Lower Bounds for Enumerative Counts of Positive-Genus Real Curves

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Complex Case

Setup

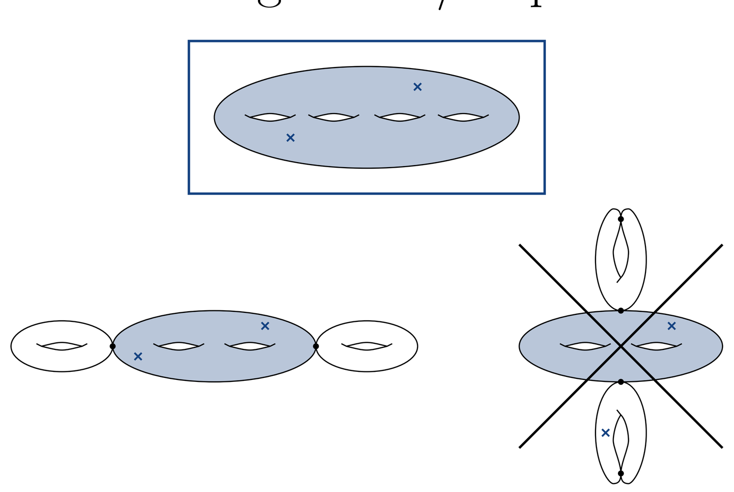
- (X,ω,J) compact (almost) Kähler 3-fold
- $B\in H_2(X;\mathbb{Z})$ "Fano" class: $\langle c_1(TX),B\rangle>0$
- $\underline{\mu}=(\mu_1,\ldots,\mu_l)\in H^*(X;\mathbb{Z})^{\oplus l}$
- $Y_1,\ldots,Y_l\subset X$ cycles representing $\mu_1,\ldots,\mu_l$

Enumerative Invariants

$\mathrm{E}^X_{g,B}(\underline{\mu})$ # smooth genus g degree B curves meeting $Y_1,\ldots,Y_l$ (counted with signs if J not integrable)

Gromov-Witten Invariants

- $\overline{\mathfrak{M}}\equiv\overline{\mathfrak{M}}_{g,l}(X,B;J)$ moduli space of degree B J-holomorphic $\Sigma_g\longrightarrow X$ with l marked points
- $\overline{\mathfrak{M}}$ has preferred orientation and virtual class
- $\mathrm{ev}_i\colon\overline{\mathfrak{M}}\longrightarrow X$ evaluation at i-th marked point
- $\mathrm{GW}^X_{g,B}(\underline{\mu})=\big\langle \prod_i \mathrm{ev}_i^*\mu_i\,,\,[\overline{\mathfrak{M}}]^{vir}\big\rangle$
- $\mathrm{GW}^X_{g,B}(\underline{\mu})\in\mathbb{Q}$ due to contributions from lower-genus curves
- Possibly contributing curves/maps:



Stratawise Contributions to GW

- $\overline{\mathfrak{M}}$ stratified by topological types of curves and distributions of marked points
- Only main "boundary" strata contribute to GW
- Each contributes # zeros of affine bundle map

  Def of curve $\longrightarrow$ Obs of map
- Direct argument used in $\mathbb{R}$ case indirect argument in $\mathbb{C}$ case

Enumerative vs GW

- "Fano" case of Gopakumar-Vafa prediction (Conjecture [5], Theorem [6]):

$$\mathrm{GW}^X_{g,B}(\underline{\mu})=\sum_{h=0}^{g} C^X_{h,B}(g-h)\,\mathrm{E}^X_{h,B}(\underline{\mu}),\qquad(\mathbb{C}1)$$

where

$$\sum_{g'=0}^{\infty} C^X_{h,B}(g')t^{2g'}=\left(\frac{\sin(t/2)}{t/2}\right)^{2h-2+c_1(B)}\qquad(\mathbb{C}2)$$

- $C^X_{h,B}(g')$'s are the degenerate contributions of [4]
- $(\mathbb{C}1)$ and $(\mathbb{C}2)$ imply that E's and GW's determine each other

Invariants of $\mathbb{P}^3$

d	1	2	3	4	5	6	7	8
$\mathrm{GW}_{0,d}$	1	0	1	4	105	2576	122129	7397760
$\mathrm{GW}_{1,d}$	$-\frac{1}{12}$	0	$-\frac{5}{12}$	$-\frac{4}{3}$	$-\frac{147}{4}$	$\frac{1496}{3}$	$\frac{1121131}{12}$	14028960
$\mathrm{GW}_{2,d}$	$\frac{1}{360}$	0	$\frac{1}{12}$	$-\frac{1}{180}$	$-\frac{49}{8}$	$-\frac{7427}{5}$	$-\frac{4905131}{45}$	-7022780
$\mathrm{GW}_{3,d}$	$-\frac{1}{20160}$	0	$-\frac{43}{4032}$	$\frac{103}{1080}$	$\frac{473}{64}$	$\frac{206873}{270}$	$\frac{283305113}{8640}$	$-\frac{110089487}{63}$
$\mathrm{GW}_{4,d}$	$\frac{1}{1814400}$	0	$\frac{713}{725760}$	$-\frac{26813}{907200}$	$-\frac{833}{320}$	$-\frac{12355247}{56700}$	$-\frac{1332337}{34560}$	$\frac{117632950}{63}$
$\mathrm{E}_{0,d}$	1	0	1	4	105	2576	122129	7397760
$\mathrm{E}_{1,d}$	0	0	0	1	42	2860	225734	23276160
$\mathrm{E}_{2,d}$	0	0	0	0	0	312	83790	18309660
$\mathrm{E}_{3,d}$	0	0	0	0	0	11	10800	6072960
$\mathrm{E}_{4,d}$	0	0	0	0	0	0	605	980100

GW- and enumerative invariants of $\mathbb{P}^3$ with point constraints

- GW's from Gathmann's *growi*
- E's from GW's by $(\mathbb{C}1)$
- 0's match Castelnuovo constraints

Real Case

Setup

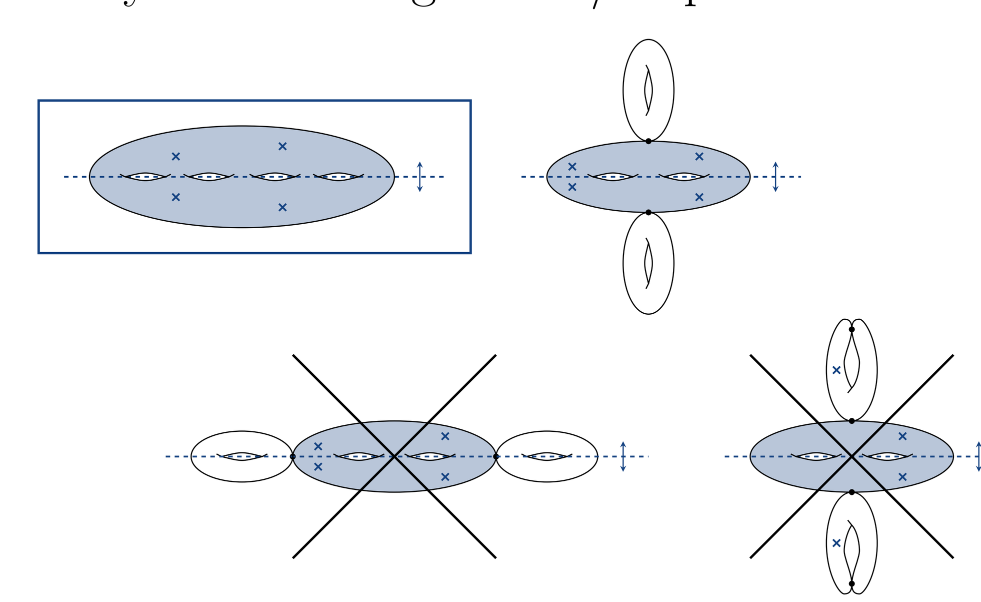
- (X,ω,J,ϕ) compact *real* (almost) Kähler 3-fold with a real orientation in the sense of [1, Dfn 1.2]
- $B\in H_2(X;\mathbb{Z})$ "Fano" class: $\langle c_1(TX),B\rangle>0$
- $\underline{\mu}=(\mu_1,\ldots,\mu_l)\in H^*(X;\mathbb{Z})^{\oplus l}$
- $Y_1,\ldots,Y_l\subset X$ cycles representing $\mu_1,\ldots,\mu_l$

Real Enumerative Invariants

$\mathrm{E}^{X,\phi}_{g,B}(\underline{\mu})$ # *real* smooth genus g degree B curves meeting $Y_1,\ldots,Y_l$ counted with signs of [1]; provide lower bounds for actual counts of such curves

Real Gromov-Witten Invariants

- $\overline{\mathfrak{M}}^{\phi}\equiv\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ moduli space of *real* degree B J-holomorphic maps $\Sigma_g\longrightarrow X$ with l conjugate pairs of marked points
- Real orientation on (X,ω,ϕ) endows $\overline{\mathfrak{M}}^{\phi}$ with orientation and virtual class
- $\mathrm{GW}^{X,\phi}_{g,B}(\underline{\mu})=\big\langle \prod_i \mathrm{ev}_i^*\mu_i\,,\,[\overline{\mathfrak{M}}^{\phi}]^{vir}\big\rangle$
- $\mathrm{GW}^{X,\phi}_{g,B}(\underline{\mu})\in\mathbb{Q}$
- Possibly contributing curves/maps :



Real Enumerative vs GW

- Real Gopakumar-Vafa for "Fano" classes [3]

$$\mathrm{GW}^{X,\phi}_{g,B}(\underline{\mu})=\sum_{\substack{0\le h\le g\\ g-h\in 2\mathbb{Z}}}\widetilde{C}^X_{h,B}\big(\tfrac{g-h}{2}\big)\mathrm{E}^{X,\phi}_{h,B}(\underline{\mu}),\qquad(\mathbb{R}1)$$

where

$$\sum_{g'=0}^{\infty}\widetilde{C}^X_{h,B}(g')t^{2g'}=\left(\frac{\sinh(t/2)}{t/2}\right)^{h-1+c_1(B)/2}\qquad(\mathbb{R}2)$$

- $(\mathbb{R}1)$ and $(\mathbb{R}2)$ imply that real E's and real GW's determine each other

Invariants of $\mathbb{P}^3$

d	1	2	3	4	5	6	7	8
$\mathrm{GW}^{\phi}_{0,d}$	1	0	-1	0	5	0	-85	0
$\mathrm{GW}^{\phi}_{1,d}$	0	0	0	-1	0	-4	0	-1000
$\mathrm{GW}^{\phi}_{2,d}$	$\frac{1}{24}$	0	$-\frac{5}{24}$	0	$\frac{15}{8}$	0	$-\frac{1345}{24}$	0
$\mathrm{GW}^{\phi}_{3,d}$	0	0	0	$-\frac{1}{3}$	0	-3	0	$-\frac{2840}{3}$
$\mathrm{GW}^{\phi}_{4,d}$	$\frac{1}{1920}$	0	$-\frac{23}{1152}$	0	$\frac{43}{128}$	0	$-\frac{2475}{128}$	0
$\mathrm{GW}^{\phi}_{5,d}$	0	0	0	$-\frac{19}{360}$	0	$-\frac{16}{15}$	0	$-\frac{1400}{3}$
$\mathrm{E}^{\phi}_{0,d}$	1	0	-1	0	5	0	-85	0
$\mathrm{E}^{\phi}_{1,d}$	0	0	0	-1	0	-4	0	-1000
$\mathrm{E}^{\phi}_{2,d}$	0	0	0	0	0	0	-10	0
$\mathrm{E}^{\phi}_{3,d}$	0	0	0	0	0	-1	0	-280
$\mathrm{E}^{\phi}_{4,d}$	0	0	0	0	0	0	-1	0
$\mathrm{E}^{\phi}_{5,d}$	0	0	0	0	0	0	0	-40

Real GW- and enum. invariants of $\mathbb{P}^3$ with point constraints

- Real GW's from $(\mathbb{C}^*)^2$-localization data of [2]
- Real E's from real GW's by $(\mathbb{R}1)$
- 0's match Castelnuovo constraints and/or "parity" vanishing of [1, Proposition 2.4]

One-partition Hodge Integral

No genus $g>0$ degree 1 curves implies

$$1+\sum_{g=1}^{\infty}t^{2g}\int_{\overline{\mathcal{M}}_{g,1}}\frac{\Lambda(x+y)\Lambda(x)\Lambda(y)}{(x+y)(x+y-\psi_1)}=\frac{\sin(t/2)}{t/2},$$

where $\Lambda(u)=\sum_{i=0}^{g}c_i(\mathbb{E}^*)u^{g-i}$ for formal variable u

References

[1] P. Georgieva and A. Zinger, *Real Gromov-Witten invariants and lower bounds in real enumerative geometry: construction*, math/1504.06617

[2] P. Georgieva and A. Zinger, *Real Gromov-Witten invariants and lower bounds in real enumerative geometry: computation*, math/1510.07568

[3] J. Niu and A. Zinger, *Lower bounds for enumerative counts of positive-genus real curves*, math/1511.02206

[4] R. Pandharipande, *Hodge integrals and degenerate contributions*, Comm. Math. Phys. 208 (1999), no. 2, 489–506

[5] R. Pandharipande, *Three questions in Gromov-Witten theory*, Proceedings of ICM, Beijing (2002), 503–512

[6] A. Zinger, *A comparison theorem for Gromov-Witten invariants in the symplectic category*, Adv. Math. 228 (2011), no. 1, 535–574