Lower Bounds for Enumerative Counts of Positive-Genus Real Curves

Jingchen Niu* and Aleksey Zinger†

September 6, 2018

Abstract

We transform the positive-genus real Gromov-Witten invariants of many real-orientable symplectic threefolds into signed counts of curves. These integer invariants provide lower bounds for counts of real curves of a given genus that pass through conjugate pairs of constraints. We conclude with some implications and related conjectures for Hodge integrals.

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*Partially supported by DMS Grants 0846978 and 1500875
†Partially supported by DMS Grant 1500875 and MPIM

1 Introduction

Gromov-Witten invariants of a symplectic manifold (X, ω) are counts of J-holomorphic curves in X; they are usually rational numbers. However, it is well-known that the genus 0 Gromov-Witten invariants of Fano manifolds are precisely counts of rational curves; this observation is key to enumerating rational curves in the complex projective space $\mathbb{P}^n$ in [11, Section 5] and [21, Section 10]. Positive-genus Gromov-Witten invariants of many symplectic threefolds are transformed into integer counts of J-holomorphic curves in [30] in the process of establishing the "Fano" case of the Gopakumar-Vafa prediction of [20, Conjecure 2(i)]. In this paper, we obtain a similar result for the real Gromov-Witten invariants defined in [7]; see Theorem 1.1. The resulting integer invariants count genus g real curves with certain signs and thus provide lower bounds for the actual counts of such curves. For example, we find that there are at least 10 genus 2 degree 7 real curves passing through 7 general pairs of conjugate points in $\mathbb{P}^3$ and at least 40 genus 5 degree 8 real curves passing through 8 general pairs of conjugate points in $\mathbb{P}^3$.

A real symplectic manifold is a triple (X, ω, ϕ) consisting of a symplectic manifold (X, ω) and an anti-symplectic involution ϕ such that $\phi^*\omega = -\omega$. A symmetric surface (Σ, σ) is a closed connected oriented, possibly nodal, surface Σ ($\dim_{\mathbb{R}} \Sigma = 2$) with an orientation-reversing involution σ. The fixed locus of σ is a disjoint union of circles. A real map

$$u : (\Sigma, \sigma) \longrightarrow (X, \phi)$$

is a smooth map $u : \Sigma \longrightarrow X$ such that $u \circ \sigma = \phi \circ u$. Let $\mathcal{J}_\omega$ be the space of ω-compatible almost complex structures on X and define

$$\mathcal{J}_\omega^\phi = \big\{ J \in \mathcal{J}_\omega :\ \phi^* J = -J \big\}.$$

For $g, l \in \mathbb{Z}^{\geqslant 0}$, $B \in H_2(X; \mathbb{Z})$, and $J \in \mathcal{J}_\omega^\phi$, we denote by $\overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi$ the moduli space of equivalence classes of stable degree B J-holomorphic real maps from genus g symmetric (possibly nodal) surfaces with l pairs of conjugate marked points; see [7, Section 1.1]. For each $i = 1, \ldots, l$, let

$$\mathrm{ev}_i : \overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi \longrightarrow X, \qquad \big[u, (z_1^+, z_1^-), \ldots, (z_l^+, z_l^-)\big] \longrightarrow u(z_i^+),$$

be the evaluation at the first point in the i-th pair of conjugate points.

Suppose (X, ω, ϕ) is a compact real symplectic threefold. By [7, Theorem 1.3], a real orientation on (X, ω, ϕ) in the sense of [7, Definition 1.2] determines an orientation on the moduli space $\overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi$ and endows it with a virtual class of dimension

$$\dim_{\mathbb{R}} \big[\overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi\big]^{\mathrm{vir}} = c_1(B) + 2l, \tag{1.1}$$

where $c_1(B) \equiv \langle c_1(TX), B \rangle$. If $\mu_1, \ldots, \mu_l \in H^*(X; \mathbb{Z})$, let

$$\mathrm{GW}_{g,B}^{X,\phi}(\mu_1, \ldots, \mu_l) \equiv \Big\langle \prod_{i=1}^{l} \mathrm{ev}_i^* \mu_i, \big[\overline{\mathfrak{M}}_{g,l}(X, B; J)^\phi\big]^{\mathrm{vir}} \Big\rangle \tag{1.2}$$

be the corresponding genus g degree B real GW-invariant of (X, ω, ϕ). By (1.1), this number is zero unless

$$\sum_{i=1}^{l} \dim_{\mathbb{R}} \mu_i = c_1(B) + 2l. \tag{1.3}$$

In general, this number is rational and includes contributions from lower-genus curves.

In this paper, we determine the lower-genus contributions to the number (1.2). We find that the real genus h curves with $h<g$ do not contribute to this number unless g and h have the same parity; see Propositions 2.2 and 2.3. If $h<g$ and g and h have the same parity, the genus h curves do contribute to (1.2); see Proposition 2.4. These contributions are closely related to the contributions from genus h curves to the genus $(g+h)/2$ GW-invariant in the complex setting computed in [19]; see Theorem 1.1. The integer invariants provided by Theorem 1.1 extract direct lower bounds for the counts of real curves in X from the real GW-invariants of [7].

If (X,ω,ϕ) is a compact real symplectic threefold, $g,l\in\mathbb{Z}^{\geqslant 0}$, $B\in H_2(X;\mathbb{Z})-0$, and $J\in\mathcal{J}_\omega^\phi$, let

$$\mathfrak{M}^*_{g,l}(X,B;J)^\phi\subset\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi$$

denote the subspace consisting of simple real maps from smooth domains; see [16, Section 2.5]. If

$$f_i:\,Y_i\longrightarrow X,\qquad 1\leqslant i\leqslant l,$$

are pseudocycle representatives for the Poincare duals of $\mu_1,\ldots,\mu_l$, let

$$\begin{split}\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J)^\phi &= \Big\{\big([u,(z_i^+,z_i^-)_{i=1}^l],(y_i)_{i=1}^l\big)\in\overline{\mathfrak{M}}_{g,l}\big(X,B;J\big)^\phi\times\prod_{i=1}^l Y_i:\\ &\qquad\qquad u(z_i^+)=f_i(y_i)\ \forall\,i\Big\},\\ \mathfrak{M}^*_{g,\mathbf{f}}(X,B;J)^\phi &= \overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J)^\phi\cap\bigg(\mathfrak{M}^*_{g,l}(X,B;J)^\phi\times\prod_{i=1}^l Y_i\bigg).\end{split}\tag{1.4}$$

If $\mathfrak{M}^*_{g,\mathbf{f}}(X,B;J)^\phi$ is cut out transversely, we denote its signed cardinality by $\mathrm{E}^{X,\phi}_{g,B}(J,\mathbf{f})$.

The expected dimension of $\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi$ is independent of the genus g if the (real) dimension of X is 6; see (1.1). Thus, one can mix curve counts of different genera passing through the same constraints. If $c_1(B)<0$, then the moduli space of unmarked maps has a negative expected dimension and all degree B GW-invariants vanish. This leaves the "Calabi-Yau" case, $c_1(B)=0$, and the "Fano" case, $c_1(B)>0$. For $g,h\in\mathbb{Z}^{\geqslant 0}$, define $\widetilde{C}^X_{h,B}(g)\in\mathbb{Q}$ by

$$\sum_{g=0}^\infty\widetilde{C}^X_{h,B}(g)t^{2g}=\bigg(\frac{\sinh(t/2)}{t/2}\bigg)^{h-1+c_1(B)/2}.\tag{1.5}$$

For example,

$$\widetilde{C}^X_{h,B}(0)=1,\quad \widetilde{C}^X_{h,B}(1)=\frac{2h-2+c_1(B)}{48},\quad \widetilde{C}^X_{h,B}(2)=\frac{\big(2h-2+c_1(B)\big)\big(5(2h-2+c_1(B))-4\big)}{23040}.$$

Theorem 1.1. *Suppose (X,ω,ϕ) is a compact real-orientable symplectic threefold with a choice of real orientation, $g,l\in\mathbb{Z}^{\geqslant 0}$, $B\in H_2(X;\mathbb{Z})$, and $\mu_i\in H^*(X;\mathbb{Z})$ for $1\leqslant i\leqslant l$ are such that $c_1(B)>0$ and (1.3) is satisfied.*

(1) There exists a subset $\mathcal{J}^\phi_{\mathrm{reg}}(g,B)\subset\mathcal{J}^\phi_\omega$ of the second category such that for all $h\leqslant g$:

- *the moduli space $\mathfrak{M}^*_{h,l}(X,B;J)^\phi$ consists of regular maps;*
- *for a generic choice of pseudocycle representatives $f_i\colon Y_i\longrightarrow X$ for μ_i, $\mathfrak{M}^*_{h,\mathbf{f}}(X,B;J)^\phi$ is a finite set of regular pairs $([u,(z_i^+,z_i^-)_{i=1}^l],(y_i)_{i=1}^l)$ such that u is an embedding.*

(2) The numbers $\mathrm{E}^{X,\phi}_{h,B}(\mathbf{f},J)$, with $h\leqslant g$, are independent of the choice of $J\in\mathcal{J}^\phi_{\mathrm{reg}}(g,B)$ and f_i and can thus be denoted $\mathrm{E}^{X,\phi}_{h,B}(\mu_1,\ldots,\mu_l)$.

(3) With $\widetilde{C}^{X,\phi}_{h,B}(g)$ defined by (1.5),

$$\mathrm{GW}^{X,\phi}_{g,B}(\mu_1,\ldots,\mu_l)=\sum_{\substack{0\leqslant h\leqslant g\\ g-h\in 2\mathbb{Z}}}\widetilde{C}^X_{h,B}\big(\tfrac{g-h}{2}\big)\mathrm{E}^{X,\phi}_{h,B}(\mu_1,\ldots,\mu_l). \tag{1.6}$$

With $\epsilon=0,1$, (1.6) gives

$$\begin{aligned}
\mathrm{GW}^{X,\phi}_{\epsilon,B}(\underline{\mu})&=\mathrm{E}^{X,\phi}_{\epsilon,B}(\underline{\mu}),\quad \mathrm{GW}^{X,\phi}_{2+\epsilon,B}(\underline{\mu})=\mathrm{E}^{X,\phi}_{2+\epsilon,B}(\underline{\mu})+\frac{c_1(B)-2+2\epsilon}{48}\,\mathrm{E}^{X,\phi}_{\epsilon,B}(\underline{\mu}),\\
\mathrm{GW}^{X,\phi}_{4+\epsilon,B}(\underline{\mu})&=\mathrm{E}^{X,\phi}_{4+\epsilon,B}(\underline{\mu})+\frac{c_1(B)+2+2\epsilon}{48}\,\mathrm{E}^{X,\phi}_{2+\epsilon,B}(\underline{\mu})\\
&\qquad+\frac{(c_1(B)-2+2\epsilon)(5c_1(B)-14+10\epsilon)}{23040}\,\mathrm{E}^{X,\phi}_{\epsilon,B}(\underline{\mu}),
\end{aligned} \tag{1.7}$$

where $\underline{\mu}=(\mu_1,\ldots,\mu_l)$. Different versions of the $g=0$ case of (1.7) go back to [23, 24, 6, 3]; the $g=1$ case is [9, Theorem 1.5].

The analogue of Theorem 1.1 in the complex setting is [30, Theorem 1.5]; it establishes the "Fano" case of Gopakumar-Vafa prediction of [20, Conjecure 2(i)]. The GW- and enumerate invariants are then related by the formula

$$\mathrm{GW}^X_{g,B}(\mu_1,\ldots,\mu_l)=\sum_{h=0}^{g}C^X_{h,B}(g-h)\mathrm{E}^X_{h,B}(\mu_1,\ldots,\mu_l), \tag{1.8}$$

with the coefficients defined by

$$\sum_{g=0}^{\infty}C^X_{h,B}(g)t^{2g}=\left(\frac{\sin(t/2)}{t/2}\right)^{2h-2+c_1(B)}.$$

The relation (1.8) is also invertible and determines the complex enumerative invariants from the GW-invariants.

The standard conjugation on $\mathbb{P}^3$ is given by

$$\tau_4:\mathbb{P}^3\longrightarrow\mathbb{P}^3,\quad [Z_1,\ldots,Z_4]\mapsto[\bar{Z}_1,\ldots,\bar{Z}_4].$$

Let $\ell\in H_2(\mathbb{P}^3;\mathbb{Z})$ denote the homology class of a line. By [8, Theorem 1.6] and (1.6), $\mathrm{E}^{\mathbb{P}^3,\tau_4}_{g,d\ell}=0$ if $d-g\in 2\mathbb{Z}$. In general, the genus g real GW-invariants with conjugate pairs of constraints can be computed using the virtual equivariant localization theorem of [10]; see [9, Section 4.2]. The standard complex structure $J_{\mathbb{P}^3}$ is regular in the sense of Theorem 1.1 and [30, Theorem 1.5] if

$d \geqslant 2g-1$; it may be regular for some smaller degrees as well. Thus, Theorem 1.1 provides lower bounds for the number of positive-genus real curves in $\mathbb{P}^3$. Tables 1 and 2 show some complex and real GW- and enumerative invariants of $\mathbb{P}^3$; the complex numbers are obtained from [5] and (1.8). The numbers in the two tables are consistent with each other as well as with the Castelnuovo bounds.

The first claim of Theorem 1.1 is standard; see the beginning of Section 2. The second claim follows immediately from (1.6) and $\widetilde{C}_{h,B}^X(0)=1$ for all $h\in\mathbb{Z}^+$. Thus, the key part of Theorem 1.1 is the last one. Its analogue in the complex setting, i.e. the last part of [30, Theorem 1.5], is deduced from

- [30, Theorem 1.2], which compares the GW-invariants of a manifold with the GW-invariants of a submanifold which contains all relevant curves, and
- [19, Theorem 3], which integrates the Euler classes of the relevant obstruction bundles over the moduli spaces of stable maps into smooth curves.

It should be fairly straightforward to establish the analogue of [30, Theorem 1.2] in the real setting using the approach of [30]. It does not appear so straightforward to establish the real analogue of [19, Theorem 3] using the approach of [19] though, as the topology of the Deligne-Mumford moduli spaces is less well understood.

We instead adapt a direct approach to the last part of Theorem 1.1 and determine the contribution of every (usually non-compact) stratum of the moduli space of the constrained maps to the number (1.2). We show that only the simplest possible boundary strata in fact contribute. The contributions of these boundary strata are given by the number of zeros of affine bundle maps from the associated virtual normal bundles to the obstruction bundles. In the process, we establish the real analogue of [19, Theorem 3], but using a different approach (as far as the applications of Hodge integrals go).

Propositions 2.2-2.4 in the next section decompose (1.2) into contributions from individual strata of the moduli space (1.4). We deduce (1.6) from these propositions and [19, (1)] at the end of Section 2. In Section 3, we describe the contribution of each curve $\mathcal{C}$ of genus $h<g$ to (1.2) in terms of the deformation-obstruction complex for each stratum of (1.4) associated with $\mathcal{C}$; see Proposition 3.1. We obtain Propositions 2.2-2.4 from Proposition 3.1 in Section 4 and establish the latter in Section 5. In Section 6.2, we compute low-degree real GW-invariants of $(\mathbb{P}^3,\tau_4)$ by equivariant localization. As a corollary of Theorem 1.1 and these computations, we obtain closed formulas for Hodge integrals and formulate related conjectures; see Proposition 6.5 and Conjectures 6.6 and 6.7. In contrast to the one- and two-partition Hodge integrals computed in [14, 15], the equivariant weights in the Hodge integrals of Section 6.2 do not satisfy the Calabi-Yau condition.

We would like to thank J. Bryan, R. Cavalieri, C.-C. Liu, and A. Oblomkov for discussions on Hodge integrals and the referee for a thorough reading of the paper and detailed comments. The second author is grateful to MPIM for its hospitality while this project was being completed.

d	1	2	3	4	5	6	7	8
$\mathrm{GW}_{0,d}$	1	0	1	4	105	2576	122129	7397760
$\mathrm{GW}_{1,d}$	$-\frac{1}{12}$	0	$-\frac{5}{12}$	$-\frac{4}{3}$	$-\frac{147}{4}$	$\frac{1496}{3}$	$\frac{1121131}{12}$	14028960
$\mathrm{GW}_{2,d}$	$\frac{1}{360}$	0	$\frac{1}{12}$	$-\frac{1}{180}$	$-\frac{49}{8}$	$-\frac{7427}{5}$	$-\frac{4905131}{45}$	-7022780
$\mathrm{GW}_{3,d}$	$-\frac{1}{20160}$	0	$-\frac{43}{4032}$	$\frac{103}{1080}$	$\frac{473}{64}$	$\frac{206873}{270}$	$\frac{283305113}{8640}$	$-\frac{110089487}{63}$
$\mathrm{GW}_{4,d}$	$\frac{1}{1814400}$	0	$\frac{713}{725760}$	$-\frac{26813}{907200}$	$-\frac{833}{320}$	$-\frac{12355247}{56700}$	$-\frac{1332337}{34560}$	$\frac{117632950}{63}$
$\mathrm{E}_{0,d}$	1	0	1	4	105	2576	122129	7397760
$\mathrm{E}_{1,d}$	0	0	0	1	42	2860	225734	23276160
$\mathrm{E}_{2,d}$	0	0	0	0	0	312	83790	18309660
$\mathrm{E}_{3,d}$	0	0	0	0	0	11	10800	6072960
$\mathrm{E}_{4,d}$	0	0	0	0	0	0	605	980100

Table 1: Complex GW- and enumerative invariants of $\mathbb{P}^3$ with point constraints

d	1	2	3	4	5	6	7	8
$\mathrm{GW}^{\phi}_{0,d}$	1	0	-1	0	5	0	-85	0
$\mathrm{GW}^{\phi}_{1,d}$	0	0	0	-1	0	-4	0	-1000
$\mathrm{GW}^{\phi}_{2,d}$	$\frac{1}{24}$	0	$-\frac{5}{24}$	0	$\frac{15}{8}$	0	$-\frac{1345}{24}$	0
$\mathrm{GW}^{\phi}_{3,d}$	0	0	0	$-\frac{1}{3}$	0	-3	0	$-\frac{2840}{3}$
$\mathrm{GW}^{\phi}_{4,d}$	$\frac{1}{1920}$	0	$-\frac{23}{1152}$	0	$\frac{43}{128}$	0	$-\frac{2475}{128}$	0
$\mathrm{GW}^{\phi}_{5,d}$	0	0	0	$-\frac{19}{360}$	0	$-\frac{16}{15}$	0	$-\frac{1400}{3}$
$\mathrm{E}^{\phi}_{0,d}$	1	0	-1	0	5	0	-85	0
$\mathrm{E}^{\phi}_{1,d}$	0	0	0	-1	0	-4	0	-1000
$\mathrm{E}^{\phi}_{2,d}$	0	0	0	0	0	0	-10	0
$\mathrm{E}^{\phi}_{3,d}$	0	0	0	0	0	-1	0	-280
$\mathrm{E}^{\phi}_{4,d}$	0	0	0	0	0	0	-1	0
$\mathrm{E}^{\phi}_{5,d}$	0	0	0	0	0	0	0	-40

Table 2: Real GW- and enumerative invariants of $(\mathbb{P}^3, \tau_4)$ with conjugate pairs of point constraints

2 Overview of the proof

Let (X,ω,ϕ) be a compact real-orientable symplectic threefold and $J\in\mathcal{J}_\omega^\phi$. If (Σ,σ) is a symmetric surface, $\mathfrak{j}$ is a complex structure on Σ such that $\sigma^*\mathfrak{j}=-\mathfrak{j}$, and $u\colon(\Sigma,\sigma)\longrightarrow(X,\phi)$ is a real map, we define

$$\Gamma(u)^{\phi,\sigma}\equiv\big\{\xi\in\Gamma(\Sigma;u^*TX)\colon\ \mathrm{d}\phi\circ\xi=\xi\circ\sigma\big\},$$
$$\Gamma^{0,1}(u)^{\phi,\sigma}\equiv\big\{\eta\in\Gamma\big(\Sigma;(T^*\Sigma,\mathfrak{j})^{0,1}\otimes_{\mathbb{C}}u^*TX\big)\colon\ \mathrm{d}\phi\circ\eta=\eta\circ\mathrm{d}\sigma\big\}.$$

If in addition u is $(J,\mathfrak{j})$-holomorphic, let

$$D_u^\phi:\ \Gamma(u)^{\phi,\sigma}\longrightarrow\Gamma^{0,1}(u)^{\phi,\sigma}\tag{2.1}$$

be the linearization of the real $\bar\partial_J$-operator at u.

We take the subset $\mathcal{J}_{\mathrm{reg}}^\phi(g,B)$ to consist of J in $\mathcal{J}_\omega^\phi$ so that D_u^ϕ is surjective for every simple real J-holomorphic map u from a genus h symmetric surface (Σ,σ) such that

$$h\leqslant g\qquad\text{and}\qquad\big\langle\omega,u_*[\Sigma]\big\rangle\leqslant\big\langle\omega,B\big\rangle.$$

By the proof of [31, Proposition 3.6(1)], which adapts the proof of [16, Theorem 3.1.5] to the action of ϕ, $\mathcal{J}_{\mathrm{reg}}^\phi(g,B)$ is of the second category in $\mathcal{J}_\omega^\phi$.

Let g, B, and $\mu_1,\ldots,\mu_l$ be as in Theorem 1.1, and $J\in\mathcal{J}_{\mathrm{reg}}^\phi(g,B)$. With $\mathbf{f}\equiv(f_1,\ldots,f_l)$ as before, define

$$\mathcal{M}_{h,\mathbf{f}}^\phi(B;J)\equiv\big\{\mathrm{Im}\,u\colon\ \big([u,(z_i^+,z_i^-)_{i=1}^l],(y_i)_{i=1}^l\big)\in\mathfrak{M}_{h,\mathbf{f}}^*(X,B;J)^\phi\big\}.\tag{2.2}$$

If $\mathbf{f}$ is chosen generically, then $\mathfrak{M}_{h,\mathbf{f}}^*(X,B;J)^\phi$ is cut out transversely and is a finite set for all $h\leqslant g$ by [30, Proposition 3.2]; see also [30, Section 1.2]. Furthermore, the collections (2.2) with $h\leqslant g$ consist of smooth disjoint curves meeting the pseudocycles f_i at distinct non-real points. Under these assumptions, we denote by $\mathrm{sgn}(\mathbf{u})$ the sign of

$$[\mathbf{u}]\equiv\big([u,(z_i^+,z_i^-)_{i=1}^l],(y_i)_{i=1}^l\big)\in\mathfrak{M}_{h,\mathbf{f}}^*(X,B;J)^\phi\tag{2.3}$$

and define $\mathrm{sgn}_{\mathbf{f}}(\mathrm{Im}\,u)=\mathrm{sgn}(\mathbf{u})$.

Under the assumptions of the previous paragraph,

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J)^\phi=\bigsqcup_{h=0}^{g}\ \bigsqcup_{\mathcal{C}\in\mathcal{M}_{h,\mathbf{f}}^\phi(B;J)}\overline{\mathfrak{M}}_{g,\mathbf{f}}(\mathcal{C})^\phi,\qquad\text{where}\quad\overline{\mathfrak{M}}_{g,\mathbf{f}}(\mathcal{C})^\phi\equiv\overline{\mathfrak{M}}_{g,\mathbf{f}}(\mathcal{C},[\mathcal{C}];J)^\phi.$$

The domain (Σ_u,σ_u) of an element $[\mathbf{u}]$ of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(\mathcal{C})^\phi$ has a unique smooth irreducible component $\Sigma_{u;0}$, called the center component, so that $u_0\equiv u|_{\Sigma_{u;0}}$ is non-constant. Each of the remaining components, called bubble components, has at most one node in common with $\Sigma_{u;0}$ because the elements of (2.2) are smooth curves in X. We denote by $\mathcal{T}_{\mathbf{u}}\equiv(\mathfrak{t}_{\mathbf{u}},\mathfrak{m}_{\mathbf{u}})$ the pair consisting of the topological type $\mathfrak{t}_{\mathbf{u}}$ of the real symmetric surface (Σ_u,σ_u) and the assignment $\mathfrak{m}_{\mathbf{u}}$ of l conjugate pairs of marked points to the irreducible components of Σ_u.

For each pair $\mathcal{T}\equiv(\mathfrak{t},\mathfrak{m})$ consisting of a topological type of genus g symmetric surfaces (Σ,σ) and an assignment of l conjugate pairs of marked points to the irreducible components of Σ, let

$$\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi} \equiv \big\{[\mathbf{u}]\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(\mathcal{C})^{\phi}:\ \mathcal{T}_{\mathbf{u}}=\mathcal{T}\big\}. \tag{2.4}$$

The subspaces $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ then partition $\overline{\mathfrak{M}}_{g,\mathbf{f}}(\mathcal{C})^{\phi}$ into smooth, but generally non-compact, strata. We denote by $\mathscr{T}_{g,l}(\mathcal{C})^{\phi}$ the collection of all pairs $\mathcal{T}\equiv(\mathfrak{t},\mathfrak{m})$ for which the space (2.4) is non-empty.

We will call a topological type $\mathfrak{t}$ as above basic if it describes nodal symmetric surfaces consisting of smooth bubble components attached directly to the center component. For any topological type $\mathfrak{t}$, let $\mathfrak{m}_0(\mathfrak{t})$ be the distribution that assigns all marked points to the center component. We denote by $\mathscr{T}^*_{g,l}(\mathcal{C})^{\phi}\subset\mathscr{T}_{g,l}(\mathcal{C})^{\phi}$ the set of all tuples $\mathcal{T}=\big(\mathfrak{t},\mathfrak{m}_0(\mathfrak{t})\big)$ such that $\mathfrak{t}$ is basic.

Given $J\in\mathcal{J}^{\phi}_{\mathrm{reg}}(g,B)$, let $\mathcal{A}^{\phi}_{g,l}(J)$ be the set of ϕ-invariant $\bar\partial_J$-admissible perturbations ν of $\bar\partial_J$, as in [6, Section 1], [9, Section 3.1], and [31, Section 3.1]; in the unstable case $(g,l)=(0,1)$, we take $\mathcal{A}^{\phi}_{g,l}(J)=\{0\}$. For $\nu\in\mathcal{A}^{\phi}_{g,l}(J)$, we denote by $\mathfrak{M}_{g,l}(X,B;J,\nu)^{\phi}$ the moduli space of equivalence classes of degree B (J,ν)-holomorphic real maps from smooth genus g symmetric surfaces with l pairs of conjugate marked points. Let

$$\begin{split}\mathfrak{M}_{g,\mathbf{f}}(X,B;J,\nu)^{\phi}=\Big\{\big([u,(z_i^+,z_i^-)_{i=1}^l],(y_i)_{i=1}^l\big)\in\mathfrak{M}_{g,l}\big(X,B;J,\nu\big)^{\phi}\times\prod_{i=1}^{l}Y_i:\\ u(z_i^+)=f_i(y_i)\ \forall\,i\Big\}.\end{split} \tag{2.5}$$

If $\mathbf{f}$ and ν are generic, this is a compact 0-dimensional manifold. A real orientation on (X,ω,ϕ) determines a sign for each element of (2.5). The signed cardinality ${}^{\pm}|\mathfrak{M}_{g,\mathbf{f}}(X,B;J,\nu)^{\phi}|$ of (2.5) is the number (1.2). We establish Theorem 1.1 by splitting (2.5) into contributions from the strata $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J)^{\phi}$ as described by Definition 2.1 and Proposition 2.2 below.

Let $\mathfrak{X}_{g,l}(X,B)^{\phi}$ be the configuration space of equivalence classes of stable degree B real maps from genus g symmetric (possibly nodal) surfaces with l pairs of conjugate marked points. This space is topologized as in [12, Section 3]. Define

$$\mathfrak{X}_{g,\mathbf{f}}(X,B)^{\phi}=\Big\{\big([u,(z_i^+,z_i^-)_{i=1}^l],(y_i)_{i=1}^l\big)\in\mathfrak{X}_{g,l}\big(X,B\big)^{\phi}\times\prod_{i=1}^{l}Y_i:\ u(z_i^+)=f_i(y_i)\ \forall\,i\Big\}.$$

Definition 2.1 ([13, Definition 2.4])**.** Let (X,ω,ϕ), B, g, l, and $\mathbf{f}$ be as in Theorem 1.1 and $J\in\mathcal{J}^{\phi}_{\mathrm{reg}}(g,B)$. A subset $\mathcal{Z}\subset\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J)^{\phi}$ is $\bar\partial_J$-regular if there exist $\mathrm{Cntr}^{\phi}_{\mathbf{f}}(\mathcal{Z})\in\mathbb{Q}$ and a dense open subset $\mathcal{A}^{\phi}_{\mathcal{Z}}(J)\subset\mathcal{A}^{\phi}_{g,l}(J)$ with the following property. For every $\nu\in\mathcal{A}^{\phi}_{\mathcal{Z}}(J)$, there exist

(a) a compact subset $K_\nu\subset\mathcal{Z}$,

(b) an open neighborhood $U_\nu(K)\subset\mathfrak{X}_{g,\mathbf{f}}(X,B)^{\phi}$ of each compact subset $K\subset\mathcal{Z}$, and

(c) $\epsilon_\nu(U)\in\mathbb{R}^+$ for each open subset U of $\mathfrak{X}_{g,\mathbf{f}}(X,B)^{\phi}$

such that

$${}^{\pm}\big|\mathfrak{M}_{g,\mathbf{f}}(X,B;J,t\nu)^{\phi}\cap U\big|=\mathrm{Cntr}^{\phi}_{\mathbf{f}}(\mathcal{Z})\qquad \text{if } t\in\big(0,\epsilon_\nu(U)\big),\ K_\nu\subset K\subset U\subset U_\nu(K).$$

If $\mathcal{Z}\subset\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J)^\phi$ is $\bar\partial_J$-regular, the corresponding number $\mathrm{Cntr}^\phi_{\mathbf{f}}(\mathcal{Z})$ is the $\bar\partial_J$-contribution of $\mathcal{Z}$ to (1.2).

Proposition 2.2. *Let (X,ω,ϕ), B, g,l, and $\mathbf{f}$ be as in Theorem 1.1, $J\in\mathcal{J}^\phi_{\mathrm{reg}}(g,B)$, and $\mathcal{C}$ be an element of $\mathcal{M}^\phi_{h,\mathbf{f}}(B;J)$. For each $\mathcal{T}\in\mathscr{T}_{g,l}(\mathcal{C})^\phi$, $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$ is a $\bar\partial_J$-regular subspace of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J)^\phi$. Furthermore, $\mathrm{Cntr}^\phi_{\mathbf{f}}(\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi)=0$ if $\mathcal{T}\notin\mathscr{T}^*_{g,l}(\mathcal{C})^\phi$.*

The implication of Proposition 2.2 is that

$$\begin{aligned}\mathrm{GW}^{X,\phi}_{g,B}\big(\mu_1,\ldots,\mu_l\big) &= \sum_{h=0}^{g}\sum_{\mathcal{C}\in\mathcal{M}^\phi_{h,\mathbf{f}}(B;J)}\sum_{\mathcal{T}\in\mathscr{T}_{g,l}(\mathcal{C})^\phi}\mathrm{Cntr}^\phi_{\mathbf{f}}\big(\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi\big)\\ &= \sum_{h=0}^{g}\sum_{\mathcal{C}\in\mathcal{M}^\phi_{h,\mathbf{f}}(B;J)}\sum_{\mathcal{T}\in\mathscr{T}^*_{g,l}(\mathcal{C})^\phi}\mathrm{Cntr}^\phi_{\mathbf{f}}\big(\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi\big)\,.\end{aligned}\tag{2.6}$$

It thus remains to reduce the last expression in (2.6) to the right-hand side of (1.6).

For a tuple $I\equiv(g_1,\ldots,g_m)$ of positive integers, let

$$\ell(I)\equiv m, \qquad |I|\equiv g_1+\ldots+g_m\,.$$

For $I=(g_1,\ldots,g_m)$ and $I'=(g'_1,\ldots,g'_{m'})$, denote by $\mathfrak{t}_{I,I'}(\mathcal{C})$ the topological type of nodal symmetric surfaces consisting of the symmetric surface $(\mathcal{C},\phi)$ as the center component,

- m pairs of smooth surfaces of genera $g_1,\ldots,g_m$ (the same genus in each pair) attached at conjugate points of $\mathcal{C}$, and
- m' smooth symmetric surfaces attached at real points of $\mathcal{C}$.

The set $\mathscr{T}^*_{g,l}(\mathcal{C})^\phi$ then consists of the pairs

$$\mathcal{T}_{I,I'}\equiv\Big(\mathfrak{t}_{I,I'}(\mathcal{C}),\mathfrak{m}_0\big(\mathfrak{t}_{I,I'}(\mathcal{C})\big)\Big)$$

such that $2|I|+|I'|=g-h$, where h is the genus of $\mathcal{C}$.

Proposition 2.3. *Let (X,ω,ϕ), B, g,l, $\mathbf{f}$, J, and $\mathcal{C}$ be as in Proposition 2.2. If $I\in(\mathbb{Z}^+)^m$ and $I'\in(\mathbb{Z}^+)^{m'}$ are such that $2|I|+|I'|=g-h$ and $m'>0$, then $\mathrm{Cntr}^\phi_{\mathbf{f}}(\mathfrak{M}_{\mathcal{T}_{I,I'},\mathbf{f}}(\mathcal{C})^\phi)=0$.*

For each $I=(g_1,\ldots,g_m)$, let $\mathcal{T}_I\equiv\mathcal{T}_{I,()}$ and denote by $\mathrm{Aut}(I)\subset S_m$ the stabilizer of I. For each $g'\in\mathbb{Z}^+$, let

$$\mathcal{P}(g')=\big\{(g_1,\ldots,g_m)\in(\mathbb{Z}^+)^m\colon\ g_1+\ldots+g_m=g',\ m\in\mathbb{Z}^{\geqslant0}\big\}.$$

By (2.6) and Proposition 2.3,

$$\mathrm{GW}^{X,\phi}_{g,B}\big(\mu_1,\ldots,\mu_l\big)=\sum_{\substack{0\leqslant h\leqslant g\\ g-h\in2\mathbb{Z}}}\sum_{\mathcal{C}\in\mathcal{M}^\phi_{h,\mathbf{f}}(B;J)}\sum_{I\in\mathcal{P}((g-h)/2)}\frac{|\mathrm{Aut}(I)|}{\ell(I)!}\mathrm{Cntr}^\phi_{\mathbf{f}}\big(\mathfrak{M}_{\mathcal{T}_I,\mathbf{f}}(\mathcal{C})^\phi\big)\,.\tag{2.7}$$

The last contribution is described Proposition 2.4 below.

For $g'\in\mathbb{Z}^+$, we denote by $\overline{\mathcal{M}}_{g',1}$ the Deligne-Mumford moduli space of stable genus g' one-marked curves. Let

$$\mathbb{E}, L \longrightarrow \overline{\mathcal{M}}_{g',1} \tag{2.8}$$

be the Hodge vector bundle of harmonic differentials and the universal tangent line bundle at the marked point, respectively. Let $\lambda_k = c_k(\mathbb{E})$, $\psi = c_1(L^*)$, and

$$\pi_{g'}, \pi_{\mathcal{C}} \colon \overline{\mathcal{M}}_{g',1}\times\mathcal{C} \longrightarrow \overline{\mathcal{M}}_{g',1},\, \mathcal{C} \tag{2.9}$$

be the component projection maps. Let

$$\alpha_{g'} = \int_{\overline{\mathcal{M}}_{g',1}} \lambda_{g'-1}\lambda_{g'}\bigg(\sum_{r=0}^{g'-1}(-1)^r\lambda_r\psi^{g'-1-r}\bigg).$$

For each $g_c\in\mathbb{Z}^{\geqslant 0}$, define

$$\widehat{C}^X_{h,B}(g_c) = \sum_{(g_1,\ldots,g_m)\in\mathcal{P}(g_c)} \frac{(2-2h-c_1(B))^m}{2^m m!}\prod_{i=1}^m\big((-1)^{g_i}\alpha_{g_i}\big)\,. \tag{2.10}$$

Proposition 2.4. *Let (X,ω,ϕ), B, g,l, $\mathbf{f}$, J, and $\mathcal{C}$ be as in Proposition 2.2. For each element $I\equiv(g_1,\ldots,g_m)$ of $\mathcal{P}((g-h)/2)$,*

$$\begin{aligned}&\mathrm{Cntr}^\phi_{\mathbf{f}}\big(\mathfrak{M}_{\mathcal{T}_I,\mathbf{f}}(\mathcal{C})^\phi\big)\\ &\quad= \frac{\mathrm{sgn}_{\mathbf{f}}(\mathcal{C})}{2^m|\mathrm{Aut}(I)|}\prod_{i=1}^m\Bigg((-1)^{g_i}\int_{\overline{\mathcal{M}}_{g_i,1}\times\mathcal{C}} c_{2g_i}\big(\pi^*_{g_i}\mathbb{E}^*\otimes\pi^*_{\mathcal{C}}\mathcal{N}_X\mathcal{C}\big)\bigg(\sum_{r=0}^{g_i-1}(-1)^r\lambda_r\psi^{g_i-1-r}\bigg)\Bigg),\end{aligned}$$

where $\mathcal{N}_X\mathcal{C}$ is the normal bundle of $\mathcal{C}$ in X.

We now proceed as at the end of [19, Section 2.3]. Since $\lambda^2_{g'}=0$ by [17, (5.3)],

$$c_{2g'}\big(\pi^*_{g'}\mathbb{E}^*\otimes\pi^*_{\mathcal{C}}\mathcal{N}_X\mathcal{C}\big) = -\pi^*_{g'}\big(\lambda_{g'-1}\lambda_{g'}\big)\cdot\pi^*_{\mathcal{C}}\,c_1(\mathcal{N}_X\mathcal{C}). \tag{2.11}$$

Since

$$\int_{\mathcal{C}} c_1(\mathcal{N}_X\mathcal{C}) = c_1(B) - (2-2h), \tag{2.12}$$

Proposition 2.4 thus gives

$$\mathrm{Cntr}^\phi_{\mathbf{f}}\big(\mathfrak{M}_{\mathcal{T}_I,\mathbf{f}}(\mathcal{C})^\phi\big) = \mathrm{sgn}_{\mathbf{f}}(\mathcal{C})\frac{(2-2h-c_1(B))^m}{2^m|\mathrm{Aut}(I)|}\prod_{i=1}^m\big((-1)^{g_i}\alpha_{g_i}\big)\,.$$

Combining this with (2.7) and (2.10), we obtain

$$\mathrm{GW}^{X,\phi}_{g,B}(\mu_1,\ldots,\mu_l) = \sum_{\substack{0\leqslant h\leqslant g\\ g-h\in2\mathbb{Z}}}\;\sum_{\mathcal{C}\in\mathcal{M}^\phi_{h,\mathbf{f}}(B;J)}\widehat{C}^X_{h,B}\big(\tfrac{g-h}{2}\big)\mathrm{sgn}_{\mathbf{f}}(\mathcal{C}) = \sum_{\substack{0\leqslant h\leqslant g\\ g-h\in2\mathbb{Z}}}\widehat{C}^X_{h,B}\big(\tfrac{g-h}{2}\big)\mathrm{E}^{X,\phi}_{h,B}(\mathbf{f},J)\,.$$

We show below that $\widehat{C}^X_{h,B}(g)$ satisfies (1.5). This implies (1.6) and thus establishes Theorem 1.1.

By the $g=0$ case of [19, (26)] and [19, (1)],

$$\exp\bigg(\sum_{g'=1}^{\infty}\alpha_{g'}t^{2g'}\bigg)=\bigg(\frac{\sin(t/2)}{t/2}\bigg)^{-1}.$$

Thus, by (2.10),

$$\begin{aligned}\sum_{g_c=0}^{\infty}\widehat{C}^X_{h,B}(g_c)t^{2g_c}&=\exp\bigg(\big(1-h-c_1(B)/2\big)\sum_{g'=1}^{\infty}\alpha_{g'}(\mathfrak{i}t)^{2g'}\bigg)\\&=\bigg(\frac{\sin(\mathfrak{i}t/2)}{\mathfrak{i}t/2}\bigg)^{h-1+c_1(B)/2}=\bigg(\frac{\sinh(t/2)}{t/2}\bigg)^{h-1+c_1(B)/2}.\end{aligned}$$

Thus, $\widehat{C}^X_{h,B}(g_c)=\widetilde{C}^X_{h,B}(g_c)$.

3 Stratawise contributions

Proposition 3.1 below relates the contribution of each stratum $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ in Proposition 2.2 to the zeros of a bundle map from the space $\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}$ of smoothing parameters to the obstruction bundle $\mathrm{Obs}^{\phi}_{\mathcal{T}}$. For the purposes of Proposition 3.1, we describe the structure of $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ in detail in terms of graphs.

3.1 Rooted decorated graphs

A graph (Ver, Edg) is pair consisting of a finite set Ver of vertices and an element

$$\mathrm{Edg}\in\mathrm{Sym}^m(\mathrm{Sym}^2\mathrm{Ver})$$

for some $m\in\mathbb{Z}^{\geqslant 0}$. We will view Edg as a collection of two-element subsets of Ver, called edges, but some of these subsets may contain the same element of Ver twice and Edg may contain several copies of the same two-element subset. Hereafter we use w and e to denote vertices and edges, respectively.

An S-marked decorated graph or simply decorated graph

$$\mathcal{T}=\big(\mathrm{Ver},\mathrm{Edg},S,\mathfrak{g},\mathfrak{m}\big) \tag{3.1}$$

consists of a graph (Ver, Edg), a finite set S, and maps

$$\mathfrak{g}\colon\ \mathrm{Ver}\longrightarrow\mathbb{Z}^{\geqslant 0}\qquad\text{and}\qquad\mathfrak{m}\colon\ S\longrightarrow\mathrm{Ver}.$$

We define the arithmetic genus $g_a(\mathcal{T})$ of $\mathcal{T}$ as in (3.1) by

$$g_a(\mathcal{T})=p_a+\sum_{w\in\mathrm{Ver}}\mathfrak{g}(w),$$

where p_a is the arithmetic genus of the graph (Ver, Edg). For each $w\in\mathrm{Ver}$, let

$$\mathrm{val}_w(\mathcal{T})=2\mathfrak{g}(w)+\big|\mathfrak{m}^{-1}(w)\big|+\big|\{e\in\mathrm{Edg}\colon\ w\in e\}\big|,$$

with each $e=(w,w)$ counted twice above. A decorated graph $\mathcal{T}$ as in (3.1) is called stable if $\mathrm{val}_w(\mathcal{T})\geqslant 3$ for every $w\in\mathrm{Ver}$.

A decorated subgraph of a decorated graph $\mathcal{T}$ as in (3.1) is a decorated graph

$$\mathcal{T}'=(\mathrm{Ver}',\mathrm{Edg}',S',\mathfrak{g}',\mathfrak{m}') \tag{3.2}$$

such that $\mathrm{Ver}'\subset\mathrm{Ver}$ and

- $\mathrm{Edg}'\subset\mathrm{Edg}$ is the subcollection of the edges with both vertices in Ver',
- S' is the disjoint union of $\mathfrak{m}^{-1}(\mathrm{Ver}')$ and the subcollection $\mathrm{Edg}'^{\bullet}\subset\mathrm{Edg}$ of the edges with one vertex in Ver' and the other in $\mathrm{Ver}-\mathrm{Ver}'$,
- $\mathfrak{g}'=\mathfrak{g}|_{\mathrm{Ver}'}$,
- $\mathfrak{m}'|_{\mathfrak{m}^{-1}(\mathrm{Ver}')}=\mathfrak{m}|_{\mathfrak{m}^{-1}(\mathrm{Ver}')}$ and $\mathfrak{m}'(e)=w$ if $e\in\mathrm{Edg}'^{\bullet}$ and $e\cap\mathrm{Ver}'=\{w\}$.

Thus, we choose the vertices $\mathrm{Ver}'\subset\mathrm{Ver}$ to be contained in $\mathcal{T}'$ and then cut the edges starting in Ver', but ending in $\mathrm{Ver}-\mathrm{Ver}'$, in half and thus convert them to marked points. We define the complement of a decorated subgraph $\mathcal{T}'$ as in (3.2) to be the decorated subgraph

$$(\mathcal{T}')^c=(\mathrm{Ver}'^c,\mathrm{Edg}'^c,S'^c,\mathfrak{g}'^c,\mathfrak{m}'^c) \tag{3.3}$$

of $\mathcal{T}$ with $\mathrm{Ver}'^c=\mathrm{Ver}-\mathrm{Ver}'$.

An involution σ on a decorated graph $\mathcal{T}$ as in (3.1) is an automorphism of the graph $(\mathrm{Ver},\mathrm{Edg})$ and the set S such that

$$\sigma\circ\sigma=\mathrm{id},\qquad \mathfrak{g}\circ\sigma=\mathfrak{g},\qquad \sigma\circ\mathfrak{m}=\mathfrak{m}\circ\sigma. \tag{3.4}$$

In such a case, let $\mathrm{V}^\sigma_{\mathbb{R}}(\mathcal{T})\subset\mathrm{Ver}$ and $\mathrm{E}^\sigma_{\mathbb{R}}(\mathcal{T})\subset\mathrm{Edg}$ be the subsets consisting of the fixed points of σ and

$$\mathrm{V}^\sigma_{\mathbb{C}}(\mathcal{T})\equiv\mathrm{Ver}-\mathrm{V}^\sigma_{\mathbb{R}}(\mathcal{T}),\qquad \mathrm{E}^\sigma_{\mathbb{C}}(\mathcal{T})\equiv\mathrm{Edg}-\mathrm{E}^\sigma_{\mathbb{R}}(\mathcal{T}).$$

A rooted decorated graph

$$\mathcal{T}=\big(\mathrm{Ver},\mathrm{Edg},S,\mathfrak{g},\mathfrak{m};\sigma,\mathrm{w}_0\big) \tag{3.5}$$

is a connected decorated graph with an involution σ and a special vertex $\mathrm{w}_0\in\mathrm{Ver}$, called the root, such that $\sigma(\mathrm{w}_0)=\mathrm{w}_0$ and

- there are no loops in $(\mathrm{Ver},\mathrm{Edg})$ passing through w_0 (i.e. removing any edge containing w_0 disconnects this graph), and
- $\mathrm{val}_w(\mathcal{T})\geqslant 3$ for all $w\in\mathrm{Ver}-\{\mathrm{w}_0\}$;

see Figure 1. In such a case, let $\mathrm{E}_0(\mathcal{T})\subset\mathrm{Edg}$ be the subset of edges containing the root w_0 and

$$\mathrm{E}^\sigma_{0;\mathbb{C}}(\mathcal{T})=\mathrm{E}_0(\mathcal{T})\cap\mathrm{E}^\sigma_{\mathbb{C}}(\mathcal{T}),\qquad \mathrm{E}^\sigma_{0;\mathbb{R}}(\mathcal{T})=\mathrm{E}_0(\mathcal{T})\cap\mathrm{E}^\sigma_{\mathbb{R}}(\mathcal{T})\,.$$

Since $\mathrm{E}_0(\mathcal{T})$ is preserved by the action of σ, so is $\mathrm{E}^\sigma_{0;\mathbb{C}}(\mathcal{T})$.

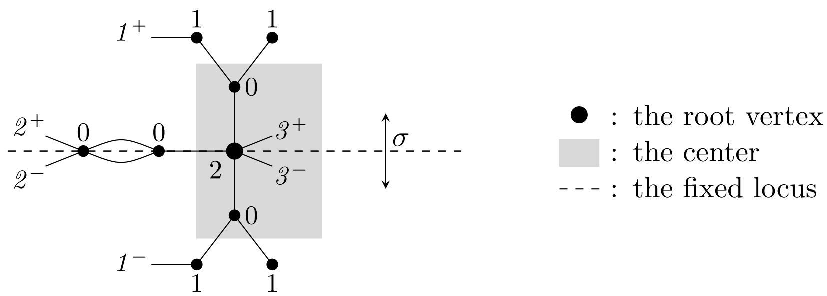

Figure 1: A rooted decorated graph representing an element $\mathcal{T}$ of $\mathscr{T}_{7,3}(\mathcal{C})^\phi$

A rooted decorated subgraph of a rooted decorated graph $\mathcal{T}$ as in (3.5) is a rooted decorated graph

$$\mathcal{T}' = \big(\mathrm{Ver}', \mathrm{Edg}', S', \mathfrak{g}', \mathfrak{m}'; \sigma', \mathrm{w}_0\big) \tag{3.6}$$

such that $(\mathrm{Ver}', \mathrm{Edg}', S', \mathfrak{g}', \mathfrak{m}')$ is a decorated subgraph of $(\mathrm{Ver}, \mathrm{Edg}, S, \mathfrak{g}, \mathfrak{m})$ and

$$\sigma' = \sigma|_{\mathrm{Ver}'\sqcup\mathrm{Edg}'\sqcup S'}.$$

The complement $(\mathcal{T}')^c$ of a rooted decorated subgraph $\mathcal{T}'$ as in (3.6) is the complement of the decorated graph $(\mathrm{Ver}', \mathrm{Edg}', S', \mathfrak{g}', \mathfrak{m}')$ in $(\mathrm{Ver}, \mathrm{Edg}, S, \mathfrak{g}, \mathfrak{m})$. This is a decorated subgraph preserved by σ.

We define the center of a rooted decorated graph $\mathcal{T}$ as in (3.5) to be the maximal rooted decorated subgraph

$$\mathcal{T}^0 \equiv \big(\mathrm{Ver}^0, \mathrm{Edg}^0, S^0, \mathfrak{g}^0, \mathfrak{m}^0; \sigma^0, \mathrm{w}_0\big) \tag{3.7}$$

such that

$$g_a(\mathcal{T}) = \mathfrak{g}(\mathrm{w}_0) + g_a\big((\mathcal{T}^0)^c\big). \tag{3.8}$$

The assumption that no loop in $(\mathrm{Ver}, \mathrm{Edg})$ contains w_0 implies that each rooted decorated graph has a well-defined center. The complement of the one-vertex decorated subgraph

$$\mathcal{T}_0 \equiv \big(\{\mathrm{w}_0\}, \varnothing, S_0, \mathfrak{g}_0, \{S_0 \to \mathrm{w}_0\}; \sigma_0, \mathrm{w}_0\big) \tag{3.9}$$

of $\mathcal{T}^0$ containing w_0 consists of connected decorated subgraphs $\mathcal{T}'_{0;e}$ of $\mathcal{T}^0$ indexed by the edges e in $\mathrm{E}_0(\mathcal{T})$. If e is in $\mathrm{E}_0(\mathcal{T}) \cap S^0$, we define $\mathcal{T}'_{0;e} = \varnothing$. The assumption that $\mathrm{val}_w(\mathcal{T}) \geqslant 3$ for all $w \in \mathrm{Ver} - \{\mathrm{w}_0\}$ and (3.8) imply that each $\mathcal{T}'_{0;e}$ is a stable graph and $g(\mathcal{T}'_{0;e}) = 0$. For each $e \in \mathrm{E}_0(\mathcal{T})$, let

$$\mathcal{T}^\sigma_{0;e} = \big(\mathrm{Ver}^\sigma_{0;e}, \mathrm{Edg}^\sigma_{0;e}, S^\sigma_{0;e}, \mathfrak{g}^\sigma_{0;e}, \mathfrak{m}^\sigma_{0;e}\big) \tag{3.10}$$

be the decorated subgraph determined by $\mathrm{Ver}^\sigma_{0;e} = \mathrm{Ver}(\mathcal{T}'_{0;e}) \cup \sigma(\mathrm{Ver}(\mathcal{T}'_{0;e}))$.

Let $\mathcal{T}$ be a rooted decorated graph as in (3.5), $\mathcal{T}^0$ be its center as in (3.7), and

$$\mathcal{T}^c \equiv \big(\mathrm{Ver}^c, \mathrm{Edg}^c, S^c, \mathfrak{g}^c, \mathfrak{m}^c\big) = \big((\mathrm{Ver}^0)^c, (\mathrm{Edg}^0)^c, (S^0)^c, (\mathfrak{g}^0)^c, (\mathfrak{m}^0)^c\big) \equiv (\mathcal{T}^0)^c \tag{3.11}$$

be its complement. Let

$$\mathrm{E}_\bullet(\mathcal{T}) \equiv \mathrm{Edg} - \big(\mathrm{Edg}^0 \sqcup \mathrm{Edg}^c\big), \quad \mathrm{E}^\sigma_{\bullet;\mathbb{C}}(\mathcal{T}) \equiv \mathrm{E}_\bullet(\mathcal{T}) \cap \mathrm{E}^\sigma_{\mathbb{C}}(\mathcal{T}), \quad \mathrm{E}^\sigma_{\bullet;\mathbb{R}}(\mathcal{T}) \equiv \mathrm{E}_\bullet(\mathcal{T}) \cap \mathrm{E}^\sigma_{\mathbb{R}}(\mathcal{T})$$

be the edges separating the center $\mathcal{T}^0$ of $\mathcal{T}$ from its complement $\mathcal{T}^c$. Thus,

$$\mathrm{Edg} = \mathrm{Edg}^0 \sqcup \mathrm{E}_\bullet(\mathcal{T}) \sqcup \mathrm{Edg}^c. \tag{3.12}$$

The decorated subgraph $\mathcal{T}^c$ is the disjoint union of the connected decorated subgraphs $\mathcal{T}'_{\bullet;e}$ indexed by the edges $e\in\mathrm{E}_\bullet(\mathcal{T})$. The assumption that $\mathrm{val}_w(\mathcal{T})\geqslant 3$ for all $w\in\mathrm{Ver}-\mathrm{w}_0$ implies that each $\mathcal{T}'_{\bullet;e}$ is a stable graph. For each $e\in\mathrm{E}_\bullet(\mathcal{T})$, let

$$\mathcal{T}^\sigma_{\bullet;e} = \big(\mathrm{Ver}^\sigma_{\bullet;e}, \mathrm{Edg}^\sigma_{\bullet;e}, S^\sigma_{\bullet;e}, \mathfrak{g}^\sigma_{\bullet;e}, \mathfrak{m}^\sigma_{\bullet;e}\big) \tag{3.13}$$

be the decorated subgraph determined by $\mathrm{Ver}^\sigma_{\bullet;e} = \mathrm{Ver}(\mathcal{T}'_{\bullet;e}) \cup \mathrm{Ver}(\sigma(\mathcal{T}'_{\bullet;e}))$. Define

$$\begin{gathered} \langle\cdot\rangle\colon \mathrm{E}_\bullet(\mathcal{T}) \longrightarrow \mathrm{E}_0(\mathcal{T}) \qquad \text{by} \qquad e \in S^\sigma_{0;\langle e\rangle} \cup \{\langle e\rangle\} \quad \forall\ e\in\mathrm{E}_\bullet(\mathcal{T})\,, \\ \mathrm{Aut}_\bullet(\mathcal{T}) = \big\{g\in\mathrm{Aut}(\mathcal{T})\colon g\cdot e = e\ \forall e\in\mathrm{E}_\bullet(\mathcal{T})\big\}, \quad \mathrm{Aut}^*(\mathcal{T}) = \mathrm{Aut}(\mathcal{T})/\mathrm{Aut}_\bullet(\mathcal{T})\,. \end{gathered} \tag{3.14}$$

3.2 The strata of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(\mathcal{C})^\phi$

We recall the notion of a nodal g_0-doublet introduced in [9]. It is a two-component nodal symmetric surface (Σ,σ) of the form

$$\Sigma \equiv \Sigma_1 \sqcup \Sigma_2 \equiv \{1\}\times\Sigma_0 \sqcup \{2\}\times\overline{\Sigma}_0, \qquad \sigma(i,z) = \big(3-i, z\big) \ \ \forall\ (i,z)\in\Sigma, \tag{3.15}$$

where Σ_0 is a connected oriented, possibly nodal, genus g_0 surface and $\overline{\Sigma}_0$ denotes Σ_0 with the opposite orientation. The arithmetic genus of a g_0-doublet is $2g_0$–1. The components of Σ are ordered.

If S_1 and S_2 are finite sets with a fixed bijection σ_S between them, let $\overline{\mathcal{M}}^\bullet_{2g_0-1,(S_1\sqcup S_2,\sigma_S)}$ denote the moduli space of stable nodal g_0-doublets with the first component carrying the S_1-marked points and with the marked points interchanged by the involution σ_S. Similarly, if S is a finite set with an involution σ_S and $g\in\mathbb{Z}^{\geqslant 0}$ is such that $2g+|S|\geqslant 3$, we denote by $\mathbb{R}\overline{\mathcal{M}}_{g,(S,\sigma_S)}$ the Deligne-Mumford space of real genus g curves with marked points indexed by S and their type (real vs. conjugate) characterized by σ_S. Let

$$L_i,\ \mathbb{E} \longrightarrow \overline{\mathcal{M}}^\bullet_{2g_0-1,(S_1\sqcup S_2,\sigma_S)},\ \mathbb{R}\overline{\mathcal{M}}_{g,(S,\sigma_S)} \tag{3.16}$$

be the universal tangent line bundle for the i-th marked point for each $i\in S_1\sqcup S_2, S$ and the Hodge vector bundles of holomorphic differentials, respectively.

For each $e\in\mathrm{E}^\sigma_{0;\mathbb{C}}(\mathcal{T}), \mathrm{E}^\sigma_{\bullet;\mathbb{C}}(\mathcal{T})$, we denote by

$$\mathcal{M}_{\mathcal{T},\sigma;e} \subset \overline{\mathcal{M}}^\bullet_{g_a(\mathcal{T}^\sigma_{0;e}),(S^\sigma_{0;e},\sigma)}, \overline{\mathcal{M}}^\bullet_{g_a(\mathcal{T}^\sigma_{\bullet;e}),(S^\sigma_{\bullet;e},\sigma)}$$

the open stratum consisting of the symmetric surfaces (Σ,σ) with σ-compatible complex structure whose dual graph is $\mathcal{T}^\sigma_{0;e}$ or $\mathcal{T}^\sigma_{\bullet;e}$ with involution σ. For each $e\in\mathrm{E}^\sigma_{0;\mathbb{R}}(\mathcal{T}), \mathrm{E}^\sigma_{\bullet;\mathbb{R}}(\mathcal{T})$, let

$$\mathcal{M}_{\mathcal{T},\sigma;e} \subset \mathbb{R}\overline{\mathcal{M}}_{g_a(\mathcal{T}^\sigma_{0;e}),(S^\sigma_{0;e},\sigma)}, \mathbb{R}\overline{\mathcal{M}}_{g_a(\mathcal{T}^\sigma_{\bullet;e}),(S^\sigma_{\bullet;e},\sigma)}$$

be the analogous open stratum. If $e\in\mathrm{E}_0(\mathcal{T})$ and $\mathcal{T}^\sigma_{0;e} = \varnothing$, we define $\mathcal{M}_{\mathcal{T},\sigma;e}$ to consist of one point.

We now return to the setting of Proposition 2.2. If $\mathbf{f}$ is generic, the smooth embedded real curve $\mathcal{C}$ in (X,ϕ) intersects the images of $f_1,\ldots,f_l$ at distinct non-real points

$$z_1^+(\mathcal{C})\equiv\mathcal{C}\cap f_1(Y_1), \qquad \ldots, \qquad z_l^+(\mathcal{C})\equiv\mathcal{C}\cap f_l(Y_l).$$

Let $z_i^-(\mathcal{C})=\phi(z_i^+(\mathcal{C}))$. The dual graph of an element $[\mathbf{u}]$ of $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$ is a rooted decorated graph which is completely determined by the pair $\mathcal{T}=(\mathfrak{t},\mathfrak{m})$. The center $\mathcal{T}^0$ of $\mathcal{T}$ corresponds to the maximal connected union Σ_u^0 of the irreducible components of the domain Σ_u of each element of $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$ such that Σ_u^0 contains the center component $\Sigma_{u;0}$. The topological components of the complement of $\Sigma_{u;0}$ in Σ_u^0 are the trees $\widehat{\Sigma}_{u;e}^0$ of spheres corresponding to the graphs $\mathcal{T}'_{0;e}$ with $e\in\mathrm{E}_0(\mathcal{T})-S^0$; we define $\widehat{\Sigma}_{u;e}^0=\varnothing$ if $e\in\mathrm{E}_0(\mathcal{T})\cap S^0$. The topological components of the complement of Σ_u^0 in Σ are the unions $\widehat{\Sigma}_{u;e}^c$ of the irreducible components of Σ_u corresponding to the graphs $\mathcal{T}'_{\bullet;e}$ with $e\in\mathrm{E}_\bullet(\mathcal{T})$; the sum of their arithmetic genera is $g-h$. For now on, we will not distinguish a pair $\mathcal{T}=(\mathfrak{t},\mathfrak{m})$ from the associated rooted decorated graph.

For each $e\in\mathrm{E}_0(\mathcal{T})$, let $S_e(\mathcal{T})\subset S$ be the subset of the marked points of $\mathcal{T}$ carried by the connected component of the graph obtained by cutting $\mathcal{T}$ at the edge e which does not contain w_0. If $\mathcal{T}\in\mathscr{T}_{g,l}(\mathcal{C})^\phi$ is as in (3.5), then

$$S=\big\{1^+,1^-,\ldots,l^+,l^-\big\}, \qquad g_a(\mathcal{T})=g, \qquad \mathfrak{g}(\mathrm{w}_0)=h, \tag{3.17}$$

$$\big|S_e(\mathcal{T})\big|\leqslant 1 \quad \forall\ e\in\mathrm{E}_{0;\mathbb{C}}^\sigma(\mathcal{T}), \quad S_e(\mathcal{T})=\varnothing \ \ \forall\ e\in\mathrm{E}_{0;\mathbb{R}}^\sigma(\mathcal{T}); \tag{3.18}$$

the last two properties hold because the points $z_1^+(\mathcal{C}),\ldots,z_l^+(\mathcal{C})$ are non-real and distinct. Define

$$\mathcal{C}_{\mathcal{T}}^*\equiv\big\{(z_e)_{e\in\mathrm{E}_0(\mathcal{T})}\in\mathcal{C}^{\mathrm{E}_0(\mathcal{T})}:\ z_e\neq z_{e'} \text{ if } e\neq e',\ z_e=z_i^\pm(\mathcal{C}) \text{ if } S_e(\mathcal{T})=\{i^\pm\}\big\}\subset\mathcal{C}^{\mathrm{E}_0(\mathcal{T})}.$$

The involution σ of $\mathcal{T}$ acts on $\mathcal{C}_{\mathcal{T}}^*$ by

$$\big(\sigma(\mathbf{z})\big)_e=\phi(z_{\sigma(e)}) \qquad \forall\ \mathbf{z}=(z_e)_{e\in\mathrm{E}_0(\mathcal{T})}\,. \tag{3.19}$$

We denote by $\mathcal{C}_{\mathcal{T}}^{*;\sigma}\subset\mathcal{C}_{\mathcal{T}}^*$ the fixed locus.

By (3.18), we can choose $\mathrm{E}_{0;+}^\sigma(\mathcal{T})\subset\mathrm{E}_{0;\mathbb{C}}^\sigma(\mathcal{T})$ such that

$$\mathrm{E}_{0;\mathbb{C}}^\sigma(\mathcal{T})=\mathrm{E}_{0;+}^\sigma(\mathcal{T})\sqcup\sigma\big(\mathrm{E}_{0;+}^\sigma(\mathcal{T})\big) \qquad \text{and} \qquad S_e(\mathcal{T})\cap\{1^-,\ldots,l^-\}=\varnothing \quad \forall\ e\in\mathrm{E}_{0;+}^\sigma(\mathcal{T}).$$

We then choose $\mathrm{E}_{\bullet;+}^\sigma(\mathcal{T})\subset\mathrm{E}_{\bullet;\mathbb{C}}^\sigma(\mathcal{T})$ such that

$$\mathrm{E}_{\bullet;\mathbb{C}}^\sigma(\mathcal{T})=\mathrm{E}_{\bullet;+}^\sigma(\mathcal{T})\sqcup\sigma\big(\mathrm{E}_{\bullet;+}^\sigma(\mathcal{T})\big) \qquad \text{and} \qquad \langle e\rangle\in\mathrm{E}_{0;+}^\sigma(\mathcal{T})\cup\mathrm{E}_{0;\mathbb{R}}^\sigma(\mathcal{T}) \quad \forall\, e\in\mathrm{E}_{\bullet;+}^\sigma(\mathcal{T})\,.$$

Let

$$\mathrm{E}_{0;\geqslant}^\sigma(\mathcal{T})\equiv\mathrm{E}_{0;+}^\sigma(\mathcal{T})\sqcup\mathrm{E}_{0;\mathbb{R}}^\sigma(\mathcal{T}), \qquad \mathrm{E}_{\bullet;\geqslant}^\sigma(\mathcal{T})\equiv\mathrm{E}_{\bullet;+}^\sigma(\mathcal{T})\sqcup\mathrm{E}_{\bullet;\mathbb{R}}^\sigma(\mathcal{T})\,.$$

We take

$$\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi=\prod_{e\in\mathrm{E}_{0;\geqslant}^\sigma(\mathcal{T})\sqcup\mathrm{E}_{\bullet;\geqslant}^\sigma(\mathcal{T})}\mathcal{M}_{\mathcal{T},\sigma;e}\times\mathcal{C}_{\mathcal{T}}^{*;\sigma}. \tag{3.20}$$

The actions of the group $\mathrm{Aut}^*(\mathcal{T})$ on $\mathrm{E}_0(\mathcal{T})$ and $\mathrm{E}_\bullet(\mathcal{T})$ and the possible changes in the subsets

$$\mathrm{E}_{0;+}^\sigma(\mathcal{T})\subset\mathrm{E}_{0;\mathbb{C}}^\sigma(\mathcal{T}) \qquad \text{and} \qquad \mathrm{E}_{\bullet;+}^\sigma(\mathcal{T})\subset\mathrm{E}_{\bullet;\mathbb{C}}^\sigma(\mathcal{T})$$

induce an action of a group $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$ on (3.20). The natural node-identifying immersion

$$\iota_{\mathcal{T},\mathbf{f}}\colon \widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi \longrightarrow \mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi \tag{3.21}$$

is $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-invariant.

For each $e\in\mathrm{E}^\sigma_{0;\geqslant}(\mathcal{T}),\mathrm{E}^\sigma_{\bullet;\geqslant}(\mathcal{T})$ and $e\in\mathrm{E}^\sigma_0(\mathcal{T})$, denote by

$$\pi_{\mathcal{T};e}\colon \widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi \longrightarrow \mathcal{M}_{\mathcal{T},\sigma;e} \qquad\text{and}\qquad \pi_{\mathcal{C};e}\colon \widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi \longrightarrow \mathcal{C}$$

the corresponding projection maps. For $e\in\sigma(\mathrm{E}^\sigma_{0;+}(\mathcal{T})),\sigma(\mathrm{E}^\sigma_{\bullet;+}(\mathcal{T}))$, let

$$\pi_{\mathcal{T};e}=\pi_{\mathcal{T};\sigma(e)}\colon \widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi \longrightarrow \mathcal{M}_{\mathcal{T},\sigma;\sigma(e)}.$$

For each $e\in\mathrm{E}^\sigma_{\bullet;\mathbb{C}}(\mathcal{T})$, let

$$\mathbb{E}_e \longrightarrow \widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi \tag{3.22}$$

be the subbundle of $\pi^*_{\mathcal{T};e}\mathbb{E}$ consisting of the holomorphic differentials supported on $\widehat{\Sigma}^c_{u;e}$. For each $e\in\mathrm{E}^\sigma_{\bullet;\mathbb{R}}(\mathcal{T})$, let

$$\mathbb{E}_e=\pi^*_{\mathcal{T};e}\mathbb{E} \longrightarrow \widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi\,. \tag{3.23}$$

The involution σ lifts to an $\mathbb{R}$-linear isomorphism

$$\widetilde{\sigma}_e\colon \mathbb{E}^*_e \longrightarrow \mathbb{E}^*_{\sigma(e)}, \qquad \big\{\widetilde{\sigma}_e(\eta)\big\}(\kappa) = \overline{\eta(\mathfrak{c}_{\mathbb{C}}\circ\kappa\circ\mathrm{d}\sigma)}\in\mathbb{C} \qquad \forall\ \eta\in\mathbb{E}^*_e,\ \kappa\in\mathbb{E}_{\sigma(e)},$$

where $\mathfrak{c}_{\mathbb{C}}\colon\mathbb{C}\longrightarrow\mathbb{C}$ is the standard conjugation. Let

$$\mathrm{Obs}_{\mathcal{T}} \equiv \bigoplus_{e\in\mathrm{E}_\bullet(\mathcal{T})} \mathbb{E}^*_e\otimes\pi^*_{\mathcal{C};\langle e\rangle}TX \longrightarrow \widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi\,. \tag{3.24}$$

The involution ϕ acts on $\mathrm{Obs}_{\mathcal{T}}$ by

$$\phi\big(\eta_e\otimes Y_e\big) = \big(\widetilde{\sigma}_e(\eta_e)\big)\otimes\big(\mathrm{d}_{z_{\langle e\rangle}}\phi(Y_e)\big)\in\mathbb{E}^*_{\sigma(e)}\otimes\pi^*_{\mathcal{C};\sigma\langle e\rangle}TX \quad \forall\, e\in\mathrm{E}_\bullet(\mathcal{T}),\, \eta_e\in\mathbb{E}^*_e,\, Y_e\in\pi^*_{\mathcal{C};\langle e\rangle}TX.$$

In other words,

$$\big\{\phi(\eta)\big\}\big((\kappa_e)_{e\in\mathrm{E}_\bullet(\mathcal{T})}\big) = \big(\mathrm{d}_{z_{\sigma(\langle e\rangle)}}\phi\big(\eta(\mathfrak{c}_{\mathbb{C}}\circ\kappa_e\circ\mathrm{d}\sigma)\big)\big)_{e\in\mathrm{E}_\bullet(\mathcal{T})} \quad \forall\,\eta\in\mathrm{Obs}_{\mathcal{T}},\ \kappa_e\in\mathrm{E}_e,\ e\in\mathrm{E}_\bullet(\mathcal{T}).$$

We take the obstruction bundle

$$\mathrm{Obs}^\phi_{\mathcal{T}}\subset\mathrm{Obs}_{\mathcal{T}} \longrightarrow \widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi \tag{3.25}$$

to be the fixed locus under this action.

Let $\widetilde{\mathbf{u}}\in\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$ with $\mathbf{u}=\iota_{\mathcal{T},\mathbf{f}}(\widetilde{\mathbf{u}})$ as in (2.3). For each $e\in\mathrm{E}_\bullet(\mathcal{T})$, u is constant on the support of the elements of $\mathbb{E}_e|_{\widetilde{\mathbf{u}}}$ (which is contained in $\widehat{\Sigma}^c_{u;e}$) and sends it to $\pi_{\mathcal{C};\langle e\rangle}(\widetilde{\mathbf{u}})\in\mathcal{C}$. Thus, we can define

$$\begin{gathered}\Theta_{\widetilde{\mathbf{u}}}\colon \Gamma\big(\Sigma;T^*\Sigma^{0,1}_u\otimes u^*(TX,J)\big) \longrightarrow \mathrm{Obs}_{\mathcal{T}}\big|_{\widetilde{\mathbf{u}}} \quad\text{by}\\ \big\{\Theta_{\widetilde{\mathbf{u}}}(\eta)\big\}\big((\kappa_e)_{e\in\mathrm{E}_\bullet(\mathcal{T})}\big) = \bigg(\frac{\mathrm{i}}{2\pi}\int_{\widehat{\Sigma}^c_{u;e}}\kappa_e\wedge\eta\bigg)_{e\in\mathrm{E}_\bullet(\mathcal{T})} \qquad \forall\ \kappa_e\in\mathbb{E}_e|_{\widetilde{\mathbf{u}}},\ e\in\mathrm{E}_\bullet(\mathcal{T})\,.\end{gathered} \tag{3.26}$$

If η is (ϕ,σ_u)-invariant, then $\Theta_{\tilde{\mathbf{u}}}(\eta)\in\mathrm{Obs}^{\phi}_{\mathcal{T}}|_{\tilde{\mathbf{u}}}$. A Ruan-Tian deformation ν for maps from genus g curves with $2l$ markings into (X,J) as in [4, Section 2] determines elements

$$\nu_{\tilde{\mathbf{u}}}\in\Gamma\big(\Sigma;T^*\Sigma_u^{0,1}\otimes u^*(TX,J)\big)\qquad\text{and}\qquad \overline{\nu}_{\mathcal{T}}(\tilde{\mathbf{u}})\equiv\Theta_{\tilde{\mathbf{u}}}(\nu_{\tilde{\mathbf{u}}})\in\mathrm{Obs}_{\mathcal{T}}\big|_{\tilde{\mathbf{u}}}.$$

If $\nu\in\mathcal{A}^{\phi}_{g,l}(J)$, then $\overline{\nu}_{\mathcal{T}}$ takes values in $\mathrm{Obs}^{\phi}_{\mathcal{T}}$. The homomorphism $\nu\longrightarrow\overline{\nu}_{\mathcal{T}}$ surjects onto the subspace of $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-invariant sections.

For each edge $e=(w_1,w_2)$ in Edg, we denote by $\widetilde{L}_e$ the tensor product of the two complex line bundles corresponding to the tangent spaces of the irreducible components $\Sigma_{u;w_1}$ and $\Sigma_{u;w_2}$ of Σ_u at the node e. Let

$$\pi_{\tilde{\mathcal{F}}\mathcal{T}}\colon \tilde{\mathcal{F}}\mathcal{T}=\Big(\bigoplus_{e\in\mathrm{Edg}}\widetilde{L}_e\Big)\Big/\mathrm{Aut}_{\bullet}(\mathcal{T})\longrightarrow\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}. \tag{3.27}$$

We fix an $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-invariant and involution-invariant metric on $\tilde{\mathcal{F}}\mathcal{T}$ and for each $\delta\in\mathbb{R}$ define

$$\tilde{\mathcal{F}}\mathcal{T}_{\delta}=\big\{\upsilon\in\tilde{\mathcal{F}}\mathcal{T}:\ |\upsilon|<\delta\big\}.$$

The involution σ acts on $\tilde{\mathcal{F}}\mathcal{T}$ by

$$\big(\sigma\big((\upsilon_{e'})_{e'\in\mathrm{Edg}}\big)\big)_e=\mathrm{d}\sigma(\upsilon_{\sigma(e)})\qquad\forall\ e\in\mathrm{Edg}\,.$$

We denote the corresponding fixed locus by $\tilde{\mathcal{F}}\mathcal{T}^{\sigma}$.

If $\mathcal{T}=(\mathfrak{t},\mathfrak{m})$ with $\mathfrak{t}$ basic, then $\mathrm{Edg}=\mathrm{E}_{\bullet}(\mathcal{T})$ and

$$\tilde{\mathcal{F}}\mathcal{T}=\bigoplus_{e\in\mathrm{E}_{\bullet}(\mathcal{T})}\pi^*_{\mathcal{T};e}L_e\otimes\pi^*_{\mathcal{C};\langle e\rangle}T\mathcal{C}\,. \tag{3.28}$$

Define

$$\mathcal{D}_{\mathcal{T}}\colon\tilde{\mathcal{F}}\mathcal{T}\longrightarrow\mathrm{Obs}_{\mathcal{T}},\quad\big\{\mathcal{D}_{\mathcal{T}}\big((\upsilon_e\otimes\upsilon_{\langle e\rangle})_{e\in\mathrm{E}_{\bullet}(\mathcal{T})}\big)\big\}\big((\kappa_e)_{e\in\mathrm{E}_{\bullet}(\mathcal{T})}\big)=\big(\kappa_e(\upsilon_e)\upsilon_{\langle e\rangle}\big)_{e\in\mathrm{E}_{\bullet}(\mathcal{T})}, \tag{3.29}$$

with $\kappa_e\in\mathbb{E}_e$. The homomorphism $\mathcal{D}_{\mathcal{T}}$ restricts to a homomorphism

$$\mathcal{D}^{\phi}_{\mathcal{T}}\colon\tilde{\mathcal{F}}\mathcal{T}^{\sigma}\longrightarrow\mathrm{Obs}^{\phi}_{\mathcal{T}}$$

between the fixed loci of the two bundles.

The bundles $\tilde{\mathcal{F}}\mathcal{T}^{\sigma}$ and $\mathrm{Obs}^{\phi}_{\mathcal{T}}$ form a deformation-obstruction complex for $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J)^{\phi}$ over $\overline{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$. The choice of real orientation on (X,ω,ϕ) orients this moduli space and thus the total space of the vector bundle

$$\pi^*_{\tilde{\mathcal{F}}\mathcal{T}^{\sigma}}\mathrm{Obs}^{\phi}_{\mathcal{T}}\longrightarrow\tilde{\mathcal{F}}\mathcal{T}^{\sigma}\,, \tag{3.30}$$

where $\pi_{\tilde{\mathcal{F}}\mathcal{T}^{\sigma}}:\tilde{\mathcal{F}}\mathcal{T}^{\sigma}\longrightarrow\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ is the bundle projection map. Since the dimension of the total space of $\tilde{\mathcal{F}}\mathcal{T}^{\sigma}$ and the rank of $\mathrm{Obs}^{\phi}_{\mathcal{T}}$ are the same, every transverse zero of a section Ψ of (3.30) thus has a well defined sign. The next structural proposition will be proved in Section 5.

Proposition 3.1. *Let (X,ω,ϕ), B, g,l, and $\mathbf{f}$ be as in Theorem 1.1, $J\in\mathcal{J}^{\phi}_{\mathrm{reg}}(g,B)$, $\mathcal{C}$ be an element of $\mathcal{M}^{\phi}_{h,\mathbf{f}}(B;J)$, $\nu\in\mathcal{A}^{\phi}_{g,l}(J)$, $\mathcal{T}\equiv(\mathfrak{t},\mathfrak{m})\in\mathscr{T}_{g,l}(\mathcal{C})^{\phi}$, and $K\subset\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ be a precompact open subset.*

(1) If $\mathfrak{t}$ is not basic and ν is generic, there exist $\delta_\nu(K)\in\mathbb{R}^+$ and an open neighborhood $U_\nu(K)$ of K in $\mathfrak{X}_{g,\mathbf{f}}(X,B)^\phi$ such that

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J,t\nu)^\phi\cap U_\nu(K)=\varnothing \qquad \forall\ t\in\big(0,\delta_\nu(K)\big). \tag{3.31}$$

(2) If $\mathfrak{t}$ is basic, there exist $\delta_\nu(K)\in\mathbb{R}^+$, a family

$$U_{\nu;\delta}(K)\subset\mathfrak{X}_{g;\mathbf{f}}(X,B)^\phi, \qquad \delta\in\big(0,\delta_\nu(K)\big),$$

of neighborhoods of K, continuous families of continuous $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-invariant maps

$$\Phi_{\mathcal{T};t\nu}\colon\widetilde{\mathcal{F}}\mathcal{T}^\sigma_{\delta_\nu(K)}\big|_{\iota^{-1}_{\mathcal{T},\mathbf{f}}(K)}\longrightarrow U_{\nu;\delta_\nu K}(K),\quad \varepsilon_{\mathcal{T};t\nu}\colon\widetilde{\mathcal{F}}\mathcal{T}^\sigma\longrightarrow\mathbb{R}^+,\quad t\in\big(-\delta_\nu(K),\delta_\nu(K)\big),$$

and a continuous family of $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-sections

$$\Psi_{\mathcal{T};t\nu}\in\Gamma\big(\widetilde{\mathcal{F}}\mathcal{T}^\sigma_{\delta_\nu(K)}\big|_{\iota^{-1}_{\mathcal{T},\mathbf{f}}(K)};\,\pi^*_{\widetilde{\mathcal{F}}\mathcal{T}^\sigma}\mathrm{Obs}^\phi_{\mathcal{T}}\big),\quad t\in\big(-\delta_\nu(K),\delta_\nu(K)\big), \tag{3.32}$$

such that $\Phi_{\mathcal{T};t\nu}$ is a degree $|\widetilde{\mathrm{Aut}}^*(\mathcal{T})|$ covering of its image,

$$\Phi_{\mathcal{T};0}\big|_{\iota^{-1}_{\mathcal{T},\mathbf{f}}(K)}=\iota_{\mathcal{T},\mathbf{f}}, \qquad \varepsilon_{\mathcal{T};0}\big|_{\iota^{-1}_{\mathcal{T},\mathbf{f}}(K)}=0, \tag{3.33}$$

$$\Big\|\Psi_{\mathcal{T};t\nu}(\upsilon)-\Big(\mathcal{D}^\phi_{\mathcal{T}}(\upsilon)+t\bar\nu_{\mathcal{T}}\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}^\sigma}(\upsilon)\big)\Big)\Big\|\leqslant\varepsilon_{\mathcal{T};t\nu}(\upsilon)\big(|\upsilon|+|t|\big), \tag{3.34}$$

$$\Phi_{\mathcal{T};t\nu}\big(\Psi^{-1}_{\mathcal{T};t\nu}(0)\cap\widetilde{\mathcal{F}}\mathcal{T}^\sigma_\delta\big)=\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J,t\nu)^\phi\cap U_{\nu;\delta}(K)\quad \forall\ t\in(-\delta,\delta),\ \delta\in\big(0,\delta_\nu(K)\big). \tag{3.35}$$

If in addition ν is generic, $t\in(-\delta,\delta)$, and $\delta\in(0,\delta_\nu(K))$, then

$$\Phi_{\mathcal{T};t\nu}\colon\Psi^{-1}_{\mathcal{T};t\nu}(0)\cap\widetilde{\mathcal{F}}\mathcal{T}^\sigma_\delta\longrightarrow\mathfrak{M}_{g,\mathbf{f}}(X,B;J,t\nu)^\phi\cap U_{\nu;\delta}(K) \tag{3.36}$$

preserves the signs of all points.

4 Proofs of Propositions 2.2-2.4

We will next deduce the three propositions of Section 2 from Proposition 3.1 following the principles laid out in [25, Sections 3] which relate stratawise degenerate contributions to zeros of affine maps between vector bundles. These principles readily apply with the manifold $\mathcal{M}$ (accidentally denoted by X in Lemma 3.5(3) and Corollary 3.6(5) in [25]) replaced by an orbifold (in the sense specified after [27, Definition 2.10]). All bundles in [25, Sections 3] are assumed to be complex so that all transverse isolated zeros of bundles sections are canonically oriented. While the former is no longer the case, the latter is; see the paragraph preceding Proposition 3.1. The rank over $\mathbb{C}$ appearing in the statements of [25, Sections 3] should be replaced by half the rank over $\mathbb{R}$. We will apply the reasoning behind [25, Corollary 3.6] with

$$\mathcal{M}=\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi, \qquad F=F^-=\widetilde{F}^-=\widetilde{\mathcal{F}}\mathcal{T}^\sigma, \qquad \mathcal{O}=\mathcal{O}^-=\mathrm{Obs}^\phi_{\mathcal{T}},$$

$F^+\subset F$ and $\mathcal{O}^+\subset\mathcal{O}$ being the trivial (zero) subbundles, and $\alpha=\mathcal{D}^\phi_{\mathcal{T}}$. We continue with the assumptions and the notation as in Propositions 2.2 and 3.1 and assume that ν is generic.

4.1 The regularity of the boundary strata

The crucial difference between the main boundary strata, i.e. those corresponding to $\mathcal{T}=(\mathfrak{t},\mathfrak{m})$ with $\mathfrak{t}$ basic, and the remaining boundary strata is that the latter are $\bar{\partial}_J$-hollow in the sense of [25, Definition 3.11] and thus automatically do not contribute to the number (1.2). In order to complete the proof of Proposition 2.2, we show that the main boundary strata with $\mathfrak{m}\neq\mathfrak{m}_0(\mathfrak{t})$ do not contribute either; the reason is more delicate in this case.

Proof of Proposition 2.2. Let $\mathcal{T}=(\mathfrak{t},\mathfrak{m})$.

Suppose $\mathfrak{t}$ is not basic. For the purposes of Definition 2.1, we then take $K_\nu=\varnothing$. Given a precompact open subset $K\subset\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$, let $\delta_\nu(K)\in\mathbb{R}^+$ and $U_\nu(K)\subset\mathfrak{X}_{g,\mathbf{f}}(X,B)^\phi$ be as in Proposition 3.1(1). Let $U\subset U_\nu(K)$ be an open neighborhood of K. By (3.31),

$$\mathfrak{M}_{g,\mathbf{f}}(X,B;J,t\nu)^\phi\cap U\subset\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J,t\nu)^\phi\cap U_\nu(K)=\varnothing.$$

This establishes Proposition 2.2 for $\mathcal{T}=(\mathfrak{t},\mathfrak{m})$ with $\mathfrak{t}$ not basic.

Suppose $\mathfrak{t}$ is basic. In particular,

$$\mathrm{E}_0(\mathcal{T})=\mathrm{E}_\bullet(\mathcal{T})=\mathrm{Edg},\quad \dim\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi+\mathrm{rk}\,\tilde{\mathcal{F}}\mathcal{T}^\sigma=\mathrm{rk}\,\mathrm{Obs}_{\mathcal{T}}^\phi.$$

We assume that $\mathcal{T}$ is given by (3.5) and satisfies (3.17) and (3.18). For each $e\in\mathrm{Edg}$, we denote by $e/w_0\in e$ the vertex different from $w_0\in e$. Let

$$S_0(\mathcal{T})=\big\{i\in\{1,\ldots,l\}:\,i^+\in\mathfrak{m}^{-1}(w_0)\big\}$$

be the pairs of marked points carried by the non-contracted curve in $\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$.

The vector bundles $\tilde{\mathcal{F}}\mathcal{T}^\sigma$ and $\mathrm{Obs}_{\mathcal{T}}^\phi$, the bundle homomorphism $\mathcal{D}_{\mathcal{T}}^\phi$, and the bundle section $\bar{\nu}_{\mathcal{T}}$ naturally extend over the compactification

$$\widehat{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi=\prod_{e\in\mathrm{E}^\sigma_{\bullet;+}(\mathcal{T})}\overline{\mathcal{M}}^\bullet_{2\mathfrak{g}(e/w_0)-1,(S^\sigma_{\bullet;e},\sigma)}\times\prod_{e\in\mathrm{E}^\sigma_{\bullet;\mathbb{R}}(\mathcal{T})}\mathbb{R}\overline{\mathcal{M}}_{\mathfrak{g}(e/w_0),(S^\sigma_{\bullet;e},\sigma)}\times\hat{\mathcal{C}}_{\mathcal{T}}\tag{4.1}$$

of $\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$, where $\hat{\mathcal{C}}_{\mathcal{T}}$ is the closure of

$$\big\{\big((z_i^+(\mathcal{C}),z_i^-(\mathcal{C}))_{i\in S_0(\mathcal{T})},(z_e)_{e\in\mathrm{E}_\bullet(\mathcal{T})}\big):\,(z_e)_{e\in\mathrm{E}_\bullet(\mathcal{T})}\in\mathcal{C}^*_{\mathcal{T}}\big\}$$

in the corresponding real moduli space of points on $\mathcal{C}$. We denote these extensions in the same way. The compactification (4.1) is a union of the quotients of the spaces $\widetilde{\mathfrak{M}}_{\mathcal{T}',\mathbf{f}}(\mathcal{C})^\phi$ by subgroups of $\widetilde{\mathrm{Aut}}^*(\mathcal{T}')$ for some rooted decorated graphs $\mathcal{T}'$. The restriction of $\bar{\nu}_{\mathcal{T}}$ to the quotient of $\widetilde{\mathfrak{M}}_{\mathcal{T}',\mathbf{f}}(\mathcal{C})^\phi$ lifts to $\bar{\nu}_{\mathcal{T}'}$ (as the sections $\bar{\nu}_{\mathcal{T}}$ and $\bar{\nu}_{\mathcal{T}'}$ are pullbacks from the closure of $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$ in $\overline{\mathfrak{M}}_{g,\mathbf{f}}(\mathcal{C})^\phi$).

The homomorphism $\mathcal{D}_{\mathcal{T}}^\phi$ does not vanish over $\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$. The pairings of tangent and cotangent vectors in (3.29) correspond to algebraic bundle sections. Thus, the homomorphism $\mathcal{D}_{\mathcal{T}}^\phi$ is a regular polynomial (linear map in this case) in the sense of [25, Definition 3.9]. By the first two parts of the proof of [25, Lemma 3.10], the zero set of the affine bundle map

$$\psi_{\mathcal{T};\nu}:\tilde{\mathcal{F}}\mathcal{T}^\sigma\longrightarrow\mathrm{Obs}_{\mathcal{T}}^\phi,\qquad \psi_{\mathcal{T};\nu}(\upsilon)=\mathcal{D}_{\mathcal{T}}^\phi(\upsilon)+\bar{\nu}_{\mathcal{T}}(\mathbf{u})\quad\forall\,\upsilon\in\tilde{\mathcal{F}}\mathcal{T}^\sigma|_{\mathbf{u}},\ \mathbf{u}\in\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi,$$

thus has finitely many zeros. The continuous extension of this map over $\widehat{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ does not vanish over the complement of $\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$. By the third part of the proof of [25, Lemma 3.10], the signed cardinality of the affine bundle map

$$\psi_{\mathcal{T};\varpi}\colon \widetilde{\mathcal{F}}\mathcal{T}^{\sigma} \longrightarrow \mathrm{Obs}_{\mathcal{T}}^{\phi}, \qquad \psi_{\mathcal{T};\varpi}(\upsilon) = \mathcal{D}_{\mathcal{T}}^{\phi}(\upsilon) + \varpi(\mathbf{u}) \quad \forall\, \upsilon\in\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}|_{\mathbf{u}},\ \mathbf{u}\in\widehat{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi},$$

does not depend on a generic choice of the section ϖ of $\mathrm{Obs}_{\mathcal{T}}^{\phi}$ over $\widehat{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$. In this case, the cobordism in the proof of [25, Lemma 3.10] may cross the fibers of $\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}$ over the codimension-one boundary strata of $\widehat{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$. However, the orientation on such a cobordism extends across the zeros in these fibers based on the same considerations as above Proposition 3.1. We denote the signed cardinality of $\psi_{\mathcal{T};\varpi}^{-1}(0)$ by $N(\mathcal{D}_{\mathcal{T}}^{\phi})$.

By the previous paragraph, (3.34), and the proofs of Lemma 3.5 and Corollary 3.6 in [25], there exist a compact subset $\widetilde{K}_{\mathcal{T};\nu}\subset\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ and $C_{\mathcal{T};\nu}\in\mathbb{R}^+$ with the following property. If $\widetilde{K}\subset\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ is a precompact open subset containing $\widetilde{K}_{\mathcal{T};\nu}$, then there exists $\delta_{\nu}(\widetilde{K})\in\mathbb{R}^+$ so that the bundle map

$$\Psi_{\mathcal{T};t\nu}\colon \widetilde{\mathcal{F}}\mathcal{T}^{\sigma}_{\delta_{\nu}(\widetilde{K})}|_{\widetilde{K}} \longrightarrow \mathrm{Obs}_{\mathcal{T}}^{\phi}$$

is defined and

$$\Psi_{\mathcal{T};t\nu}^{-1}(0) \subset \widetilde{\mathcal{F}}\mathcal{T}^{\sigma}_{C_{\mathcal{T};\nu}t}|_{\widetilde{K}_{\mathcal{T};\nu}}, \quad {}^{\pm}\big|\Psi_{\mathcal{T};t\nu}^{-1}(0)\big| = N\big(\mathcal{D}_{\mathcal{T}}^{\phi}\big) \qquad \forall\, t\in\big(0,\delta_{\nu}(\widetilde{K})\big). \tag{4.2}$$

Let $K_{\nu}\subset\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ be a compact subset such that

$$\widetilde{K}_{\mathcal{T};\nu} \subset \widetilde{K}_{\nu} \equiv \iota_{\mathcal{T};\mathbf{f}}^{-1}(K_{\nu})$$

and set

$$\mathrm{Cntr}_{\mathbf{f}}^{\phi}\big(\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}\big) = \frac{1}{|\widetilde{\mathrm{Aut}}^{*}(\mathcal{T})|} N\big(\mathcal{D}_{\mathcal{T}}^{\phi}\big)\,. \tag{4.3}$$

We verify below that K_{ν} and the number (4.3) satisfy the conditions of Definition 2.1.

Let $K\subset\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ be a precompact open subset containing K_{ν}, $\widetilde{K}=\iota_{\mathcal{T},\mathbf{f}}^{-1}(K)$, and

$$\delta_{\nu}(K) \in \big(0,\delta_{\nu}(\widetilde{K})\big) \qquad \text{and} \qquad U_{\nu}(K) \equiv U_{\nu;\delta_{\nu}(K)}(K)$$

be as in Proposition 3.1(2), and $U\subset U_{\nu}(K)$ be a neighborhood of K. Since K_{ν} is compact, there exists $\epsilon_{\nu}(U)\in(0,\delta_{\nu}(K))$ such that

$$\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}_{C_{\mathcal{T};\nu}\epsilon_{\nu}(U)}\big|_{\widetilde{K}_{\nu}} \subset \Phi_{\mathcal{T};t\nu}^{-1}(U)\,.$$

Suppose $t\in(0,\epsilon_{\nu}(U))$. By (3.35), the first statement in (4.2), and the last condition,

$$\begin{aligned}\Psi_{\mathcal{T};t\nu}^{-1}(0)\cap\Phi_{\mathcal{T};t\nu}^{-1}(U) &\subset \Psi_{\mathcal{T};t\nu}^{-1}(0)\cap\Phi_{\mathcal{T};t\nu}^{-1}\big(U_{\nu}(K)\big) = \Psi_{\mathcal{T};t\nu}^{-1}(0)\cap\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}_{\delta_{\nu}(K)}|_{\widetilde{K}}\\ &\subset \Psi_{\mathcal{T};t\nu}^{-1}(0)\cap\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}_{C_{\mathcal{T};\nu}\epsilon_{\nu}(U)}|_{\widetilde{K}_{\mathcal{T};\nu}} \subset \Psi_{\mathcal{T};t\nu}^{-1}(0)\cap\Phi_{\mathcal{T};t\nu}^{-1}(U).\end{aligned}$$

This implies that

$$\Psi_{\mathcal{T};t\nu}^{-1}(0)\cap\Phi_{\mathcal{T};t\nu}^{-1}(U) = \Psi_{\mathcal{T};t\nu}^{-1}(0)\cap\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}_{\delta_{\nu}(K)}|_{K}\,.$$

Combining this conclusion with (4.2), we obtain

$$^{\pm}\big|\Psi_{\mathcal{T};t\nu}^{-1}(0)\cap\Phi_{\mathcal{T};t\nu}^{-1}(U)\big| = {}^{\pm}\big|\Psi_{\mathcal{T};t\nu}^{-1}(0)\cap\widetilde{\mathcal{F}}\mathcal{T}_{\delta_\nu(K)}^{\sigma}|_K\big| = N\big(\mathcal{D}_{\mathcal{T}}^{\phi}\big)\,.$$

The last two statements and (3.36) give the first statement of Proposition 2.2 for $\mathcal{T}=(\mathfrak{t},\mathfrak{m})$ with $\mathfrak{t}$ basic and the contribution given by (4.3).

It remains to show that $N(\mathcal{D}_{\mathcal{T}}^{\phi})=0$ if $\mathcal{T}=(\mathfrak{t},\mathfrak{m})$ with $\mathfrak{t}$ basic and $\mathfrak{m}\neq\mathfrak{m}_0(\mathfrak{t})$. The last condition implies that at least one conjugate pair (z_i^+,z_i^-) of marked points is carried by the contracted components of the elements in $\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$. For $e\in\mathrm{E}_{\bullet;\geqslant}^{\sigma}(\mathcal{T})$, let

$$S_{\bullet;e}'^{\sigma}\equiv\big\{e,\sigma(e)\big\}\subset S_{\bullet;e}^{\sigma}$$

be the complement of the markings $\{1^+,1^-,\ldots,l^+,l^-\}$. Let

$$\widehat{\mathfrak{M}}'_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}=\prod_{e\in\mathrm{E}_{\bullet;+}^{\sigma}(\mathcal{T})}\overline{\mathcal{M}}^{\bullet}_{2\mathfrak{g}(e/w_0)-1,(S_{\bullet;e}'^{\sigma},\sigma)}\times\prod_{e\in\mathrm{E}_{\bullet;\mathbb{R}}^{\sigma}(\mathcal{T})}\mathbb{R}\overline{\mathcal{M}}_{\mathfrak{g}(e/w_0),(S_{\bullet;e}'^{\sigma},\sigma)}\times\hat{\mathcal{C}}_{\mathcal{T}}\,.$$

We denote by

$$\mathcal{D}_{\mathcal{T}}'^{\phi}\colon\widetilde{\mathcal{F}}'\mathcal{T}^{\sigma}\longrightarrow\mathrm{Obs}_{\mathcal{T}}'^{\phi}$$

the vector bundle homomorphism over $\widehat{\mathfrak{M}}'_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ defined analogously to the vector bundle homomorphism $\mathcal{D}_{\mathcal{T}}^{\phi}$ over $\widehat{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$.

Let

$$\mathfrak{f}\colon\widehat{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}\longrightarrow\widehat{\mathfrak{M}}'_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$$

be the morphism dropping the points marked by $1^+,1^-,\ldots,l^+,l^-$ that are carried by the contracted components. Thus,

$$\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}|_{\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}}=\mathfrak{f}^*\widetilde{\mathcal{F}}'\mathcal{T}^{\sigma}|_{\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}},\quad\mathrm{Obs}_{\mathcal{T}}^{\phi}=\mathfrak{f}^*\mathrm{Obs}_{\mathcal{T}}'^{\phi},\quad\mathcal{D}_{\mathcal{T}}^{\phi}|_{\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}}=\mathfrak{f}^*\mathcal{D}_{\mathcal{T}}'^{\phi}|_{\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}}\,.$$

This implies that $N(\mathcal{D}_{\mathcal{T}}^{\phi})=N(\mathfrak{f}^*\mathcal{D}_{\mathcal{T}}'^{\phi})$, with the second number defined analogous to the former. If $\mathfrak{m}\neq\mathfrak{m}_0(\mathfrak{t})$, then

$$\dim\widehat{\mathfrak{M}}'_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}+\mathrm{rk}\,\widetilde{\mathcal{F}}'\mathcal{T}^{\sigma}<\dim\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}+\mathrm{rk}\,\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}=\mathrm{rk}\,\mathrm{Obs}_{\mathcal{T}}^{\phi}=\mathrm{rk}\,\mathrm{Obs}_{\mathcal{T}}'^{\phi}\,.$$

Thus, there exists a section ϖ' of $\mathrm{Obs}_{\mathcal{T}}'^{\phi}$ so that the affine bundle map

$$\psi'_{\mathcal{T};\varpi}\colon\widetilde{\mathcal{F}}'\mathcal{T}^{\sigma}\longrightarrow\mathrm{Obs}_{\mathcal{T}}'^{\phi},\qquad\psi'_{\mathcal{T};\varpi}(\upsilon)=\mathcal{D}_{\mathcal{T}}'^{\phi}(\upsilon)+\varpi(\mathbf{u})\quad\forall\ \upsilon\in\widetilde{\mathcal{F}}'\mathcal{T}^{\sigma}|_{\mathbf{u}},\ \mathbf{u}\in\widehat{\mathfrak{M}}'_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi},$$

over $\widehat{\mathfrak{M}}'_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ does not vanish. This implies that the affine bundle map

$$\mathfrak{f}^*\psi'_{\mathcal{T};\varpi}\colon\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}\longrightarrow\mathrm{Obs}_{\mathcal{T}}^{\phi},\quad\big\{\mathfrak{f}^*\psi'_{\mathcal{T};\varpi}\big\}(\upsilon)=\big\{\mathfrak{f}^*\mathcal{D}_{\mathcal{T}}'^{\phi}\big\}(\upsilon)+\big\{\mathfrak{f}^*\varpi\big\}(\mathbf{u})\ \ \forall\ \upsilon\in\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}|_{\mathbf{u}},\ \mathbf{u}\in\widehat{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi},$$

does not vanish and so

$$N(\mathcal{D}_{\mathcal{T}}^{\phi})=N(\mathfrak{f}^*\mathcal{D}_{\mathcal{T}}'^{\phi})=0.$$

This establishes the last claim of Proposition 2.2. □

4.2 The contributions of the main boundary strata

We continue with the setup and notation in the proof of Proposition 2.2 and set

$$\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi} = \prod_{e\in \mathrm{E}^{\sigma}_{\bullet;+}(\mathcal{T})} \overline{\mathcal{M}}^{\bullet}_{2\mathfrak{g}(e/w_0)-1,(S^{\sigma}_{\bullet;e},\sigma)} \times \prod_{e\in \mathrm{E}^{\sigma}_{\bullet;\mathbb{R}}(\mathcal{T})} \mathbb{R}\overline{\mathcal{M}}_{\mathfrak{g}(e/w_0),(S^{\sigma}_{\bullet;e},\sigma)} \times \mathcal{C}_{\mathcal{T}},$$

where $\mathcal{C}_{\mathcal{T}}\subset\mathcal{C}^{\mathrm{E}_{\bullet}(\mathcal{T})}$ is the closure of $\mathcal{C}^{*}_{\mathcal{T}}$. Thus,

$$\mathcal{C}_{\mathcal{T}} = \big(\mathcal{C}^{\phi}\big)^{\mathrm{E}^{\sigma}_{\bullet;\mathbb{R}}(\mathcal{T})} \times \mathcal{C}_{\mathcal{T};\mathbb{C}}\,,$$

where $\mathcal{C}_{\mathcal{T};\mathbb{C}}\subset\mathcal{C}^{\mathrm{E}^{\sigma}_{\bullet;\mathbb{C}}(\mathcal{T})}$ is the fixed locus of the involution given by (3.19) with $\mathrm{E}_0(\mathcal{T})$ replaced by $\mathrm{E}^{\sigma}_{\bullet;\mathbb{C}}(\mathcal{T})$. We denote by

$$\breve{\mathcal{D}}^{\phi}_{\mathcal{T}}\colon \breve{\mathcal{F}}\mathcal{T}^{\sigma} \longrightarrow \widetilde{\mathrm{Obs}}^{\phi}_{\mathcal{T}}$$

the vector bundle homomorphism over $\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ defined analogously to the vector bundle homomorphism $\mathcal{D}^{\phi}_{\mathcal{T}}$ over $\widehat{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$.

Let

$$q\colon \widehat{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi} \longrightarrow \widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$$

be the map induced by the natural projection $\widehat{\mathcal{C}}_{\mathcal{T}}\longrightarrow\mathcal{C}_{\mathcal{T}}$. Since

$$\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}\big|_{\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}} = q^{*}\breve{\mathcal{F}}\mathcal{T}^{\sigma}\big|_{\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}}, \quad \mathrm{Obs}^{\phi}_{\mathcal{T}} = q^{*}\widetilde{\mathrm{Obs}}^{\phi}_{\mathcal{T}}\,, \quad \mathcal{D}^{\phi}_{\mathcal{T}}\big|_{\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}} = q^{*}\breve{\mathcal{D}}^{\phi}_{\mathcal{T}}\big|_{\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}},$$

and q is the identity on $\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$,

$$N\big(\mathcal{D}^{\phi}_{\mathcal{T}}\big) = N\big(q^{*}\breve{\mathcal{D}}^{\phi}_{\mathcal{T}}\big) = N\big(\breve{\mathcal{D}}^{\phi}_{\mathcal{T}}\big). \tag{4.4}$$

Since the contributions of the boundary strata of Propositions 2.3 and 2.4 are given by (4.3), it remains to determine the number $N(\breve{\mathcal{D}}^{\phi}_{\mathcal{T}})$.

Proof of Proposition 2.3. Let

$$\mathfrak{f}\colon \widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi} \longrightarrow \widetilde{\mathfrak{M}}'_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi} \equiv \prod_{e\in \mathrm{E}^{\sigma}_{\bullet;+}(\mathcal{T})} \overline{\mathcal{M}}^{\bullet}_{2\mathfrak{g}(e/w_0)-1,(S^{\sigma}_{\bullet;e},\sigma)} \times \prod_{e\in \mathrm{E}^{\sigma}_{\bullet;\mathbb{R}}(\mathcal{T})} \mathbb{R}\overline{\mathcal{M}}_{\mathfrak{g}(e/w_0),(S^{\sigma}_{\bullet;e},\sigma)} \times \mathcal{C}_{\mathcal{T};\mathbb{C}}$$

be the natural projection. We denote by

$$\breve{\mathcal{D}}'^{\phi}_{\mathcal{T}}\colon \breve{\mathcal{F}}'\mathcal{T}^{\sigma} \longrightarrow \widetilde{\mathrm{Obs}}'^{\phi}_{\mathcal{T}}$$

the vector bundle homomorphism over $\widetilde{\mathfrak{M}}'_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ defined analogously to the vector bundle homomorphism $\mathcal{D}^{\phi}_{\mathcal{T}}$ over $\widehat{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$, but for each $e\in\mathrm{E}^{\sigma}_{\bullet;\mathbb{R}}(\mathcal{T})$ we replace the factor $\pi^{*}_{\mathcal{C};\langle e\rangle}T\mathcal{C}$ in (3.28) by the trivial complex line bundle (with the standard conjugation) and the factor $\pi^{*}_{\mathcal{C};\langle e\rangle}TX$ in (3.24) by the trivial rank 3 complex vector bundle. Since $\mathcal{C}^{\phi}$ is a disjoint union of circles and the restriction of every orientable vector bundle to a circle is trivial,

$$\breve{\mathcal{F}}\mathcal{T}^{\sigma} = \mathfrak{f}^{*}\breve{\mathcal{F}}'\mathcal{T}^{\sigma}\big|_{\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}}, \quad \widetilde{\mathrm{Obs}}^{\phi}_{\mathcal{T}} = \mathfrak{f}^{*}\widetilde{\mathrm{Obs}}'^{\phi}_{\mathcal{T}}\,, \quad \breve{\mathcal{D}}^{\phi}_{\mathcal{T}} = \mathfrak{f}^{*}\breve{\mathcal{D}}'^{\phi}_{\mathcal{T}}$$

and thus $N(\check{\mathcal{D}}_{\mathcal{T}}^{\phi})=N(\mathfrak{f}^*\check{\mathcal{D}}_{\mathcal{T}}'^{\phi})$. If $\mathrm{E}_{\bullet;\mathbb{R}}^{\sigma}(\mathcal{T})\neq\varnothing$, then

$$\dim\widetilde{\mathfrak{M}}'_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}+\mathrm{rk}\,\check{\mathcal{F}}'\mathcal{T}^{\sigma}<\dim\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}+\mathrm{rk}\,\tilde{\mathcal{F}}\mathcal{T}^{\sigma}=\mathrm{rk}\,\mathrm{Obs}_{\mathcal{T}}^{\phi}=\mathrm{rk}\,\widetilde{\mathrm{Obs}}_{\mathcal{T}}'^{\phi}.$$

As in the last paragraph of the proof of Proposition 2.2, this implies that $N(\mathfrak{f}^*\check{\mathcal{D}}_{\mathcal{T}}'^{\phi})=0$. Proposition 2.3 now follows from (4.3) and (4.4). □

Proof of Proposition 2.4. We continue with the setup and notation in the proof of Proposition 2.3. We now assume that $\mathfrak{t}$ is a basic topological type, $\mathcal{T}=(\mathfrak{t},\mathfrak{m}_0(\mathfrak{t}))$, and $\mathrm{E}_{\bullet;\mathbb{R}}^{\sigma}(\mathcal{T})=\varnothing$. Let $m=|\mathrm{E}_{\bullet;+}^{\sigma}(\mathcal{T})|$ so that

$$\big|\widetilde{\mathrm{Aut}}^*(\mathcal{T})\big|=2^m\big|\mathrm{Aut}^*(\mathcal{T})\big|, \tag{4.5}$$

with the groups $\mathrm{Aut}^*(\mathcal{T})$ and $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$ as in Section 3.2. With the notation as in (2.8) and (2.9), we define the bundle homomorphism

$$\mathcal{D}_{g'}:\pi_{g'}^*L\otimes\pi_{\mathcal{C}}^*T\mathcal{C}\longrightarrow\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*TX$$

over $\overline{\mathcal{M}}_{g',1}\times\mathcal{C}$ analogously to (3.29).

For each $e\in\mathrm{E}_{\bullet;+}^{\sigma}(\mathcal{T})$, let

$$\overline{\mathcal{M}}_{\mathcal{T};e}^{\bullet}(\mathcal{C})=\overline{\mathcal{M}}_{2\mathfrak{g}(e/w_0)-1,(\{e,\sigma(e)\},\sigma)}^{\bullet}\times\big\{(z^+,z^-)\in\mathcal{C}^2:\,z^+=\phi(z^-)\big\}$$

and denote by

$$\pi_{\mathcal{T};e},\pi_{\mathcal{C};+},\pi_{\mathcal{C};-}:\overline{\mathcal{M}}_{\mathcal{T};e}^{\bullet}(\mathcal{C})\longrightarrow\overline{\mathcal{M}}_{2\mathfrak{g}(e/w_0)-1,(\{e,\sigma(e)\},\sigma)}^{\bullet},\mathcal{C},\mathcal{C}$$

the three component projection maps. Let

$$L_+,L_-,\mathbb{E}_+,\mathbb{E}_-\longrightarrow\overline{\mathcal{M}}_{2\mathfrak{g}(e/w_0)-1,(\{e,\sigma(e)\},\sigma)}^{\bullet}$$

be the universal tangent line bundles at the marked points indexed by e and $\sigma(e)$ and the subbundles of the Hodge bundle $\mathbb{E}$ consisting of the holomorphic differentials supported on the first and second component of each doublet (the first component carries the marked point indexed by e). Denote by

$$\begin{aligned}\mathcal{F}_e^{\sigma}&\subset\pi_{\mathcal{T};e}^*L_+\otimes\pi_{\mathcal{C};+}^*T\mathcal{C}\oplus\pi_{\mathcal{T};e}^*L_-\otimes\pi_{\mathcal{C};-}^*T\mathcal{C}\qquad\text{and}\\ \mathrm{Obs}_e^{\phi}&\subset\pi_{\mathcal{T};e}^*\mathbb{E}_+^*\otimes\pi_{\mathcal{C};+}^*TX\oplus\pi_{\mathcal{T};e}^*\mathbb{E}_-^*\otimes\pi_{\mathcal{C};-}^*TX\end{aligned} \tag{4.6}$$

the fixed loci of the conjugations over the identity induced by the involution ϕ as in Section 3.2 and by

$$\mathcal{D}_e^{\phi}:\mathcal{F}_e^{\sigma}\longrightarrow\mathrm{Obs}_e^{\phi}$$

the vector bundle homomorphism over $\overline{\mathcal{M}}_{\mathcal{T};e}^{\bullet}(\mathcal{C})$ defined in the same way as before.

The projections

$$\overline{\mathcal{M}}_{2\mathfrak{g}(e/w_0)-1,(\{e,\sigma(e)\},\sigma)}^{\bullet}\longrightarrow\overline{\mathcal{M}}_{\mathfrak{g}(e/w_0),1}\quad\text{and}\quad\big\{(z^+,z^-)\in\mathcal{C}^2:\,z^+=\phi(z^-)\big\}\longrightarrow\mathcal{C}$$

on the first component in both cases are diffeomorphisms and induce orientations on their domains from the complex orientations on their targets; we will call the former orientations the complex orientations. The map

$$\overline{\mathcal{M}}_{\mathcal{T};e}^{\bullet}(\mathcal{C})\longrightarrow\overline{\mathcal{M}}_{\mathfrak{g}(e/w_0),1}\times\mathcal{C}$$

induced by the above projections naturally lifts to vector bundle isomorphisms

$$\mathcal{F}_e^\sigma \longrightarrow \pi_{\mathfrak{g}(e/w_0)}^* L \otimes \pi_{\mathcal{C}}^* T\mathcal{C} \qquad \text{and} \qquad \mathrm{Obs}_e^\phi \longrightarrow \pi_{\mathfrak{g}(e/w_0)}^* \mathbb{E}^* \otimes \pi_{\mathcal{C}}^* TX \tag{4.7}$$

obtained by projecting to the first components in (4.6). These projections induce orientations on their domains from the complex orientations on their targets; we will call the former orientations the complex orientations. Since the isomorphisms (4.7) intertwine the vector bundle homomorphisms $\mathcal{D}_e^\phi$ and $\mathcal{D}_{\mathfrak{g}(e/w_0)}$,

$$N\big(\mathcal{D}_e^\phi\big)_{\mathbb{C}} = N\big(\mathcal{D}_{\mathfrak{g}(e/w_0)}\big), \tag{4.8}$$

where the two sides denote the numbers of zeros of generic affine bundle maps associated with $\mathcal{D}_e^\phi$ and $\mathcal{D}_{\mathfrak{g}(e/w_0)}$ with respect to the complex orientations on their domain and target vector bundles and the bases of these vector bundles.

In the case of Proposition 2.4,

$$\breve{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi = \prod_{e\in \mathrm{E}^\sigma_{\bullet;+}(\mathcal{T})} \overline{\mathcal{M}}^\bullet_{2\mathfrak{g}(e/w_0)-1,(\{e,\sigma(e)\},\sigma)} \times \mathcal{C}_{\mathcal{T}} = \prod_{e\in \mathrm{E}^\sigma_{\bullet;+}(\mathcal{T})} \overline{\mathcal{M}}^\bullet_{\mathcal{T};e}(\mathcal{C})\,. \tag{4.9}$$

With

$$\pi_e \colon \breve{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi \longrightarrow \overline{\mathcal{M}}^\bullet_{\mathcal{T};e}(\mathcal{C})$$

denoting the component projection map,

$$\breve{\mathcal{D}}_{\mathcal{T}}^\phi = \bigoplus_{e\in \mathrm{E}^\sigma_{\bullet;+}(\mathcal{T})} \pi_e^* \mathcal{D}_e^\phi \colon \breve{\mathcal{F}}\mathcal{T}^\sigma = \bigoplus_{e\in \mathrm{E}^\sigma_{\bullet;+}(\mathcal{T})} \pi_e^* \mathcal{F}_e^\sigma \longrightarrow \widetilde{\mathrm{Obs}}_{\mathcal{T}}^\phi = \bigoplus_{e\in \mathrm{E}^\sigma_{\bullet;+}(\mathcal{T})} \pi_e^* \mathrm{Obs}_e^\phi\,. \tag{4.10}$$

The identifications in (4.9) and (4.10) induce orientations on their domains from the complex orientations of their targets defined in the previous paragraph. By (4.8) and the first equality in (4.10),

$$N\big(\breve{\mathcal{D}}_{\mathcal{T}}^\phi\big)_{\mathbb{C}} = \prod_{e\in \mathrm{E}^\sigma_{\bullet;+}(\mathcal{T})} N\big(\mathcal{D}_{\mathfrak{g}(e/w_0)}\big), \tag{4.11}$$

where the left-hand side denotes the number of zeros of a generic affine bundle map associated with $\breve{\mathcal{D}}_{\mathcal{T}}^\phi$ with respect to the complex orientations on its domain and target vector bundles and the base of these vector bundles.

The curve $\mathcal{C} \subset X$ corresponds to an element $[\mathbf{u}]$ of $\mathfrak{M}_{h,\mathbf{f}}(X,B;J)^\phi$ as around (2.3). The real orientation on (X,ω,ϕ) fixed in Section 1 determines the sign,

$$\mathrm{sgn}_{\mathbf{f}}(\mathcal{C}) = \mathrm{sgn}(\mathbf{u}) \in \big\{\pm 1\big\},$$

of $[\mathbf{u}]$. The open subspace $\mathcal{C}^*_{\mathcal{T}}$ of $\mathcal{C}_{\mathcal{T}}$ corresponds to the family

$$\big\{\big[\mathbf{u},(z_e)_{e\in \mathrm{E}_\bullet(\mathcal{T})}\big] \colon (z_e)_{e\in \mathrm{E}_\bullet(\mathcal{T})} \in \mathcal{C}^*_{\mathcal{T}}\big\}$$

of marked maps meeting the constraints $\mathbf{f}$. The real orientation on (X,ω,ϕ) and the choice of the subset $\mathrm{E}^\sigma_{\bullet;+}(\mathcal{T})$ of $\mathrm{E}_\bullet(\mathcal{T})$ determine an orientation on $\mathcal{C}_{\mathcal{T}}$. It differs from the complex orientation by $\mathrm{sgn}_{\mathbf{f}}(\mathcal{C})$.

For each $g'\in\mathbb{Z}^+$, the real orientation on (X,ω,ϕ) determines an orientation on the moduli space $\overline{\mathfrak{M}}^{\bullet}_{2g'-1,(1^+,1^-)}(X,0;J)^\phi$ of J-holomorphic degree 0 maps from g'-doublets with one conjugate pair of marked points. As topological spaces

$$\overline{\mathfrak{M}}^{\bullet}_{2g'-1,(1^+,1^-)}(X,0;J)^\phi \approx \overline{\mathcal{M}}^{\bullet}_{2g'-1,(1^+,1^-)}\times\big\{(x^+,x^-)\in X^2:\ x_+=\phi(x_-)\big\}, \tag{4.12}$$

but the moduli space on the left-hand side comes with an associated obstruction bundle

$$\mathrm{Obs}_X^\phi \longrightarrow \overline{\mathcal{M}}^{\bullet}_{2g'-1,(1^+,1^-)}\times\big\{(x,\phi(x)):\ x\in X\big\}; \tag{4.13}$$

the latter is the fixed locus of the involution on

$$\pi_{g'}^*\mathbb{E}_+^*\otimes\pi_+^*TX\oplus\pi_{g'}^*\mathbb{E}_-^*\otimes\pi_-^*TX \longrightarrow \overline{\mathcal{M}}^{\bullet}_{2g'-1,(1^+,1^-)}\times\big\{(x^+,x^-)\in X^2:\ x_+=\phi(x_-)\big\}$$

induced by ϕ as in Section 3.2. In particular, the projection to the first component

$$\mathrm{Obs}_X^\phi \longrightarrow \pi_{g'}^*\mathbb{E}_+^*\otimes\pi_+^*TX \tag{4.14}$$

is an isomorphism. The last factors in (4.12) and (4.14) are oriented from the (almost) complex orientation of X (by projecting to the first component in the case of (4.12)). Along with the complex orientations on the first factors on the right-hand sides of (4.12) and (4.14), these orientations on the last factors induce an orientation on the total space of the vector bundle in (4.13); we will call this orientation the complex orientation. By [8, Theorem 1.3], the orientation on the total space of this vector bundle induced by the real orientation on (X,ω,ϕ) differs from the complex orientation by $(-1)^{g'-1}$.

By [8, Theorem 1.2], the immersion (3.21) is orientation-preserving with respect to the orientations on its domain and target induced by a real orientation on (X,ω,ϕ) if and only if $m\in2\mathbb{Z}$. Combining this with (4.11) and the last two paragraphs, we conclude that

$$\begin{aligned} N\big(\check{\mathcal{D}}^\phi_{\mathcal{T}}\big) &= (-1)^m\mathrm{sgn}_{\mathbf{f}}(\mathcal{C})\prod_{e\in\mathrm{E}^\sigma_{\bullet;+}(\mathcal{T})}\Big((-1)^{\mathfrak{g}(e/w_0)-1}N\big(\mathcal{D}_{\mathfrak{g}(e/w_0)}\big)\Big)\\ &= \mathrm{sgn}_{\mathbf{f}}(\mathcal{C})\prod_{e\in\mathrm{E}^\sigma_{\bullet;+}(\mathcal{T})}\Big((-1)^{\mathfrak{g}(e/w_0)}N\big(\mathcal{D}_{\mathfrak{g}(e/w_0)}\big)\Big). \end{aligned}$$

The claim now follows from (4.3), (4.5), and Lemma 4.1 below. □

Lemma 4.1. *With notation as above,*

$$N(\mathcal{D}_{g'}) = \int_{\overline{\mathcal{M}}_{g',1}\times\mathcal{C}} c_{2g'}\big(\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*\mathcal{N}_X\mathcal{C}\big)\bigg(\sum_{r=0}^{g'-1}(-1)^r\lambda_r\psi^{g'-1-r}\bigg).$$

Proof. By definition $N(\mathcal{D}_{g'})$ is the signed cardinality of the zero set of the affine bundle map

$$\begin{aligned} &\psi_{g';\varpi}\colon \pi_{g'}^*L\otimes\pi_{\mathcal{C}}^*T\mathcal{C}\longrightarrow\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*TX,\\ &\psi_{g';\varpi}(\upsilon)=\mathcal{D}_{g'}(\upsilon)+\varpi(\mathbf{u})\quad\forall\ \upsilon\in\pi_{g'}^*L\otimes\pi_{\mathcal{C}}^*T\mathcal{C}\big|_{\mathbf{u}},\ \mathbf{u}\in\overline{\mathcal{M}}_{g',1}\times\mathcal{C}, \end{aligned}$$

with respect to the complex orientations on the domain and target vector bundles and the complex orientation on their base, for a generic section ϖ of the target bundle. Since ϖ is generic, it spans a trivial complex subbundle $\mathbb{C}\varpi$ in $\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*TX$. The composition of ϖ with the tensor product of the identity and the natural projection $TX|_{\mathcal{C}}\longrightarrow\mathcal{N}_X\mathcal{C}$ induces a section ϖ' of the vector bundle

$$\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*\mathcal{N}_X\mathcal{C}\longrightarrow\overline{\mathcal{M}}_{g',1}\times\mathcal{C}\,.$$

Since ϖ is generic, the zero set $\mathcal{Z}_{\varpi'}$ of this section is a smooth oriented sub-orbifold.

We denote by $\Delta_{g'}\subset\overline{\mathcal{M}}_{g',1}$ the subspace of marked curves so that the irreducible component containing the marked point is a smooth rational curve. The vector bundle homomorphism $\mathcal{D}_{g'}$ corresponds to a section of the bundle

$$\big(\pi_{g'}^*L\otimes\pi_{\mathcal{C}}^*T\mathcal{C}\big)^*\otimes\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*TX\longrightarrow\overline{\mathcal{M}}_{g',1}\times\mathcal{C},$$

which we denote in the same way. We note that

$$\mathcal{D}_{g'}^{-1}(0)=\Delta_{g'}\times\mathcal{C}\subset\overline{\mathcal{M}}_{g',1}\times\mathcal{C}\,. \tag{4.15}$$

Along with the projection

$$\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*TX\longrightarrow\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*TX/\mathbb{C}\varpi\,,$$

the section $\mathcal{D}_{g'}$ induces a section of the bundle

$$\big(\pi_{g'}^*L\otimes\pi_{\mathcal{C}}^*T\mathcal{C}\big)^*\otimes\big(\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*TX/\mathbb{C}\varpi\big)\longrightarrow\overline{\mathcal{M}}_{g',1}\times\mathcal{C}.$$

Since the image of $\mathcal{D}_{g'}$ lies in $\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*T\mathcal{C}$, the latter restricts to a section of the bundle

$$V\equiv\big(\pi_{g'}^*L\otimes\pi_{\mathcal{C}}^*T\mathcal{C}\big)^*\otimes\big(\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*T\mathcal{C}/\mathbb{C}\varpi\big)\longrightarrow\mathcal{Z}_{\varpi'}\,;$$

we denote it by $\mathcal{D}_{g'}^{\perp}$.

Since the image of $\mathcal{D}_{g'}$ lies in $\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*T\mathcal{C}$, $N(\mathcal{D}_{g'})$ is the signed cardinality of the zero set of the affine bundle map

$$\psi'_{g';\varpi}\colon\pi_{g'}^*L\otimes\pi_{\mathcal{C}}^*T\mathcal{C}\longrightarrow\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*T\mathcal{C},\quad\psi'_{g';\varpi}(\upsilon)=\psi_{g';\varpi}(\upsilon)\ \ \forall\ \upsilon\in\pi_{g'}^*L\otimes\pi_{\mathcal{C}}^*T\mathcal{C}\big|_{\mathcal{Z}_{\varpi'}}.$$

Since the linear part of this bundle map is holomorphic, it is a regular polynomial in the sense of [25, Definition 3.9]. By [25, Lemma 3.14],

$$N(\mathcal{D}_{g'})=\big\langle e(V),\big[\mathcal{Z}_{\varpi'}\big]\big\rangle-\mathrm{Cntr}_{\mathcal{D}_{g'}^{-1}(0)\cap\mathcal{Z}_{\varpi'}}\big(\mathcal{D}_{g'}^{\perp}\big). \tag{4.16}$$

The last term above is the $\mathcal{D}_{g'}^{\perp}$-contribution to the middle term from the subspace $\mathcal{D}_{g'}^{-1}(0)\cap\mathcal{Z}_{\varpi'}$ of $\mathcal{Z}_{\varpi'}$; this notion is the direct analogue of Definition 2.1 in this setting. By [27, Propositions 2.18A,B] and (4.15),

$$\mathrm{Cntr}_{\mathcal{D}_{g'}^{-1}(0)\cap\mathcal{Z}_{\varpi'}}\big(\mathcal{D}_{g'}^{\perp}\big)=\sum_i d_i\big\langle e(V_i),\big[(\Delta_{g';i}\times\mathcal{C})\cap\mathcal{Z}_{\varpi'}\big]\big\rangle \tag{4.17}$$

for some $d_i\in\mathbb{Z}$, closed subvarieties $\Delta_{g';i}\subset\Delta_{g'}$, and vector bundles V_i over $\Delta_{g';i}\times\mathcal{C}$; these are determined by the scheme structure of $\mathcal{D}_{g'}^{-1}(0)\subset\overline{\mathcal{M}}_{g',1}$.

By the Poincare Duality, (2.11), and (2.12),

$$\begin{aligned}\big\langle e(V_i),\big[(\Delta_{g';i}\times\mathcal{C})\cap\mathcal{Z}_{\varpi'}\big]\big\rangle &= \big\langle e(V_i)e\big(\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*\mathcal{N}_X\mathcal{C}\big),\big[\Delta_{g';i}\times\mathcal{C}\big]\big\rangle\\ &= \big(2-2h-c_1(B)\big)\big\langle e(V_i)\lambda_{g'-1}\lambda_{g'},\big[\Delta_{g';i}\big]\big\rangle.\end{aligned}\tag{4.18}$$

By [1, Lemma 1], $\lambda_{g'-1}\lambda_{g'}$ vanishes on $\overline{\mathcal{M}}_{g',1}-\mathcal{M}_{g',1}$ and thus on $\Delta_{g';i}$. Combining this observation with (4.16), (4.17), and (4.18), we conclude that

$$\begin{aligned}N(\mathcal{D}_{g'}) &= \big\langle e\big((\pi_{g'}^*L\otimes\pi_{\mathcal{C}}^*T\mathcal{C})^*\otimes(\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*T\mathcal{C}/\mathbb{C}\varpi)\big),\big[\mathcal{Z}_{\varpi'}\big]\big\rangle\\ &= \big\langle e\big((\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*T\mathcal{C})/(\pi_{g'}^*L\otimes\pi_{\mathcal{C}}^*T\mathcal{C})\big),\big[\mathcal{Z}_{\varpi'}\big]\big\rangle = \big\langle e\big((\pi_{g'}^*\mathbb{E}^*)/(\pi_{g'}^*L)\big),\big[\mathcal{Z}_{\varpi'}\big]\big\rangle\\ &= \sum_{r=0}^{g'-1}\big\langle(-1)^r\lambda_r\psi^{g'-1-r},\big[\mathcal{Z}_{\varpi'}\big]\big\rangle = \sum_{r=0}^{g'-1}\int_{\overline{\mathcal{M}}_{g',1}\times\mathcal{C}}(-1)^r\lambda_r\psi^{g'-1-r}c_{2g'}\big(\pi_{g'}^*\mathbb{E}^*\otimes\pi_{\mathcal{C}}^*\mathcal{N}_X\mathcal{C}\big).\end{aligned}$$

The third equality above holds because $\pi_{\mathcal{C}}^*c_1(T\mathcal{C})$ vanishes on $\mathcal{Z}_{\varpi'}$ by (2.11) and the Poincare Duality; the last equality holds by the Poincare Duality. □

5 Proof of Proposition 3.1

Continuing with the notation of Section 3, we set up a two-step gluing construction to prove Proposition 3.1. For each $\mathcal{T}\in\mathscr{T}_{g,l}(\mathcal{C})^{\phi}$ as in (3.5), we first smooth out all nodes corresponding to Edg^0 in (3.7) and Edg^c in (3.11). This step is not obstructed. At the second stage, we smooth out the remaining nodes, i.e. the nodes corresponding to $\mathrm{E}_{\bullet}(\mathcal{T})$; see (3.12). This step is obstructed, i.e. not every J-holomorphic map in $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ can be smoothed out to a nearby element of $\mathfrak{M}_{g,\mathbf{f}}(X,B;J,t\nu)^{\phi}$. We show that the elements of $\mathfrak{M}_{g,\mathbf{f}}(X,B;J,t\nu)^{\phi}$ that lie near $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ (and thus are smoothings of some elements of $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$) correspond to the small zeros of the (non-linear) Banach bundle map $\Xi_{\mathcal{T};t\nu}$ in (5.12). This map has no small zeros if ν is generic and $\mathcal{T}$ is as in Proposition 3.1(1). If $\mathcal{T}$ is as in Proposition 3.1(2), the small zeros of $\Xi_{\mathcal{T};t\nu}$ correspond to the small zeros of the map $\Psi_{\mathcal{T};t\nu}$ in (3.32) and (5.30) between finite-rank vector bundles over $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$. We use the same strategy as in [28, Sections 3.2,4.2,4.3] and [29, Section 3.3].

For a generic choice of the pseudocycles $\mathbf{f}$, the projection

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J)^{\phi}\longrightarrow\overline{\mathfrak{M}}_{g,l}\big(X,B;J\big)^{\phi}$$

to the moduli space component in (1.4) is an embedding. Thus, we can view $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ as the subset of the elements of $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ which map to the curve $\mathcal{C}\subset X$ and meet the pseudocycles $\mathbf{f}$ at the marked points.

Let (S_0,σ_0) be as in (3.9). We denote by $\mathfrak{M}_{h,(S_0,\sigma_0)}(B)^{\phi}$ the moduli space of degree B J-holomorphic real maps from smooth genus h symmetric surfaces with the marked points indexed by the set S_0 with the involution σ_0. Analogously to (3.20), let

$$\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^{\phi}=\prod_{e\in\mathrm{E}^{\sigma}_{0;\geqslant}(\mathcal{T})\sqcup\mathrm{E}^{\sigma}_{\bullet;\geqslant}(\mathcal{T})}\mathcal{M}_{\mathcal{T},\sigma;e}\times\mathfrak{M}_{h,(S_0,\sigma_0)}(B)^{\phi}\tag{5.1}$$

and

$$\iota_{\mathcal{T},\mathbf{f}}\colon\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^{\phi}\longrightarrow\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$$

be the natural $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-invariant node-identifying immersion extending (3.21). The image of this immersion is the stratum

$$\mathfrak{M}_{\mathcal{T}}(X)^{\phi} \subset \overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$$

of stable maps of the topological type $\mathcal{T}$ (not necessarily passing through the constrains). For each element $\mathbf{u}$ of $\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^{\phi}$, a vertex $w\in\mathrm{Ver}$ corresponds to an irreducible component $\Sigma_{\mathbf{u};w}$ of the domain $\Sigma_{\mathbf{u}}$ of the map u in the image of this immersion. Each edge $e=(w_e^0,w_e^c)$ in $\mathrm{E}_{\bullet}(\mathcal{T})\subset\mathrm{Edg}$ corresponds to a node $x_e(\mathbf{u})\in\Sigma_u$ formed by joining a point $x_e^0(\mathbf{u})$ in $\Sigma^0_{\mathbf{u};e}\equiv\Sigma_{\mathbf{u};w_e^0}$ with a point $x_e^c(\mathbf{u})$ in $\Sigma^c_{\mathbf{u};e}\equiv\Sigma_{\mathbf{u};w_e^c}$.

We denote the natural extensions of the vector bundles in (3.24), (3.25), and (3.27) and of the bundle homomorphism in (3.26) to $\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^{\phi}$ in the same way. The center component $\Sigma_{\mathbf{u};0}=\Sigma_{\mathbf{u};\mathrm{w}_0}$ is no longer identified with $\mathcal{C}$. The bundle $\pi^*_{\mathcal{C};\langle e\rangle}TX$ in (3.24) should thus be replaced by $\pi^*_{\mathfrak{M}}\mathrm{ev}^*_{\langle e\rangle}TX$, where $\pi_{\mathfrak{M}}$ is the projection onto the last factor in (5.1). With notation as in (3.27), let

$$\widetilde{\mathcal{F}}_1\mathcal{T} = \Big(\bigoplus_{e\in\mathrm{Edg}^0\sqcup\mathrm{Edg}^c}\widetilde{L}_e\Big)\Big/\mathrm{Aut}_{\bullet}(\mathcal{T}) \longrightarrow \widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^{\phi},$$

$$\widetilde{\mathfrak{F}}\mathcal{T} = \bigoplus_{e\in\mathrm{E}_{\bullet}(\mathcal{T})}\pi^*_{\mathcal{T};e}L_e\otimes\pi^*_{\mathfrak{M}}L_{\langle e\rangle} \longrightarrow \widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^{\phi}.$$

We denote by $\upsilon_1\in\widetilde{\mathcal{F}}_1\mathcal{T}$ the image of an element υ of $\widetilde{\mathcal{F}}\mathcal{T}$ under the natural projection

$$\widetilde{\mathcal{F}}\mathcal{T} \longrightarrow \widetilde{\mathcal{F}}_1\mathcal{T} \subset \widetilde{\mathcal{F}}\mathcal{T}.$$

The involution σ acts on $\widetilde{\mathfrak{F}}\mathcal{T}$ by

$$\big(\sigma\big((\upsilon_{e'}\otimes\upsilon_{\langle e'\rangle})_{e'\in\mathrm{E}_{\bullet}(\mathcal{T})}\big)\big)_e = \mathrm{d}\sigma(\upsilon_{\sigma(e)})\otimes\mathrm{d}\sigma(\upsilon_{\sigma\langle e\rangle}).$$

We denote the corresponding fixed locus by $\widetilde{\mathfrak{F}}\mathcal{T}^{\sigma}$.

For each $e\in\mathrm{E}_{\bullet}(\mathcal{T})$, let $\mathrm{Edg}'_{0,\bullet;e}\subset\mathrm{Edg}'_{0;\langle e\rangle}$ be the subset of edges whose removal separates e and $\langle e\rangle$. This subset is empty if $e=\langle e\rangle$. Let

$$\mathrm{Edg}_{0,\bullet;e} = \mathrm{Edg}'_{0,\bullet;e}\cup\{e\}\cup\{\langle e\rangle\}\subset\mathrm{Edg} \qquad \forall\ e\in\mathrm{E}_{\bullet}(\mathcal{T}).$$

Since $\Sigma_{\mathbf{u};w}\approx\mathbb{P}^1$ for every $w\in e$ with $e\in\mathrm{Edg}'_{0,\bullet;e}$, there is a natural isomorphism

$$F_e\colon \bigotimes_{e'\in\mathrm{Edg}_{0,\bullet;e}}\widetilde{L}_{e'} \xrightarrow{\ \cong\ } \pi^*_{\mathcal{T};e}L_e\otimes\pi^*_{\mathcal{C};\langle e\rangle}T\mathcal{C}; \tag{5.2}$$

see [22, (2.3)]. Define

$$\rho_{\mathcal{T}}\colon\widetilde{\mathcal{F}}\mathcal{T}\longrightarrow\widetilde{\mathfrak{F}}\mathcal{T}, \qquad \rho_{\mathcal{T}}(\upsilon)\big(\rho_e(\upsilon)\big)_{e\in\mathrm{E}_{\bullet}(\mathcal{T})} = \bigg(F_e\Big(\bigotimes_{e'\in\mathrm{Edg}_{0,\bullet;e}}\upsilon_{e'}\Big)\bigg)_{e\in\mathrm{E}_{\bullet}(\mathcal{T})},$$

$$\mathcal{D}_{\mathcal{T}}\colon\widetilde{\mathfrak{F}}\mathcal{T}\longrightarrow\mathrm{Obs}_{\mathcal{T}}, \quad \big\{\mathcal{D}_{\mathcal{T}}\big((\upsilon_e\otimes\upsilon_{\langle e\rangle})_{e\in\mathrm{E}_{\bullet}(\mathcal{T})}\big)\big\}\big((\kappa_e)_{e\in\mathrm{E}_{\bullet}(\mathcal{T})}\big) = \Big(\kappa_e(\upsilon_e)\big(\mathrm{d}_{x^0_{\langle e\rangle}(\mathbf{u})}u_0(\upsilon_{\langle e\rangle})\big)\Big)_{e\in\mathrm{E}_{\bullet}(\mathcal{T})},$$

where $u_0=u|_{\Sigma_{\mathbf{u};0}}$ and $x^0_{\langle e\rangle}(\mathbf{u})\in\Sigma_{\mathbf{u};0}$ is the point forming the node of $\Sigma_{\mathbf{u}}$ corresponding to the edge $\langle e\rangle\in\mathrm{E}_0(\mathcal{T})$. The bundle maps $\mathcal{D}_{\mathcal{T}}$ and $\rho_{\mathcal{T}}$ restrict to

$$\mathcal{D}^{\phi}_{\mathcal{T}}\colon\widetilde{\mathfrak{F}}\mathcal{T}^{\sigma}\longrightarrow\mathrm{Obs}^{\phi}_{\mathcal{T}} \qquad\text{and}\qquad \rho^{\sigma}_{\mathcal{T}}\colon\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}\longrightarrow\widetilde{\mathfrak{F}}\mathcal{T}^{\sigma},$$

respectively. If $\mathcal{T}=(\mathfrak{t},\mathfrak{m})$ with $\mathfrak{t}$ basic, then $\widetilde{\mathfrak{F}}\mathcal{T}^{\sigma}=\widetilde{\mathcal{F}}\mathcal{T}^{\sigma}$, $\rho^{\sigma}_{\mathcal{T}}=\mathrm{id}$, and the restriction of $\mathcal{D}^{\phi}_{\mathcal{T}}$ to $\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$ is as below (3.29).

5.1 The unobstructed gluing step

Let $\mathfrak{U}\longrightarrow\Delta^\sigma$ be a family of deformations of the domains of the elements of $\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi$ for a small neighborhood $\Delta^\sigma\subset\widetilde{\mathcal{F}}\mathcal{T}^\sigma$ of the zero section. It is the restriction to $\widetilde{\mathcal{F}}\mathcal{T}^\sigma\subset\widetilde{\mathcal{F}}\mathcal{T}$ of the analogous family in the complex setting. The fiber of $\mathfrak{U}\longrightarrow\Delta^\sigma$ over $\upsilon\in\Delta^\sigma$ is a nodal symmetric surface $(\Sigma_\upsilon,\sigma_\upsilon)$. Its dual graph $\mathcal{T}_\upsilon$ is a rooted decorated graph obtained from $\mathcal{T}$ by deleting the edges corresponding to the non-zero components of υ and identifying the vertices of each deleted edge. In particular,

$$\mathrm{E}_\bullet(\mathcal{T}_\upsilon)=\mathrm{E}_\bullet(\mathcal{T})\qquad\forall\,\upsilon\in\Delta^\sigma\cap\widetilde{\mathcal{F}}_1\mathcal{T};$$

we will not distinguish between these two sets below. It can be assumed that $\upsilon_1\in\Delta^\sigma$ for all $\upsilon\in\Delta^\sigma$.

We fix an involution-invariant Riemannian metric on $\mathfrak{U}$ and denote its restriction to Σ_υ by g_υ. For any $\upsilon\in\Delta^\sigma$, $e\in\mathrm{Edg}$ with $\upsilon_e=0$, and $\delta\in\mathbb{R}$, let $\Sigma_{\upsilon;e}(\delta)\subset\Sigma_\upsilon$ denote the g_υ-ball of radius δ centered at the node $x_e(\upsilon)$ corresponding to e. If in addition $\upsilon\in\widetilde{\mathcal{F}}_1\mathcal{T}$ and $e\in\mathrm{E}_\bullet(\mathcal{T})$, define

$$\Sigma^0_{\upsilon;e}(\delta)=\Sigma_{\upsilon;e}(\delta)\cap\Sigma^0_{\upsilon;e}\,,\qquad\Sigma^c_{\upsilon;e}(\delta)=\Sigma_{\upsilon;e}(\delta)\cap\Sigma^c_{\upsilon;e}\,,\qquad\widehat{\Sigma}^c_{\upsilon;e}(\delta)=\widehat{\Sigma}^c_{\upsilon;e}\cup\Sigma_{\upsilon;e}(\delta)\,.$$

If $e\in\mathrm{E}_0(\mathcal{T})$, let $\widehat{\Sigma}^0_{\mathbf{u};e}(\delta)=\widehat{\Sigma}^0_{\mathbf{u};e}\cup\Sigma_{\mathbf{u};e}(\delta)$. The subset $\Delta^\sigma\subset\widetilde{\mathcal{F}}\mathcal{T}^\sigma$ contains $\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi$ as the zero section. There are continuous fiber-preserving retractions q and $q_\bullet$ respecting the involutions so that the diagram

$$\begin{array}{ccccc}
\mathfrak{U} & \xrightarrow{q_\bullet} & \mathfrak{U}|_{\Delta^\sigma\cap\widetilde{\mathcal{F}}_1\mathcal{T}^\sigma} & \xrightarrow{q} & \mathfrak{U}|_{\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi}\\
\downarrow & & \downarrow & & \downarrow\\
\Delta^\sigma & \longrightarrow & \Delta^\sigma\cap\widetilde{\mathcal{F}}_1\mathcal{T}^\sigma & \longrightarrow & \widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi
\end{array}\qquad(\text{with } q:\mathfrak{U}\longrightarrow\mathfrak{U}|_{\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi})$$

commutes. We denote by

$$q_\upsilon\colon\Sigma_\upsilon\longrightarrow\Sigma_{\mathbf{u}}\quad\text{and}\quad q_{\bullet;\upsilon}\colon\Sigma_\upsilon\longrightarrow\Sigma_{\upsilon_1},\qquad\upsilon\in\Delta^\sigma|_{\mathbf{u}},\ \mathbf{u}\in\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi\,,\tag{5.3}$$

the restrictions of q and $q_\bullet$ to Σ_υ.

After possibly shrinking Δ^σ, the Riemannian metric on $\mathfrak{U}$ and the map q can be chosen so that

- q is smooth on each stratum of $\mathfrak{U}$,
- $q(z_i^\pm(\upsilon))=z_i^\pm\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}^\sigma}(\upsilon)\big)$ for all $\upsilon\in\Delta^\sigma$ and $i=1,\ldots,l$,
- for each $\upsilon\in\Delta^\sigma$, q_υ is biholomorphic on the complement of the subspaces $q_\upsilon^{-1}(\Sigma_{\mathbf{u};e}(2\sqrt{|\upsilon_e|}))$ with $e\in\mathrm{Edg}$,
- there exists a continuous function $\delta_q:\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi\longrightarrow\mathbb{R}^+$ such that every restriction (5.3) is a $(g_\upsilon,g_{\mathbf{u}})$-isometry on the complement of the subspaces $q_\upsilon^{-1}(\Sigma_{\mathbf{u};e}(\delta_q(\mathbf{u})))$ with $e\in\mathrm{Edg}$ and $\upsilon_e\neq0$,
- there exist a continuous function $\delta_{\bar\partial}\colon\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi\longrightarrow\mathbb{R}^+$ and holomorphic functions

$$z_e,z_e^c\colon\bigcup_{\mathbf{u}\in\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi}q^{-1}\big(\Sigma_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))\big)\longrightarrow\mathbb{C},\qquad e\in\mathrm{E}_\bullet(\mathcal{T}),$$

such that $z_{e;\mathbf{u}}\equiv z_e|_{\Sigma^0_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))}$ and $z^c_{e;\mathbf{u}}\equiv z^c_e|_{\Sigma^c_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))}$ are unitary coordinates centered at $x_e(\mathbf{u})$ and

$$\big(z_ez_e^c\big)\big|_{q_\upsilon^{-1}(\Sigma_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u})))}\,\frac{\partial}{\partial z_{e;\mathbf{u}}}\bigg|_{x^0_e(\mathbf{u})}\otimes\frac{\partial}{\partial z^c_{e;\mathbf{u}}}\bigg|_{x^c_e(\mathbf{u})}=\upsilon_e\,,\quad \big|z_e\big|_{q_\upsilon^{-1}(x_e(\mathbf{u}))}\big|=\big|z^c_e\big|_{q_\upsilon^{-1}(x_e(\mathbf{u}))}\big|\,,$$
$$z_e\big|_{q_\upsilon^{-1}(\Sigma^0_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))-\Sigma_{\mathbf{u};e}(2\sqrt{|\upsilon_e|}))}=z_{e;\mathbf{u}}\circ q_\upsilon\big|_{q_\upsilon^{-1}(\Sigma^0_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))-\Sigma_{\mathbf{u};e}(2\sqrt{|\upsilon_e|}))}\,,$$
$$z^c_e\big|_{q_\upsilon^{-1}(\Sigma^c_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))-\Sigma_{\mathbf{u};e}(2\sqrt{|\upsilon_e|}))}=z^c_{e;\mathbf{u}}\circ q_\upsilon\big|_{q_\upsilon^{-1}(\Sigma^c_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))-\Sigma_{\mathbf{u};e}(2\sqrt{|\upsilon_e|}))}$$

for all $\upsilon\in\Delta^\sigma\big|_{\mathbf{u}}$ and $\mathbf{u}\in\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi$.

We can assume that $2\delta_q(\mathbf{u})$ is less than the minimal distance between the nodal and marked points of $\Sigma_{\mathbf{u}}$ for every $\mathbf{u}\in\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi$, $\delta_{\bar\partial}<\delta_q<1$, and

$$16|\upsilon|<\delta_{\bar\partial}\big(\pi_{\tilde{\mathcal{F}}\mathcal{T}}(\upsilon)\big)^2\qquad\forall\ \upsilon\in\Delta^\sigma\,.$$

The functions δ_q, $\delta_{\bar\partial}$, z_e, and z_e^c can be chosen compatibly with the $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-actions and with the involution on $\mathfrak{U}$.

With $e\in\mathrm{E}_\bullet(\mathcal{T})$ and F_e as in (5.2), define

$$\rho_{e;\mathbf{u}}\colon\mathbb{C}\longrightarrow T_{x_{\langle e\rangle}(\mathbf{u})}\Sigma_{\mathbf{u};0}\qquad\text{by}$$
$$v_e\otimes\rho_{e;\mathbf{u}}(c)=F_e\Bigg(v_e\otimes c\frac{\partial}{\partial z_{e;\mathbf{u}}}\bigg|_{x^0_e(\mathbf{u})}\otimes\bigotimes_{\substack{e'\in\mathrm{Edg}_{0,\bullet;e}\\ e'\neq e}}v_{e'}\Bigg)\quad\forall\ c\in\mathbb{C},\ v_e\in\pi^*_{\mathcal{T};e}L_e\big|_{\mathbf{u}}\,.$$

By the $\mathbb{C}$-linearity of F_e, $\rho_{e;\mathbf{u}}$ is a well-defined $\mathbb{C}$-linear isomorphism.

Let g_X be a ϕ-invariant metric on X, ∇^X be its Levi-Civita connection, and exp be the exponential map of ∇^X. Fix a number $p>2$. For each $\upsilon\in\Delta^\sigma$ and a smooth real map $f\colon\Sigma_\upsilon\longrightarrow X$, let

$$D^\phi_f\colon\Gamma(f)^\phi\equiv\Gamma\big(\Sigma_\upsilon;f^*TX\big)^{\phi,\sigma_\upsilon}\longrightarrow\Gamma^{0,1}(f)^\phi\equiv\Gamma^{0,1}\big(\Sigma_\upsilon;T^*\Sigma^{0,1}_\upsilon\otimes f^*TX\big)^{\phi,\sigma_\upsilon}\tag{5.4}$$

denote the linearization of the $\bar\partial_J$-operator at f defined via the connection ∇^X as in [16, Section 3.1]. The construction of [12, Section 3] provides modified Sobolev norms $\|\cdot\|_{\upsilon,p,1}$ and $\|\cdot\|_{\upsilon,p}$ on the domain and target of the homomorphism (5.4) as well as L^2-inner products $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\upsilon,2}$ on these spaces. These homomorphisms satisfy elliptic estimates involving the $\|\cdot\|_{\upsilon,p,1}$ and $\|\cdot\|_{\upsilon,p}$ norms with coefficients that depend only on $\pi_{\tilde{\mathcal{F}}\mathcal{T}}(\upsilon)\in\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi$ and $\|\mathrm{d}f\|_{\upsilon,p}\in\mathbb{R}$.

For $\upsilon\in\Delta^\sigma\cap\tilde{\mathcal{F}}_1\mathcal{T}|_{\mathbf{u}}$, Σ_υ is obtained from $\Sigma_{\mathbf{u}}$ by smoothing the nodes corresponding to the subsets Edg^0 and Edg^c of the edges of $\mathcal{T}$. Since the restrictions of u to the irreducible components of $\Sigma_{\mathbf{u}}$ sharing the nodes indexed by Edg^c are constant, the smoothings of these nodes of $\Sigma_{\mathbf{u}}$ extend to the deformations of the map $\mathbf{u}$ into X. The nodes indexed by Edg^0 are the nodes of the center $\Sigma^0_{\mathbf{u}}$ of $\Sigma_{\mathbf{u}}$, which consists of the non-contracted smooth genus h curve $\Sigma_{\mathbf{u};0}$ with contracted rational tails $\widehat\Sigma^0_{\mathbf{u};e}$ attached. If $\mathbf{u}\in\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$, then $u|_{\Sigma^0_{\mathbf{u}}}$ is a regular J-holomorphic map. Therefore, there exists a neighborhood

$$\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})\subset\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi$$

of $\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$ such that the family of the deformations

$$\mathfrak{U}\big|_{\Delta^\sigma_{\mathcal{C}}\cap\tilde{\mathcal{F}}_1\mathcal{T}} \longrightarrow \Delta^\sigma_{\mathcal{C}}\cap\tilde{\mathcal{F}}_1\mathcal{T}\,, \qquad \text{where} \quad \Delta^\sigma_{\mathcal{C}} = \Delta^\sigma \cap \tilde{\mathcal{F}}\mathcal{T}^\sigma\big|_{\tilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})}\,,$$

of the domains of the elements of $\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})$ extends to a continuous $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-invariant family

$$\tilde{u}_\upsilon\colon \Sigma_\upsilon \longrightarrow X, \qquad \upsilon\in\Delta^\sigma_{\mathcal{C}}\cap\tilde{\mathcal{F}}_1\mathcal{T}\,, \tag{5.5}$$

of J-holomorphic maps. This family can be chosen so that it is smooth on each stratum of $\Delta^\sigma_{\mathcal{C}}\cap\tilde{\mathcal{F}}_1\mathcal{T}$. The smoothings of the maps $\mathbf{u}$ with a fixed image curve arise algebraically.

By shrinking $\delta_{\bar\partial}$, we can assume that the diameter of $u(\Sigma^0_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u})))$ is at most half the injectivity radius of g_X for every $\mathbf{u}\in\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi$ and $e\in\mathrm{E}_0(\mathcal{T})$. Since $\tilde{u}_\upsilon\longrightarrow u$ as $\upsilon\longrightarrow\mathbf{u}$, we can also assume that the diameter of

$$u\big(\Sigma^0_{\mathbf{u};\langle e\rangle}\big(\delta_{\bar\partial}(\mathbf{u})\big)\big)\cup\tilde{u}_\upsilon\big(q_\upsilon^{-1}\big(\hat{\Sigma}^0_{\mathbf{u};\langle e\rangle}(\delta_{\bar\partial}(\mathbf{u}))\big)\big)\subset X$$

is less than the injectivity radius for all $\upsilon\in\Delta^\sigma_{\mathcal{C}}\cap\tilde{\mathcal{F}}_1\mathcal{T}|_{\mathbf{u}}$ and $e\in\mathrm{E}_\bullet(\mathcal{T})$. For $\upsilon\in\Delta^\sigma_{\mathcal{C}}\cap\tilde{\mathcal{F}}_1\mathcal{T}|_{\mathbf{u}}$ and $e\in\mathrm{E}_\bullet(\mathcal{T})$, we can thus define a smooth function

$$\zeta_{\upsilon;e}\colon q_\upsilon^{-1}\big(\hat{\Sigma}^0_{\mathbf{u};\langle e\rangle}(\delta_{\bar\partial}(\mathbf{u}))\big)\cup\hat{\Sigma}^c_{\upsilon;e}\longrightarrow T_{u(x_{\langle e\rangle}(\mathbf{u}))}X \quad \text{s.t.} \quad \exp_{u(x_{\langle e\rangle}(\mathbf{u}))}\zeta_{\upsilon;e}=\tilde{u}_\upsilon \tag{5.6}$$

by requiring that $\zeta_{\upsilon;e}|_{\hat{\Sigma}^c_{\upsilon;e}}=0$.

For each $\upsilon\in\Delta^\sigma_{\mathcal{C}}|_{\mathbf{u}}\cap\tilde{\mathcal{F}}_1\mathcal{T}$, let

$$u_\upsilon = u\circ q_\upsilon\colon \Sigma_\upsilon\longrightarrow X.$$

Since $\tilde{u}_\upsilon\longrightarrow u$ as $\upsilon\longrightarrow\mathbf{u}$,

$$\tilde{u}_\upsilon \equiv \exp_{u_\upsilon}\zeta_\upsilon \tag{5.7}$$

for some small vector field $\zeta_\upsilon\in\Gamma(u_\upsilon)^\phi$, provided Δ^σ and $\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})$ are sufficiently small. Since the map q_υ is holomorphic on $\Sigma_{\upsilon;e}(\delta_{\bar\partial}(\mathbf{u}))$ for each $e\in\mathrm{E}_\bullet(\mathcal{T})$,

$$z_{e;\upsilon}\equiv z_e\big|_{\Sigma^0_{\upsilon;e}(\delta_{\bar\partial}(\mathbf{u}))} = z_{\mathbf{u};e}\circ q_\upsilon\big|_{\Sigma^0_{\upsilon;e}(\delta_{\bar\partial}(\mathbf{u}))}\colon \Sigma^0_{\upsilon;e}\big(\delta_{\bar\partial}(\mathbf{u})\big)\longrightarrow\mathbb{C}$$

is a holomorphic coordinate centered at $x^0_e(\upsilon)=q_\upsilon^{-1}(x^0_e(\mathbf{u}))$.

Lemma 5.1. *The families of $\mathfrak{U}\longrightarrow\Delta^\sigma$ and (5.5) can be parametrized so that the following holds. There exist smooth functions*

$$C\colon \widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})\longrightarrow\mathbb{R}^+ \qquad \textit{and}$$

$$\mathfrak{R}_{\upsilon;e}\colon \Sigma_{\upsilon;e}\big(\delta_{\bar\partial}(\mathbf{u})\big)\longrightarrow T_{u(x_{\langle e\rangle}(\mathbf{u}))}X \quad \textit{with } \upsilon\in\Delta^\sigma_{\mathcal{C}}\cap\tilde{\mathcal{F}}_1\mathcal{T}|_{\mathbf{u}},\ \mathbf{u}\in\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C}),\ e\in\mathrm{E}_\bullet(\mathcal{T})$$

respecting the $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-actions such that

$$\zeta_{\upsilon;e}\big(z_{e;\upsilon}\big) = \zeta_{\upsilon;e}(0) + \mathrm{d}_{x^0_{\langle e\rangle}(\mathbf{u})}u_0\big(\rho_{e;\mathbf{u}}(z_{e;\upsilon})\big) + \mathfrak{R}_{\upsilon;e}\big(z_{e;\upsilon}\big)\,, \tag{5.8}$$

$$\|\zeta_\upsilon\|_{\upsilon,p,1}\leqslant C(\mathbf{u})|\upsilon|^{1/p}, \quad \big|\mathfrak{R}_{\upsilon;e}(z_{e;\upsilon})\big|\leqslant C(\mathbf{u})\big|\rho_{e;\mathbf{u}}\big(z_{e;\upsilon}\big)\big||z_{e;\upsilon}|, \quad \big|\mathrm{d}_{z_{e;\upsilon}}\mathfrak{R}_{\upsilon;e}\big|\leqslant C(\mathbf{u})\big|\rho_{e;\mathbf{u}}\big(z_{e;\upsilon}\big)\big|$$

for all $z_{e;\upsilon}\in\Sigma_{\upsilon;e}(\delta_{\bar\partial}(\mathbf{u}))$, $\upsilon\in\Delta^\sigma_{\mathcal{C}}\cap\tilde{\mathcal{F}}_1\mathcal{T}|_{\mathbf{u}}$, $\mathbf{u}\in\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})$, and $e\in\mathrm{E}_\bullet(\mathcal{T})$.

Proof. Since $\widetilde{u}_v|_{\widehat{\Sigma}_v^c}$ is constant, this lemma concerns only the smoothings of the nodes of $\Sigma_{\mathbf{u}}^0$ and the restrictions of q_v and $\widetilde{u}_v$ to $\Sigma_v^0\subset\Sigma_v$. The restriction $q_v|_{\Sigma_v^0}$ can be written as the basic gluing map of [26, Section 2.2]. Since $u|_{\Sigma_{\mathbf{u}}^0}$ is a regular map, [28, Lemma 3.2] provides a family of maps (5.5) satisfying the first bound in Lemma 5.1.

The smooth function $\mathfrak{R}_{v;e}$ is defined by (5.8). The two bounds on $\mathfrak{R}_{v;e}$ follow directly from the proofs of [28, Lemmas 3.4,3.5]. The only difference is the presence of the 0-th order term $\zeta_{v;e}(0)$ in the current setting; it does not appear in [28, (3.27)] because of the vanishing condition imposed on the elements of $\Gamma_+(v)$ in [28, (3.2)]. By [26, Lemma 3.5] and the first bound of Lemma 5.1,

$$\|\zeta_v\|_{v,C^0} \leqslant C_1(\mathbf{u})|v|^{1/p}\,.$$

Along with (5.7) and (5.6), this implies that

$$\big|\zeta_{v;e}(0)\big| \leqslant C_2(\mathbf{u})|v|^{1/p}\,. \tag{5.9}$$

Thus, all estimates from [28, Lemmas 3.4,3.5] apply in the current setting. □

5.2 The obstructed gluing step

We now move to the second stage of the gluing construction. For each $v\in\Delta_{\mathcal{C}}^{\sigma}$ sufficiently small, we will construct an approximately J-holomorphic map

$$u_{\bullet;v}\colon\Sigma_v\longrightarrow X \qquad \text{s.t.} \quad \big\|\bar\partial_J u_{\bullet;v}\big\|_{v,p}\leqslant C_0\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}}(v)\big)\big|\rho_{\mathcal{T}}^{\sigma}(v)\big| \tag{5.10}$$

for some continuous function $C_0\colon\widetilde{U}_{\mathcal{T},\mathbf{f}}^{\phi}(\mathcal{C})\longrightarrow\mathbb{R}^+$. This map is analogous to the approximately J-holomorphic map constructed just before [12, Lemma 3.5]. It can be written explicitly in this case as each map $\widetilde{u}_{v_1}$ is non-constant on only one irreducible component $\Sigma_{v_1;0}$ of its domain and $\Sigma_{v_1;0}$ is smooth.

Let $\beta\colon\mathbb{R}^+\longrightarrow[0,1]$ be a smooth cutoff function such that

$$\beta(r)=\begin{cases}1, & \text{if } r\leqslant 1/2;\\ 0, & \text{if } r\geqslant 1.\end{cases}$$

For each $\epsilon\in\mathbb{R}^+$, we define $\beta_\epsilon\in C^\infty(\mathbb{R};\mathbb{R})$ by $\beta_\epsilon(r)=\beta(r/\epsilon)$. For $v\in\Delta^\sigma|_{\mathbf{u}}$ and $e\in\mathrm{E}_\bullet(\mathcal{T})$, define

$$\widetilde{q}_{v;e}\colon q_v^{-1}\big(\Sigma_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))\big)\longrightarrow q_{v_1}^{-1}\big(\Sigma_{\mathbf{u};e}^0(\delta_{\bar\partial}(\mathbf{u}))\big) \quad \text{by} \quad z_e\big(\widetilde{q}_{v;e}(x)\big)=\beta_{\delta_{\bar\partial}(\mathbf{u})}(|z_e^c(x)|)z_e(x);$$

in particular,

$$\widetilde{q}_{v;e}(x)=\begin{cases}q_{\bullet;v}(x), & \text{if } x\in q_v^{-1}(\Sigma_{\mathbf{u};e}^0(\delta_{\bar\partial}(\mathbf{u}))-\Sigma_{\mathbf{u};e}(2\sqrt{|v_e|}));\\ x_e(v_1), & \text{if } x\in q_v^{-1}(\partial\Sigma_{\mathbf{u};e}^c(\delta_{\bar\partial}(\mathbf{u}))).\end{cases}$$

If in addition $\mathbf{u}\in\widetilde{U}_{\mathcal{T},\mathbf{f}}^{\phi}(\mathcal{C})$, define

$$u_{\bullet;v}\colon\Sigma_v\longrightarrow X,\quad u_{\bullet;v}(x)=\begin{cases}\widetilde{u}_{v_1}(q_{\bullet;v}(x)), & \text{if } x\in q_v^{-1}(\Sigma_{\mathbf{u};e}^0),\ x\notin q_v^{-1}(\Sigma_{\mathbf{u};e}(2\sqrt{|v_e|}))\ \forall\, e\in\mathrm{E}_\bullet(\mathcal{T});\\ \widetilde{u}_{v_1}(\widetilde{q}_{v;e}(x)), & \text{if } x\in q_v^{-1}(\Sigma_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))),\ e\in\mathrm{E}_\bullet(\mathcal{T});\\ \widetilde{u}_{v_1}(x_e(v_1)), & \text{if } x\in q_v^{-1}(\Sigma_{\mathbf{u};e}^c-\Sigma_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))),\ e\in\mathrm{E}_\bullet(\mathcal{T}).\end{cases}$$

The support of $\bar{\partial}_J u_{\bullet;\upsilon}$ is contained in the annuli $q_\upsilon^{-1}(\Sigma^c_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u})))$ with $e\in \mathrm{E}_\bullet(\mathcal{T})$. On these annuli, the map $\widetilde{q}_{\upsilon;e}$ is equivalent to the modified gluing map $\widetilde{q}_{\upsilon0;2}$ of [28, Section 4.2]. The bound in (5.10) thus follows from Lemma 5.1 as in the proof of the first estimate in [28, Lemma 4.4(3)].

For each $\upsilon\in\Delta^\sigma_{\mathcal{C}}$ and $\xi\in\ker D^\phi_{\widetilde{u}_{\upsilon_1}}$, we define $R_\upsilon\xi\in\Gamma(u_{\bullet;\upsilon})^\phi$ as in the above construction of $u_{\bullet;\upsilon}$. Let

$$\Gamma_-\big(u_{\bullet;\upsilon}\big)^\phi=\big\{R_\upsilon\xi\colon\ \xi\in\ker D^\phi_{\widetilde{u}_{\upsilon_1}}\big\}$$

and denote by

$$\Gamma_+\big(u_{\bullet;\upsilon}\big)^\phi\subset\Gamma\big(u_{\bullet;\upsilon}\big)^\phi$$

the L^2-orthogonal complement of $\Gamma_-(u_{\bullet;\upsilon})^\phi$. Analogously to the third estimate in [28, Lemma 4.4(3)],

$$C_1\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}}(\upsilon)\big)^{-1}\|\zeta\|_{\upsilon,p,1}\leqslant\big\|D^\phi_{u_{\bullet;\upsilon}}\zeta\big\|_{\upsilon,p}\leqslant C_1\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}}(\upsilon)\big)\|\zeta\|_{\upsilon,p,1}\qquad\forall\ \zeta\in\Gamma_+\big(u_{\bullet;\upsilon}\big)^\phi \tag{5.11}$$

for some continuous function $C_1\colon \widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})\longrightarrow\mathbb{R}^+$. The second estimate in [28, Lemma 4.4(3)] also applies in this case for the same reasons as before.

Lemma 5.2. *If Δ^σ, $\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})$, and $\delta_\bullet\colon\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})\longrightarrow\mathbb{R}^+$ are sufficiently small, then*

$$\begin{gathered}\Phi_{\mathcal{T}}\colon\big\{(\upsilon,\zeta)\colon\ \upsilon\in\Delta^\sigma_{\mathcal{C}},\ \zeta\in\Gamma_+\big(u_{\bullet;\upsilon}\big)^\phi,\ \|\zeta\|_{\upsilon,p,1}<\delta_\bullet\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}^\sigma}(\upsilon)\big)\big\}\longrightarrow\mathfrak{X}_{g,l}(X,B)^\phi,\\ (\upsilon,\zeta)\longrightarrow\big[\upsilon(\zeta)\big]\equiv\big[\exp_{u_{\bullet;\upsilon}}\zeta,\big(z_i^+(\upsilon),z_i^-(\upsilon)\big)_{i=1}^l\big],\end{gathered}$$

is an $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-invariant degree $|\widetilde{\mathrm{Aut}}^*(\mathcal{T})|$ covering of an open neighborhood $\mathfrak{X}_{\mathcal{T};\mathbf{f}}(\mathcal{C})$ of $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$.

Proof. By the choice of $\Gamma_-(u_{\bullet;\upsilon})^\phi$, this is essentially an inverse function theorem. The continuity, injectivity, and surjectivity of the induced map on the quotient by $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$ are proved by arguments similar to Sections 4.1, 4.2, and 4.3-4.5 in [26], respectively; see also the paragraph following Lemma 4.4 in [28]. □

Let ∇^J be the J-linear connection corresponding to ∇^X. For each $\upsilon\in\Delta^\sigma_{\mathcal{C}}$ and $\zeta\in\Gamma(u_{\bullet;\upsilon})^\phi$, let

$$\Pi_\zeta\colon\Gamma^{0,1}\big(u_{\bullet;\upsilon}\big)^\phi\longrightarrow\Gamma^{0,1}_{\mathfrak{j}_\upsilon}\big(\exp_{u_{\bullet;\upsilon}}\zeta\big)^{\phi,\sigma_\upsilon}$$

be the isomorphism induced by the ∇^J-parallel transport along the ∇^X-geodesics

$$s\longrightarrow\exp_{u_{\bullet;\upsilon}}\big(s\zeta\big),\qquad s\in[0,1].$$

With ν as in Proposition 3.1, define

$$\Xi_{\mathcal{T};t\nu}(\upsilon,\cdot)\colon\Gamma\big(u_{\bullet;\upsilon}\big)^\phi\longrightarrow\Gamma^{0,1}\big(u_{\bullet;\upsilon}\big)^\phi\quad\text{by}\quad\Xi_{\mathcal{T};t\nu}(\upsilon,\zeta)=\Pi_\zeta^{-1}\circ\big(\{\bar\partial_J+t\nu\}\exp_{u_{\bullet;\upsilon}}\zeta\big)\,. \tag{5.12}$$

Similarly to [26, (3.9),(3.10)],

$$\Xi_{\mathcal{T};t\nu}(\upsilon,\zeta)=\bar\partial_J u_{\bullet;\upsilon}+t\nu_{u_{\bullet;\upsilon}}+D^\phi_{u_{\bullet;\upsilon}}\zeta+N^\phi_{\upsilon;t}(\zeta), \tag{5.13}$$

with the quadratic term $N_{\upsilon;t}$ satisfying

$$\begin{gathered}N^\phi_{\upsilon;t}(0)=0,\quad\big\|N^\phi_{\upsilon;t}(\zeta)-N^\phi_{\upsilon;t}(\zeta')\big\|_{\upsilon,p}\leqslant C_2\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}}(\upsilon)\big)\big(|t|+\|\zeta\|_{\upsilon,p,1}+\|\zeta'\|_{\upsilon,p,1}\big)\big\|\zeta-\zeta'\big\|_{\upsilon,p,1}\\ \forall\quad t\in\mathbb{R},\ \zeta,\zeta'\in\Gamma\big(u_{\bullet;\upsilon}\big)^\phi\ \text{ s.t. }\ \|\zeta\|_{\upsilon,p,1},\|\zeta'\|_{\upsilon,p,1}\leqslant\delta_\bullet\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}}(\upsilon)\big)\end{gathered} \tag{5.14}$$

for some continuous function $C_2\colon \widetilde{U}^{\phi}_{\mathcal{T},\mathbf{f}}(\mathcal{C})\longrightarrow\mathbb{R}^+$.

We will show that $\Xi_{\mathcal{T};t\nu}$ does not vanish if ν is generic, $\mathcal{T}=(\mathfrak{t},\mathfrak{m})$ with $\mathfrak{t}$ not basic, $[\upsilon(\zeta)]$ is sufficiently close to $\mathfrak{M}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^{\phi}$, and $t\in\mathbb{R}^+$ is sufficiently small. In the case $\mathfrak{t}$ is basic, we will construct a section $\Psi_{\mathcal{T};t\nu}$ as in (3.32) so that its vanishing locus corresponds to the vanishing locus of $\Xi_{\mathcal{T};t\nu}$, after the constraints $\mathbf{f}$ are taken into account.

For any functions $\delta_{\mathcal{T}},\epsilon_{\mathcal{T}}\colon \widetilde{U}^{\phi}_{\mathcal{T},\mathbf{f}}(\mathcal{C})\longrightarrow\mathbb{R}^+$, let

$$\Omega_{\delta_{\mathcal{T}},\epsilon_{\mathcal{T}}}=\big\{(\upsilon,\zeta)\colon\ \upsilon\in\Delta^{\sigma}_{\mathcal{C}},\ \zeta\in\Gamma_+\big(u_{\bullet;\upsilon}\big)^{\phi},\ |\upsilon|<\delta_{\mathcal{T}}\big(\pi_{\tilde{\mathcal{F}}\mathcal{T}^{\sigma}}(\upsilon)\big),\ \|\zeta\|_{\upsilon,p,1}<\epsilon_{\mathcal{T}}\big(\pi_{\tilde{\mathcal{F}}\mathcal{T}^{\sigma}}(\upsilon)\big)\big\}.$$

By (5.13), (5.10), (5.11), and (5.14), there exist continuous functions

$$\delta_{\mathcal{T}},\epsilon_{\mathcal{T}},C_{\mathcal{T}}\colon \widetilde{U}^{\phi}_{\mathcal{T},\mathbf{f}}(\mathcal{C})\longrightarrow\mathbb{R}^+$$

such that

$$\begin{split}&\big\{(t;\upsilon,\zeta)\in\mathbb{R}\times\Omega_{\delta_{\mathcal{T}},\epsilon_{\mathcal{T}}}\colon |t|<\delta_{\mathcal{T}}\big(\pi_{\tilde{\mathcal{F}}\mathcal{T}^{\sigma}}(\upsilon)\big),\ \Xi_{\mathcal{T};t\nu}(\upsilon,\zeta)=0\big\}\\ &\qquad\subset\big\{(t;\upsilon,\zeta)\in\mathbb{R}\times\Omega_{\delta_{\mathcal{T}},\epsilon_{\mathcal{T}}}\colon\ \|\zeta\|_{\upsilon,p,1}\leqslant C_{\mathcal{T}}\big(\pi_{\tilde{\mathcal{F}}\mathcal{T}^{\sigma}}(\upsilon)\big)\big(|\rho^{\sigma}_{\mathcal{T}}(\upsilon)|+|t|\big)\big\}.\end{split}\tag{5.15}$$

We can assume that

$$\big\{\upsilon\in\tilde{\mathcal{F}}\mathcal{T}^{\phi}\big|_{\widetilde{U}^{\phi}_{\mathcal{T},\mathbf{f}}(\mathcal{C})}\colon |\upsilon|<\delta_{\mathcal{T}}\big(\pi_{\tilde{\mathcal{F}}\mathcal{T}^{\sigma}}(\upsilon)\big)\big\}\subset\Delta^{\sigma}_{\mathcal{C}},\quad \epsilon_{\mathcal{T}}<\delta_{\bullet},\quad \delta_{\mathcal{T}},\epsilon_{\mathcal{T}}<1,\quad 2C_{\mathcal{T}}\delta_{\mathcal{T}}<\epsilon_{\mathcal{T}}\,,$$

and that the functions $\delta_{\mathcal{T}},\epsilon_{\mathcal{T}},C_{\mathcal{T}}$ are $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-invariant.

Let $e\in\mathrm{E}_{\bullet}(\mathcal{T})$, $\mathbf{u}\in\widetilde{U}^{\phi}_{\mathcal{T},\mathbf{f}}(\mathcal{C})$, and $\upsilon\in\Delta^{\sigma}_{\mathcal{C}}|_{\mathbf{u}}$. We define a holomorphic map

$$q_{\upsilon;e}\colon\ \widehat{\Sigma}^c_{\upsilon;e}\big(\delta_{\bar\partial}(\mathbf{u})\big)\equiv q_{\upsilon}^{-1}\big(\widehat{\Sigma}^c_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))\big)\longrightarrow\widehat{\Sigma}^c_{\upsilon_1;e}\subset\Sigma_{\upsilon_1}\qquad\text{by}$$
$$z^c_e\circ q_{\upsilon;e}\big|_{q_{\upsilon}^{-1}(\Sigma_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u})))}=z^c_e\big|_{q_{\upsilon}^{-1}(\Sigma_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u})))},\quad q_{\upsilon;e}\big|_{\widehat{\Sigma}^c_{\upsilon;e}-q_{\upsilon}^{-1}(\Sigma_{\mathbf{u};e}(2\sqrt{|\upsilon_e|}))}=q_{\bullet;\upsilon}\big|_{\widehat{\Sigma}^c_{\upsilon;e}-q_{\upsilon}^{-1}(\Sigma_{\mathbf{u};e}(2\sqrt{|\upsilon_e|}))}.$$

For each holomorphic $(1,0)$-differential $\kappa\in\mathbb{E}_e|_{\Sigma_{\upsilon_1}}$ supported on $\widehat{\Sigma}^c_{\upsilon_1;e}$, we define a $(1,0)$-differential $R_{\bullet;\upsilon}\kappa$ on Σ_{υ} by

$$R_{\bullet;\upsilon}\kappa\big|_x=\begin{cases}\kappa\circ\mathrm{d}_xq_{\upsilon;e}, & \text{if } x\in q_{\upsilon}^{-1}(\widehat{\Sigma}^c_{\mathbf{u};e}-\Sigma_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u})));\\ \beta_{\delta_{\bar\partial}(\mathbf{u})}(|z_e(x)|)\kappa\circ\mathrm{d}_xq_{\upsilon;e}, & \text{if } x\in q_{\upsilon}^{-1}(\Sigma_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u})));\\ 0, & \text{if } x\in\Sigma_{\upsilon}-q_{\upsilon}^{-1}(\widehat{\Sigma}^c_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))).\end{cases}$$

Combining this construction with a continuous collection of isomorphisms

$$R_{\upsilon_1;e}\colon\mathbb{E}_e|_{\Sigma_{\mathbf{u}}}\longrightarrow\mathbb{E}_e|_{\Sigma_{\upsilon_1}}$$

as above [28, (4.9)], for each $\kappa\in\mathbb{E}_e|_{\mathbf{u}}$ we obtain a $(1,0)$-differential $R_{\bullet;\upsilon}\kappa$ on Σ_{υ}. Analogously to [28, (4.10)], for each $q\in[1,2)$ there exists a continuous function $C_q\colon\widetilde{U}^{\phi}_{\mathcal{T},\mathbf{f}}(\mathcal{C})\longrightarrow\mathbb{R}^+$ such that

$$\big\|R_{\upsilon}\kappa\big\|_{\upsilon,q}\leqslant C_q\big(\mathbf{u}\big)\|\kappa\|\qquad\forall\ \upsilon\in\Delta^{\sigma}_{\mathcal{C}}|_{\mathbf{u}},\ \kappa\in\mathbb{E}_e|_{\mathbf{u}},\ e\in\mathrm{E}_{\bullet}(\mathcal{T})\,.\tag{5.16}$$

The norm on the left-hand side of (5.16) is the usual Sobolev norm with respect to the metric g_υ; the norm on the right-hand side of (5.16) is any fixed norm on the finite-rank vector bundle $\mathbb{E}_e\longrightarrow\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})$.

With e, $\mathbf{u}$, and $\upsilon\in\Delta^\sigma_{\mathcal{C}}|_{\mathbf{u}}$ as above and $\zeta_{\upsilon1;e}$ as in (5.6), let

$$\widetilde{\zeta}_{\upsilon;e}=\zeta_{\upsilon1;e}\circ\widetilde{q}_{\upsilon;e}(x)\colon\widehat{\Sigma}^c_{\upsilon;e}\big(\delta_{\bar\partial}(\mathbf{u})\big)\longrightarrow T_{u(x_{\langle e\rangle}(\mathbf{u}))}X.$$

Define

$$\Theta_\upsilon\colon\Gamma^{0,1}\big(u_{\bullet;\upsilon}\big)^\phi\longrightarrow\mathrm{Obs}^\phi_{\mathcal{T}}\big|_{\mathbf{u}}\qquad\text{by}$$
$$\big\{\Theta_\upsilon(\eta)\big\}\big((\kappa_e)_{e\in\mathrm{E}_\bullet(\mathcal{T})}\big)=\bigg(\frac{\mathfrak{i}}{2\pi}\int_{\widehat{\Sigma}^c_{\upsilon;e}\delta_{\bar\partial}(\mathbf{u})}\big(R_\upsilon\kappa_e\big)\wedge\big(\Pi^{-1}_{\widetilde{\zeta}_{\upsilon;e}}\eta\big)\bigg)_{e\in\mathrm{E}_\bullet(\mathcal{T})}\qquad\forall\;\kappa_e\in\mathbb{E}_e|_{\mathbf{u}},\;e\in\mathrm{E}_\bullet(\mathcal{T})\,.$$

By Hölder's inequality and (5.16), the above integral is finite and

$$\big\|\Theta_\upsilon(\eta)\big\|\leqslant C_\Theta(\mathbf{u})\|\eta\|_{\upsilon,p}\qquad\forall\;\eta\in\Gamma^{0,1}\big(u_{\bullet;\upsilon}\big)^\phi\tag{5.17}$$

for some continuous function $C_\Theta\colon\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})\longrightarrow\mathbb{R}^+$.

It is immediate that

$$\big\|\Theta_\upsilon\big(\nu_{u_{\bullet;\upsilon}}\big)-\overline{\nu}_{\mathcal{T}}\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}}(\upsilon)\big)\big\|\leqslant\epsilon_\nu(\upsilon)\tag{5.18}$$

for some continuous function $\epsilon_\nu:\Delta^\sigma_{\mathcal{C}}\longrightarrow\mathbb{R}^+$ that vanishes along $\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})$. By the proof of [28, Lemma 4.4(7)], there exists a continuous function $\epsilon_1\colon\Delta^\sigma_{\mathcal{C}}\longrightarrow\mathbb{R}^+$ vanishing along $\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})$ such that

$$\big\|\Theta_\upsilon\big(D^\phi_{u_{\bullet;\upsilon}}\zeta\big)\big\|\leqslant\epsilon_1(\upsilon)\|\zeta\|_{\upsilon,p,1}\qquad\forall\;\zeta\in\Gamma(u_{\bullet;\upsilon})^\phi\,.\tag{5.19}$$

By Lemma 5.1 and the proof of [28, Lemma 4.4(6)], there exists a continuous function $\epsilon_{\bar\partial}\colon\Delta^\sigma_{\mathcal{C}}\longrightarrow\mathbb{R}^+$ vanishing along $\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})$ such that

$$\big\|\Theta_\upsilon\big(\bar\partial_J u_{\bullet;\upsilon}\big)-\mathcal{D}^\phi_{\mathcal{T}}\rho^\sigma_{\mathcal{T}}(\upsilon)\big\|\leqslant\epsilon_{\bar\partial}(\upsilon)\big|\rho^\sigma_{\mathcal{T}}(\upsilon)\big|\,;\tag{5.20}$$

the 0-th order term $\zeta_{\upsilon;e}(0)$ appearing in the expansion of Lemma 5.1 has no effect on [28, (4.24)]. By (5.13), (5.14), and (5.17)-(5.20), there exist continuous functions

$$C_\Xi\colon\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})\longrightarrow\mathbb{R}^+\qquad\text{and}\qquad\epsilon_\Xi\colon\Delta^\sigma_{\mathcal{C}}\longrightarrow\mathbb{R}^+$$

with ϵ_Ξ vanishing along $\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})$ such that

$$\begin{split}\big\|\Theta_\upsilon\big(\Xi_{\mathcal{T};t\nu}(\upsilon,\zeta)\big)-\big(\mathcal{D}^\phi_{\mathcal{T}}\rho^\sigma_{\mathcal{T}}(\upsilon)+t\overline{\nu}_{\mathcal{T}}(\pi_{\widetilde{\mathcal{F}}\mathcal{T}^\sigma}(\upsilon))\big)\big\|\leqslant C_\Xi\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}}(\upsilon)\big)\big(|t|+\|\zeta\|_{\upsilon,p,1}\big)\|\zeta\|_{\upsilon,p,1}\\+\epsilon_\Xi(t,\upsilon)\big(\big|\rho^\sigma_{\mathcal{T}}(\upsilon)\big|+|t|+\|\zeta\|_{\upsilon,p,1}\big)\end{split}\tag{5.21}$$

for all $t\in\mathbb{R}$, $\upsilon\in\Delta^\sigma_{\mathcal{C}}$, and $\zeta\in\Gamma(u_{\bullet;\upsilon})^\phi$ with $\|\zeta\|_{\upsilon,p,1}\leqslant\delta_\bullet(\pi_{\widetilde{\mathcal{F}}\mathcal{T}}(\upsilon))$.

By (5.15) and (5.21), there exists a continuous function

$$\epsilon_{\mathcal{T};\nu}\colon\mathbb{R}\times\Delta^\sigma_{\mathcal{C}}\longrightarrow\mathbb{R}^+$$

vanishing along $\{0\}\times\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})$ such that

$$\begin{split}\big\|\mathcal{D}^\phi_{\mathcal{T}}\rho^\sigma_{\mathcal{T}}(\upsilon)+t\overline{\nu}_{\mathcal{T}}\big(\pi_{\tilde{\mathcal{F}}\mathcal{T}^\sigma}(\upsilon)\big)\big\| \leqslant \epsilon_{\mathcal{T};\nu}(t,\nu)\big(\big|\rho^\sigma_{\mathcal{T}}(\upsilon)\big|+|t|\big)\\ \forall\ (t;\upsilon,\zeta)\in\mathbb{R}\times\Omega_{\mathcal{T};\delta_{\mathcal{T}},\epsilon_{\mathcal{T}}}\ \text{ s.t. }\ |t|<\delta_{\mathcal{T}}\big(\pi_{\tilde{\mathcal{F}}\mathcal{T}^\sigma}(\upsilon)\big),\ \Xi_{\mathcal{T};t\nu}(\upsilon,\zeta)=0.\end{split} \tag{5.22}$$

Suppose $\mathcal{T}=(\mathfrak{t},\mathfrak{m})$ and $\mathfrak{t}$ is not basic so that

$$\dim\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi+\operatorname{rk}\tilde{\mathfrak{F}}\mathcal{T}^\sigma<\dim\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi+\operatorname{rk}\tilde{\mathcal{F}}\mathcal{T}^\sigma=\operatorname{rk}\operatorname{Obs}^\phi_{\mathcal{T}}.$$

Thus, the bundle map

$$\tilde{\mathcal{F}}\mathcal{T}^\sigma\longrightarrow\operatorname{Obs}^\phi_{\mathcal{T}},\qquad \upsilon\longrightarrow\mathcal{D}^\phi_{\mathcal{T}}\rho^\sigma_{\mathcal{T}}(\upsilon)+\overline{\nu}_{\mathcal{T}}\big(\pi_{\tilde{\mathcal{F}}\mathcal{T}}(\upsilon)\big),$$

has no zeros over $\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$ for a generic choice of $\nu\in\mathcal{A}^\phi_{g,l}(J)$. By (5.22) and the proof of [25, Lemma 3.2], for every precompact open subset $\widetilde{K}\subset\widetilde{\mathfrak{M}}_{\mathcal{T},\mathbf{f}}(\mathcal{C})^\phi$ there thus exist $\delta_\nu(\widetilde{K})\in\mathbb{R}^+$ and an open neighborhood $\Omega_\nu(\widetilde{K})$ of $\widetilde{K}$ in $\Omega_{\delta_{\mathcal{T}},\epsilon_{\mathcal{T}}}$ such that $\Xi_{\mathcal{T};t\nu}$ does not vanish on $\Omega_\nu(\widetilde{K})$. By Lemma 5.2 and (5.12),

$$U_\nu(\widetilde{K})\equiv\mathfrak{X}_{g,\mathbf{f}}(X,B)^\phi\cap\bigg(\Phi_{\mathcal{T}}\big(\Omega_\nu(\widetilde{K})\big)\times\prod_{i=1}^{l}Y_i\bigg)$$

is an open neighborhood of $K=\iota_{\mathcal{T};\mathbf{f}}(\widetilde{K})$ in $\mathfrak{X}_{g,\mathbf{f}}(X,B)^\phi$ satisfying (3.31) with K replaced by $\widetilde{K}$. This establishes the first statement of Proposition 3.1.

From now on we assume that $\mathcal{T}=(\mathfrak{t},\mathfrak{m})$ with $\mathfrak{t}$ basic. Thus, the domain of each element of $\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi$ consists of a smooth genus h curve $\Sigma_{\mathbf{u};0}$ with smooth positive-genus curves

$$\widehat{\Sigma}^c_{\mathbf{u};e}=\Sigma^c_{\mathbf{u};e}\qquad\text{for}\quad e\in\mathrm{E}_\bullet(\mathcal{T})=\mathrm{E}_0(\mathcal{T})$$

attached at distinct points. Furthermore, $\tilde{\mathfrak{F}}\mathcal{T}=\tilde{\mathcal{F}}\mathcal{T}$ and $\rho^\sigma_{\mathcal{T}}(\upsilon)=\upsilon$. The gluing construction now consists of only the second, obstructed step.

For each $\mathbf{u}\in\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi$, let

$$\Gamma^{0,1}_-(u)\subset\Gamma\big(\Sigma;T^*\Sigma^{0,1}_u\otimes u^*(TX,J)\big)\qquad\text{and}\qquad\Gamma^{0,1}_-(u)^\phi\equiv\Gamma^{0,1}_-(u)\cap\Gamma^{0,1}(u)^\phi$$

denote the subspace of harmonic $(0,1)$-forms and the subspace of ϕ-invariant harmonic $(0,1)$-forms. The former is generated by harmonic forms each of which is supported on $\widehat{\Sigma}^c_{\mathbf{u};e}$ for some $e\in\mathrm{E}_\bullet(\mathcal{T})$. The homomorphism (3.26) restricts to an isomorphism

$$\Theta_{\mathbf{u}}\colon\Gamma^{0,1}_-(u)^\phi\longrightarrow\operatorname{Obs}^\phi_{\mathcal{T}}\big|_{\mathbf{u}}\,.$$

For each $\upsilon\in\Delta^\sigma_{\mathcal{C}}|_{\mathbf{u}}$, $e\in\mathrm{E}_\bullet(\mathcal{T})$, and $\eta\in\Gamma^{0,1}_-(u)$ supported on $\Sigma^c_{\mathbf{u};e}$, we define an element $R_\upsilon\eta$ in $\Gamma^{0,1}(u_{\bullet;\upsilon})$ by

$$R_\upsilon\eta\big|_x=\begin{cases}\eta\circ\mathrm{d}_xq_{\upsilon;e}, & \text{if } x\in q_\upsilon^{-1}(\Sigma^c_{\mathbf{u};e}-\Sigma_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u})));\\ \beta_{\delta_{\bar\partial}(\mathbf{u})}(|z_e(x)|)\Pi_{\zeta_{\upsilon;e}(x)}\circ\eta\circ\mathrm{d}_xq_{\upsilon;e}, & \text{if } x\in q_\upsilon^{-1}(\Sigma_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u})));\\ 0, & \text{if } x\in\Sigma_\upsilon-q_\upsilon^{-1}(\widehat{\Sigma}^c_{\mathbf{u};e}(\delta_{\bar\partial}(\mathbf{u}))).\end{cases}$$

This definition induces an injective homomorphism

$$R_\upsilon\colon \Gamma_-^{0,1}(u)^\phi \longrightarrow \Gamma^{0,1}\big(u_{\bullet;\upsilon}\big)$$

that satisfies

$$\big\|R_\upsilon\eta\big\|_{\upsilon,p} \leqslant C_R(\mathbf{u})\|\eta\|, \quad \big\|\Theta_\upsilon(R_\upsilon\eta)-\Theta_{\mathbf{u}}(\eta)\big\| \leqslant C_R(\mathbf{u})|\upsilon|^2\|\eta\| \qquad \forall\ \eta\in\Gamma_-^{0,1}(u) \tag{5.23}$$

for some continuous function $C_R\colon \widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})\longrightarrow\mathbb{R}$.

For each $\upsilon\in\Delta^\sigma_{\mathcal{C}}$, let

$$\Gamma_-^{0,1}\big(u_{\bullet;\upsilon}\big)^\phi=\big\{R_\upsilon\eta\colon\ \eta\in\Gamma_-^{0,1}(u)^\phi\big\} \quad\text{and}\quad \Gamma_+^{0,1}\big(u_{\bullet;\upsilon}\big)^\phi=\big\{\eta\in\Gamma^{0,1}\big(u_{\bullet;\upsilon}\big)\colon\ \Theta_\upsilon(\eta)=0\big\}.$$

By (5.23),

$$\Gamma^{0,1}\big(u_{\bullet;\upsilon}\big)=\Gamma_-^{0,1}\big(u_{\bullet;\upsilon}\big)^\phi\oplus\Gamma_+^{0,1}\big(u_{\bullet;\upsilon}\big)^\phi \tag{5.24}$$

for all υ sufficiently small; by shrinking $\Delta^\sigma_{\mathcal{C}}$, we can assume that (5.24) holds for all $\upsilon\in\Delta^\sigma_{\mathcal{C}}$. Denote by

$$\pi^{0,1}_{\upsilon;+}\colon\Gamma^{0,1}\big(u_{\bullet;\upsilon}\big)\longrightarrow\Gamma_+^{0,1}\big(u_{\bullet;\upsilon}\big)^\phi$$

the projection to the second component in the decomposition (5.24). By (5.17) and (5.23), there exists a continuous function $C_+\colon\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})\longrightarrow\mathbb{R}^+$ such that

$$\big\|\pi^{0,1}_{\upsilon;+}\eta\big\|_{\upsilon,p}\leqslant C_+\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}^\sigma}(\upsilon)\big)\|\eta\|_{\upsilon,p} \qquad \forall\ \eta\in\Gamma^{0,1}\big(u_{\bullet;\upsilon}\big)\,. \tag{5.25}$$

By (5.11), (5.19), (5.23), and (5.25), there exists a continuous function such that $\widetilde{C}_1\colon\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})\longrightarrow\mathbb{R}^+$ such that

$$\widetilde{C}_1\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}}(\upsilon)\big)^{-1}\|\zeta\|_{\upsilon,p,1}\leqslant\big\|\pi^{0,1}_{\upsilon;+}D^\phi_{u_{\bullet;\upsilon}}\zeta\big\|_{\upsilon,p}\leqslant\widetilde{C}_1\big(\pi_{\widetilde{\mathcal{F}}\mathcal{T}}(\upsilon)\big)\|\zeta\|_{\upsilon,p,1} \quad \forall\ \zeta\in\Gamma_+(\upsilon)^\phi \tag{5.26}$$

for all υ sufficiently small; by shrinking $\Delta^\sigma_{\mathcal{C}}$, we can assume that (5.26) holds for all $\upsilon\in\Delta^\sigma_{\mathcal{C}}$.

By (5.13), the identity $\Xi_{\mathcal{T};t\nu}(\upsilon,\zeta)=0$ is equivalent to the system of equations

$$\begin{cases}\pi^{0,1}_{\upsilon;+}D^\phi_{u_{\bullet;\upsilon}}\zeta+\pi^{0,1}_{\upsilon;+}N^\phi_{\upsilon;t}(\zeta)=-\pi^{0,1}_{\upsilon;+}\bar\partial_J u_{\bullet;\upsilon}-t\pi^{0,1}_{\upsilon;+}\nu_{u_{\bullet;\upsilon}}\in\Gamma_+^{0,1}\big(u_{\bullet;\upsilon}\big)^\phi\,,\\ \Theta_\upsilon\big(\Xi_{\mathcal{T};t\nu}(\upsilon,\zeta)\big)=0\in\mathrm{Obs}^\phi_{\mathcal{T}}\big|_{\mathbf{u}}.\end{cases} \tag{5.27}$$

By (5.26) and index considerations, the homomorphism

$$\pi^{0,1}_{\upsilon;+}D^\phi_{u_{\bullet;\upsilon}}\colon\Gamma_+\big(u_{\bullet;\upsilon}\big)^\phi\longrightarrow\Gamma_+^{0,1}\big(u_{\bullet;\upsilon}\big)^\phi$$

is an isomorphism; its norm and the norm of its inverse are bounded depending only on $\pi_{\widetilde{\mathcal{F}}\mathcal{T}}(\upsilon)$. In light of (5.10), (5.14), and (5.25), we can thus apply the Contraction Principle to the first equation in (5.27) provided $|\upsilon|$ and $|t|$ are sufficiently small (depending on $\pi_{\widetilde{\mathcal{F}}\mathcal{T}}(\upsilon)$). More precisely, for every precompact open subset $\widetilde{K}\subset\widetilde{U}^\phi_{\mathcal{T},\mathbf{f}}(\mathcal{C})$ there exist $\delta_\nu(\widetilde{K}),\epsilon_\nu(\widetilde{K}),C(\widetilde{K})\in\mathbb{R}^+$ such that

$$\delta_\nu(\widetilde{K})<\inf_{\widetilde{K}}\delta_{\mathcal{T}}\,,\quad \epsilon_\nu(\widetilde{K})<\inf_{\widetilde{K}}\epsilon_{\mathcal{T}}\,,\quad 2C_{\mathcal{T}}\delta_\nu(\widetilde{K})<\epsilon_\nu(\widetilde{K})\,,$$

and for all $t\in\mathbb{R}$ and $\upsilon\in\tilde{\mathcal{F}}\mathcal{T}|_{\tilde{K}}$ with $|t|,|\upsilon|<\delta_\nu(\tilde{K})$ the first equation in (5.27) has a unique solution $\zeta_{t\nu}(\upsilon)\in\Gamma_+(u_{\bullet;\upsilon})^\phi$ with $\|\zeta_{t\nu}(\upsilon)\|_{\upsilon,p,1}<\epsilon_\nu(\tilde{K})$ and this solution satisfies

$$\big\|\zeta_{t\nu}(\upsilon)\big\|_{\upsilon,p,1}<C(\tilde{K})\big(|\upsilon|+|t|\big). \tag{5.28}$$

If $\tilde{K}$ is preserved by the $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-action, then the function $\upsilon\longrightarrow\zeta_{t\nu}(\upsilon)$ is $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-equivariant.

The maps

$$\tilde{\gamma}_{\mathcal{T};t\nu}\colon\tilde{\mathcal{F}}\mathcal{T}^\sigma_{\delta_\nu(\tilde{K})}\big|_{\tilde{K}}\longrightarrow\mathfrak{X}_{g,l}(X,B)^\phi,\quad \upsilon\longrightarrow\big[\upsilon(\zeta_{t\nu}(\upsilon))\big]\equiv\big[\exp_{u_\bullet;\upsilon}\zeta_{t\nu}(\upsilon),\big(z_i^+(\upsilon),z_i^-(\upsilon)\big)_{i=1}^l\big],$$

with $|t|<\delta_\nu(\tilde{K})$ are the analogues of [26, (3.16)] in the present situation. Let

$$\widetilde{\mathfrak{M}}^{(t)}_{g,l}(\tilde{K})^\phi\subset\mathfrak{X}_{g,l}(X,B)^\phi$$

denote the image of $\tilde{\gamma}_{\mathcal{T};t\nu}$. Since the evaluation map

$$\mathrm{ev}\colon\mathfrak{M}_{\mathcal{T}}(X)^\phi\longrightarrow X^l$$

is transverse to the pseudocycles $\mathbf{f}$ along $\mathfrak{M}_{\mathcal{T};\mathbf{f}}(\mathcal{C})^\phi$, the maps $\tilde{\gamma}_{\mathcal{T};t\nu}$ can be adjusted for the constraints as in the proof of [26, Lemma 3.28]; this adjustment is summarized below.

We denote by

$$\pi_{\tilde{\mathcal{N}}^\mu}\colon\tilde{\mathcal{N}}^\mu\longrightarrow\mathfrak{M}_{\mathcal{T};\mathbf{f}}(\mathcal{C})^\phi$$

the normal bundle of $\mathfrak{M}_{\mathcal{T};\mathbf{f}}(\mathcal{C})^\phi$ in $\mathfrak{M}_{\mathcal{T}}(X)^\phi$. Fix an $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-equivariant identification

$$\tilde{\phi}^\mu_{\mathcal{T}}\colon\tilde{\mathcal{N}}^\mu\longrightarrow\widetilde{\mathfrak{M}}_{\mathcal{T}}(X)^\phi$$

between neighborhoods of $\mathfrak{M}_{\mathcal{T};\mathbf{f}}(\mathcal{C})^\phi$ in $\tilde{\mathcal{N}}^\mu$ and in $\mathfrak{M}_{\mathcal{T}}(X)^\phi$ restricting to the identity over $\mathfrak{M}_{\mathcal{T};\mathbf{f}}(\mathcal{C})^\phi$ and identifications

$$\tilde{\Phi}^\mu_{\mathcal{T}}\colon\pi^*_{\tilde{\mathcal{F}}\mathcal{T}^\sigma}\tilde{\mathcal{N}}^\mu=\pi^*_{\tilde{\mathcal{N}}^\mu}\tilde{\mathcal{F}}\mathcal{T}^\sigma\longrightarrow\tilde{\mathcal{F}}\mathcal{T}^\sigma\quad\text{and}\quad\Pi^\mu_{\mathcal{T}}\colon\pi^*_{\tilde{\mathcal{N}}^\mu}\mathrm{Obs}^\phi_{\mathcal{T}}\longrightarrow\mathrm{Obs}^\phi_{\mathcal{T}}$$

of vector bundles lifting $\tilde{\phi}^\mu_{\mathcal{T}}$, restricting to the identity over $\mathfrak{M}_{\mathcal{T};\mathbf{f}}(\mathcal{C})^\phi$, and respecting the splittings.

Let $K\subset\mathfrak{M}_{\mathcal{T};\mathbf{f}}(\mathcal{C})^\phi$ be an open subset such that $\iota^{-1}_{\mathcal{T},\mathbf{f}}(K)\subset\tilde{K}$ is precompact. By Lemma 5.2, (5.12), (5.28), and the proof of [26, Lemma 3.28], there exist a neighborhood $U_\nu(K)$ of K in $\mathfrak{X}_{g,\mathbf{f}}(X,B)^\phi$, $\delta_\nu(K),C_\nu(K)\in\mathbb{R}^+$, and a unique section

$$\tilde{\varphi}^\mu_{t\nu}\in\Gamma\big(\tilde{\mathcal{F}}\mathcal{T}^\phi_{\delta_\nu(K)}\big|_{\iota^{-1}_{\mathcal{T},\mathbf{f}}(K)};\tilde{\pi}^*_{\tilde{\mathcal{F}}\mathcal{T}^\phi}\tilde{\mathcal{N}}^\mu\big)$$

such that

$$\big\|\tilde{\varphi}^\mu_{t\nu}(\upsilon)\big\|<C_\nu(K)\big(|\upsilon|+|t|\big) \tag{5.29}$$

and the map

$$\Phi_{\mathcal{T};t\nu}\equiv\tilde{\gamma}_{\mathcal{T};t\nu}\circ\tilde{\Phi}^\mu_{\mathcal{T}}\circ\tilde{\varphi}^\mu_{t\nu}\colon\tilde{\mathcal{F}}\mathcal{T}^\sigma_{\delta_\nu(K)}\big|_{\iota^{-1}_{\mathcal{T},\mathbf{f}}(K)}\longrightarrow\widetilde{\mathfrak{M}}^{(t)}_{g,l}(\tilde{K})^\phi\cap U_\nu(K)$$

is an $\widetilde{\mathrm{Aut}}^*(\mathcal{T})$-invariant covering of degree $|\widetilde{\mathrm{Aut}}^*(\mathcal{T})|$. By (5.28) and (5.29), $\Phi_{\mathcal{T};t\nu}$ satisfies the first condition in (3.33).

We define the section in (3.32) by

$$\Psi_{\mathcal{T};t\nu}(\upsilon) = \big\{\Pi^{\mu}_{\mathcal{T}}|_{\widetilde{\varphi}^{\mu}_{t\nu}(\upsilon)}\big\}^{-1}\Big(\Theta_{\widetilde{\Phi}^{\mu}_{\mathcal{T}}(\widetilde{\varphi}^{\mu}_{t\nu}(\upsilon))}\big(\Xi_{\mathcal{T};t\nu}\big(\widetilde{\Phi}^{\mu}_{\mathcal{T}}(\widetilde{\varphi}^{\mu}_{t\nu}(\upsilon)),\zeta_{t\nu}\big(\widetilde{\Phi}^{\mu}_{\mathcal{T}}(\widetilde{\varphi}^{\mu}_{t\nu}(\upsilon))\big)\big)\big)\Big). \tag{5.30}$$

By (5.21), (5.28), and (5.29), $\Psi_{\mathcal{T};t\nu}(\upsilon)$ satisfies (3.34) with $\varepsilon_{\mathcal{T};t\nu}$ satisfying the second condition in (3.33). By (5.12), (5.27), and the last sentence of the previous paragraph, (3.35) is satisfied as well. If ν is generic, then

$$\mathfrak{M}_{g,\mathbf{f}}(X,B;J,t\nu)^{\phi} = \overline{\mathfrak{M}}_{g,\mathbf{f}}(X,B;J,t\nu)^{\phi}\,.$$

The last statement of Proposition 3.1 is obtained as in the proof of [26, Corollary 3.26]. The crucial point in both cases is that the signs of the elements of $\Psi^{-1}_{\mathcal{T};t\nu}(0)$ are determined from the orientation of the deformation-obstruction complex over each stratum associated with the moduli space $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$.

6 Computations and applications

In this section, we determine the genus g degree $d=1,3,4$ real GW- and enumerative invariants of $(\mathbb{P}^3,\tau_4)$ with d conjugate pairs of point insertion; the latter is readily obtained from the former via (1.7). By [9, Theorem 1.6] and (1.6), the genus g degree d real GW- and enumerative invariants vanish whenever $d-g\in 2\mathbb{Z}$. Examples 6.2-6.4 apply [9, Theorem 4.6] to compute the genus g degree $d=1,3,4$ real GW-invariants with $d-g\notin 2\mathbb{Z}$ by equivariant localization with the standard $(\mathbb{C}^*)^2$-action on $(\mathbb{P}^3,\tau_4)$. In [18], these computations are carried out for $d=5,6,7,8$. The degree 2 GW- and enumerative invariants vanish because there are no conics passing through two generic conjugate pairs of points in $\mathbb{P}^3$; the vanishing of the GW-invariants in this case is shown by equivariant localization in [9, Example 4.10]. The results of these computations are summarized in Table 2. We conclude by combining Theorem 1.1 with Castelnuovo bounds to obtain conclusions about one- and two-partition Hodge integrals.

6.1 Preliminaries

Let $g,k\in\mathbb{Z}^{\geqslant 0}$ with $2g+k\geqslant 3$. We denote by

$$\mathbb{E}\longrightarrow\overline{\mathcal{M}}_{g,k}$$

the Hodge vector bundle of harmonic differentials over the Deligne-Mumford moduli space of (complex) genus g curves with k marked points. For each $i=1,\ldots,k$, let

$$\psi_i\in H^2\big(\overline{\mathcal{M}}_{g,k}\big)$$

be the first Chern class of the universal cotangent line bundle at the i-th marked point. For a formal variable u, we define

$$\Lambda(u)=\sum_{r=0}^{g}c_r(\mathbb{E}^*)u^{g-r}\in H^*\big(\overline{\mathcal{M}}_{g,k}\big)[u].$$

This is the equivariant Euler class of $\mathbb{E}^*$ tensored with the trivial line bundle with the equivariant first Chern class equal to u.

For formal variables u_1, u_2, and u_3, define

$$I_{1;0}(u_1,u_2,u_3)=1, \qquad I_{1;g}(u_1,u_2,u_3)=\int_{\overline{\mathcal{M}}_{g,1}}\frac{\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)}{u_1(u_1-\psi_1)} \qquad \forall\, g\in\mathbb{Z}^+,$$
$$I_{2;0}(u_1,u_2,u_3)=\frac{(u_1+u_2)u_3}{u_1u_2}, \quad I_{2;g}(u_1,u_2,u_3)=\frac{(u_1+u_2)^2u_3}{u_1u_2}\int_{\overline{\mathcal{M}}_{g,2}}\frac{\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)}{(u_1-\psi_1)(u_2-\psi_2)} \quad \forall\, g\in\mathbb{Z}^+.$$

These integrals are known in the literature as one- and two-partition Hodge integrals. We note that

$$I_{1;g}(u_1,u_2,u_3)=I_{1;g}(u_1,u_3,u_2), \qquad I_{2;g}(u_1,u_2,u_3)=I_{2;g}(u_2,u_1,u_3), \tag{6.1}$$
$$I_{1;g}(u_1,u_2,u_3)=I_{1;g}(-u_1,-u_2,-u_3), \qquad I_{2;g}(u_1,u_2,u_3)=I_{2;g}(-u_1,-u_2,-u_3) \tag{6.2}$$

for all $g\in\mathbb{Z}^{\geqslant 0}$.

Lemma 6.1. *For $g,k\in\mathbb{Z}^{\geqslant 0}$,*

$$\int_{\overline{\mathcal{M}}_{g,k}}\frac{\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)}{u_1(u_1-\psi_1)}=\big(u_1^{-1}\big)^{k-1}I_{1;g}(u_1,u_2,u_3) \qquad \text{if } 2g+k\geqslant 3,\ k\geqslant 1,$$
$$\frac{(u_1+u_2)^2u_3}{u_1u_2}\int_{\overline{\mathcal{M}}_{g,k}}\frac{\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)}{(u_1-\psi_1)(u_2-\psi_2)}=\big(u_1^{-1}+u_2^{-1}\big)^{k-2}I_{2;g}(u_1,u_2,u_3) \quad \text{if } 2g+k\geqslant 3,\ k\geqslant 2.$$

Proof. For $(g,k)=(0,3)$, both identities follow from the moduli space $\overline{\mathcal{M}}_{0,3}$ being a single point. In the remaining cases, they are consequences of the dilaton relation as shown below. If $k\geqslant 2$ and $2g+k\geqslant 4$,

$$\begin{aligned}\int_{\overline{\mathcal{M}}_{g,k}}\frac{\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)}{u_1(u_1-\psi_1)}&=\sum_{s=0}^{\infty}\int_{\overline{\mathcal{M}}_{g,k}}\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)\frac{\psi_1^s}{u_1^{s+2}}=\sum_{s=1}^{\infty}\int_{\overline{\mathcal{M}}_{g,k-1}}\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)\frac{\psi_1^{s-1}}{u_1^{s+2}}\\&=\sum_{s=0}^{\infty}\int_{\overline{\mathcal{M}}_{g,k-1}}\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)\frac{u_1^{-1}\psi_1^s}{u_1^{s+2}}=u_1^{-1}\int_{\overline{\mathcal{M}}_{g,k-1}}\frac{\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)}{u_1(u_1-\psi_1)}\,.\end{aligned}$$

If $k\geqslant 3$ and $2g+k\geqslant 4$,

$$\begin{aligned}\int_{\overline{\mathcal{M}}_{g,k}}\frac{\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)}{(u_1-\psi_1)(u_2-\psi_2)}&=\sum_{s_1,s_2\geqslant 0}\int_{\overline{\mathcal{M}}_{g,k}}\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)\frac{\psi_1^{s_1}\psi_1^{s_2}}{u_1^{s_1+1}u_2^{s_2+1}}\\&=\sum_{s_1,s_2\geqslant 0}\int_{\overline{\mathcal{M}}_{g,k-1}}\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)\frac{(u_1^{-1}+u_2^{-1})\psi_1^{s_1}\psi_1^{s_2}}{u_1^{s_1+1}u_2^{s_2+1}}\\&=\big(u_1^{-1}+u_2^{-1}\big)\int_{\overline{\mathcal{M}}_{g,k-1}}\frac{\Lambda(u_1)\Lambda(u_2)\Lambda(u_3)}{(u_1-\psi_1)(u_2-\psi_2)}\,.\end{aligned}$$

Both of the claimed identities now follow by induction from the base cases. □

In the next three examples, we denote by $H\in H^2(\mathbb{P}^3;\mathbb{Q})$ the usual hyperplane class and use the standard $(\mathbb{C}^*)^2$-action on $(\mathbb{P}^3,\tau_4)$. The fixed points of this action are

$$P_1=[1,0,0,0], \quad P_2=[0,1,0,0], \quad P_3=[0,0,1,0], \quad P_4=[0,0,0,1];$$

they satisfy $P_1=\tau_4(P_2)$ and $P_3=\tau_4(P_4)$. Define

$$\tau_4\colon \big\{1,2,3,4\big\} \longrightarrow \big\{1,2,3,4\big\} \qquad \text{by} \quad P_{\tau_4(i)} = \tau_4(P_i)\,.$$

We denote by

$$\alpha_1 = -\alpha_2 \qquad \text{and} \qquad \alpha_3 = -\alpha_4$$

the weights of the standard $(\mathbb{C}^*)^2$-action and by

$$\mathbf{x}, \mathbf{e}\big(T\mathbb{P}^3\big), \mathbf{c}\big(T\mathbb{P}^3\big) \in H^*_{\mathbb{T}^2}(\mathbb{P}^3;\mathbb{Q})$$

the equivariant hyperplane class and the equivariant Euler and Chern classes of $T\mathbb{P}^3$. Thus,

$$\mathbf{x}|_{P_i} = \alpha_i, \quad \mathbf{c}\big(T\mathbb{P}^3\big)\big|_{P_i} = \prod_{j\neq i}\big(1+\mathbf{x}-\alpha_j\big) \qquad \forall\ i=1,2,3,4. \tag{6.3}$$

For $i\in\mathbb{Z}$, let

$$\langle i\rangle = \begin{cases} 1, & \text{if } i\notin 2\mathbb{Z};\\ 3, & \text{if } i\in 2\mathbb{Z}. \end{cases}$$

The genus g degree d real GW-invariant of $(\mathbb{P}^3,\tau_4)$ with d pairs of conjugate point constraints is given by

$$\mathrm{GW}_{g,d}^{\mathbb{P}^3,\tau_4}\big(\underbrace{H^3,\ldots,H^3}_{d}\big) = \int_{[\overline{\mathfrak{M}}_{g,d}(\mathbb{P}^3,d)^{\tau_4}]^{\mathrm{vir}}} \prod_{i=1}^{d}\bigg(\mathrm{ev}_i^*\prod_{j\neq\langle i\rangle}(\mathbf{x}-\alpha_j)\bigg). \tag{6.4}$$

By [9, Theorem 4.6], this integral is the sum over contributions $\mathrm{Cntr}_{\Gamma,\sigma}(3^d)$ from the $(\mathbb{C}^*)^2$-fixed loci $\mathcal{Z}_{\Gamma,\sigma}$ corresponding to the elements (Γ,σ) of the collection $\mathcal{A}_{g,d}(4,d)$ of admissible pairs; these are reviewed below.

Each element (Γ,σ) of $\mathcal{A}_{g,d}(4,d)$ consists of a genus g S-marked $[4]$-labeled decorated graph Γ with S as in (3.17) for $l=d$ and an involution σ on Γ. The former consists of an S-marked decorated graph $\mathcal{T}$ as in (3.1) and additional functions

$$\vartheta\colon \mathrm{Ver} \longrightarrow \big\{1,2,3,4\big\} \qquad \text{and} \qquad \mathfrak{d}\colon \mathrm{Edg}\longrightarrow \mathbb{Z}^+$$

such that $\vartheta(v_1)\neq\vartheta(v_2)$ for every edge $e=\{v_1,v_2\}$ and the values of $\mathfrak{d}$ add up to d. The bijection σ is an involution on $\mathcal{T}$ as in (3.4) such that $\vartheta\circ\sigma=\tau_4\circ\vartheta$ and $\mathfrak{d}\circ\sigma=\mathfrak{d}$. The first condition implies that $\mathrm{V}^\sigma_{\mathbb{R}}(\Gamma)=\varnothing$. A pair (Γ,σ) is admissible if $\mathfrak{d}(e)\notin 2\mathbb{Z}$ for every $e\in\mathrm{E}^\sigma_{\mathbb{R}}(\Gamma)$. Since $\mathrm{ev}_i|_{\mathcal{Z}_{\Gamma,\sigma}}$ is the constant map with value $P_{\vartheta(\mathfrak{m}(i))}$, (6.3) implies that

$$\mathrm{ev}_i^*\prod_{j\neq\langle i\rangle}\big(\mathbf{x}-\alpha_j\big) = \prod_{j\neq\langle i\rangle}\big(\alpha_{\vartheta(\mathfrak{m}(i^+))}-\alpha_j\big) = \begin{cases} \mathbf{e}(T_{P_{\langle i\rangle}}\mathbb{P}^3), & \text{if } \vartheta(\mathfrak{m}(i^+))=\langle i\rangle;\\ 0, & \text{if } \vartheta(\mathfrak{m}(i^+))\neq\langle i\rangle; \end{cases} \tag{6.5}$$

for each $i=1,\ldots,d$.

For each edge $e=\{v_1,v_2\}$, let

$$\psi_{e;v_1} = \frac{\alpha_{v_2}-\alpha_{v_1}}{\mathfrak{d}(e)}\,.$$

For each $v\in\mathrm{Ver}$, let

$$S_v=\mathrm{E}_v(\Gamma)\sqcup\mathfrak{m}^{-1}(v)$$

be the disjoint union of the edges leaving v and the marked points carried by v. If in addition $\mathrm{val}_v(\Gamma)\geqslant 3$ and $e\in\mathrm{E}_v(\Gamma)$, let

$$\psi_{v;e}\in H^2\big(\overline{\mathcal{M}}_{\mathfrak{g}(v),S_v}\big)$$

denote the first Chern class of the universal cotangent line bundle associated with the marked point indexed by e.

If $\vartheta(\mathfrak{m}(i^+))=\langle i\rangle$ and $\mathrm{val}_v(\Gamma)\geqslant 3$, the contribution of the vertex v to $\mathrm{Cntr}_{\Gamma,\sigma}(3^d)$ is given by [9, (4.18)] as

$$(-1)^{\mathfrak{s}_v}\mathrm{Cntr}_{\Gamma,\sigma;v}(3^d)=-(-1)^{\mathfrak{g}(v)+|\mathrm{E}_v(\Gamma)|}\mathbf{e}\big(T_{P_{\vartheta(v)}}\mathbb{P}^3\big)^{|S_v|-1}\int_{\overline{\mathcal{M}}_{\mathfrak{g}(v),S_v}}\frac{\mathbf{e}(\mathbb{E}^*\otimes T_{P_{\vartheta(v)}}\mathbb{P}^3)}{\prod\limits_{e\in\mathrm{E}_v(\Gamma)}\psi_{e;v}(\psi_{e;v}+\psi_{v;e})}\,. \tag{6.6}$$

By the second statement in (6.3),

$$\mathbf{e}\big(\mathbb{E}^*\otimes T_{P_{\vartheta(v)}}\mathbb{P}^3\big)=\prod_{j\neq\vartheta(v)}\Lambda\big(\alpha_{\vartheta(v)}-\alpha_j\big)\,. \tag{6.7}$$

If $\vartheta(\mathfrak{m}(i^+))=\langle i\rangle$ and $\mathrm{val}_v(\Gamma)\in\{1,2\}$, the contribution of the vertex v to $\mathrm{Cntr}_{\Gamma,\sigma}(3^d)$ is given by [9, (4.19)] as

$$\begin{split}(-1)^{\mathfrak{s}_v}\mathrm{Cntr}_{\Gamma,\sigma;v}(3^d)=(-1)^{|\mathfrak{m}^{-1}(v)|}\mathbf{e}\big(T_{P_{\vartheta(v)}}\mathbb{P}^3\big)^{|S_v|-1}\\ \times\Big(\prod_{e\in\mathrm{E}_v(\Gamma)}\psi_{e;v}\Big)^{-1}\Big(\sum_{e\in\mathrm{E}_v(\Gamma)}\psi_{e;v}\Big)^{3-|S_v|-|\mathrm{E}_v(\Gamma)|}\,.\end{split} \tag{6.8}$$

If $\vartheta(\mathfrak{m}(i^+))\neq\langle i\rangle$, then $(-1)^{\mathfrak{s}_v}\mathrm{Cntr}_{\Gamma,\sigma;v}(3^d)=0$ by the second case of (6.5).

If $e\in\mathrm{E}_{\mathbb{C}}^{\sigma}(\mathcal{T})$, the contribution of the edge e to $\mathrm{Cntr}_{\Gamma,\sigma}(3^d)$ is given by [9, (4.22)] as

$$\mathrm{Cntr}_{\Gamma,\sigma;e}=\frac{(-1)^{\mathfrak{d}(e)}}{\mathfrak{d}(e)\,(\mathfrak{d}(e)!)^2}\frac{1}{\Big(\frac{\alpha_{\vartheta(v_1)}-\alpha_{\vartheta(v_2)}}{\mathfrak{d}(e)}\Big)^{2\mathfrak{d}(e)-2}\prod\limits_{j\neq\vartheta(v_1),\vartheta(v_2)}\prod\limits_{r=0}^{\mathfrak{d}(e)}\Big(\frac{(\mathfrak{d}(e)-r)\alpha_{\vartheta(v_1)}+r\alpha_{\vartheta(v_2)}}{\mathfrak{d}(e)}-\alpha_j\Big)}\,. \tag{6.9}$$

If $e\in\mathrm{E}_{\mathbb{R}}^{\sigma}(\Gamma)$, the contribution of the edge e to $\mathrm{Cntr}_{\Gamma,\sigma}(3^d)$ is given by [9, (4.23)] as

$$\mathrm{Cntr}_{\Gamma,\sigma;e}=\frac{(-1)^{\frac{\mathfrak{d}(e)-1}{2}}}{\mathfrak{d}(e)\mathfrak{d}(e)!}\frac{1}{\Big(2\frac{\alpha_{\vartheta(v_1)}}{\mathfrak{d}(e)}\Big)^{\mathfrak{d}(e)-1}\prod\limits_{j\neq\vartheta(v_1),\vartheta(v_2)}\prod\limits_{r=0}^{(\mathfrak{d}(e)-1)/2}\Big(\frac{(\mathfrak{d}(e)-2r)\alpha_{\vartheta(v_1)}}{\mathfrak{d}(e)}-\alpha_j\Big)}. \tag{6.10}$$

The description of the contribution $\mathrm{Cntr}_{\Gamma,\sigma}(3^d)$ of $\mathcal{Z}_{\Gamma,\sigma}$ to (6.4) in [9, Theorem 4.6] involves picking any subsets

$$\mathrm{V}_+^{\sigma}(\Gamma)\subset\mathrm{Ver}\quad\text{and}\quad\mathrm{E}_+^{\sigma}(\Gamma)\subset\mathrm{E}_{\mathbb{C}}^{\sigma}(\Gamma)\qquad\text{s.t.}$$
$$\mathrm{Ver}=\mathrm{V}_+^{\sigma}(\Gamma)\sqcup\sigma\big(\mathrm{V}_+^{\sigma}(\Gamma)\big)\qquad\text{and}\qquad\mathrm{Edg}=\mathrm{E}_{\mathbb{R}}^{\sigma}(\Gamma)\sqcup\mathrm{E}_+^{\sigma}(\Gamma)\sqcup\sigma\big(\mathrm{E}_+^{\sigma}(\Gamma)\big).$$

By [9, (4.26)],

$$\mathrm{Cntr}_{\Gamma,\sigma}(3^d) = \frac{1}{|\mathrm{Aut}(\Gamma,\sigma)|} \prod_{v\in \mathrm{V}_+^\sigma(\Gamma)} (-1)^{\mathfrak{s}_v}\mathrm{Cntr}_{\Gamma,\sigma;v}(3^d) \prod_{e\in \mathrm{E}_\mathbb{R}^\sigma(\Gamma)\sqcup \mathrm{E}_+^\sigma(\Gamma)} \mathrm{Cntr}_{\Gamma,\sigma;e}\,. \tag{6.11}$$

We will call an element (Γ,σ) of $\mathcal{A}_{g,d}(4,d)$ contributing if $\vartheta(\mathfrak{m}(i^+))=\langle i\rangle$ for every $i=1,\ldots,d$. By [9, Theorem 4.6], the number (6.4) is the sum of the rational fractions (6.11) over the contributing elements of $\mathcal{A}_{g,d}(4,d)$.

6.2 Low-degree real GW-invariants

We next apply the above setup to evaluate the integral in (6.4) for $d=1,3,4$ and $g\in\mathbb{Z}^{\geqslant 0}$ such that $d-g\notin 2\mathbb{Z}$.

Example 6.2 ($d=1$)**.** We compute the genus g degree 1 real GW-invariant of $(\mathbb{P}^3,\tau_4)$ with 1 conjugate pair of point constraints for every $g\in 2\mathbb{Z}^{\geqslant 0}$. This invariant is given by (6.4) with $d=1$. If (Γ,σ) is an element of $\mathcal{A}_{g,1}(4,1)$, then Γ consists of a single edge $e=\{v_1,v_2\}$ with

$$e\in \mathrm{E}_\mathbb{R}^\sigma, \qquad \mathfrak{d}(e)=1, \qquad \mathfrak{g}(v_1)=\mathfrak{g}(v_2)=\frac{g}{2}, \qquad \mathfrak{m}(1^+)=1, \qquad \mathfrak{m}(1^-)=2\,.$$

If (Γ,σ) is a contributing pair, then

$$\vartheta(v_1)=1, \quad \vartheta(v_2)=2, \quad \psi_{e;v_1}=-2\alpha_i, \quad \mathrm{Cntr}_{\Gamma,\sigma;e}=\frac{1}{\alpha_1^2-\alpha_3^2}\,,$$
$$(-1)^{\mathfrak{s}_{v_1}}\mathrm{Cntr}_{\Gamma,\sigma;v_1}(3^1)=\alpha_1^2-\alpha_3^2 \quad \text{if } g=0\,;$$

the last two statements follow from (6.10) and (6.8). If $g=2g'>0$, then (6.6), (6.7), and the first statement of Lemma 6.1 give

$$\begin{aligned}(-1)^{\mathfrak{s}_{v_1}}\mathrm{Cntr}_{\Gamma,\sigma;v_1}(3^1) &= (-1)^{g'}2\alpha_1\big(\alpha_1^2-\alpha_3^2\big)\int_{\overline{\mathcal{M}}_{g',2}}\frac{\Lambda(2\alpha_1)\Lambda(\alpha_1-\alpha_3)\Lambda(\alpha_1+\alpha_3)}{2\alpha_1(2\alpha_1-\psi_1)}\\ &= (-1)^{g'}\big(\alpha_1^2-\alpha_3^2\big)I_{1;g'}\big(2\alpha_1,\alpha_1-\alpha_3,\alpha_1+\alpha_3\big).\end{aligned}$$

Combining the last three equations with (6.11), we obtain

$$\mathrm{GW}_{g,1}^{\mathbb{P}^3,\tau_4}\big(H^3\big) = (-1)^{g/2}I_{1;g/2}\big(2\alpha_1,\alpha_1-\alpha_3,\alpha_1+\alpha_3\big) \qquad \forall\, g\in 2\mathbb{Z}^{\geqslant 0}\,. \tag{6.12}$$

In particular, we obtain the numbers $\mathrm{GW}_{0,d}^\phi$, $\mathrm{GW}_{2,d}^\phi$, and $\mathrm{GW}_{4,d}^\phi$ listed in the $d=1$ column of Table 2.

Example 6.3 ($d=3$)**.** Let $g\in 2\mathbb{Z}^{\geqslant 0}$. The genus g degree 3 real GW-invariant of $(\mathbb{P}^3,\tau_4)$ with 3 conjugate pairs of point constraints is given by (6.4) with $d=3$. If $(\Gamma,\sigma)\in\mathcal{A}_{g,3}(4,3)$ is a contributing pair, then

$$\vartheta\big(\mathfrak{m}(1^+)\big),\vartheta\big(\mathfrak{m}(3^+)\big)=1, \quad \vartheta\big(\mathfrak{m}(1^-)\big),\vartheta\big(\mathfrak{m}(3^-)\big)=2, \quad \vartheta\big(\mathfrak{m}(2^+)\big)=3, \quad \vartheta\big(\mathfrak{m}(2^-)\big)=4. \tag{6.13}$$

There are 4 types of admissible pairs satisfying (6.13); see Figure 2. The number next to each vertex v in this figure, i.e. 1,2,3, or 4, is $\vartheta(v)$, the number next to each edge is $\mathfrak{d}(e)$, and the arrows

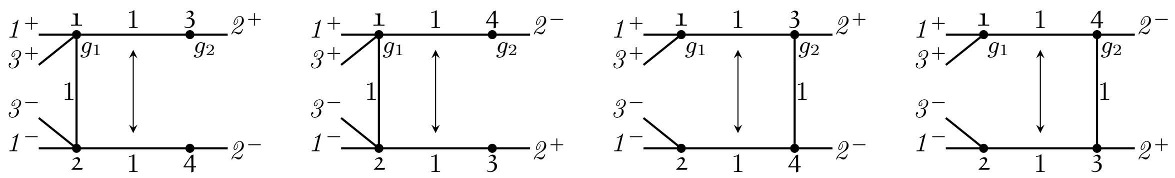

Figure 2: Admissible pairs potentially contributing to (6.4) with $d\!=\!3$

indicate the involutions. The labels g_1 and g_2 next to the top vertices are the values of $\mathfrak{g}$ on these vertices. Since the values of $\mathfrak{g}$ are preserved by the involution, $g_1+g_2=g/2$. We can take $\mathrm{V}_+^\sigma(\Gamma)$ to consist of the two top vertices and $\mathrm{E}_+^\sigma(\Gamma)$ of the top edge e_1. We denote by v_1 the left top vertex, by v_2 the right top vertex, and by e_2 the vertical edge. In the case of the first diagram in Figure 2, (6.6)-(6.10) and Lemma 6.1 give

$$\begin{gathered}\mathrm{Cntr}_{\Gamma_1,\sigma_1;e_1}=\frac{-1}{4\alpha_1\alpha_3(\alpha_1+\alpha_3)^2}\,,\qquad \mathrm{Cntr}_{\Gamma_1,\sigma_1;e_2}=\frac{1}{\alpha_1^2-\alpha_3^2}\,,\\ (-1)^{\mathfrak{s}_{v_1}}\mathrm{Cntr}_{\Gamma_1,\sigma_1;v_1}(3^3)=-(-1)^{g_1}2\alpha_1\big(\alpha_1^2-\alpha_3^2\big)\big(\alpha_1+\alpha_3\big)I_{2;g_1}\big(\alpha_1-\alpha_3,2\alpha_1,\alpha_1+\alpha_3\big),\\ (-1)^{\mathfrak{s}_{v_2}}\mathrm{Cntr}_{\Gamma_1,\sigma_1;v_2}(3^3)=(-1)^{g_2}2\alpha_3\big(\alpha_3+\alpha_1\big)I_{1;g_2}\big(\alpha_3-\alpha_1,2\alpha_3,\alpha_3+\alpha_1\big).\end{gathered}$$

By (6.11), the contributions of the first two diagrams to (6.4) with $d\!=\!3$ are thus

$$\begin{aligned}\mathrm{Cntr}_{\Gamma_1,\sigma_1}(3^3)&=(-1)^{g/2}I_{2;g_1}\big(\alpha_1-\alpha_3,2\alpha_1,\alpha_1+\alpha_3\big)I_{1;g_2}\big(\alpha_3-\alpha_1,2\alpha_3,\alpha_1+\alpha_3\big),\\ \mathrm{Cntr}_{\Gamma_2,\sigma_2}(3^3)&=(-1)^{g/2}I_{2;g_1}\big(\alpha_1+\alpha_3,2\alpha_1,\alpha_1-\alpha_3\big)I_{1;g_2}\big(\alpha_1+\alpha_3,2\alpha_3,\alpha_3-\alpha_1\big);\end{aligned}\tag{6.14}$$

the second expression is obtained from the first by replacing α_3 by $-\alpha_3$ (which corresponds to interchanging 3 and 4) and using the second identity in (6.2). In the case of the third diagram in Figure 2, (6.6)-(6.10) and Lemma 6.1 give

$$\begin{gathered}\mathrm{Cntr}_{\Gamma_3,\sigma_3;e_1}=\frac{-1}{4\alpha_1\alpha_3(\alpha_1+\alpha_3)^2}\,,\qquad \mathrm{Cntr}_{\Gamma_3,\sigma_3;e_2}=\frac{1}{\alpha_3^2-\alpha_1^2}\,,\\ (-1)^{\mathfrak{s}_{v_1}}\mathrm{Cntr}_{\Gamma_3,\sigma_3;v_1}(3^3)=(-1)^{g_1}4\alpha_1^2\big(\alpha_1+\alpha_3\big)^2I_{1;g_1}\big(\alpha_1-\alpha_3,2\alpha_1,\alpha_1+\alpha_3\big),\\ (-1)^{\mathfrak{s}_{v_2}}\mathrm{Cntr}_{\Gamma_3,\sigma_3;v_2}(3^3)=-(-1)^{g_2}\frac{2\alpha_3(\alpha_3^2-\alpha_1^2)}{3\alpha_3-\alpha_1}I_{2;g_2}\big(\alpha_3-\alpha_1,2\alpha_3,\alpha_3+\alpha_1\big).\end{gathered}$$

By (6.11), the contributions of the last two diagrams to (6.4) with $d\!=\!3$ are thus

$$\begin{aligned}\mathrm{Cntr}_{\Gamma_3,\sigma_3}(3^3)&=\frac{(-1)^{g/2}2\alpha_1}{3\alpha_3-\alpha_1}I_{1;g_1}\big(\alpha_1-\alpha_3,2\alpha_1,\alpha_1+\alpha_3\big)I_{2;g_2}\big(\alpha_3-\alpha_1,2\alpha_3,\alpha_1+\alpha_3\big),\\ \mathrm{Cntr}_{\Gamma_4,\sigma_4}(3^3)&=\frac{-(-1)^{g/2}2\alpha_1}{3\alpha_3+\alpha_1}I_{1;g_1}\big(\alpha_1+\alpha_3,2\alpha_1,\alpha_1-\alpha_3\big)I_{2;g_2}\big(\alpha_1+\alpha_3,2\alpha_3,\alpha_3-\alpha_1\big).\end{aligned}\tag{6.15}$$

The genus g degree 3 real GW-invariant of $(\mathbb{P}^3,\tau_4)$ with 3 conjugate pairs of point constraints is the sum of the four fractions in (6.14) and (6.15) over all $g_1,g_2\in\mathbb{Z}^{\geqslant0}$ with $g_1+g_2=g/2$. In particular, we obtain the numbers $\mathrm{GW}_{0,d}^\phi$, $\mathrm{GW}_{2,d}^\phi$, and $\mathrm{GW}_{4,d}^\phi$ listed in the $d\!=\!3$ column of Table 2.

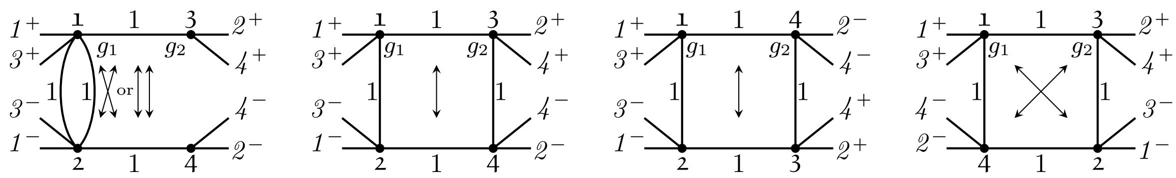

Figure 3: Admissible pairs potentially contributing to (6.4) with $d=4$

Example 6.4 ($d=4$)**.** Let $g\in\mathbb{Z}^+-2\mathbb{Z}$. The genus g degree 4 real GW-invariant of $(\mathbb{P}^3,\tau_4)$ with 3 conjugate pairs of point constraints is given by (6.4) with $d=4$. If $(\Gamma,\sigma)\in\mathcal{A}_{g,4}(4,4)$ is a contributing pair, then

$$\vartheta\big(\mathfrak{m}(1^+)\big),\vartheta\big(\mathfrak{m}(3^+)\big)=1,\qquad \vartheta\big(\mathfrak{m}(2^+)\big),\vartheta\big(\mathfrak{m}(4^+)\big)=3. \tag{6.16}$$

There are 11 types of admissible pairs satisfying (6.13): the first diagram of Figure 3, with the 4 possible ways of labeling the vertices and the 2 possible involutions on the loop of edges, and the remaining 3 diagrams in this figure. All 8 versions of the first diagram in Figure 3 have the same automorphism group $\mathbb{Z}_2$ and all edges of degree 1; their contributions thus cancel in pairs by [9, Corollary 4.8]. For each of the remaining diagrams, we denote the top left and right vertices by v_1 and v_2, respectively, and the top, left, and right edges by e_1, e_2, and e_3, respectively. We take $\mathrm{V}_+^\sigma(\Gamma)=\{v_1,v_2\}$ in all three cases, $\mathrm{E}_+^\sigma(\Gamma)=\{e_1\}$ for the middle two diagrams, and $\mathrm{E}_+^\sigma(\Gamma)=\{e_1,e_2\}$ for the last diagram. In the case of the second diagram in Figure 3, (6.9), (6.10), (6.6), (6.7), and Lemma 6.1 give

$$\begin{gathered}\mathrm{Cntr}_{\Gamma_2,\sigma_2;e_1}=\frac{-1}{4\alpha_1\alpha_3(\alpha_1+\alpha_3)^2},\quad \mathrm{Cntr}_{\Gamma_2,\sigma_2;e_2}=\frac{1}{\alpha_1^2-\alpha_3^2},\quad \mathrm{Cntr}_{\Gamma_2,\sigma_2;e_3}=\frac{1}{\alpha_3^2-\alpha_1^2},\\ (-1)^{\mathfrak{s}_{v_1}}\mathrm{Cntr}_{\Gamma_2,\sigma_2;v_1}(3^4)=-(-1)^{g_1}2\alpha_1\big(\alpha_1^2-\alpha_3^2\big)\big(\alpha_1+\alpha_3\big)I_{2;g_1}\big(\alpha_1-\alpha_3,2\alpha_1,\alpha_1+\alpha_3\big),\\ (-1)^{\mathfrak{s}_{v_2}}\mathrm{Cntr}_{\Gamma_2,\sigma_2;v_2}(3^4)=-(-1)^{g_2}2\alpha_3\big(\alpha_3^2-\alpha_1^2\big)\big(\alpha_3+\alpha_1\big)I_{2;g_2}\big(\alpha_3-\alpha_1,2\alpha_3,\alpha_3+\alpha_1\big).\end{gathered}$$

By (6.11), the contributions of the middle two diagrams to (6.4) with $d=4$ are thus

$$\begin{split}\mathrm{Cntr}_{\Gamma_2,\sigma_2}(3^4)&=-(-1)^{(g-1)/2}I_{2;g_1}\big(\alpha_1-\alpha_3,2\alpha_1,\alpha_1+\alpha_3\big)I_{2;g_2}\big(\alpha_3-\alpha_1,2\alpha_3,\alpha_1+\alpha_3\big),\\ \mathrm{Cntr}_{\Gamma_3,\sigma_3}(3^4)&=-(-1)^{(g-1)/2}I_{2;g_1}\big(\alpha_1+\alpha_3,2\alpha_1,\alpha_1-\alpha_3\big)I_{2;g_2}\big(\alpha_1+\alpha_3,2\alpha_3,\alpha_3-\alpha_1\big);\end{split} \tag{6.17}$$

the second expression is obtained from the first by replacing α_3 by $-\alpha_3$ and using the second identity in (6.2). In the case of the last diagram in Figure 3, (6.9), (6.6), (6.7), and Lemma 6.1 give

$$\begin{gathered}\mathrm{Cntr}_{\Gamma_4,\sigma_4;e_1}=\frac{-1}{4\alpha_1\alpha_3(\alpha_1+\alpha_3)^2},\qquad \mathrm{Cntr}_{\Gamma_4,\sigma_4;e_2}=\frac{1}{4\alpha_1\alpha_3(\alpha_1-\alpha_3)^2},\\ (-1)^{\mathfrak{s}_{v_1}}\mathrm{Cntr}_{\Gamma_4,\sigma_4;v_1}(3^4)=-(-1)^{g_1}4\alpha_1^2\big(\alpha_1^2-\alpha_3^2\big)I_{2;g_1}\big(\alpha_1-\alpha_3,\alpha_1+\alpha_3,2\alpha_1\big),\\ (-1)^{\mathfrak{s}_{v_2}}\mathrm{Cntr}_{\Gamma_4,\sigma_4;v_2}(3^4)=-(-1)^{g_2}4\alpha_3^2\big(\alpha_3^2-\alpha_1^2\big)I_{2;g_2}\big(\alpha_3-\alpha_1,\alpha_3+\alpha_1,2\alpha_3\big).\end{gathered}$$

By (6.11), the contribution of the last diagram to (6.4) with $d=4$ is thus

$$\mathrm{Cntr}_{\Gamma_4,\sigma_4}(3^4)=(-1)^{(g-1)/2}I_{2;g_1}\big(\alpha_1-\alpha_3,\alpha_1+\alpha_3,2\alpha_1\big)I_{2;g_2}\big(\alpha_3-\alpha_1,\alpha_1+\alpha_3,2\alpha_3\big). \tag{6.18}$$

The genus g degree 4 real GW-invariant of $(\mathbb{P}^3,\tau_4)$ with 4 conjugate pairs of point constraints is the sum of the three expressions in (6.17) and (6.18) over all $g_1,g_2\in\mathbb{Z}^{\geqslant 0}$ with $g_1+g_2=(g-1)/2$. In particular, we obtain the numbers $\mathrm{GW}^{\phi}_{1,d}$, $\mathrm{GW}^{\phi}_{3,d}$, and $\mathrm{GW}^{\phi}_{5,d}$ listed in the $d=4$ column of Table 2.

6.3 Implications for Hodge integrals

We put the Hodge integrals $I_{1;g}$ and $I_{2;g}$ into the generating functions

$$F_1(u_1,u_2,u_3;t)=\sum_{g=0}^{\infty}I_{1;g}(u_1,u_2,u_3)t^{2g},\qquad F_2(u_1,u_2,u_3;t)=\sum_{g=0}^{\infty}I_{2;g}(u_1,u_2,u_3)t^{2g}\,.$$

Proposition 6.5. *With notation as above,*

$$F_1(x+y,x,y;t)=\frac{\sin(t/2)}{t/2}\,, \tag{6.19}$$

$$\begin{split}\tfrac{x+y}{2x-y}F_1(x,x+y,y;t)F_2(x,x-y,-y;t)+\tfrac{x+y}{2y-x}F_1(y,x+y,x;t)F_2(-y,x-y,x;t)\\ -F_1(x,x-y,-y;t)F_2(x,x+y,y;t)-F_1(-y,x-y,x;t)F_2(y,x+y,x;t)=\bigg(\frac{\sin(t/2)}{t/2}\bigg)^5,\end{split} \tag{6.20}$$

$$\begin{split}F_2(x+y,x,y;t)F_2(x-y,x,-y;t)+F_2(x+y,y,x;t)F_2(x-y,-y,x;t)\\ -F_2(x,y,x+y;t)F_2(x,-y,x-y;t)=\bigg(\frac{\sin(t/2)}{t/2}\bigg)^8.\end{split} \tag{6.21}$$

Proof. By the $n=1$ case of [3, Examples 6.3] or the $g=0$ case of Example 6.2,

$$\mathrm{E}^{\mathbb{P}^3,\tau_4}_{0,1}(H^3)=\mathrm{GW}^{\mathbb{P}^3,\tau_4}_{0,1}(H^3)=1.$$

Since every degree 1 curve in $\mathbb{P}^3$ is of genus 0, $\mathrm{E}^{\mathbb{P}^3,\tau_4}_{g,1}(H^3)=0$ for every $g\in\mathbb{Z}^+$. Along with (1.6) and (1.5), these two observations give

$$\sum_{g=0}^{\infty}(-1)^g\mathrm{GW}^{\mathbb{P}^3,\tau_4}_{2g,1}(H^3)t^{2g}=\sum_{g=0}^{\infty}\widetilde{C}^{\mathbb{P}^3}_{0,\ell}(g)(\mathrm{i}t)^{2g}=\bigg(\frac{\sin(t/2)}{t/2}\bigg)^{0-1+4/2}.$$

Combining this statement with (6.12) and setting $x=\alpha_1+\alpha_3$ and $y=\alpha_1-\alpha_3$, we obtain (6.19).

By the $n=1$ case of [3, Examples 6.4] or the $g=0$ case of Example 6.3,

$$\mathrm{E}^{\mathbb{P}^3,\tau_4}_{0,3}(H^3,H^3,H^3)=\mathrm{GW}^{\mathbb{P}^3,\tau_4}_{0,3}(H^3,H^3,H^3)=-1.$$

By [9, Example 4.11] or the $g=1$ case of Example 6.4,

$$\mathrm{E}^{\mathbb{P}^3,\tau_4}_{1,4}(H^3,H^3,H^3,H^3)=\mathrm{GW}^{\mathbb{P}^3,\tau_4}_{1,4}(H^3,H^3,H^3,H^3)=-1.$$

By the Castelnuovo bound, every degree 3, 4 curve of genus 2 or higher lies in a $\mathbb{P}^2$ and so

$$\mathrm{E}^{\mathbb{P}^3,\tau_4}_{2g,3}(H^3,H^3,H^3),\mathrm{E}^{\mathbb{P}^3,\tau_4}_{2g+1,4}(H^3,H^3,H^3,H^3)=0\qquad\forall\ g\in\mathbb{Z}^+.$$

Along with (1.6) and (1.5), these three observations give

$$\sum_{g=0}^{\infty}(-1)^g\mathrm{GW}^{\mathbb{P}^3,\tau_4}_{2g,3}(H^3,H^3,H^3)t^{2g} = -\sum_{g=0}^{\infty}\widetilde{C}^{\mathbb{P}^3}_{0,3\ell}(g)(\mathfrak{i}t)^{2g} = -\left(\frac{\sin(t/2)}{t/2}\right)^{0-1+12/2},$$

$$\sum_{g=0}^{\infty}(-1)^g\mathrm{GW}^{\mathbb{P}^3,\tau_4}_{1+2g,4}(H^3,H^3,H^3,H^3)t^{2g} = -\sum_{g=0}^{\infty}\widetilde{C}^{\mathbb{P}^3}_{1,4\ell}(g)(\mathfrak{i}t)^{2g} = -\left(\frac{\sin(t/2)}{t/2}\right)^{1-1+16/2}.$$

Combining these two statements with the conclusions of Examples 6.3 and 6.4 and setting $x=\alpha_1+\alpha_3$ and $y=\alpha_1-\alpha_3$, we obtain (6.20) and (6.21). □

The strategy in the proof of Proposition 6.5 can be applied with complex GW-invariants using [30, Theorem 1.5]. The $d=1$ case would then give

$$F_1(u,x,y;t)F_1(u,u-x,u-y;t) = \left(\frac{\sin(t/2)}{t/2}\right)^2. \tag{6.22}$$

The $u=x+y$ case of this identity recovers (6.19).

It is straightforward to see that

$$F_1\big(u_1,u_2,u_3;t\big) = F_1\big(1,u_2/u_1,u_3/u_1;t\big), \qquad F_2\big(u_1,u_2,u_3;t\big) = F_2\big(u_1/u_3,u_2/u_3,1;t\big)$$

By Mumford's relation [17, (5.3)],

$$F_1(-x,x,-2x;t) = 1+\sum_{g=1}^{\infty}(-1)^g t^{2g}\int_{\overline{\mathcal{M}}_{g,1}}\frac{\Lambda(-2)}{(-1)(-1-\psi_1)} = 1+\sum_{g=1}^{\infty}\sum_{i=0}^{g}(-2)^i\int_{\overline{\mathcal{M}}_{g,1}}\psi_i^{2g-2+i}\lambda_{g-i}\,.$$

Thus, the $y=-2x$ case of (6.19) is equivalent to the $k=-2$ case of [2, Theorem 2].

The coefficients of $F_1(1,x,y;t)$ are rational symmetric functions in x,y. Thus, $F_1(1,x,y;t)$ is a function of $x+y$, xy, and t. However, it does not appear to depend on xy.

Conjecture 6.6. *The function $F_1=F_1(u_1,u_2,u_3;t)$ depends only on u_1, u_2+u_3, and t.*

If this is the case, (6.22) would imply that

$$F_1(u_1,u_2,u_3;t) = \frac{\sin(t/2)}{t/2}\exp\Big(f_1\big((u_2+u_3)/u_1-1;t\big)\Big)$$

for some $f_1\in\mathbb{Q}(z)[[t^2]]$ such that $f_1(-z;t)=-f_1(z;t)$.

The coefficients of $F_2(x,y,1;t)$ are rational symmetric functions in x,y. However, they appear to depend on $x+y$ and xy and not just up to a fixed uniform scaling factor. On the other hand, the next statement appears plausible.

Conjecture 6.7. *With notation as above,*

$$F_2(x,y,x+y;t)F_2(x,-y,x-y;t) = -\left(\frac{\sin(t/2)}{t/2}\right)^8\frac{(x^2-y^2)^2}{x^2y^2}\,. \tag{6.23}$$

Along with (6.21), (6.23) would imply that

$$F_2(x+y,x,y;t)F_2(x-y,x,-y;t) + F_2(x+y,y,x;t)F_2(x-y,-y,x;t) = -\left(\frac{\sin(t/2)}{t/2}\right)^8 \frac{x^4-3x^2y^2+y^4}{x^2y^2}\,.$$

Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651
Current Address: *Department of Mathematics, The University of Arizona, Tucson, AZ 85721-0089*
jniu@math.arizona.edu

Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651
azinger@math.stonybrook.edu

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