ^b as follows. Let $S^1\times\mathbb{I}\longrightarrow U$ be a parametrization of a (closed) neighborhood U of $(\partial\Sigma^b)_i\subset\Sigma^b$ with coordinates (θ, s) and define

$$G_t : \Sigma^b \longrightarrow \mathrm{SU}(n) \qquad \text{by} \quad G_t(z) = \begin{cases} f_{(1-s)t}(\theta)\cdot f^{-1}(\theta), & \text{if } z = (\theta, s) \in U \approx S^1\times\mathbb{I}; \\ I_n, & \text{if } z \in \Sigma^b - U. \end{cases}$$

Since $G_t((\theta,1)) = I_n$ for all t, this map is continuous. Moreover, $G_0(z) = I_n$ for all $z \in \Sigma^b$ and

$$G_t((\theta,0)) = f_t(\theta)\cdot f^{-1}(\theta)$$

is a homotopy between Id and f^{-1}. Thus, $H_t = G_t \cdot f$ is a homotopy over Σ^b extending f_t.

By the above, we may assume that f is the constant map Id on the boundary of Σ^b. Choose arcs in Σ^b with endpoints on $\partial\Sigma^b$ which cut Σ^b into a disk. Each such arc defines an element of $\pi_1(\mathrm{SU}(n), I_n) = 0$. Thus, we can homotope f to Id over the arcs while keeping it fixed at the endpoints. Similarly to the above, this homotopy extends over Σ^b. Thus, we may assume that f is the constant map Id over the boundary of the disk obtained from cutting Σ^b along the arcs. Since $\pi_2(\mathrm{SU}(n), I_n) = 0$,

$$f\colon (D^2, S^1) \longrightarrow \big(SU(n), I_n\big)$$

can be homotoped to Id as a relative map. Doubling such a homotopy f_t by the requirement that $f_t(\sigma(z))=\overline{f_t(z)}$ for all $z\in\Sigma$, we obtain the desired homotopy from f to Id over all of Σ. $\square$

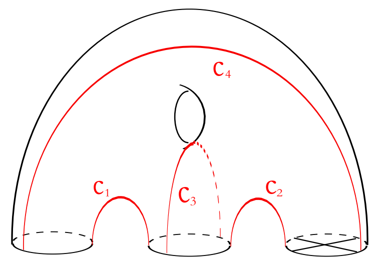

Figure 3: The arcs $c_1, \ldots, c_4$ cut Σ^b to a disk.

Proposition 3.3. *Let (Σ, σ) be a symmetric surface, $G\colon (\Sigma,\sigma) \longrightarrow (\Sigma,\sigma)$ be a real orientation-preserving diffeomorphism, and $(W,\tilde{\phi})$ be a rank n real bundle pair over (M_G, σ_G). If $n \geq 2$ and $w_2(W^{\tilde{\phi}}) = w_1(W^{\tilde{\phi}})^2$, then there is an isomorphism*

$$(W,\tilde{\phi}) \oplus \big(\Lambda^{\mathrm{top}}_{\mathbb{C}}(W,\tilde{\phi})\big)^* \approx (n+1)(M_G \times \mathbb{C}, \phi_G \times \mathfrak{c}_{\mathbb{C}})$$

of real bundle pairs.

Proof. Applying Proposition 3.2, we obtain

$$\begin{aligned}(W,\tilde{\phi}) \oplus \big(\Lambda^{\mathrm{top}}_{\mathbb{C}}(W,\tilde{\phi})\big)^* &\approx \Lambda^{\mathrm{top}}_{\mathbb{C}}\big((W,\tilde{\phi}) \oplus \big(\Lambda^{\mathrm{top}}_{\mathbb{C}}(W,\tilde{\phi})\big)^*\big) \oplus n(M_G \times \mathbb{C}, \phi_G \times \mathfrak{c}_{\mathbb{C}})\\ &= \Lambda^{\mathrm{top}}_{\mathbb{C}}(W,\tilde{\phi}) \otimes \big(\Lambda^{\mathrm{top}}_{\mathbb{C}}(W,\tilde{\phi})\big)^* \oplus n(M_G \times \mathbb{C}, \phi_G \times \mathfrak{c}_{\mathbb{C}})\\ &= (n+1)(M_G \times \mathbb{C}, \phi_G \times \mathfrak{c}_{\mathbb{C}})\,.\end{aligned}$$

This establishes the claim. □

The importance of these propositions to the orientability problem lies in the implication that they give rise to isomorphisms of the determinant bundles of $\bar{\partial}$-operators on the two sides, inducing equality on their first Stiefel-Whitney classes. This equality may still hold even without the splittings of the bundles provided by the above propositions; see the appendix.

4 Proofs of main statements

In this section, we use Proposition 3.3 to show that the first Stiefel-Whitney classes of $\mathcal{M}^{\sigma}_{g,l}$ and of the determinant line bundle for the trivial rank 1 real bundle pair over it are the same; see Proposition 4.1. This result is also obtained in [5]; see Corollaire 1.2, Corollaire 1.1, Proposition 1.4, Lemme 1.3, and Lemme 1.4 in [5]. We then use Proposition 4.1 along with Proposition 3.3 to establish Theorem 1.2. We conclude this section by deducing Corollary 1.4 from Theorem 1.2.

Proposition 4.1. *Let $g, l \in \mathbb{Z}^{\geq 0}$ be such that $g \geq 2$ and $g+2l \geq 4$. If (Σ,σ) is a genus g symmetric surface,*

$$w_1(\mathcal{M}^{\sigma}_{g,l}) = w_1(\det \bar{\partial}_{\mathbb{C}}).$$

Proof. Via the Kodaira-Spencer map, $T_{[\mathfrak{j}]}\mathcal{M}_g^\sigma$ is canonically isomorphic to $H^1_{\mathfrak{j}}(\Sigma;T\Sigma)^\sigma$; see [17, Section 3.1.2]. By Serre duality [13, p153], there is a canonical isomorphism

$$H^1_{\mathfrak{j}}(\Sigma;T\Sigma)^\sigma \approx \big(H^0_{\mathfrak{j}}(\Sigma;T^*\Sigma^{\otimes 2})^\sigma\big)^*.^1$$

Since the genus of Σ is at least 2 under our assumptions,

$$H^0_{\mathfrak{j}}(\Sigma;T^*\Sigma^{\otimes 2})^\sigma \cong \ker \bar\partial_{(T^*\Sigma,\mathrm{d}\sigma^*)^{\otimes 2}}.$$

The forgetful map

$$f: \mathcal{M}_{g,l}^\sigma \longrightarrow \mathcal{M}_g^\sigma$$

with fiber isomorphic to an open subset of Σ^l, determined by the positions of the first elements in the l pairs of conjugate points, induces an isomorphism

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}(T\mathcal{M}_{g,l}^\sigma) \approx \Lambda^{\mathrm{top}}_{\mathbb{R}}(f^{\mathrm{Vert}}) \otimes \Lambda^{\mathrm{top}}_{\mathbb{R}}(f^*T\mathcal{M}_g^\sigma).$$

Since the elements of $\mathcal{D}_\sigma$ preserve the orientation of Σ, the bundle $\Lambda^{\mathrm{top}}_{\mathbb{R}}(f^{\mathrm{Vert}})$ is orientable. Thus,

$$w_1(\mathcal{M}_{g,l}^\sigma) = w_1(f^*\ker \bar\partial_{(T^*\Sigma,\mathrm{d}\sigma^*)^{\otimes 2}}).$$

Let γ be a loop in $\mathcal{M}_{g,l}^\sigma$. Under our assumptions, γ can be taken in the smooth locus and thus lifts to a mapping torus (M_G,σ_G) for some real diffeomorphism $G\colon(\Sigma,\sigma)\longrightarrow(\Sigma,\sigma)$. Let

$$(W,\tilde\phi) = \mathbb{I}\times T\Sigma/\!\sim \longrightarrow (M_G,\sigma_G), \qquad (1,v)\sim(0,\mathrm{d}G(v))\ \forall\, v\in T\Sigma,$$

with the complex structure in the fiber of W over $s\times\Sigma$ being $\mathfrak{j}_s$. We note that

$$w_2\big(W^{\tilde\phi}\oplus W^{\tilde\phi}\big) = w_1\big(W^{\tilde\phi}\big)^2 = 0;$$

the second equality holds for the following reason. Every topological component of $M_G^{\sigma_G}$ is either a torus or a Klein bottle. The square of any class on the former is zero. In the second case, the torsion element of H_1 is represented by a fixed component Σ_i^σ in a fiber of $M_G\longrightarrow S^1$. Since $T\Sigma^\sigma$ is orientable, the restriction of $w_1(W^{\tilde\phi})$ to this class vanishes, which implies that $w_1(W^{\tilde\phi})^2=0$; see [11, Lemma 2.2], for example. Since $w_1(W^{\tilde\phi}\oplus W^{\tilde\phi})$ also vanishes, Proposition 3.3 gives

$$(W,\tilde\phi)\oplus(W,\tilde\phi)\oplus(W,\tilde\phi)^{*\otimes 2} \approx 3(M_G\times\mathbb{C},\sigma_G\times\mathfrak{c}_{\mathbb{C}}).$$

Since the indices of $\bar\partial$-operators on $2(W,\tilde\phi)$ and $2(M_G\times\mathbb{C},\sigma_G\times\mathfrak{c}_{\mathbb{C}})$ are canonically oriented,

$$w_1(\det\bar\partial_{(T^*\Sigma,\sigma^*)^{\otimes 2}}) = w_1(\det\bar\partial_{\mathbb{C}}),$$

proving the claim. □

Proof of Theorem 1.2. First assume that l is sufficiently large so that $\mathcal{M}_{g,l}^\sigma$ is a manifold; see Remark 2.3. If the moduli space $\mathfrak{M}_{g,l}(X,B)^{\phi,\sigma}$ is cut transversely, the forgetful map

$$\mathfrak{f}: \mathfrak{M}_{g,l}(X,B)^{\phi,\sigma} \longrightarrow \mathcal{M}_{g,l}^\sigma$$

[1] The real part of the Serre duality identifies the spaces of invariant sections on one side with the space of anti-invariant sections on the other; the latter is isomorphic to the space of invariant sections by multiplication by $\mathfrak{i}$.

induces the equality of first Stiefel-Whitney classes

$$w_1(\mathfrak{M}_{g,l}(X,B)^{\phi,\sigma}) = w_1(\det D_{(TX,\mathrm{d}\phi)}) + \mathfrak{f}^* w_1(\mathcal{M}_{g,l}^{\sigma}).$$

Thus, it suffices to show that

$$w_1(\det D_{(TX,\mathrm{d}\phi)}) = n\, \mathfrak{f}^* w_1(\mathcal{M}_{g,l}^{\sigma})$$

over $\mathcal{H}_{g,l}(X,B)^{\phi,\sigma}$.

By (2.3), there is a canonical isomorphism

$$\det D_{(TX\oplus 2L,\mathrm{d}\phi\oplus 2\tilde{\phi})} \approx \det D_{(TX,\mathrm{d}\phi)} \otimes \big(\det D_{(L,\tilde{\phi})}\big)^{\otimes 2}$$

and thus

$$w_1(\det D_{(TX\oplus 2L,\mathrm{d}\phi\oplus 2\tilde{\phi})}) = w_1(\det D_{(TX,\mathrm{d}\phi)}).$$

Let γ be a loop in $\mathfrak{M}_{g,l}(X,B)^{\phi,\sigma}$. Under the assumption of Remark 2.3, the projection

$$\mathfrak{B}_{g,l}(X,B)^{\phi,\sigma} \times \mathcal{J}_{\sigma} \longrightarrow \mathcal{H}_{g,l}(X,B)^{\phi,\sigma}$$

admits local slices. Thus, there exists a path $\widetilde{\gamma}_t = (u_t, j_t)$ in $\mathfrak{B}_g(X,B)^{\phi,\sigma} \times \mathcal{J}_{\sigma}$ lifting γ and a real diffeomorphism $G \in \mathcal{D}_{\sigma}$ such that $\widetilde{\gamma}_1 = G \cdot \widetilde{\gamma}_0$. Let (M_G, σ_G) be the corresponding mapping torus. By Proposition 3.3,

$$\mathrm{ev}^*(TX \oplus 2L, \mathrm{d}\phi \oplus 2\tilde{\phi}) \oplus \mathrm{ev}^*(\Lambda_{\mathbb{C}}^{\mathrm{top}}(TX \oplus 2L, \mathrm{d}\phi \oplus 2\tilde{\phi}))^* \approx (n+3)\,(M_G \times \mathbb{C}, \sigma_G \times \mathfrak{c}_{\mathbb{C}}), \tag{4.1}$$

where $\mathrm{ev}\colon M_G \longrightarrow X$ is the natural evaluation map determined by $\widetilde{\gamma}_t$. Note that

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}(TX \oplus 2L, \mathrm{d}\phi \oplus 2\tilde{\phi}) \approx \big((L,\tilde{\phi})^{\otimes 2}\big)^{\otimes 2}$$

as real bundle pairs over (X,ϕ). By Proposition 3.3,

$$2\,\mathrm{ev}^*(L,\tilde{\phi})^{\otimes 2} \oplus \big(\Lambda_{\mathbb{C}}^{\mathrm{top}}\big(2\,\mathrm{ev}^*(L,\tilde{\phi})^{\otimes 2}\big)\big)^* \approx 3(M_G \times \mathbb{C}, \sigma_G \times \mathfrak{c}_{\mathbb{C}}).$$

Since the determinant line bundles of $\bar{\partial}$-operators on $2\,\mathrm{ev}^*(L,\tilde{\phi})^{\otimes 2}$ and $2(M_G \times \mathbb{C}, \sigma_G \times \mathfrak{c}_{\mathbb{C}})$ are canonically oriented,

$$w_1\big(\det \bar{\partial}_{(\Lambda_{\mathbb{C}}^{\mathrm{top}}(TX\oplus 2L,\mathrm{d}\phi\oplus 2\tilde{\phi}))^*}\big) = w_1\big(\det \bar{\partial}_{((L,\tilde{\phi})^{\otimes 4})^*}\big) = w_1(\det \bar{\partial}_{\mathbb{C}}).$$

By (4.1),

$$w_1\big(\det \bar{\partial}_{(TX\oplus 2L,\mathrm{d}\phi\oplus 2\tilde{\phi})}\big) + w_1(\det \bar{\partial}_{\Lambda_{\mathbb{C}}^{\mathrm{top}}(TX\oplus 2L,\mathrm{d}\phi\oplus 2\tilde{\phi})}) = (n+3)\, w_1(\det \bar{\partial}_{\mathbb{C}}),$$

and thus

$$w_1(\det \bar{\partial}_{(TX\oplus 2L,\mathrm{d}\phi\oplus 2\tilde{\phi})}) = n\,(w_1(\det \bar{\partial}_{\mathbb{C}})) = n\, \mathfrak{f}^* w_1(\mathcal{M}_{g,l}^{\sigma}),$$

where the last equality holds by Proposition 4.1.

As in the proof of Propositions 4.1, the vertical bundle of the map

$$f\colon \mathfrak{M}_{g,l}(X,B)^{\phi,\sigma} \longrightarrow \mathfrak{M}_{g,l'}(X,B)^{\phi,\sigma}$$

forgetting the last $l-l'$ marked points is orientable. Thus, $\mathfrak{M}_{g,l'}(X,B)^{\phi,\sigma}$ is as orientable as $\mathfrak{M}_{g,l}(X,B)^{\phi,\sigma}$. The condition $g+2l \geq 4$ in (2) of Theorem 1.2 ensures that $w_1(\mathcal{M}_{g,l}^{\sigma})$ is defined. □

Proof of Corollary 1.4. The involutions τ_n on $\mathbb{P}^{n-1}$ and η_{2n} on $\mathbb{P}^{2n-1}$ naturally lift to involutions $\tilde{\tau}_n$ on $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ and $\tilde{\eta}_{2n}$ on $2\mathcal{O}_{\mathbb{P}^{2n-1}}(1)$ so that the usual Euler sequences for $\mathbb{P}^{n-1}$ and $\mathbb{P}^{2n-1}$ become short exact sequences of real bundle pairs:

$$0 \longrightarrow (\mathbb{P}^{n-1}\times\mathbb{C}, \tau_n\times\mathrm{id}_{\mathbb{C}}) \longrightarrow n\big(\mathcal{O}_{\mathbb{P}^{n-1}}(1), \tilde{\tau}_n\big) \longrightarrow (T\mathbb{P}^{n-1}, \mathrm{d}\tau_n) \longrightarrow 0,$$
$$0 \longrightarrow (\mathbb{P}^{2n-1}\times\mathbb{C}, \eta_{2n}\times\mathrm{id}_{\mathbb{C}}) \longrightarrow n\big(2\mathcal{O}_{\mathbb{P}^{2n-1}}(1), \tilde{\eta}_{2n}\big) \longrightarrow (T\mathbb{P}^{2n-1}, \mathrm{d}\eta_{2n}) \longrightarrow 0\,.$$

If $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ and $X_{2n;\mathbf{a}}\subset\mathbb{P}^{2n-1}$ are complete intersections preserved by the involutions τ_n and η_{2n}, respectively, the sequences

$$0 \longrightarrow (TX_{n;\mathbf{a}}, \mathrm{d}\tau_{n;\mathbf{a}}) \longrightarrow (T\mathbb{P}^{n-1}, \mathrm{d}\tau_n)\big|_{X_{n;\mathbf{a}}} \longrightarrow \bigoplus_{i=1}^{k}\big(\mathcal{O}_{\mathbb{P}^{n-1}}(1), \tilde{\tau}_n\big)^{\otimes a_i}\big|_{X_{n;\mathbf{a}}} \longrightarrow 0,$$
$$0 \longrightarrow (TX_{2n;\mathbf{a}}, \mathrm{d}\eta_{2n;\mathbf{a}}) \longrightarrow (T\mathbb{P}^{2n-1}, \mathrm{d}\eta_{2n})\big|_{X_{2n;\mathbf{a}}}$$
$$\longrightarrow \bigoplus_{a_i\in 2\mathbb{Z}}\big(\Lambda^{\mathrm{top}}_{\mathbb{C}}(\mathcal{O}_{\mathbb{P}^{2n-1}}(2), \tilde{\eta}_{2n})\big)^{\otimes(a_i/2)}\big|_{X_{2n;\mathbf{a}}} \oplus \bigoplus_{a_i'\notin 2\mathbb{Z}}\big(2\mathcal{O}_{\mathbb{P}^{2n-1}}(a_i'), \tilde{\eta}_{2n})\big)\big|_{X_{2n;\mathbf{a}}} \longrightarrow 0$$

are also short exact sequences of real bundle pairs.[2] Thus, under the assumptions of Corollary 1.4,

$$\Lambda^{\mathrm{top}}_{\mathbb{C}}\big(TX_{n;\mathbf{a}}, \mathrm{d}\tau_{n;\mathbf{a}}\big) \approx \big(\big(\mathcal{O}_{\mathbb{P}^{n-1}}(1), \tilde{\tau}_n\big)^{\otimes((n-|\mathbf{a}|)/2)}\big|_{X_{n;\mathbf{a}}}\big)^{\otimes 2},$$
$$\Lambda^{\mathrm{top}}_{\mathbb{C}}\big(TX_{2n;\mathbf{a}}, \mathrm{d}\eta_{2n;\mathbf{a}}\big) \approx \big(\big(\Lambda^{\mathrm{top}}_{\mathbb{C}}(\mathcal{O}_{\mathbb{P}^{2n-1}}(2), \tilde{\eta}_{2n})\big)^{\otimes((2n-|\mathbf{a}|)/4)}\big|_{X_{2n;\mathbf{a}}}\big)^{\otimes 2},$$

where $|\mathbf{a}|=a_1+\ldots+a_k$. We denote the rank 1 real bundle pairs being squared above (before the square is taken) by $(L_\tau,\tilde{\phi}_\tau)$ and $(L_\eta,\tilde{\phi}_\eta)$. Since $w_1(X^{\tau_n}_{n;\mathbf{a}})=0$ under the assumptions of Corollary 1.4(1) and $X^{\eta_{2n}}_{2n;\mathbf{a}}=\emptyset$,

$$w_2(X^{\tau_n}_{n;\mathbf{a}}) = \left(\binom{n}{2} - \sum_{i<j} a_i a_j\right)x^2 = \left(\frac{n-|\mathbf{a}|}{2}\right)^2 x^2 = w_1(L_\tau^{\tilde{\phi}_\tau})^2\,, \quad w_2(X^{\eta_{2n}}_{2n;\mathbf{a}}) = 0 = w_1(L_\eta^{\tilde{\phi}_\eta})^2,$$

where x is the restriction of the generator of $H^1(\mathbb{R}\mathbb{P}^{n-1};\mathbb{Z}_2)$ to $X^{\tau_n}_{n;\mathbf{a}}$ in the first case; the middle equality in the first case above follows from the numerical assumptions in Corollary 1.4(1). Therefore, $(X_{n;\mathbf{a}},\omega_{n;\mathbf{a}},\tau_{n;\mathbf{a}})$ and $(X_{2n;\mathbf{a}},\omega_{2n;\mathbf{a}},\eta_{2n;\mathbf{a}})$ are real-orientable in the sense of Definition 1.1 under the assumptions in (1) and (2), respectively, of Corollary 1.4. The claim thus follows from Theorem 1.2 if the genus of Σ is at least 2 and from [12, Theorems 1.1, 1.2] otherwise. $\square$

5 Examples: real Hurwitz covers

For $d\in\mathbb{Z}^+$, a degree d Hurwitz cover of a closed connected Riemann surface $(\Sigma_0,\mathfrak{j}_0)$ is a holomorphic map $u\colon(\Sigma,\mathfrak{j})\longrightarrow(\Sigma_0,\mathfrak{j}_0)$, where $(\Sigma,\mathfrak{j})$ is another connected Riemann surface, such that

$$u_*[\Sigma]_{\mathbb{Z}} = d[\Sigma_0]_{\mathbb{Z}} \in H_2(\Sigma_0;\mathbb{Z}).$$

By Riemann-Hurwitz [13, p218], such a map u has

$$2b \equiv 2(d-1+g-dg_0)\,, \tag{5.1}$$

[2] In the second case, the odd degrees a_i come in pairs; the second sum is taken over one $a_i'=a_i$ for each such pair.

branched points, counting multiplicity. If σ_0 is an anti-holomorphic involution on $(\Sigma_0, \mathfrak{j}_0)$, a real Hurwitz cover of $(\Sigma_0, \sigma_0, \mathfrak{j}_0)$ is a real holomorphic map

$$u\colon (\Sigma, \sigma, \mathfrak{j}) \longrightarrow (\Sigma_0, \sigma_0, \mathfrak{j}_0), \tag{5.2}$$

i.e. $u\circ\sigma = \sigma_0\circ u$. In this section, we show that many moduli spaces $\mathfrak{M}(\Sigma_0, d; \mathfrak{j}_0)^{\sigma_0,\sigma}$ of Hurwitz covers, with a fixed complex target and a fixed topological domain, are orientable; see Theorem 5.3.

Example 5.1. Let $(\Sigma_0, \sigma_0, \mathfrak{j}_0)$ be a genus g_0 symmetric Riemann surface such that $\Sigma_0^{\sigma_0} = \emptyset$ and (Σ, σ) be a genus g symmetric surface. If there exists a degree d Hurwitz cover as in (5.2), then

$$d - 1 + g - dg_0 \in 2\mathbb{Z}.$$

In particular, if g is even, then g_0 is even and d is odd.

Proof. (1) Suppose first that g_0 is odd. By [20, Theorem 1.2], there are two disjoint circles $C_1', C_2' \subset \Sigma_0$ that are preserved by σ_0 and split Σ_0 into bordered surfaces interchanged by σ_0. Similarly to the case $\Sigma_0 = \mathbb{T}$, these two circles can be replaced by two circles $C_1, C_2 \subset \Sigma_0$ that are interchanged by σ_0 and still split Σ_0 into bordered surfaces Σ_0^+ and Σ_0^- interchanged by σ_0. The preimages of Σ_0^+ and Σ_0^- split Σ into bordered surfaces $\Sigma_1^+, \ldots, \Sigma_k^+$ and $\Sigma_1^-, \ldots, \Sigma_k^-$, which are interchanged by σ. Let m_i^+ be the number of boundary components of Σ_i^+. The set of boundary components of all these surfaces can be grouped into quadruples: pairs of them are identified and mapped to C_1 and conjugate pairs of them are mapped to C_2 by σ. Thus, the Euler characteristic of Σ,

$$\chi(\Sigma) = 2\big(\chi(\Sigma_1^+) + \ldots + \chi(\Sigma_k^+)\big) = 4\big(1-g(\Sigma_1^+) + \ldots + 1-g(\Sigma_k^+)\big) - 2\big(m_1^+ + \ldots + m_k^+\big),$$

is divisible by 4. This establishes the claim for g_0 odd.

(2) Suppose that g_0 is even. By [20, Theorem 1.2], there is a circle $C \subset \Sigma_0$ which is preserved by σ_0 and splits Σ_0 into bordered surfaces Σ_0^+ and Σ_0^- interchanged by σ_0. The preimages of Σ_0^+ and Σ_0^- split Σ into bordered surfaces $\Sigma_1^+, \ldots, \Sigma_k^+$ and $\Sigma_1^-, \ldots, \Sigma_k^-$, which are interchanged by σ. The set of boundary components of all these bordered surfaces can be grouped into sets of two types:

(−) pairs, in which each element corresponds to a circle in Σ preserved by σ;

(+) quadruples, in which two elements correspond to a circle in Σ not preserved by σ and the other two elements correspond to its image under σ.

We denote the number of pairs of the first type by m^- and the number of quadruples of the second type by m^+. The restriction of u to a boundary in (−) is a map $S^1 \longrightarrow S^1$ commuting with the antipodal involution and must be of odd degree. The restrictions of u to boundaries in (+) in the same quadruple are maps of the same degree. Thus, $d \equiv m^- \mod 2$. Since

$$\chi(\Sigma) = 2\big(\chi(\Sigma_1^+) + \ldots + \chi(\Sigma_k^+)\big) = 4\big(1-g(\Sigma_1^+) + \ldots + 1-g(\Sigma_k^+)\big) - 2m^- + 4m^+,$$

the genus of Σ is odd if m^- and d are even; the genus of Σ is even if m^- and d are odd. This establishes the claim for g_0 odd. $\square$

Example 5.2. We now describe the real Hurwitz double covers of $\mathbb{P}^1$. Let

$$\tau \equiv \tau_1, \eta \equiv \eta_1 \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \qquad \tau(z)=\bar{z}, \quad \eta(z)=-1/\bar{z}\,,$$

be the standard representatives of the two equivalence classes of anti-holomorphic involutions on $\mathbb{P}^1$. Denote by $\mathring{D}^2 \subset D^2$ the interior of D^2 and by $\bar{\mathbb{R}} \subset \mathbb{P}^1$ the closure of $\mathbb{R} \subset \mathbb{C}$. For each $m \in \mathbb{Z}^{\ge 0}$, set

$$U_m(\eta)=\big\{(z_1,\ldots,z_m)\in(\mathbb{P}^1)^m \colon\; z_i\neq z_j, \eta(z_j)\ \forall\, i\neq j\big\},$$
$$U_m=\big\{(z_1,\ldots,z_m)\in(\mathring{D}^2)^m \colon\; z_i\neq z_j\ \forall\, i\neq j\big\},$$

and let

$$U_m^{\mathbb{R}} \subset \big\{(x_1,\ldots,x_{2m})\in\bar{\mathbb{R}}^{2m} \colon\; x_i\neq x_j\ \forall\, i\neq j\big\}$$

be the subset of $2m$-tuples so that x_2 follows x_1, x_3 follows x_2, etc., with respect to the positive direction on $\bar{\mathbb{R}}$. The m-th symmetric group $\mathbb{S}_m$ acts freely on U_m by interchanging the coordinates. The m-th cyclic group $\mathbb{Z}_m$ acts on $U_m^{\mathbb{R}}$ by cyclically permuting pairs of consequence coordinates, i.e.

$$(x_1,\ldots,x_{2m}) \longrightarrow (x_3,x_4,\ldots,x_{2m-1},x_{2m},x_1,x_2).$$

Let $\mathbb{S}'_m$ be the group of automorphisms of $U_m(\eta)$ generated by the interchanges of the coordinates and the maps $z_i \longrightarrow -1/\bar{z}_i$ on each coordinate separately and $\mathfrak{U}_m(\eta)$ be the orientation double of $U_m(\eta)/\mathbb{S}'_m$. If (Σ,σ) is a genus g symmetric surface,

$$\mathfrak{M}(\mathbb{P}^1,2;\mathrm{j}_0)^{\tau,\sigma} \approx \begin{cases} U_1^{\mathbb{R}}\times(U_g/\mathbb{S}_g)\sqcup U_{g+1}/\mathbb{S}_{g+1}, & \text{if } |\pi_0(\Sigma^\sigma)|=1,\ g\in 2\mathbb{Z} \\ (U_2^{\mathbb{R}}/\mathbb{Z}_2)\times(U_{g-1}/\mathbb{S}_{g-1})\sqcup U_{g+1}/\mathbb{S}_{g+1}, & \text{if } |\pi_0(\Sigma^\sigma)|=2,\ g\notin 2\mathbb{Z} \\ (U_k^{\mathbb{R}}/\mathbb{Z}_k)\times(U_{g+1-k}/\mathbb{S}_{g+1-k}), & \text{if } k\equiv|\pi_0(\Sigma^\sigma)| \text{ otherwise;} \end{cases} \tag{5.3}$$

$$\mathfrak{M}(\mathbb{P}^1,2;\mathrm{j}_0)^{\eta,\sigma} \approx \begin{cases} \mathfrak{U}_{g+1}(\eta), & \text{if } \Sigma^\sigma=\emptyset,\ g\notin 2\mathbb{Z}, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{5.4}$$

In particular, all these moduli spaces are orientable.

Proof. By [13, p254], every degree 2 Hurwitz cover $\Sigma \longrightarrow \mathbb{P}^1$ not branched over $\infty \in \mathbb{P}^1$ can be written as

$$u\colon \Sigma'=\big\{(z,w)\in\mathbb{C}^2 \,:\; w^2=(z-z_1)\ldots(z-z_{2g+2})\big\} \longrightarrow \mathbb{C}, \qquad (z,w)\longrightarrow z,$$

where $z_1,\ldots,z_{2g+2}\in\mathbb{C}$ are the distinct branched points. This punctured Riemann surface Σ' is completed to a closed surface Σ by gluing two small disks

$$D^2_\pm \equiv \big\{y_\pm\in\mathbb{C}\colon\, |y'_\pm|<\delta\big\}$$

above $\infty\in\mathbb{P}^1$ by the biholomorphic maps

$$D^2_\pm-\{0\}\longrightarrow\Sigma', \qquad y_\pm \longrightarrow \big(y_\pm^{-1}, \pm y_\pm^{-(g+1)}\sqrt{(1-z_1y_\pm)\ldots(1-z_{2g+2}y_\pm)}\big), \tag{5.5}$$

with the square root defined near 1 by $\sqrt{1}=1$.

(1) If $u\colon (\Sigma,\sigma,\mathfrak{j})\longrightarrow(\mathbb{P}^1,\tau,\mathfrak{j}_0)$ is a Hurwitz double cover, the set of branched points $z_1,\ldots,z_{2g+2}\in\mathbb{C}$ is preserved by the involution τ. Thus, we can assume that

$$(z_1,\ldots,z_{2k})\in U_k^{\mathbb{R}} \qquad \text{and} \qquad (z_{2k+1},z_{2k+3},\ldots,z_{2g+1})\in U_{g+1-k}$$

for some $k=0,1,\ldots,g+1$. There are two lifts of τ to an involution on Σ,

$$\tau_\pm\colon \Sigma'\longrightarrow\Sigma', \qquad (z,w)\longrightarrow(\bar{z},\pm\bar{w}),$$

with extension over $D^2_\pm$ given by

$$\begin{aligned} \tau_+\colon D^2_\pm\longrightarrow D^2_\pm, &\qquad y_\pm\longrightarrow\bar{y}_\pm,\\ \tau_-\colon D^2_\pm\longrightarrow D^2_\mp, &\qquad y_\pm\longrightarrow\bar{y}_\pm. \end{aligned}$$

The fixed locus of the involution $\tau_\pm$ is the set of solutions of

$$\pm w^2=\big(z-z_1\big)\ldots\big(z-z_{2k}\big)\prod_{i=1}^{g+1-k}\!\big((\operatorname{Im} z_{2k-1+2i})^2+(z-\operatorname{Re} z_{2k-1+2i})^2\big)\,, \qquad z,w\in\bar{\mathbb{R}}.$$

If $k\in\mathbb{Z}^+$, the fixed locus of the involution τ_+ consists of k circles containing $\{z_{2k},z_1\}$ and $\{z_{2i},z_{2i+1}\}$ with $i=1,\ldots,k-1$, while the fixed locus of the involution τ_- consists of k circles containing $\{z_{2i-1},z_{2i+1}\}$ with $i=1,\ldots,k$. If $k=0$, $\Sigma^{\tau_-}=\emptyset$, while Σ^{τ_+} consists of circles distinguished by the sign of w if $g+1$ is even and one circle if $g+1$ is odd; see (5.5). This establishes (5.3). Since the actions of $\mathbb{S}_m$ on U_m and of $\mathbb{Z}_m$ on $U_m^{\mathbb{R}}$ are free and orientation-preserving, the quotients in (5.3) are orientable.

(2) If $u\colon (\Sigma,\sigma,\mathfrak{j})\longrightarrow(\mathbb{P}^1,\eta,\mathfrak{j}_0)$ is a Hurwitz double cover, the set of branched points $z_1,\ldots,z_{2g+2}\in\mathbb{C}$ is preserved by the involution τ. Thus, we can assume that

$$(z_1,z_2,\ldots,z_{g+1})\in U_{g+1}(\eta)\,, \qquad z_{g+1+i}=-1/\bar{z}_i \quad \forall\, i=1,\ldots,g+1\,.$$

There are precisely two lifts of η to an automorphism on Σ:

$$\eta_\pm\colon \Sigma\longrightarrow\Sigma, \qquad (z,w)\longrightarrow\big(-1/\bar{z},(z_1\ldots z_{2g+2})^{1/2}\bar{w}/\bar{z}^{g+1}\big). \tag{5.6}$$

These automorphisms are of order 4 if g is even and are involutions if g is odd. This establishes the second case in (5.4) and shows that $\mathfrak{M}(\mathbb{P}^1,2;\mathfrak{j}_0)^{\eta,\sigma}$ is some double cover of $U_{g+1}(\eta)/\mathbb{S}'_{g+1}$ in the first case in (5.4).

The automorphisms of $U_{g+1}(\eta)$ interchanging the coordinates are orientation-preserving, while those conjugating them are orientation-reversing. Thus, w_1 of $U_{g+1}(\eta)/\mathbb{S}'_{g+1}$ is supported on the loops generated by paths in $U_{g+1}(\eta)$ from z_i to $-1/\bar{z}_i$, such as

$$\mathbb{I}\longrightarrow U_{g+1}(\eta), \qquad t\longrightarrow\big(z_1,\ldots,z_{i-1},(1-t+t/|z_i|^2)\mathrm{e}^{\pi\mathrm{i}t}z_i,z_{i+1},\ldots,z_{g+1}\big).$$

The involution (5.6) along this loop is given by

$$(z,w)\longrightarrow\big(-1/\bar{z},\mathrm{e}^{\pi\mathrm{i}t}(z_1\ldots z_{2g+2})^{1/2}\bar{w}/\bar{z}^{g+1}\big), \qquad t\in\mathbb{I},$$

i.e. this loop lifts to a non-closed path in $\mathfrak{M}(\mathbb{P}^1,2;\mathfrak{j}_0)^{\eta,\sigma}$. Thus, $\mathfrak{M}(\mathbb{P}^1,2;\mathfrak{j}_0)^{\eta,\sigma}$ is orientable, and so is the orientation double cover of $U_{2g+2}/\mathbb{S}'_{2g+2}$. $\square$

Theorem 5.3. *Let* (Σ_0,σ_0) *and* (Σ,σ) *be symmetric surfaces and* $\mathfrak{j}_0$ *be a complex structure on* Σ_0 *such that* $\sigma_0^*\mathfrak{j}_0=-\mathfrak{j}_0$. *The moduli space* $\mathfrak{M}(\Sigma_0,d;\mathfrak{j}_0)^{\sigma_0,\sigma}$ *of degree* d *real Hurwitz covers*

$$(\Sigma,\sigma,\mathfrak{j})\longrightarrow(\Sigma_0,\sigma_0,\mathfrak{j}_0)$$

is orientable if either $(\Sigma_0,\sigma_0)=(\mathbb{P}^1,\tau)$, *or* $\Sigma_0=\mathbb{T}$, *or* $(\Sigma_0,\sigma_0)=(\mathbb{P}^1,\eta)$ *and* $d\le 2$.

Proof. (1) Suppose $(\Sigma_0,\sigma_0)=(\mathbb{P}^1,\tau)$. The involution τ naturally lifts to an involution $\tilde\tau$ on $\mathcal{O}_{\mathbb{P}^1}(1)$, so that the usual Euler sequence for $\mathbb{P}^1$ becomes a short exact sequence of real bundle pairs:

$$0\longrightarrow(\mathbb{P}^1\times\mathbb{C},\tau\times\mathrm{id}_{\mathbb{C}})\longrightarrow 2\big(\mathcal{O}_{\mathbb{P}^1}(1),\tilde\tau\big)\longrightarrow(T\mathbb{P}^1,\mathrm{d}\tau)\longrightarrow 0,$$

where $\mathfrak{c}_{\mathbb{C}}$ is the standard conjugation on $\mathbb{C}$. Thus,

$$\Lambda^{\mathrm{top}}_{\mathbb{C}}(T\mathbb{P}^1,\mathrm{d}\tau)\approx\big(\mathcal{O}_{\mathbb{P}^1}(1),\tilde\tau\big)^{\otimes 2}.$$

Since $(\mathbb{P}^1)^\tau=S^1$ is one-dimensional, $(\mathbb{P}^1)^\tau$ is orientable and

$$w_2(T(\mathbb{P}^1)^\tau)=0=w_1\big(\mathcal{O}_{\mathbb{P}^1}(1)^{\tilde\tau}\big)^2\in H^2((\mathbb{P}^1)^\tau;\mathbb{Z}_2)=\{0\}.$$

Thus, the conclusion in this case follows from [12, Theorems 1.1, 1.2] and Theorem 1.2.

(2) Suppose $\Sigma_0=\mathbb{T}=\mathbb{R}^2/\mathbb{Z}^2$. Since any complex structure on Σ_0 is Kahler, the line bundle $(T\mathbb{T},\mathrm{d}\sigma_0)$ admits a real square root; see [6, Proposition 1.5]. In this case, this can be explicitly seen as follows. There are three equivalence classes of orientation-reversing involutions on $\mathbb{T}$:

$$\mathfrak{t}_0,\mathfrak{t}_1,\mathfrak{t}_2\colon\mathbb{T}\longrightarrow\mathbb{T},\qquad \mathfrak{t}_0(u,v)=(u,-v),\quad \mathfrak{t}_1(u,v)=(v,u),\quad \mathfrak{t}_2(u,v)=(u+\frac{1}{2},-v);$$

see [1, Section 9], for example. In all three cases, the tangent bundle is trivial as a real bundle pair:

$$\begin{aligned}
&(T\mathbb{T},\mathrm{d}\mathfrak{t}_0)\longrightarrow(\mathbb{T}\times\mathbb{C},\mathfrak{t}_0\times\mathfrak{c}_{\mathbb{C}}), && (u,v,u',v')\longrightarrow(u,v,u'+\mathfrak{i}v'),\\
&(T\mathbb{T},\mathrm{d}\mathfrak{t}_1)\longrightarrow(\mathbb{T}\times\mathbb{C},\mathfrak{t}_1\times\mathfrak{c}_{\mathbb{C}}), && (u,v,u',v')\longrightarrow(u,v,(u'+v')+\mathfrak{i}(u'-v')),\\
&(T\mathbb{T},\mathrm{d}\mathfrak{t}_2)\longrightarrow(\mathbb{T}\times\mathbb{C},\mathfrak{t}_2\times\mathfrak{c}_{\mathbb{C}}), && (u,v,u',v')\longrightarrow(u,v,u'+\mathfrak{i}v').
\end{aligned}$$

In particular,

$$(T\mathbb{T},\mathrm{d}\mathfrak{t}_k)\approx(\mathbb{T}\times\mathbb{C},\mathfrak{t}_k\times\mathfrak{c}_{\mathbb{C}})^{\otimes 2}\qquad\forall\,k=0,1,2.$$

Since $\mathbb{T}^{\mathfrak{t}_k}$ is one-dimensional (and consists of $2-k$ circles), $\mathbb{T}^{\mathfrak{t}_k}$ is orientable and

$$w_2(T\mathbb{T}^{\mathfrak{t}_k})=0=w_1\big((\mathbb{T}\times\mathbb{C})^{\mathfrak{t}_k\times\mathfrak{c}_{\mathbb{C}}}\big)^2\in H^2(\mathbb{T}^{\mathfrak{t}_k};\mathbb{Z}_2)=\{0\}.$$

Thus, the conclusion in this case also follows from [12, Theorems 1.1, 1.2] and Theorem 1.2.

(3) A degree 1 map $(\Sigma,\sigma,\mathfrak{j})\longrightarrow(\mathbb{P}^1,\eta,\mathfrak{j}_0)$ is an isomorphism, and so it is sufficient to assume that $(\Sigma,\sigma)=(\mathbb{P}^1,\eta)$ in the degree 1 case. By the explicit description in [6, Appendix A.1],

$$\mathfrak{M}(\mathbb{P}^1,1;\mathfrak{j}_0)^{\eta,\eta}\approx\mathbb{RP}^3\,.$$

The $d=2$ case for $(\Sigma_0,\sigma_0)=(\mathbb{P}^1,\eta)$ is addressed by Example 5.2. □

Remark 5.4. The conclusions of Theorem 5.3 for $\Sigma=\mathbb{P}^1$ and $\sigma_0,\sigma=\tau,\eta$, without any degree restrictions, are implied by [12, Theorem 1.1] and are obtained in [6, Appendix A.1] by explicitly describing $\mathfrak{M}(\mathbb{P}^1,d;\mathfrak{j}_0)^{\sigma_0,\sigma}$. At this point, we are unaware of any non-orientable moduli spaces $\mathfrak{M}(\Sigma,d;\mathfrak{j}_0)^{\sigma_0,\sigma}$. It would be interesting to know which of the spaces $\mathfrak{M}(\Sigma,d;\mathfrak{j}_0)^{\sigma_0,\sigma}$ are orientable (if not all of them are) and which of them are empty and to obtain analogues of Theorem 5.3 and Example 5.1, respectively, in the most general situation. This appears to be a purely combinatorial problem about Hurwitz covers.

A Extensions of Theorem 1.2

In this appendix, we describe an extension of Theorem 1.2; see Theorem A.1 below. We make use of what can be seen as an alternative formulation of [5, Proposition 1.2], which appears to have broader applications to the orientability problem than Proposition 3.2; see Lemma A.2 below. The proof of Lemma A.2 consists of two main parts. The first reduces the relevant sign computation for a vector bundle isomorphism over an arbitrary diffeomorphism of (Σ,σ) to a sign computation for an isomorphism over the identity on Σ; the idea behind this step comes entirely from [5]. The second part of the proof is handled in completely different ways in [5] and below: the argument in [5] relies on a technical computation at the heart of [4], while ours makes use of a more topological sign computation in [12].

It is not clear to us at this point how useful the extension described in this appendix is. In particular, it does not enlarge the class of the complete intersections $X_{n;\mathbf{a}}\subset\mathbb{P}^n$ to which Corollary 1.4 applies for all $B\in H_2(X;\mathbb{Z})$. For the classes of the form $B=2B'$, with $B'\in H_2(X;\mathbb{Z})$, Theorem A.1 does extend the conclusion of Corollary 1.4(1) to all complete intersections with $a_1+\ldots+a_k\equiv n$ mod 2. However, [10, Section 2] implies that natural partial compactifications of these spaces are not generally orientable in the new cases of Corollary 1.4(1) provided by Theorem A.1.

Theorem A.1. *Let (X,ω) be a symplectic $2n$-manifold with an involution ϕ and B, J, l, and (Σ,σ) be as in the statement of Theorem 1.2. If there exist a real bundle pair $(E,\tilde\phi_E)\longrightarrow(X,\phi)$ such that*

$$w_2(TX^\phi)=w_1(E^{\tilde\phi_E})^2 \qquad \text{and} \qquad \frac{1}{2}\langle c_1(TX),B\rangle+\langle c_1(E),B\rangle\in 2\mathbb{Z} \tag{A.1}$$

and a rank 1 real bundle pair $(L,\tilde\phi_L)\longrightarrow(X,\phi)$ such that

$$\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX,\mathrm{d}\phi)=(L,\tilde\phi_L)^{\otimes 2}, \tag{A.2}$$

then the two conclusions of Theorem 1.2 still hold. Furthermore, (A.1) alone suffices if $\Sigma-\Sigma^\sigma$ is disconnected, while (A.2) alone suffices if $\Sigma^\sigma=\emptyset$.

Lemma A.2 ([5, Proposition 1.2])**.** *Let (Σ,σ) be a symmetric surface, $G\colon(\Sigma,\sigma)\longrightarrow(\Sigma,\sigma)$ be a real orientation-preserving diffeomorphism, $(W,\tilde\phi)$ be a rank n real bundle pair over (M_G,σ_G), and $(M_G^{\sigma_G})_i$, for $i=1,\ldots,m$, be the fixed components of σ_G. If $c_1(W)|_{\Sigma_s}=0$ for any $s\in\mathbb{I}$ and*

$$\sum_{i=1}^{m}\big\langle w_2(W^{\tilde\phi}),[(M_G^{\sigma_G})_i]_{\mathbb{Z}_2}\big\rangle=0, \tag{A.3}$$

then

$$w_1(\det\bar\partial_{(W,\tilde\phi)})=w_1(\det\bar\partial_{\Lambda^{\mathrm{top}}_{\mathbb{C}}(W,\tilde\phi)})+(n-1)\,w_1(\det\bar\partial_{(M_G\times\mathbb{C},\sigma_G\times\mathfrak{c}_{\mathbb{C}})}). \tag{A.4}$$

Proof. By [2, Propositions 4.1, 4.2],

$$(W,\tilde\phi)=\big(\mathbb{I}\times\Sigma\times\mathbb{C}^n,\mathrm{id}_{\mathbb{I}}\times\sigma\times\mathfrak{c}_{\mathbb{C}^n}\big)/\!\sim_{(G,g)}, \quad \text{where} \quad (1,z,v)\sim_{(G,g)}\big(0,G(z),g(z)v\big) \quad \forall\,(z,v)\in\Sigma\times\mathbb{C}^n,$$

for some map $g\colon\Sigma\longrightarrow\mathrm{U}(n)$ such that $g(\sigma(z))=\overline{g(z)}$. Let

$$\begin{aligned}\tilde G\colon\{1\}\times\Sigma\times\mathbb{C}^n&\longrightarrow\{0\}\times\Sigma\times\mathbb{C}^n, &\quad \tilde G(1,z,v)&=\big(0,G(z),g(z)v\big),\\ \det\tilde G\colon\{1\}\times\Sigma\times\mathbb{C}&\longrightarrow\{0\}\times\Sigma\times\mathbb{C}, &\quad \det\tilde G(1,z,v)&=\big(0,G(z),(\det g(z))v\big).\end{aligned}$$

Similarly to the proof of [5, Proposition 1.2], we write

$$\tilde{G} = \big\{ \det \tilde{G} \oplus G\times \mathrm{id}_{\mathbb{C}^{n-1}} \big\} \circ \big\{ \det \tilde{G} \oplus G\times \mathrm{id}_{\mathbb{C}^{n-1}} \big\}^{-1} \circ \tilde{G}\,.$$

Choose an orientation on the determinant bundle of $\bar\partial$ on $W_{|\mathbb{I}\times\Sigma}$ by a choice of trivializations as in [12, Section 4.1]. The (exponent of the) sign of the isomorphism induced by $\tilde{G}$ between the determinant lines of $\bar\partial_{(W,\tilde\phi)}$ over $s=1$ and $s=0$ is the sum of the the signs induced by the isomorphisms

$$\begin{aligned} \det \tilde{G} \oplus G\times \mathrm{id}_{\mathbb{C}^{n-1}} \colon \{1\}\times\Sigma\times\mathbb{C}^n &\longrightarrow \{0\}\times\Sigma\times\mathbb{C}^n \qquad \text{and}\\ \big\{ \det \tilde{G} \oplus G\times \mathrm{id}_{\mathbb{C}^{n-1}} \big\}^{-1} \circ \tilde{G} \colon \{1\}\times\Sigma\times\mathbb{C}^n &\longrightarrow \{1\}\times\Sigma\times\mathbb{C}^n. \end{aligned} \tag{A.5}$$

The latter map covers the identity and can be written as

$$(1,z,v) \longrightarrow \big(1,z,h(z)v\big)$$

for some $h:\Sigma\longrightarrow \mathrm{SU}(n)$ such that $h(\sigma(z))=\overline{h(z)}$. By [12, Proposition 4.2] applied with

$$(X,\phi) = \big(S^1\times\Sigma, \mathrm{id}_{S^1}\times\sigma\big), \qquad (V,\tilde\phi) = \big(\mathbb{I}\times\Sigma\times\mathbb{C}^n/\!\sim_{(\mathrm{id}_\Sigma,h)}, \sigma\times\mathfrak{c}_{\mathbb{C}^n}\big)$$

and (A.3), the sign induced by this map equals

$$\sum_{i=1}^{m} \big\langle w_2(W^{\tilde\phi}), (M_G^{\sigma_G})_i \big\rangle = 0\,;$$

the equivariant w_2 in [12, Proposition 4.2] vanishes by [6, Lemma 2.3], since h takes values in $\mathrm{SU}(n)$. The sign induced by the first map in (A.5) gives (A.4). □

Corollary A.3. *Let (Σ,σ) be a symmetric surface, $G:(\Sigma,\sigma)\longrightarrow(\Sigma,\sigma)$ be a real orientation-preserving diffeomorphism, $(W,\tilde\phi)$ be a real bundle pair over (M_G,σ_G), and $(M_G^{\sigma_G})_i$, for $i=1,\ldots,m$, be the fixed components of σ_G. If*

$$\sum_{i=1}^{m} \big\langle w_2(W^{\tilde\phi})+w_1(W^{\tilde\phi})^2, [(M_G^{\sigma_G})_i]_{\mathbb{Z}_2} \big\rangle = 0,$$

then

$$w_1(\det\bar\partial_{(W,\tilde\phi)}) + w_1(\det\bar\partial_{(\Lambda^{\mathrm{top}}_{\mathbb{C}}(W,\tilde\phi))^*}) = (n{+}1)\, w_1(\det\bar\partial_{(M_G\times\mathbb{C},\sigma_G\times\mathfrak{c}_{\mathbb{C}})}).$$

Proof. Applying Lemma A.2 with $W \oplus \big(\Lambda^{\mathrm{top}}_{\mathbb{C}}(W,\tilde\phi)\big)^*$, we obtain the result as in the proof of Proposition 3.3. □

Proof of Theorem A.1. We follow the proof of Theorem 1.2 with the bundle $TX\oplus 2E$ in place of $TX\oplus 2L$. The only difference in the proof is showing that the first Stiefel-Whitney class of the determinant bundle of a $\bar\partial$-operator on the pull-back of

$$\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX\oplus 2E, \mathrm{d}\phi\oplus 2\tilde\phi_E) = (L\otimes\Lambda^{\mathrm{top}}_{\mathbb{C}}E, \tilde\phi_L\otimes\Lambda^{\mathrm{top}}_{\mathbb{C}}\tilde\phi_E)^{\otimes 2} \equiv (W,\tilde\phi_W)$$

equals that on the trivial rank 1 real bundle pair. As shown in the proof of [12, Corollary 4.4],

$$\begin{aligned}\sum_{i=1}^{m}\big\langle w_2(W^{\tilde{\phi}_W})+w_1(W^{\tilde{\phi}_W})^2,[(M_G^{\sigma_G})_i]_{\mathbb{Z}_2}\big\rangle &= \sum_{i=1}^{m}\big\langle w_1((L\otimes\Lambda_{\mathbb{C}}^{\mathrm{top}}E)^{\tilde{\phi}_L\otimes\Lambda_{\mathbb{C}}^{\mathrm{top}}\tilde{\phi}_E})^2,[(M_G^{\sigma_G})_i]_{\mathbb{Z}_2}\big\rangle\\ &= \sum_{i=1}^{m'}\big\langle w_1((L\otimes\Lambda_{\mathbb{C}}^{\mathrm{top}}E)^{\tilde{\phi}_L\otimes\Lambda_{\mathbb{C}}^{\mathrm{top}}\tilde{\phi}_E}),[(\Sigma^\sigma)_i]_{\mathbb{Z}_2}\big\rangle,\end{aligned}$$

where Σ_i^σ, for $i=1,\ldots,m'$, are the components of Σ^σ. By [2, Propositions 4.1,4.2], the last expression is 0 if $2|c_1(L\otimes E)$. Corollary A.3 now completes the proof. □

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