A Recursion for Counts of Real Curves in $\mathbb{CP}^{2n-1}$: Another Proof

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Abstract

In a recent paper, we obtained a WDVV-type relation for real genus 0 Gromov-Witten invariants with conjugate pairs of insertions; it specializes to a complete recursion in the case of odd-dimensional projective spaces. This note provides another, more complex-geometric, proof of the latter. The main part of this approach readily extends to real symplectic manifolds with empty real locus, but not to the general case.

1 Introduction

The classical problem of enumerating (complex) rational curves in a complex projective space $\mathbb{P}^n$ is solved in [11, 13] using the WDVV relation of Gromov-Witten theory. Over the past decade, significant progress has been made in real enumerative geometry and real Gromov-Witten theory. Invariant signed counts of real rational curves with point constraints in real surfaces and in many real threefolds are defined in [16] and [17], respectively. An approach to interpreting these counts in the style of Gromov-Witten theory, i.e. as counts of parametrizations of such curves, is presented in [2, 14]. Signed counts of real curves with conjugate pairs of arbitrary (not necessarily point) constraints in arbitrary dimensions are defined in [5] and extended to more general settings in [3]. Two different WDVV-type relations for the real Gromov-Witten invariants of real surfaces as defined in [2, 14], along with the ideas behind them, are stated in [15]; they yield complete recursions for counts of real rational curves in $\mathbb{P}^2$ as defined in [16]. Other recursions for counts of real curves in some real surfaces have since been established by completely different methods in [4, 1, 8, 9].

In [7], we obtain a WDVV-type relation for real genus 0 Gromov-Witten invariants with conjugate pairs of constraints without restricting to low-dimensional real symplectic manifolds. In the case of $\mathbb{P}^{2n-1}$, it specializes to the complete recursions of Theorem 1.1 and Corollary 1.2. These recursions are sufficiently simple to characterize the cases when the aforementioned real invariants are nonzero and thus the existence of real rational curves passing through the specified constraints is guaranteed; see [7, Corollary 1.3]. The main proof of the WDVV-type relation in [7] is based on establishing a homology relation on the three-dimensional Deligne-Mumford space $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ of genus 0 real curves with 3 conjugate pairs of marked points. We also give an alternative proof in [7] which is closer to the proof of [13, Theorem 10.4], but makes use of a conjugate marking.

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In this note, we describe a more complex-geometric variation of the second approach in [7]. In order to focus on the approach itself, we restrict to $\mathbb{P}^{2n-1}$, but it can be applied in some other cases as well; see Remark 2.2. We work with the explicit system of orientations on the moduli spaces of real maps to $\mathbb{P}^{2n-1}$ defined in [3, Appendix A.1] from an algebro-geometric point of view; the orientations used in [7] are described from the point of view of symplectic topology. The analysis of the sign of the key gluing map of Lemma 3.1 is carried out in Section 4 using polynomials. The primary motivations for this note are to make the proof of Theorem 1.1 and Corollary 1.2 more accessible, in particular to algebraic geometers who may have no interest in the general case of the real WDVV relation of [7, Theorem 2.1], and to highlight the difficulties eliminated by the homology relation of [7, Proposition 3.3].

Each odd-dimensional projective space $\mathbb{P}^{2n-1}$ has two standard anti-holomorphic involutions (automorphisms of order 2):

$$\tau_{2n}\colon \mathbb{P}^{2n-1} \longrightarrow \mathbb{P}^{2n-1}, \quad [z_1,\ldots,z_{2n}] \longrightarrow [\bar{z}_2,\bar{z}_1,\ldots,\bar{z}_{2n},\bar{z}_{2n-1}], \tag{1.1}$$
$$\eta_{2n}\colon \mathbb{P}^{2n-1} \longrightarrow \mathbb{P}^{2n-1}, \quad [z_1,\ldots,z_{2n}] \longrightarrow [-\bar{z}_2,\bar{z}_1,\ldots,-\bar{z}_{2n},\bar{z}_{2n-1}]. \tag{1.2}$$

The fixed locus of the first involution is $\mathbb{RP}^{2n-1}$, while the fixed locus of the second involution is empty. Let

$$\tau=\tau_2,\ \eta=\eta_2 : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 .$$

For $\phi=\tau_{2n},\eta_{2n}$ and $c=\tau,\eta$, a map $u:\mathbb{P}^1\longrightarrow\mathbb{P}^{2n-1}$ is (ϕ,c)-real if $u\circ c=\phi\circ u$. For $k\in\mathbb{Z}^{\ge 0}$, a k-marked (ϕ,c)-real map is a tuple

$$\big(u,(z_1^+,z_1^-),\ldots,(z_k^+,z_k^-)\big),$$

where $z_1^+,z_1^-,\ldots,z_k^+,z_k^-\in\mathbb{P}^1$ are distinct points with $z_i^+=c(z_i^-)$ and u is a (ϕ,c)-real map. Such a tuple is c-equivalent to another k-marked (ϕ,c)-real map

$$\big(u',(z_1'^+,z_1'^-),\ldots,(z_k'^+,z_k'^-)\big)$$

if there exists a biholomorphic map $h\colon\mathbb{P}^1\longrightarrow\mathbb{P}^1$ such that

$$h\circ c=c\circ h, \qquad u'=u\circ h, \quad \text{and} \quad z_i^\pm=h(z_i'^\pm)\ \ \forall\, i=1,\ldots,k.$$

If in addition $d\in\mathbb{Z}^+$, denote by $\mathfrak{M}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi,c}$ the moduli space of c-equivalence classes of k-marked degree d holomorphic (ϕ,c)-real maps.

By [5, Theorem 6.5], a natural compactification

$$\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\tau_{2n},\tau}\supset\mathfrak{M}_{0,k}(\mathbb{P}^{2n-1},d)^{\tau_{2n},\tau}$$

is orientable. If $d\notin2\mathbb{Z}$, $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\tau_{2n},\tau}$ has no boundary and thus carries a $\mathbb{Z}$-homology class; see [5, Theorem 1.6]. By [3, Lemma 1.9],

$$\mathfrak{M}_{0,k}(\mathbb{P}^{2n-1},d)^{\eta_{2n},\tau}=\emptyset \quad \forall\ d\in\mathbb{Z}, \qquad \mathfrak{M}_{0,k}(\mathbb{P}^{2n-1},d)^{\tau_{2n},\eta}=\emptyset \quad \forall\ d\notin2\mathbb{Z},$$

and a natural compactification

$$\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi,\eta}\supset\mathfrak{M}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi,\eta}$$

is orientable for $\phi=\tau_{2n},\eta_{2n}$. If $d\notin 2\mathbb{Z}$, $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\eta_{2n},\eta}$ has no boundary and thus carries a $\mathbb{Z}$-homology class; see [3, Proposition 1.1]. If $d\in 2\mathbb{Z}$, a glued moduli space

$$\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi} \equiv \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi,\tau} \cup \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi,\eta} \tag{1.3}$$

is orientable and has no boundary; see [3, Theorem 1.7] and [3, Remark 1.11].

The glued compactified moduli spaces come with natural evaluation maps

$$\mathrm{ev}_i\colon \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi} \longrightarrow \mathbb{P}^{2n-1}, \qquad \big[u,(z_1^+,z_1^-),\ldots,(z_k^+,z_k^-)\big] \longrightarrow u(z_i^+).$$

Thus, for $c_1,\ldots,c_k\in\mathbb{Z}^+$, we define

$$N_d^{\phi}(c_1,\ldots,c_k) = \int_{\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi}} \mathrm{ev}_1^*H^{c_1}\,\ldots\,\mathrm{ev}_k^*H^{c_k} \in \mathbb{Z}\,, \tag{1.4}$$

where $H\in H^2(\mathbb{P}^{2n-1})$ is the hyperplane class. For dimensional reasons,

$$N_d^{\phi}(c_1,\ldots,c_k)\neq 0 \qquad \Longrightarrow \qquad c_1+\ldots+c_k = n(d+1)-2+k\,. \tag{1.5}$$

Similarly to [13, Lemma 10.1], the numbers (1.4) are enumerative counts of real curves in $\mathbb{P}^{2n-1}$, i.e. of curves preserved by ϕ, but now with some sign. They satisfy the usual divisor relation [10, Section 26.3]. By [3, Theorem 1.10], the numbers (1.4) with $\phi=\tau_{2n},\eta_{2n}$ vanish if either d or any c_i is even; see also [3, Remark 1.11], [7, Corollary 2.6], and [7, Theorem 2.2].

The nonzero numbers (1.4) depend on the chosen orientation of the moduli space and are thus well-defined only up to sign, a priori depending on the degree d. With the choices in [3],

$$N_d^{\tau_{2n}}(c_1,\ldots,c_k) = -N_d^{\eta_{2n}}(c_1,\ldots,c_k)\,; \tag{1.6}$$

see [3, Theorem 1.10]. Thus, it is sufficient to compute the numbers

$$\big\langle c_1,\ldots,c_k\big\rangle_d^{\phi} \equiv (-1)^{n(d-1)/2} N_d^{\phi}(c_1,\ldots,c_k) \tag{1.7}$$

with $\phi=\eta_{2n}$, $d\geq 1$ odd, and $c_i\geq 3$ odd; we comment on the sign modification in Remark 1.3. For any $d,c_1,\ldots,c_k\in\mathbb{Z}^+$, let

$$\big\langle c_1,\ldots,c_k\big\rangle_d^{\mathbb{P}^{2n-1}} = \int_{\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)} \mathrm{ev}_1^*H^{c_1}\,\ldots\,\mathrm{ev}_k^*H^{c_k} \in \mathbb{Z}^{\geq 0}\,,$$

where $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)$ is the usual moduli space of stable (complex) k-marked genus 0 degree d holomorphic maps to $\mathbb{P}^{2n-1}$, denote the (complex) genus 0 Gromov-Witten invariants of $\mathbb{P}^{2n-1}$; they are computed in [13, Theorem 10.4]. Finally, if $c_1,\ldots,c_k\in\mathbb{Z}$ and $I\subset\{1,\ldots,k\}$, let c_I denote a tuple with the entries c_i with $i\in I$, in some order.

Theorem 1.1. *Let $\phi=\tau_{2n},\eta_{2n}$ and $d,k,n,c,c_1,\ldots,c_k\in\mathbb{Z}^+$. If $k\geq 2$ and $c_1,\ldots,c_k\notin 2\mathbb{Z}$,*

$$\begin{split}\big\langle c_1,c_2+2c,c_3,\ldots,c_k\big\rangle_d^{\phi} - \big\langle c_1+2c,c_2,c_3,\ldots,c_k\big\rangle_d^{\phi} = \sum_{\substack{2d_1+d_2=d\\ d_1,d_2\geq 1}}\ \sum_{I\sqcup J=\{3,\ldots,k\}}\ \sum_{\substack{2i+j=2n-1\\ i,j\geq 1}} 2^{|I|}\Bigg(\\ \big\langle 2c,c_1,c_I,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_2,c_J,j\big\rangle_{d_2}^{\phi} - \big\langle 2c,c_2,c_I,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_1,c_J,j\big\rangle_{d_2}^{\phi}\Bigg).\end{split}$$

Corollary 1.2. *Let $\phi=\tau_{2n},\eta_{2n}$ and $d,k,n,c_1,\ldots,c_k\in\mathbb{Z}^+$. If $d\in 2\mathbb{Z}$ or $c_i\in 2\mathbb{Z}$ for some i,*

$$\langle c_1,c_2,\ldots,c_k\rangle_d^{\phi}=0.$$

If $k\ge 2$ and $c_1,\ldots,c_k\notin 2\mathbb{Z}$,

$$\begin{aligned}\langle c_1,c_2,c_3,\ldots,c_k\rangle_d^{\phi}=d\langle c_1+c_2-1,c_3,\ldots,c_k\rangle_d^{\phi}+\sum_{\substack{2d_1+d_2=d\\ d_1,d_2\ge 1}}\sum_{I\sqcup J=\{3,\ldots,k\}}\sum_{\substack{2i+j=2n-1\\ i,j\ge 1}}2^{|I|}\Bigg(\\ d_2\langle c_1-1,c_2,c_I,2i\rangle_{d_1}^{\mathbb{P}^{2n-1}}\langle c_J,j\rangle_{d_2}^{\phi}-d_1\langle c_1-1,c_I,2i\rangle_{d_1}^{\mathbb{P}^{2n-1}}\langle c_2,c_J,j\rangle_{d_2}^{\phi}\Bigg).\end{aligned}$$

The formula of Theorem 1.1 immediately implies the recursion of Corollary 1.2, which in turn determines all numbers $N_d^{\phi}(c_1,\ldots,c_k)$, with $\phi=\tau_{2n},\eta_{2n}$, from the single number

$$\langle 2n-1\rangle_1^{\tau_{2n}}=N_1^{\tau_{2n}}(2n-1),$$

i.e. the number of τ_{2n}-real lines through a point in $\mathbb{P}^{2n-1}$. The absolute value of this number is of course 1. With the choice of the orientations as in [3, Section 5.2],

$$\langle 2n-1\rangle_1^{\tau_{2n}}=(-1)^{n-1}; \tag{1.8}$$

see [3, Corollary 5.4]. Taking $d=1$ in Corollary 1.2, we obtain

$$\langle c_1,\ldots,c_k\rangle_1^{\tau_{2n}}=\langle 2n-1\rangle_1^{\tau_{2n}}=(-1)^{n-1}$$

whenever $c_1,\ldots,c_k\in\mathbb{Z}^+$ are odd and $c_1+\ldots+c_k=2n-2+k$. Some other numbers obtained from Corollary 1.2 are shown in [7, Tables 1,2].

Theorem 1.1 follows from Corollary 1.2 by interchanging c_2 and c_3, which has no effect on the left-hand side of the formula in Corollary 1.2, and setting the two right-hand sides equal. Starting as in the proof of [13, Theorem 10.4], we establish the $\phi=\eta_{2n}$ case of the recursion of Corollary 1.2 in Section 2 as follows. Denote by $\overline{\mathcal{M}}_{0,4}$ the Deligne-Mumford moduli space of stable (complex) 4-marked rational curves. Let

$$\begin{aligned} f_{0123}\colon \overline{\mathfrak{M}}_{0,k+1}(\mathbb{P}^{2n-1},d)^{\eta_{2n}} &\longrightarrow \overline{\mathcal{M}}_{0,4}\,,\\ \big[(z_0^+,z_0^-),\ldots,(z_k^+,z_k^-),u\big] &\longrightarrow [\dot z_0^+,\dot z_1^+,\dot z_2^+,\dot z_3^+], \end{aligned} \tag{1.9}$$

where $[\dot z_0^+,\dot z_1^+,\dot z_2^+,\dot z_3^+]\in\overline{\mathcal{M}}_{0,4}$ is the stabilization of the domain of the stable map with the marked points z_0^+,z_1^+,z_2^+,z_3^+ only, be the morphism forgetting the map to $\mathbb{P}^{2n-1}$ and all marked points other than z_0^+,z_1^+,z_2^+,z_3^+. By adding in $c_3=1$ if necessary, it can be assumed that $k\ge 3$ in Corollary 1.2. In Section 2, we compare two expressions for the integral of the pull-back of the orientation class on $\overline{\mathcal{M}}_{0,4}$ by f_{0123} over the two-dimensional space of maps passing through the constraints $H^{c_1-1},H^1,H^{c_2},\ldots,H^{c_k}$; see (2.1). As in the proof of [13, Theorem 10.4], we consider the preimages of two different representatives of the point class (the Poincare dual of the orientation class): nodal two-component curves, with one of them having the 0-th and 1st marked points on a common component and the other having the 0-th and 2nd marked points on a common component; see Figure 1, where $\mathcal{U}\longrightarrow\overline{\mathcal{M}}_{0,4}$ denotes the universal curve and π can be viewed as the cross-ratio

$$\pi\big([z_0,z_1,z_2,z_3]\big)=\frac{z_0-z_2}{z_0-z_3}:\frac{z_1-z_2}{z_1-z_3}\,.$$

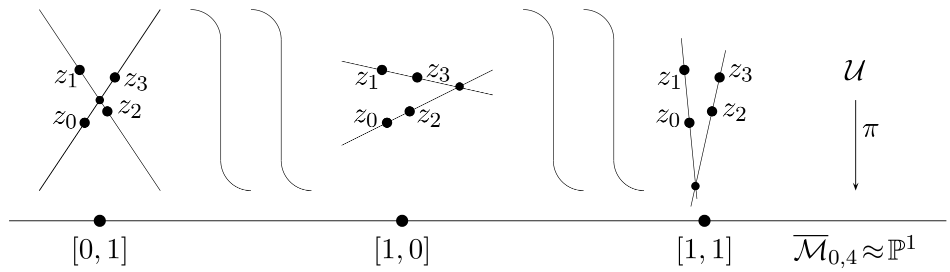

Figure 1: The universal curve $\mathcal{U}\longrightarrow\overline{\mathcal{M}}_{0,4}$

The domains of the preimages of these representatives now have three components, though each preimage is still encoded by just two of the components. The number of possible types of the preimages in this case is 7, instead of 1 as in [13]; see Figures 2 and 3. In contrast to the proof of [13, Theorem 10.4], the sign of the contribution of each element in the preimage must be carefully considered; see Proposition 2.1. With the exception of one case (the rightmost diagram in Figures 2 and 3), each element in the preimage is regular with respect to the restriction of f_{0123} to the space of maps meeting the constraints, with f_{0123} locally of the form

$$\mathbb{C}\longrightarrow\mathbb{C},\qquad \upsilon\longrightarrow\upsilon\quad\text{or}\quad\upsilon\longrightarrow\bar{\upsilon}\,,$$

with respect to a standard gluing parameter $\upsilon\in\mathbb{C}$. In the exceptional case, each element is the zero set of the map $\upsilon\longrightarrow|\upsilon|^2$ in some coordinates and so does not contribute to the curve count. Setting the sums of all contributions from each of the two degenerations equal, we obtain Corollary 1.2. This approach can also be used to prove [7, Theorem 2.1] whenever the fixed locus of the anti-symplectic involution is empty; see Remark 2.2.

Remark 1.3. There are several systematic ways of orienting the moduli spaces $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi,c}$, one of which is more natural from the point of view of algebraic geometry and the others from the point of view of symplectic topology. In [3, Section 5.2], these moduli spaces are oriented using coefficients of polynomials describing holomorphic maps $\mathbb{P}^1\longrightarrow\mathbb{P}^{2n-1}$; we use these orientations to define the numbers (1.4) with $\phi=\tau_{2n}$ and the opposite orientations to define the numbers (1.4) with $\phi=\eta_{2n}$ (as needed to orient the glued space (1.3) if $d\in2\mathbb{Z}$). This choice introduces a sign into the statement of Lemma 3.1, as compared to [7, Lemma 5.1]; the sign shifts in (1.7) offset the sign of Lemma 3.1. The orientations of moduli spaces used in [7] are induced from various pinching constructions of symplectic topology, which do not appear as natural in the context of counting curves in projective spaces. The two systems of orientations on the moduli spaces $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi,c}$ agree (up to a sign independent of d) if and only if n is even. As explained in Remark 3.2, the sign shifts in (1.7) indirectly switch the two systems of orientations so that [7, Theorem 2.1] applies to the numbers (1.7). This difference between the two systems of orientations is related to a subtle sign issue missed in the description of the localization data for real maps to $\mathbb{P}^{4n+1}$ in the first three versions of [3]; see Remark 3.2 for more details.

In Section 3, we compare different orientations of moduli spaces of constrained real maps and establish Lemma 3.3. It leads to Corollary 3.4, which implies Proposition 2.1, the key step in the proof of Corollary 1.2 in Section 2.

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2 Proof of Corollary 1.2

By (1.6) and the vanishing of the real invariants for $d\in 2\mathbb{Z}$, it is sufficient to assume that d in Theorem 1.1 is odd and $\phi=\eta_{2n}$. Let

$$\overline{\mathfrak{M}}_k^{\mathbb{C}}(d)=\overline{\mathfrak{M}}_{0,k}\big(\mathbb{P}^{2n-1},d\big),\qquad \overline{\mathfrak{M}}_k^{\mathbb{R}}(d)=\overline{\mathfrak{M}}_{0,k}\big(\mathbb{P}^{2n-1},d\big)^{\eta_{2n}};$$

we use the same conventions for the uncompactified moduli spaces. We assume that $k\ge 3$ and $c_1,\ldots,c_k\in\mathbb{Z}^+$ are odd and satisfy the equation on the right-hand side of (1.5). Let

$$f_{0123}\colon \overline{\mathfrak{M}}_{k+1}^{\mathbb{R}}(d)\longrightarrow \overline{\mathcal{M}}_{0,4}$$

denote the forgetful morphism in (1.9), with the marked points on the left-hand side indexed by $0,1,\ldots,k$. Let $\Omega_{0,4}\in H^2(\overline{\mathcal{M}}_{0,4})$ be the Poincare dual of the point class and define

$$\widetilde{N}_d^{\phi}(c_1,\ldots,c_k)=(-1)^{\frac{n(d-1)}{2}}\int_{\overline{\mathfrak{M}}_{k+1}^{\mathbb{R}}(d)} f_{0123}^*\Omega_{0,4}\,\mathrm{ev}_0^*H^{c_1-1}\,\mathrm{ev}_1^*H\,\mathrm{ev}_2^*H^{c_2}\,\ldots\,\mathrm{ev}_k^*H^{c_k}\,. \tag{2.1}$$

Choose a generic collection of linear subspaces $H_0,\ldots,H_k\subset\mathbb{P}^{2n-1}$ of complex codimensions $c_1-1,1,c_2,\ldots,c_k$, respectively. For any $\lambda\in\mathcal{M}_{0,4}$, let

$$Z_\lambda=\big\{u\in f_{0123}^{-1}(\lambda)\colon\ \mathrm{ev}_i(u)\in H_i\ \forall\, i=0,1,\ldots,k\big\}\subset\overline{\mathfrak{M}}_{k+1}^{\mathbb{R}}(d)\,.$$

This set is a compact oriented 0-dimensional submanifold of $\overline{\mathfrak{M}}_{k+1}^{\mathbb{R}}(d)$, i.e. a finite set of signed points, if λ is generic. The number (2.1) is the signed cardinality ${}^{\pm}|Z_\lambda|$ of this set.

We prove Corollary 1.2 by explicitly describing the elements of $Z_{[1,1]}$ and $Z_{[1,0]}$, with notation as in Figure 1, and determining their contribution to the number (2.1). The domain Σ_u of each element u of $Z_{[1,1]}$ and $Z_{[1,0]}$ consists of at least two irreducible components. Since the fixed point locus of the involution η_{2n} on $\mathbb{P}^{2n-1}$ is empty,

$$\overline{\mathfrak{M}}_{k+1}^{\mathbb{R}}(d)=\overline{\mathfrak{M}}_{0,k+1}(\mathbb{P}^{2n-1},d)^{\eta_{2n},\eta}$$

and Σ_u has an odd number of irreducible components; the involution η_u associated with u restricts to η on one of the components and interchanges the others in pairs. For dimensional reasons, the number of irreducible components of Σ_u cannot be greater than 3 and thus must be precisely 3. Each map u with its marked points is completely determined by its restriction $u^{\mathbb{R}}$ to the component $\Sigma_u^{\mathbb{R}}$ of Σ_u preserved by η_u and its restriction $u^{\mathbb{C}}$ to either of the other components.

We depict all possibilities for the elements of $Z_{[1,1]}$ and $Z_{[1,0]}$ in Figures 2 and 3, respectively. In each of the diagrams, the vertical line represents the irreducible component $\Sigma_u^{\mathbb{R}}$ of Σ_u preserved by η_u, while the two horizontal lines represent the components of Σ_u interchanged by η_u; the integers next to the lines specify the degrees of u on the corresponding components. The larger dots on the three lines indicate the locations of the marked points z_0^+,z_1^+,z_2^+,z_3^+; we label them by

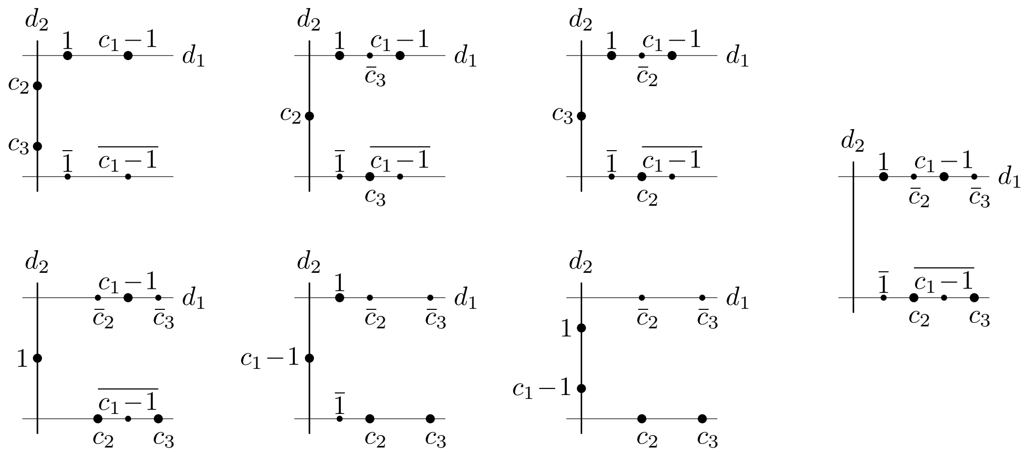

Figure 2: Domains of elements of $Z_{[1,1]}$

the codimensions of the constraints they map to, i.e. $c_1-1, 1, c_2, c_3$, in order to make the connection with the expression in Corollary 1.2 more apparent. If a marked point z_i^+ lies on the bottom component, its conjugate z_i^- lies on the top component. In such a case, we indicate the conjugate point by a small dot on the upper component and label it with $\bar{c}_i$; the restriction of u to the upper component maps this point to the linear subspace

$$\overline{H_i} \equiv \eta_{2n}(H_i) \subset \mathbb{P}^{2n-1}.$$

By the definition of $Z_{[1,1]}$, each diagram in Figure 2 contains a node separating the marked points z_0^+, z_1^+ (i.e. the larger dots labeled by $c_1-1, 1$) from the marked points z_2^+, z_3^+ (i.e. the larger dots labeled by c_2, c_3). Similarly, each diagram in Figure 3 contains a node separating the marked points z_0^+, z_2^+ from the marked points z_1^+, z_3^+. We arrange the diagrams in both cases so that the pair of marked points containing 0 lies above the other pair. The remaining marked points, $z_4^\pm, \ldots, z_k^\pm$, are distributed between the three components in some way.

Each element u of $Z_{[1,1]}$ and $Z_{[1,0]}$ corresponds, via the restriction to $\Sigma_u^{\mathbb{R}}$ and the upper component, to a pair $(u^{\mathbb{C}}, u^{\mathbb{R}})$, with

$$[u^{\mathbb{C}}] \in \mathfrak{M}_{k_1+1}(d_1), \quad [u^{\mathbb{R}}] \in \mathfrak{M}^{\mathbb{R}}_{k_2+1}(d_2), \quad 2d_1+d_2=d, \quad k_1+k_2=k+1,$$

such that $u^{\mathbb{R}}$ and $u^{\mathbb{C}}$ meet at the pair of extra marked points and pass through $H_0, \ldots, H_k$ or their conjugates as required by the distribution of the marked points. Each such pair $u=(u^{\mathbb{C}}, u^{\mathbb{R}})$ is an isolated element of

$$\mathfrak{M}_{0,k_1+1}(\mathbb{P}^{2n-1}, d_1) \times \mathfrak{M}^{\mathbb{R}}_{k_2+1}(n, d_2)$$

and has a well-defined contribution $\varepsilon(u)$ to the number (2.1), i.e. the signed number of nearby elements of Z_λ, with $\lambda \in \mathcal{M}_{0,4}$. By the next proposition,

$$\varepsilon(u) = (-1)^{n(d_2-1)/2}$$

for all elements u represented by the three diagrams in the first rows of Figures 2 and 3,

$$\varepsilon(u) = -(-1)^{n(d_2-1)/2}$$

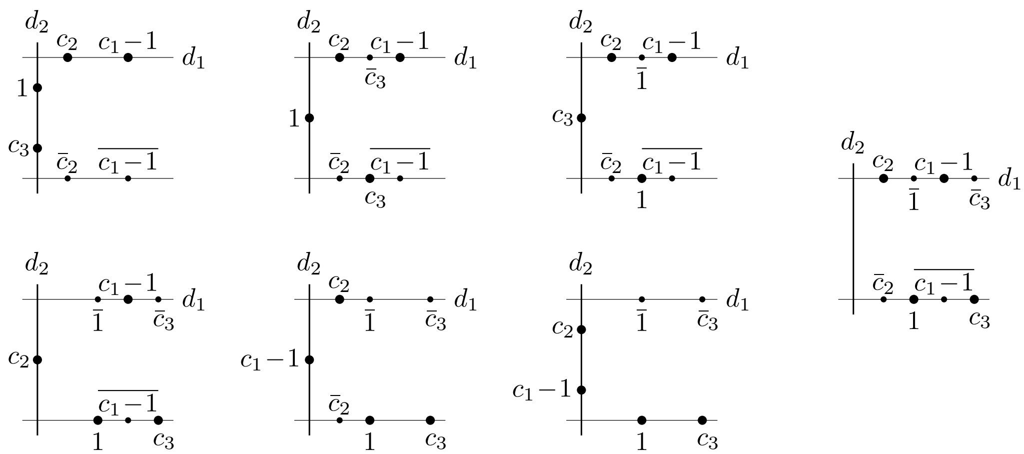

Figure 3: Domains of elements of $Z_{[1,0]}$

for the three diagrams in the second rows in these figures, and $\varepsilon(u)=0$ for the remaining, right-most diagram in each of the figures. Even if there were a contribution from the right-most diagram, it would have been the same for $Z_{[1,1]}$ and $Z_{[1,0]}$ and so would have had no effect on the recursion of Theorem 1.1.

Proposition 2.1. *Suppose* $u\in Z_{[1,1]}$.

(1) If $\Sigma_u^{\mathbb{R}}$ *contains either of the marked points* z_2^+, z_3^+, *then* $\varepsilon(u)=(-1)^{n(d_2-1)/2}$.

(2) If $\Sigma_u^{\mathbb{R}}$ *contains either of the marked points* z_0^+, z_1^+, *then* $\varepsilon(u)=-(-1)^{n(d_2-1)/2}$.

(3) If $\Sigma_u^{\mathbb{R}}$ *contains neither of the marked points* $z_0^+, z_1^+, z_2^+, z_3^+$, *then* $\varepsilon(u)=0$.

The same statements with 1 and 2 interchanged hold for $u\in Z_{[1,0]}$.

Proof. We apply Corollary 3.4 with k replaced by $k+1$, $\{1,2,3,4\}$ by $\{0,1,2,3\}$, and with linear subspaces of codimensions $c_1-1, 1, c_2, \ldots, c_k$. We take $J\subset\{0,1,\ldots,k\}$ to be the subset indexing the pairs of marked points of u that lie on the central component $\Sigma^{\mathbb{R}}$, I^+ to be the subset indexing the pairs with the first marked point on the upper component, i.e. the domain of $u^{\mathbb{C}}$, and I^- to be the complement of $I^+\sqcup J$ in $\{1,\ldots,k\}$. Since $c_i\notin 2\mathbb{Z}$ and $0\notin I^-$, the set on the left-hand side of (3.6) is empty. Since $\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})$ is 0-dimensional in this case, Corollary 3.4 compares the sign of the elements of $\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})$ with the sign of the nearby elements of Z_λ.

Since the first case above corresponds to the first case on the right-hand side of (3.6), the two signs differ by $(-1)^{nd_1}$. Taking into account the extra sign in (2.1), we obtain the first claim of the proposition. Since the second case above corresponds to the second case on the right-hand side of (3.6), we similarly obtain the second claim. The final claim of the proposition follows from the last statement of Corollary 3.4. □

Proof of Corollary 1.2. We determine the number of elements represented by each diagram in Figures 2 and 3. Splitting the set $\{4,\ldots,k\}$ into subsets I and J in all possible ways, we put the pairs of marked points indexed by J on the central component $\Sigma_u^{\mathbb{R}}$, one point of each pair indexed by I on the top component, and thus the other point in the pair on the bottom component. This gives $2^{|I|}$ choices of the distribution and requires $u^{\mathbb{C}}$ to pass through either H_i, with $i\in I$, or the conjugate complex hyperplane $\overline{H}_i$. By Proposition 2.1, the contribution $\varepsilon(u)$ is independent of this choice. Thus, we can simply multiply the number for one of these distributions by $2^{|I|}$. With the constraints completely distributed, we replace the node condition by the usual splitting of the diagonal, i.e. an extra constraint of H^i for $u^{\mathbb{C}}$ and of H^j for $u^{\mathbb{R}}$ with all possible i and j so that $i+j=2n-1$. Thus, the contribution to the number (2.1) from each diagram in Figures 2 and 3, each partition $\{4,\ldots,k\}=I\sqcup J$, and each partition $2n-1=i+j$ is

$$(-1)^{n(d_2-1)/2}\varepsilon\big\langle c_{\hat{I}},i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}N_{d_2}^{\eta_{2n}}\big(c_{\hat{J}},j\big)=\varepsilon\big\langle c_{\hat{I}},i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_{\hat{J}},j\big\rangle_{d_2}^{\eta_{2n}}, \tag{2.2}$$

where

- $\varepsilon=1$ for the three diagrams in the first rows of the figures, $\varepsilon=-1$ for the second rows, and $\varepsilon=0$ for the right-most diagrams;
- $\hat{I}$ is the union of I and the subset of $\{0,1,2,3\}$ indexing the pairs of marked points on the top and bottom components (e.g. $\{0,1,3\}$ for the second diagram in the first row of Figure 2);
- $\hat{J}$ is the union of J and the subset of $\{0,1,2,3\}$ indexing the pairs of marked points on the vertical component (e.g. $\{2\}$ for the second diagram in the first row of Figure 2);
- $c_0=c_1-1$, with c_1 as in (2.1), and $c_1=1$ for the purposes of the definitions of $c_{\hat{I}}$ and $c_{\hat{J}}$ in (2.2).

Since $c_1-1\in 2\mathbb{Z}$, the last factor in (2.2) vanishes in the case of the last two diagrams in the second rows of both figures; see [7, Theorem 2.2], which is an immediate consequence of Lemma 3.3 in this case. Furthermore, if $d_1=0$ and the complex invariant in (2.2) is nonzero, then $|\hat{I}|=2$ for dimensional reasons; thus, the only $d_1=0$ contributions arise from the first diagrams in Figures 2 and 3 with $I=\emptyset$.

By the previous paragraph, only the three diagrams in the first row and the first diagram in the second row of each figure contribute to the number (2.1). The contribution from Figure 2, which corresponds to $\lambda=[1,1]$ in $\overline{\mathcal{M}}_{0,4}$, thus equals

$$\begin{aligned}\big\langle c_1,\ldots,c_k\big\rangle_d^{\eta_{2n}}+\sum_{\substack{2d_1+d_2=d\\ d_1,d_2\ge1}}\sum_{I\sqcup J=\{4,\ldots,k\}}\sum_{\substack{2i+j=2n-1\\ i,j\ge1}}2^{|I|}\Big(&d_1\big\langle c_1-1,c_I,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_2,c_3,c_J,j\big\rangle_{d_2}^{\eta_{2n}}\\ &+d_1\big\langle c_1-1,c_3,c_I,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_2,c_J,j\big\rangle_{d_2}^{\eta_{2n}}+d_1\big\langle c_1-1,c_2,c_I,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_3,c_J,j\big\rangle_{d_2}^{\eta_{2n}}\\ &-d_2\big\langle c_1-1,c_2,c_3,c_I,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_J,j\big\rangle_{d_2}^{\eta_{2n}}\Big).\end{aligned}$$

The contribution from Figure 3, which corresponds to $\lambda=[1,0]$ in $\overline{\mathcal{M}}_{0,4}$, similarly equals

$$\begin{aligned}d\big\langle c_1+c_2-1,c_3,\ldots,c_k\big\rangle_d^{\eta_{2n}}+\sum_{\substack{2d_1+d_2=d\\ d_1,d_2\ge1}}\sum_{I\sqcup J=\{4,\ldots,k\}}\sum_{\substack{2i+j=2n-1\\ i,j\ge1}}2^{|I|}\Big(&\\ d_2\big\langle c_1-1,c_2,c_I,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_3,c_J,j\big\rangle_{d_2}^{\eta_{2n}}&+d_2\big\langle c_1-1,c_2,c_3,c_I,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_J,j\big\rangle_{d_2}^{\eta_{2n}}\\ +d_1\big\langle c_1-1,c_2,c_I,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_3,c_J,j\big\rangle_{d_2}^{\eta_{2n}}&-d_1\big\langle c_1-1,c_3,c_I,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_2,c_J,j\big\rangle_{d_2}^{\eta_{2n}}\Big).\end{aligned}$$

Setting the two expressions equal and solving for $\langle c_1,\ldots,c_k\rangle_d^{\eta_{2n}}$, we obtain

$$\begin{split}\langle c_1,\ldots,c_k\rangle_d^{\eta_{2n}} = d\langle c_1{+}c_2{-}1,c_3,\ldots,c_k\rangle_d^{\eta_{2n}} + \sum_{\substack{2d_1+d_2=d\\ d_1,d_2\ge1}}\sum_{I\sqcup J=\{4,\ldots,k\}}\sum_{\substack{2i+j=2n-1\\ i,j\ge1}} 2^{|I|}\bigg(\\ 2d_2\langle c_1{-}1,c_2,c_3,c_I,2i\rangle_{d_1}^{\mathbb{P}^{2n-1}}\langle c_J,j\rangle_{d_2}^{\eta_{2n}} - d_1\langle c_1{-}1,c_I,2i\rangle_{d_1}^{\mathbb{P}^{2n-1}}\langle c_2,c_3,c_J,j\rangle_{d_2}^{\eta_{2n}}\\ + d_2\langle c_1{-}1,c_2,c_I,2i\rangle_{d_1}^{\mathbb{P}^{2n-1}}\langle c_3,c_J,j\rangle_{d_2}^{\eta_{2n}} - 2d_1\langle c_1{-}1,c_3,c_I,2i\rangle_{d_1}^{\mathbb{P}^{2n-1}}\langle c_2,c_J,j\rangle_{d_2}^{\eta_{2n}}\bigg);\end{split}$$

this formula simplifies to the statement of Corollary 1.2. □

Remark 2.2. The above extends directly to real symplectic manifolds (X,ω,ϕ) such that the fixed locus X^ϕ of ϕ is empty. If $X^\phi\neq\emptyset$, the spaces $Z_{[1,1]}$ and $Z_{[1,0]}$ defined in this section could also contain two-component maps (u_1,u_2) of two types:

(1) the involution on the domain interchanges the two components of the domain and fixes the node (this corresponds to sphere bubbling in open GW-theory);

(2) the involution on the domain restricts to τ on each component of the domain and fixes the node (this corresponds to disk bubbling in open GW-theory).

As the above degenerations are of real codimension one, their intersections with $Z_{[1,1]}$ and $Z_{[1,0]}$ are one-dimensional. Bubble maps of the first type appear in the second proof of [7, Theorem 2.1] and do not contribute to the number in [7, (6.2)] by [7, Lemma 6.2]. By the same reasoning, these bubble maps would not contribute to the analogue of (2.1) for general real symplectic manifolds as in [7, Theorem 2.1]. However, the proof of [7, Lemma 6.2] does not apply to the spaces of two-component bubble maps of the second type above, because they can further degenerate into three real bubbles and the function $u\longrightarrow z_1(u)$ in the proof of [7, Lemma 6.2] vanishes along some of these degenerations; the problem degenerations correspond to the intersections of the closures of different strata of two-component maps. The conclusion of [7, Lemma 6.2] can fail for the closures of the individual strata of two-component maps, though it may perhaps hold for the connected components of the union of such closures under the assumptions of [7, Theorem 2.1]. For real symplectic manifolds of dimension 6 (and thus with fixed locus of dimension 3), this issue may be related to some linking phenomena to which an allusion is made in [15, Remark 4]; these phenomena do not effect the WDVV relation of [7, Theorem 2.1] though.

3 Sign computations

For $d\in\mathbb{Z}^+$, denote by $\mathcal{N}_d\subset\overline{\mathfrak{M}}_0^{\mathbb{R}}(d)$ the sub-orbifold of maps from domains consisting of precisely three components and let

$$\widetilde{\mathcal{N}}_d=\bigsqcup_{\substack{2d_1+d_2=d\\ d_1\ge0,d_2>0}}\widetilde{\mathcal{N}}_{d_1,d_2}\,,\qquad\text{where}$$
$$\widetilde{\mathcal{N}}_{d_1,d_2}=\big\{(u^{\mathbb{C}},u^{\mathbb{R}})\in\mathfrak{M}_1^{\mathbb{C}}(d_1)\times\mathfrak{M}_1^{\mathbb{R}}(d_2)\colon \mathrm{ev}_0(u^{\mathbb{C}})=\mathrm{ev}_0(u^{\mathbb{R}})\big\},$$

with the marked points indexed by 0. Identifying the marked point $z_0^{\mathbb{C}}$ of the domain of $u^{\mathbb{C}}$ with the first marked point $z_0^{\mathbb{R}}$ of the domain of $u^{\mathbb{R}}$ and the marked point $\eta(z_0^{\mathbb{C}})$ of the map $\eta_{2n}\circ u^{\mathbb{C}}\circ\eta$ with $\eta(z_0^{\mathbb{R}})$, we obtain a double covering

$$q\colon \widetilde{\mathcal{N}}_d \longrightarrow \mathcal{N}_d\,.$$

The canonical orientations of $\mathbb{P}^{2n-1}$ and $\mathfrak{M}_1^{\mathbb{C}}(d_1)$ and the chosen orientation of $\mathfrak{M}_1^{\mathbb{R}}(d_2)$ induce an orientation on $\widetilde{\mathcal{N}}_d$. The actions of the deck transformation on the moduli spaces $\mathfrak{M}_1^{\mathbb{R}}(d_2)$ and $\mathfrak{M}_1^{\mathbb{C}}(d_1)$ and the condition $\mathrm{ev}_0(u^{\mathbb{C}})=\mathrm{ev}_0(u^{\mathbb{R}})$ are all orientation-reversing because

- the first action is the conjugation of the marked point;
- the second action corresponds to the complex conjugation on $\mathfrak{M}_0^{\mathbb{C}}(d_1)$, which is of even complex dimension, and to the conjugation of the marked point;
- the third action corresponds to the complex conjugation on $\mathbb{P}^{2n-1}$.

In particular, $\mathcal{N}_d$ is not orientable ($\widetilde{\mathcal{N}}_d$ is connected by [3, Appendix A.1]).

Let $L^{\mathbb{C}}\longrightarrow\mathfrak{M}_1^{\mathbb{C}}(d_1)$ and $L^{\mathbb{R}}\longrightarrow\mathfrak{M}_1^{\mathbb{R}}(d)$ be the tautological line bundles and

$$\tilde{L}=\pi_1^*L^{\mathbb{C}}\otimes_{\mathbb{C}}\pi_2^*L^{\mathbb{R}}\longrightarrow\widetilde{\mathcal{N}}_d\,,$$

where π_1,π_2 are the component projection maps. The action of the natural lift of the deck transformation on $\tilde{L}$ is $\mathbb{C}$-antilinear on each fiber and induces a vector bundle L over $\mathcal{N}_d$. This is the normal line bundle of $\mathcal{N}_d$ in $\overline{\mathfrak{M}}_0^{\mathbb{R}}(d)$: as described in Section 4, there is a gluing map

$$\Phi\colon U\longrightarrow\overline{\mathfrak{M}}_0^{\mathbb{R}}(d), \tag{3.1}$$

where $U\subset L$ is a neighborhood of the zero set in L. The orientation on the total space of $\tilde{L}$ descends to an orientation on the quotient vector bundle of L.

Lemma 3.1. *The restriction of the gluing map (3.1) to $\mathcal{N}_{d_1,d_2}\equiv q(\widetilde{\mathcal{N}}_{d_1,d_2})$ is orientation-preserving with respect to the orientation on the total space of L described above if and only if $nd_1\in2\mathbb{Z}$.*

This lemma is proved in Section 4. In Corollary 3.4, Lemma 3.1 is applied to marked moduli spaces over topological components $\mathcal{N}_{d_1,d_2;I^+,J,I^-}$ of $\mathcal{N}_d$ on which the two conjugate bubbles can be systematically distinguished. These topological components are thus oriented by the choice of which conjugate bubble is distinguished. In the case of the diagrams in Figures 2 and 3, we take the bubble corresponding to the upper line segment to be the distinguished one.

Remark 3.2. Moduli spaces in GW-theory are normally oriented by orienting the index of the linearized $\bar{\partial}$-operator via a pinching off construction; see the proof of [7, Lemma 5.2]. In complex GW-theory, the standard orientations of the index bundles are essentially complex and gluing maps like (3.1) are automatically orientation-preserving. In similarity with the situation in complex GW-theory, analogous gluing maps are assumed to be orientation-preserving in [12, (11)] without any comment; this was also assumed to be the case in the early versions of [3]. In [3, Section 5.2], the moduli spaces $\overline{\mathfrak{M}}_0^{\mathbb{R}}(d)$ are oriented directly; this is the orientation used in this paper as described in Section 4. If $n\in2\mathbb{Z}$, the resulting orientation agrees with the orientation induced by a real square root of $\Lambda_{\mathbb{C}}^{\mathrm{top}}(T\mathbb{P}^{2n-1},\mathrm{d}\eta_{2n})$ via the pinching off construction of [3, Section 2.1]; see the paragraph above Remark 6.9 in [3]. Thus, in this case, the gluing map (3.1) is orientation-preserving, as Lemma 3.1 states. If $n\notin 2\mathbb{Z}$, $\Lambda_{\mathbb{C}}^{\mathrm{top}}(T\mathbb{P}^{2n-1},\mathrm{d}\eta_{2n})$ does not admit a real square root. In this case, Lemma 3.1 implies that the chosen orientations of $\overline{\mathfrak{M}}_0^{\mathbb{R}}(d)$ differ from orientations arising via a systematic pinching off construction, as in [6, Sections 4,6], by $(-1)^{n(d-1)/2}$ for d odd. The implication of this subtle issue for purely computational purposes is that the Euler class of the normal bundle of a fixed locus in $\overline{\mathfrak{M}}_k^{\mathbb{R}}(d)$ described in [3, Section 6.2] should be multiplied by $(-1)^{n(d-d_0)/2}$, where d_0 is the degree on the central component.

For $k\in\mathbb{Z}^+$, there is a fibration

$$\pi\colon \mathfrak{M}_k^{\mathbb{R}}(d)\longrightarrow \mathfrak{M}_0^{\mathbb{R}}(d)$$

obtained by forgetting the k pairs of conjugate marked points. The fiber of π over any point $[u]$ of the base is isomorphic to an open subspace of $(\mathbb{P}^1)^k$ by the map

$$\begin{aligned}\iota_{k,u}\colon \big\{(z_1,\ldots,z_k)\in(\mathbb{P}^1)^k\colon\ z_i\neq z_j,\eta(z_j)\ \forall\, i\neq j\big\}&\longrightarrow \mathfrak{M}_k^{\mathbb{R}}(d),\\ (z_1,\ldots,z_k)&\longrightarrow \big[(z_1,\eta(z_1)),\ldots,(z_k,\eta(z_k)),u\big].\end{aligned}$$

For each subset $I\subset\{1,\ldots,k\}$, let $\iota_{k,u;I}$ denote the modification of $\iota_{k,u}$ taking the i-th component z_i of $(z_1,\ldots,z_k)$ to the second element in the i-th conjugate pair whenever $i\in I$; thus, $\iota_{k,u;\emptyset}=\iota_{k,u}$. The canonical orientations of $\mathfrak{M}_0^{\mathbb{R}}(d)$ and of $\mathbb{P}^1$ induce via $\iota_{k,u;I}$ an orientation on $\mathfrak{M}_k^{\mathbb{R}}(d)$ and thus on $\overline{\mathfrak{M}}_k^{\mathbb{R}}(d)$, which we will call the I-orientation. The orientation on this space, which is used to define the numbers (1.4), is the $\emptyset$-orientation. Since η is an orientation-reversing involution on $\mathbb{P}^1$, the I-orientation agrees with the canonical orientation if and only if $|I|$ is even.

For each $i=1,\ldots,k$, let

$$\overline{\mathrm{ev}}_i\equiv \eta_{2n}\circ\mathrm{ev}_i\colon \overline{\mathfrak{M}}_k^{\mathbb{R}}(d)\longrightarrow \mathbb{P}^{2n-1}$$

be the evaluation map at the second point in the i-th conjugate pair. Denote by

$$\mathrm{ev}\equiv\mathrm{ev}_1\times\ldots\times\mathrm{ev}_k\colon \overline{\mathfrak{M}}_k^{\mathbb{R}}(d)\longrightarrow (\mathbb{P}^{2n-1})^k \tag{3.2}$$

the total evaluation map at the first point in each conjugation pair. For each $I\subset\{1,\ldots,k\}$, let

$$\mathrm{ev}_I\colon \overline{\mathfrak{M}}_k^{\mathbb{R}}(d)\longrightarrow (\mathbb{P}^{2n-1})^k$$

be the modification of ev obtained by replacing ev_i with $\overline{\mathrm{ev}}_i$ whenever $i\in I$.

For any subspace $H\subset\mathbb{P}^{2n-1}$, let $\overline{H}=\eta_{2n}(H)$ as before. If $\mathbf{H}=(H_1,\ldots,H_k)$ is a tuple of subspaces of $\mathbb{P}^{2n-1}$, let

$$\langle\mathbf{H}\rangle = H_1\times\ldots\times H_k\subset(\mathbb{P}^{2n-1})^k\,.$$

For each $I\subset\{1,\ldots,k\}$, denote by $\langle\mathbf{H}\rangle_I\subset(\mathbb{P}^{2n-1})^k$ the modification of $\langle\mathbf{H}\rangle$ obtained by replacing the i-th component H_i with $\overline{H_i}$ whenever $i\in I$. We define an involution on $(\mathbb{P}^{2n-1})^k$ by

$$\begin{gathered}\Theta_I\colon(\mathbb{P}^{2n-1})^k\longrightarrow(\mathbb{P}^{2n-1})^k,\qquad \big(x_1,\ldots,x_k\big)\longrightarrow\big(\Theta_{I;1}(x_1),\ldots,\Theta_{I;k}(x_k)\big),\\ \text{where}\qquad \Theta_{I;i}(x)=\begin{cases}x, & \text{if } i\notin I;\\ \eta_{2n}(x), & \text{if } i\in I.\end{cases}\end{gathered}$$

Let

$$\overline{\mathfrak{M}}_d^{\mathbb{R}}(\mathbf{H})_I = \big\{u\in\overline{\mathfrak{M}}_k^{\mathbb{R}}(d)_I:\ \mathrm{ev}_I(u)\in\langle\mathbf{H}\rangle_I\big\}.$$

This subspace does not depend on the choice of I, but its orientation imposed below does in general.

Suppose $\mathbf{H}=(H_1,\ldots,H_k)$ is a tuple of complex linear subspaces of $\mathbb{P}^{2n-1}$ that are in general position, i.e. so that the restriction of the total evaluation map (3.2) to every stratum of the moduli space (consisting of maps from domains of a fixed topological type) is transverse to $\langle\mathbf{H}\rangle$ in $(\mathbb{P}^{n-1})^k$. If $I\subset\{1,\ldots,k\}$, $\overline{\mathfrak{M}}_d^{\mathbb{R}}(\mathbf{H})_I$ is then a smooth manifold. The complex orientations on $\langle\mathbf{H}\rangle_I$ and $(\mathbb{P}^{2n-1})^k$, the I-orientation on $\overline{\mathfrak{M}}_k^{\mathbb{R}}(d)$, and the map ev_I induce an orientation on $\overline{\mathfrak{M}}_d^{\mathbb{R}}(\mathbf{H})_I$.

Lemma 3.3. *Let $d,k,n\in\mathbb{Z}^+$, $\mathbf{H}=(H_1,\ldots,H_k)$ be a general tuple of complex linear subspaces of $\mathbb{P}^{2n-1}$ of complex codimensions $c_1,\ldots,c_k$, respectively, and $I\subset\{1,\ldots,k\}$. The orientations of $\overline{\mathfrak{M}}_d^{\mathbb{R}}(\mathbf{H})\equiv\overline{\mathfrak{M}}_d^{\mathbb{R}}(\mathbf{H})_\emptyset$ and $\overline{\mathfrak{M}}_d^{\mathbb{R}}(\mathbf{H})_I$ are the same if and only if the set $\{i\in I:\ c_i\in2\mathbb{Z}\}$ is of even cardinality.*

Proof. By the transversality assumption,

$$\mathrm{dev}_I\colon \frac{T(\mathfrak{M}_k^{\mathbb{R}}(d)_I)|_{\mathfrak{M}_d^{\mathbb{R}}(\mathbf{H})_I}}{T(\mathfrak{M}_d^{\mathbb{R}}(\mathbf{H})_I)}\longrightarrow \mathrm{ev}_I^*\frac{T((\mathbb{P}^{2n-1})^k)|_{\langle\mathbf{H}\rangle_I}}{T(\langle\mathbf{H}\rangle_I)} \tag{3.3}$$

is an isomorphism of vector bundles. The orientation on the right-hand side of (3.3) induced by the complex orientations of $(\mathbb{P}^{2n-1})^k$ and $\langle\mathbf{H}\rangle_I$ induce an orientation on the left-hand side of (3.3). Along with the I-orientation on $\mathfrak{M}_k^{\mathbb{R}}(d)$, the latter induces an orientation on $\mathfrak{M}_d^{\mathbb{R}}(\mathbf{H})_I$. By the Chain Rule, $\mathrm{dev}_I=\mathrm{d}\Theta_I\circ\mathrm{dev}$. The sign of the isomorphism

$$\mathrm{d}\Theta_I\colon \frac{(T(\mathbb{P}^{2n-1})^k)|_{\langle\mathbf{H}\rangle}}{T(\langle\mathbf{H}\rangle)}\longrightarrow \Theta_I^*\frac{T((\mathbb{P}^{2n-1})^k)|_{\langle\mathbf{H}\rangle_I}}{T(\langle\mathbf{H}\rangle_I)}$$

is (-1) to the cardinality of the set $\{i\in I:\ c_i\notin2\mathbb{Z}\}$. The I-orientation on $\overline{\mathfrak{M}}_k^{\mathbb{R}}(d)$ differs from the canonical one by $(-1)^{|I|}$. Combining the two signs, we obtain the claim. $\square$

If $d=2d_1+d_2$ and $\{1,\ldots,k\}=I^+\sqcup J\sqcup I^-$, let

$$\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})\subset\mathcal{N}_{d_1,d_2}\cap\overline{\mathfrak{M}}_d^{\mathbb{R}}(\mathbf{H})$$

be the subset consisting of maps from marked three-component domains so that the central component carries the marked points in the pairs indexed by J, one of the other components carries the first points in the pairs indexed by I^+, and the third component carries the first points in the pairs indexed by I^-. With notation as at the beginning of this section, we will associate I^+ with the space of bubble components $u^{\mathbb{C}}$ used to orient $\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})$; these bubble components now carry marked points indexed by $I^+\sqcup I^-$, in addition to the marked point corresponding to the node. As in complex GW-theory, a small modification of the gluing map (3.1) gives rise to a gluing map

$$\Phi_{\mathbf{H}}\colon U_{\mathbf{H}}\longrightarrow\overline{\mathfrak{M}}_d^{\mathbb{R}}(\mathbf{H}),$$

where $U_{\mathbf{H}}\subset L$ is a neighborhood of the zero section in $L\longrightarrow\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})$. If $k\ge4$, let

$$f_{1234}\colon\overline{\mathfrak{M}}_k^{\mathbb{R}}(d)\longrightarrow\overline{\mathcal{M}}_{0,4}$$

be the projection onto the first marked points in the first four conjugate pairs.

Corollary 3.4. *Let $d,k,n\in\mathbb{Z}^+$ be such that $k\ge 4$ and $\mathbf{H}=(H_1,\ldots,H_k)$ be a general tuple of complex linear subspaces of $\mathbb{P}^{2n-1}$ of complex codimensions $c_1,\ldots,c_k$, respectively. Suppose $d_1\in\mathbb{Z}^{\ge 0}$ and $d_2\in\mathbb{Z}^+$ are such that $d=2d_1+d_2$ and $I^+,I^-,J\subset\{1,\ldots,k\}$ form a partition of $\{1,\ldots,k\}$ such that*

$$\big|I^+\cap\{1,2,3,4\}\big|=2 \qquad or \qquad \big|I^-\cap\{1,2,3,4\}\big|=2. \tag{3.4}$$

Let $\mathcal{N}_{d_1,d_2;I^+,J,I^-}$ be oriented as in the paragraph after Lemma 3.1.

(1) If $J\cap\{1,2,3,4\}\neq\emptyset$, the sequence

$$0\longrightarrow T\big(\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})\big)\longrightarrow T\big(\overline{\mathfrak{M}}_d^{\mathbb{R}}(\mathbf{H})\big)\big|_{\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})}\stackrel{\mathrm{d}f_{1234}}{\longrightarrow} f_{1234}^*T\overline{\mathcal{M}}_{0,4}\longrightarrow 0 \tag{3.5}$$

of vector bundles over $\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})$ is exact; it is compatible with the canonical orientations if and only if

$$\big|\{i\in I^-:\, c_i\in 2\mathbb{Z}\}\big|+nd_1\ \begin{cases}\in 2\mathbb{Z}, & \textit{if } |I^+\cap\{1,2,3,4\}|=2;\\ \notin 2\mathbb{Z}, & \textit{if } |I^-\cap\{1,2,3,4\}|=2.\end{cases} \tag{3.6}$$

(2) If $J\cap\{1,2,3,4\}=\emptyset$, the image of a fiber of $U\longrightarrow\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})$ under $f_{1234}\circ\Phi_{\mathbf{H}}$ is of real dimension 1.

Remark 3.5. The requirement (3.4) insures that f_{1234} is constant along $\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})$ and so the composition of the two arrows in (3.5) is trivial. The conclusion of Corollary 3.4 is compatible with changing the distinguisged conjugate component in the paragraph after Lemma 3.1 (which interchanges I^+ and I^- and thus the two cases on the right-hand side of (3.6)) for the following reason. Let

$$\overline{\mathfrak{M}}_{d_1}^{\mathbb{C}}(\mathbf{H};I^+,I^-)=\big\{u\in\overline{\mathfrak{M}}_{\{0\}\sqcup I^+\sqcup I^-}^{\mathbb{C}}(d_1):\ \mathrm{ev}_i(u)\in H_i\ \forall\, i\in I^+,\ \mathrm{ev}_i(u)\in\overline{H_i}\ \forall\, i\in I^-\big\}.$$

The space $\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})$ is oriented as the preimage of the cycle

$$\mathrm{ev}_0\colon\overline{\mathfrak{M}}_{d_1}^{\mathbb{C}}(\mathbf{H};I^+,I^-)\longrightarrow\mathbb{P}^{2n-1}$$

by the evaluation map at the marked point of $\mathfrak{M}_{\{0\}\sqcup J}^{\mathbb{R}}(d_2)$ corresponding to the chosen node. Interchanging I^+ and I^- replaces this cycle and the evaluation map with their conjugates, as before Lemma 3.3. If the cardinalities of the sets $\{i\in I^{\pm}:\ c_i\in 2\mathbb{Z}\}$ are of the same parity, the complex dimension of $\overline{\mathfrak{M}}_{d_1}^{\mathbb{C}}(\mathbf{H};I^+,I^-)$ is odd and so the codimension of the cycle ev_0 above is even. By the same argument as in the proof of Lemma 3.3, the orientation of $\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})$ thus changes, as expected from the change in the validity of (3.6) in this case. If the cardinalities of the sets $\{i\in I^{\pm}:\ c_i\in 2\mathbb{Z}\}$ are of different parities, the codimension of the cycle ev_0 above is odd. Interchanging I^+ and I^- then does not change the orientation of $\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})$, as expected from the validity of (3.6) not changing in this case.

Proof of Corollary 3.4. (1) By Lemma 3.1, the gluing map

$$\Phi_{\mathbf{H}}\colon U_{\mathbf{H}}\longrightarrow\overline{\mathfrak{M}}_d^{\mathbb{R}}(\mathbf{H})_{I^-}$$

is orientation-preserving if and only if nd_1 is even. The differential

$$\mathrm{d}(f_{1234}\circ\Phi_{\mathbf{H}})\colon L\longrightarrow\big\{f_{1234}\circ\Phi_{\mathbf{H}}\big\}^*T\mathcal{M}_{0,4} \tag{3.7}$$

is the composition of the differential for smoothing the nodes in $\overline{\mathfrak{M}}_k^{\mathbb{C}}(d)$,

$$\mathrm{d}(f_{1234}\circ\Phi^{\mathbb{C}})\colon L\oplus L' \longrightarrow \big\{f_{1234}\circ\Phi^{\mathbb{C}}\big\}^*T\mathcal{M}_{0,4}\,,$$

where L' is the analogue of L for the second node, with the embedding

$$L\longrightarrow L\oplus L', \qquad v\longrightarrow \big(v,\mathrm{d}\eta_u(v)\big);$$

see the last part of Section 4. The restriction of the latter differential to the component, L or L', corresponding to the node separating off two of the marked points $\{1,2,3,4\}$ is a $\mathbb{C}$-linear isomorphism, while the restriction to the other component is trivial. If $|I^+\cap\{1,2,3,4\}|=2$, the former component is L and (3.7) is an orientation-preserving map. If $|I^-\cap\{1,2,3,4\}|=2$, the former component is L' and (3.7) is an orientation-reversing map. Combining these two observations, we find that the sequence

$$0\longrightarrow T\big(\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})\big)\longrightarrow T\big(\overline{\mathfrak{M}}_d^{\mathbb{R}}(\mathbf{H})_{I^-}\big)\big|_{\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})}\stackrel{\mathrm{d}f_{1234}}{\longrightarrow} f_{1234}^*T\overline{\mathcal{M}}_{0,4}\longrightarrow 0$$

of vector bundles over $\mathcal{N}_{d_1,d_2;I^+,J,I^-}(\mathbf{H})$ is exact; it is compatible with the orientations if and only if

$$nd_1\begin{cases}\in 2\mathbb{Z}, & \text{if } |I^+\cap\{1,2,3,4\}|=2;\\ \notin 2\mathbb{Z}, & \text{if } |I^-\cap\{1,2,3,4\}|=2.\end{cases}$$

Combining this with Lemma 3.3, we obtain the first claim of Corollary 3.4.

(2) If $J\cap\{1,2,3,4\}=\emptyset$, the morphism

$$f_{1234}\colon \overline{\mathfrak{M}}_{2k}^{\mathbb{C}}(d)\longrightarrow \overline{\mathcal{M}}_{0,4}$$

is locally of the form

$$L\oplus L'\longrightarrow \overline{\mathcal{M}}_{0,4}, \qquad (v,v')\longrightarrow avv',$$

for some a dependent only on $\mathcal{N}_{d_1,d_2}$. Thus, the restriction of f to $\overline{\mathfrak{M}}_k^{\mathbb{R}}(d)$ is locally of the form

$$L\longrightarrow \overline{\mathcal{M}}_{0,4}, \qquad v\longrightarrow av\bar{v},$$

which implies the last claim of Corollary 3.4. $\square$

4 Comparison of orientations

We now verify Lemma 3.1 by explicitly describing and comparing the relevant orientations. This argument is fundamentally different from the proof of [7, Lemma 5.1].

Let Σ be the nodal surface consisting of three components:

(1) $\Sigma_0=\mathbb{P}^1$ with nodes at $[c,1]$ and $[1,-c']$ for some $c,c'\in\mathbb{C}^*$ with $cc'\neq -1$;

(2) $\Sigma^+=\mathbb{P}^1$ with the node at $[1,0]$, which is joined to Σ_0 at $[c,1]$, and

(3) $\Sigma^-=\mathbb{P}^1$ with the node at $[0,1]$, which is joined to Σ_0 at $[1,-c']$.

A holomorphic map $u\colon \Sigma \longrightarrow \mathbb{P}^{m-1}$ corresponds to three maps:

(1) $u_0\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^{m-1}$ typically given by

$$[x,y] \longrightarrow \bigg[A_1\prod_{r=1}^{d_0}(x-a_{1;r}y),\ldots,A_m\prod_{r=1}^{d_0}(x-a_{m;r}y)\bigg],$$

(2) $u^+\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^{m-1}$ typically given by

$$[x,y] \longrightarrow \bigg[B_1\prod_{r=1}^{d^+}(x-b_{1;r}y),\ldots,B_m\prod_{r=1}^{d^+}(x-b_{m;r}y)\bigg],$$

(3) $u^-\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^{m-1}$ typically given by

$$[x,y] \longrightarrow \bigg[B_1'\prod_{r=1}^{d^-}(b_{1;r}'x+y),\ldots,B_m'\prod_{r=1}^{d^-}(b_{m;r}'x+y)\bigg],$$

for some $A_i,B_i,B_i',a_{i;r},b_{i;r},b_{i;r}'\in\mathbb{C}^*$ such that

$$\bigcap_{r=1}^{m}\{a_{i;r}\colon r=1,\ldots,d_0\},\ \bigcap_{r=1}^{m}\{b_{i;r}\colon r=1,\ldots,d^+\},\ \bigcap_{r=1}^{m}\{b_{i;r}'\colon r=1,\ldots,d^-\}=\emptyset,$$

$$[B_1,\ldots,B_m]=\bigg[A_1\prod_{r=1}^{d_0}(c-a_{1;r}),\ldots,A_m\prod_{r=1}^{d_0}(c-a_{m;r})\bigg],\qquad\text{and}$$

$$[B_1',\ldots,B_m']=\bigg[A_1\prod_{r=1}^{d_0}(1+a_{1;r}c'),\ldots,A_m\prod_{r=1}^{d_0}(1+a_{m;r}c')\bigg].$$

The intersection conditions above are equivalent to the condition that the polynomials in each of the three sets describing u_0,u^+,u^- have no common factor; the other two conditions are equivalent to $u^+([1,0])=u_0([c,1])$ and $u^-([0,1])=u_0([1,-c'])$.

Deformations of maps of the form u above are described by the holomorphic maps

$$\begin{aligned}[x,y] \longrightarrow \bigg[&A_1\prod_{r=1}^{d_0}(x-a_{1;r}y)\prod_{r=1}^{d^+}\big(x-(c+b_{1;r}\upsilon)y\big)\prod_{r=1}^{d^-}\big((c'+b_{1;r}'\upsilon')x+y\big),\ldots,\\ &A_m\prod_{r=1}^{d_0}(x-a_{m;r}y)\prod_{r=1}^{d^+}\big(x-(c+b_{m;r}\upsilon)y\big)\prod_{r=1}^{d^-}\big((c'+b_{m;r}'\upsilon')x+y\big)\bigg],\end{aligned}$$

with $\upsilon,\upsilon'\in\mathbb{C}^*$ corresponding to the smoothings of the two nodes.

We next take $m=2n$. For $d\in\mathbb{Z}^+$, let

$$\Delta_{n,d}^{\eta}=\bigg\{\big([a_{1;1},\ldots,a_{1;d}],\ldots,[a_{n;1},\ldots,a_{n;d}]\big)\in(\mathrm{Sym}^d\mathbb{C})^n\colon \bigcap_{r=1}^{n}\{a_{i;r}\colon r=1,\ldots,d\}\cap\bigcap_{r=1}^{n}\{-1/\overline{a_{i;r}}\colon r=1,\ldots,d\}=\emptyset\bigg\}.$$

A typical (η_{2n},η)-real degree $d_0=d_2$ holomorphic map $u^{\mathbb R}\equiv u_0$ from $\mathbb P^1$ to $\mathbb P^{2n-1}$ is of the form

$$[x,y]\longrightarrow\Big[A_1\prod_{r=1}^{d_2}(x-a_{1;r}y),\bar A_1\prod_{r=1}^{d_2}(\overline{a_{1;r}}x+y),\ldots,A_n\prod_{r=1}^{d_2}(x-a_{n;r}y),\bar A_n\prod_{r=1}^{d_2}(\overline{a_{n;r}}x+y)\Big]$$

for some $A_i,a_{i;r}\in\mathbb C^*$ with

$$\big([a_{1;1},\ldots,a_{1;d_2}],\ldots,[a_{n;1},\ldots,a_{n;d_2}]\big)\in(\mathrm{Sym}^{d_2}\mathbb C)^n-\Delta^{\eta}_{n,d_2}\,.$$

If $u^{\mathbb C}\equiv u^+$ is described as in the first paragraph of this section with $d^+=d_1$, $u^-=\eta_{2n}\circ u^{\mathbb C}\circ\eta$ is given by

$$[x,y]\longrightarrow\Big[-\overline{B_2}\prod_{r=1}^{d_1}(\overline{b_{2;r}}x+y),\overline{B_1}\prod_{r=1}^{d_1}(\overline{b_{1;r}}x+y),\ldots,-\overline{B_{2n}}\prod_{r=1}^{d_1}(\overline{b_{2n;r}}x+y),\overline{B_{2n-1}}\prod_{r=1}^{d_1}(\overline{b_{2n-1;r}}x+y)\Big].$$

The resulting map $u\colon\Sigma\longrightarrow\mathbb P^{2n-1}$ is (η_{2n},η)-real if $c'=\bar c$. In such a case, the restriction of the gluing map (3.1) to an open subspace of U can be taken to be

$$\begin{aligned}\upsilon\xrightarrow{\ \Phi\ }\Big[&A_1\prod_{r=1}^{d_2}(x-a_{1;r}y)\prod_{r=1}^{d_1}\big((x-(c+b_{1;r}\upsilon)y)\big(\overline{(c+b_{2;r})\upsilon}\,x+y\big)\big),\\ &\bar A_1\prod_{r=1}^{d_2}(\overline{a_{1;r}}x+y)\prod_{r=1}^{d_1}\big((x-(c+b_{2;r}\upsilon)y)\big(\overline{(c+b_{1;r}\upsilon)}\,x+y\big)\big),\ldots\Big],\end{aligned}$$

with $\upsilon\in\mathbb C^*$ corresponding to an element of L (based on the complex case in the previous paragraph).

As explained in [3, Section 2.1], an orientation on $\mathfrak M_0^{\mathbb R}(d)$ is equivalent to an orientation on the space $\widetilde{\mathfrak M}_0^{\mathbb R}(d)$ of parametrized real maps. The latter is determined by the map

$$\begin{gathered}\big((\mathrm{Sym}^d\mathbb C)^n-\Delta^{\eta}_{n,d}\big)\times\mathbb{RP}^{2n-1}\longrightarrow\widetilde{\mathfrak M}_0^{\mathbb R}(d)\,,\\ \big([a_{1;1},\ldots,a_{1;d}],\ldots,[a_{n;1},\ldots,a_{n;d}],[A_1,\ldots,A_n]\big)\longrightarrow\\ \Big[A_1\prod_{r=1}^{d}(x-a_{1;r}y),\bar A_1\prod_{r=1}^{d}(\overline{a_{1;r}}x+y),\ldots,A_n\prod_{r=1}^{d}(x-a_{n;r}y),\bar A_n\prod_{r=1}^{d}(\overline{a_{n;r}}x+y)\Big],\end{gathered}$$

where $\mathbb{RP}^{2n-1}\equiv(\mathbb C^n-\{0\})/\mathbb R^*$ and $[x,y]\in\mathbb P^1$. This map is an isomorphism over the open subset of $\widetilde{\mathfrak M}_0^{\mathbb R}(d)$ consisting of maps u such that $u([1,0])$ does not lie in any of the coordinate subspaces of $\mathbb P^{2n-1}$; see [3, Section 5.2].

For $c\in\mathbb C^*$ as above, $i=1,2$, and $b\in\mathbb C^*$ with $|b|<|c|$, let

$$h_{c;i}(b)=\begin{cases}c+b, & \text{if } i\notin2\mathbb Z;\\ \overline{(c+b)}^{\,-1}, & \text{if } i\in2\mathbb Z.\end{cases}$$

The explicit gluing map Φ described above locally corresponds to the map

$$\widetilde\Phi=(\tilde\Phi_1,\tilde\Phi_2)\colon\big(\mathbb C^{d_1}\big)^{2n}\times\big(\mathbb C^{d_2}\big)^n\times\mathbb{RP}^{2n-1}\longrightarrow\big(\mathbb C^{d}\big)^n\times\mathbb{RP}^{2n-1}\,,$$

where $d\!=\!2d_1\!+\!d_2$, given by

$$\widetilde\Phi_{1;i;r}\big((b_{j;s})_{\substack{j\le 2n\\ s\le d_1}},(a_{j;s})_{\substack{j\le n\\ s\le d_2}},[A_1,\ldots,A_n]\big)=\begin{cases}h_{c;1}(b_{2i-1;(r+1)/2}), & \text{if } r\le 2d_1,\ r\notin 2\mathbb{Z};\\ h_{c;2}(b_{2i;r/2}), & \text{if } r\le 2d_1,\ r\in 2\mathbb{Z};\\ a_{i;r-2d_1}, & \text{if } r>2d_1;\end{cases}$$

$$\widetilde\Phi_{2;i}\big((b_{j;s})_{\substack{j\le 2n\\ s\le d_1}},(a_{j;s})_{\substack{j\le n\\ s\le d_2}},[A_1,\ldots,A_n]\big)=A_i\Big/\prod_{r=1}^{d_1}h_{c;2}(b_{2i;r}).$$

The sign of $\widetilde\Phi$ is $(-1)^{nd_1}$, which establishes Lemma 3.1.

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