# Enumeration of Real Curves in $\mathbb{CP}^{2n-1}$ and a WDVV Relation for Real Gromov-Witten Invariants

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**Abstract**

We establish a homology relation for the Deligne-Mumford moduli spaces of real curves which lifts to a WDVV-type relation for a class of real Gromov-Witten invariants of real symplectic manifolds; we also obtain a vanishing theorem for these invariants. For many real symplectic manifolds, these results reduce all genus 0 real invariants with conjugate pairs of constraints to genus 0 invariants with a single conjugate pair of constraints. In particular, we give a complete recursion for counts of real rational curves in odd-dimensional projective spaces with conjugate pairs of constraints and specify all cases when they are nonzero and thus provide non-trivial lower bounds in high-dimensional real algebraic geometry. We also show that the real invariants of the three-dimensional projective space with conjugate point constraints are congruent to their complex analogues modulo 4.

## Contents



## 1 Introduction

The classical problem of enumerating (complex) rational curves in a complex projective space $\mathbb{P}^n$ is solved in [22, 27] using the WDVV relation of Gromov-Witten theory. Over the past decade, significant progress has been made in real enumerative geometry and real Gromov-Witten theory.

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Invariant signed counts of real rational curves with point constraints in real surfaces and in many real threefolds are defined in [31] and [32], respectively. An approach to interpreting these counts in the style of Gromov-Witten theory, i.e. as counts of parametrizations of such curves, is presented in [4, 28]. Signed counts of real curves with conjugate pairs of arbitrary (not necessarily point) constraints in arbitrary dimensions are defined in [10] and extended to more general settings in [5]. Two different WDVV-type relations for the real Gromov-Witten invariants of real surfaces as defined in [4, 28], along with the ideas behind them, are stated in [29]; they yield complete recursions for counts of real rational curves in $\mathbb{P}^2$ as defined in [31]. Other recursions for counts of real curves in some real surfaces have since been established by completely different methods in [18, 2, 19, 20].

In this paper, we establish a homology relation between geometric classes on the Deligne-Mumford moduli space $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ of real genus 0 curves with 3 conjugate pairs of marked points and use it to obtain a WDVV-type relation for the real Gromov-Witten invariants of [10, 5]; see Proposition 3.3 and Theorem 2.1. This relation yields a complete recursion for counts of real rational curves with conjugate pairs of arbitrary constraints in $\mathbb{P}^{2n-1}$; see Theorem 1.2 and Corollary 1.3. It is sufficiently simple to characterize the cases when these invariants are nonzero and thus the existence of real rational curves passing through specified types of constraints is guaranteed; see Corollary 1.4. We also show that the real genus 0 Gromov-Witten invariants of $\mathbb{P}^3$ with conjugate pairs of point constraints are congruent to their complex analogues modulo 4, as expected for the real curve counts of [31, 32], and that this congruence does not persist in higher dimensions or with other types of constraints; see Corollary 1.5 and the paragraph right after it.

Each odd-dimensional projective space $\mathbb{P}^{2n-1}$ has two standard anti-holomorphic involutions (automorphisms of order 2):

$$\tau_{2n}\colon \mathbb{P}^{2n-1} \longrightarrow \mathbb{P}^{2n-1}, \quad [z_1,\ldots,z_{2n}] \longrightarrow [\bar{z}_2,\bar{z}_1,\ldots,\bar{z}_{2n},\bar{z}_{2n-1}], \tag{1.1}$$
$$\eta_{2n}\colon \mathbb{P}^{2n-1} \longrightarrow \mathbb{P}^{2n-1}, \quad [z_1,\ldots,z_{2n}] \longrightarrow [-\bar{z}_2,\bar{z}_1,\ldots,-\bar{z}_{2n},\bar{z}_{2n-1}]. \tag{1.2}$$

The fixed locus of the first involution is $\mathbb{RP}^{2n-1}$, while the fixed locus of the second involution is empty. Let

$$\tau=\tau_2,\ \eta=\eta_2 : \mathbb{P}^1 \longrightarrow \mathbb{P}^1\,.$$

For $\phi=\tau_{2n},\eta_{2n}$ and $c=\tau,\eta$, a map $u\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^{2n-1}$ is $(\phi,c)$-real if $u\circ c=\phi\circ u$. For $k\in\mathbb{Z}^{\geqslant 0}$, a $k$-marked $(\phi,c)$-real map is a tuple

$$\big(u,(z_1^+,z_1^-),\ldots,(z_k^+,z_k^-)\big),$$

where $z_1^+,z_1^-,\ldots,z_k^+,z_k^-\in\mathbb{P}^1$ are distinct points with $z_i^+=c(z_i^-)$ and $u$ is a $(\phi,c)$-real map. Such a tuple is $c$-equivalent to another $k$-marked $(\phi,c)$-real map

$$\big(u',(z_1'^+,z_1'^-),\ldots,(z_k'^+,z_k'^-)\big)$$

if there exists a biholomorphic map $h\colon \mathbb{P}^1\longrightarrow\mathbb{P}^1$ such that

$$h\circ c=c\circ h, \qquad u'=u\circ h, \quad \text{and} \quad z_i^\pm=h(z_i'^\pm)\ \ \forall\ i=1,\ldots,k.$$

If in addition $d\in\mathbb{Z}^+$, denote by

$$\mathfrak{M}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi,c}\subset\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi,c}$$

the moduli space of $c$-equivalence classes of $k$-marked degree $d$ holomorphic $(\phi,c)$-real maps and its natural compactification consisting of stable real maps from nodal domains. As in [5, Section 3], let

$$\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi} \equiv \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi,\tau} \cup \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi,\eta} \tag{1.3}$$

be the space obtained by identifying the two moduli spaces on the right-hand side along their common boundary. The glued space has no codimension 1 boundary.

By [5, Lemma 1.9] and its proof,

$$\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\tau_{2n},\eta} = \varnothing \quad \forall\, d\notin 2\mathbb{Z}\,,$$
$$\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\eta_{2n},\eta} = \varnothing \quad \forall\, d\in 2\mathbb{Z}\,, \qquad \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\eta_{2n},\tau} = \varnothing \quad \forall\, d\in \mathbb{Z}\,.$$

By [10, Theorem 6.5], $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\tau_{2n},\tau}$ is orientable for every $d\in\mathbb{Z}$. By [5, Section 5.2], the spaces

$$\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\tau_{2n},\eta} \;\text{ with } d\in 2\mathbb{Z} \quad\text{ and }\quad \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\eta_{2n},\eta} \;\text{ with } d\notin 2\mathbb{Z}$$

are orientable as well. If $\phi=\tau_{2n}$ and $d\in 2\mathbb{Z}$, the orientations on the two moduli spaces on the right-hand side of (1.3) can be chosen so that they extend across the common boundary; see [5, Proposition 5.5]. In the remaining three cases, at most one of the spaces on the right-hand side of (1.3) is not empty. Thus, the glued moduli space (1.3) is orientable and carries a fundamental class.

The glued compactified moduli spaces come with natural evaluation maps

$$\mathrm{ev}_i\colon \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi} \longrightarrow \mathbb{P}^{2n-1}, \qquad \big[u,(z_1^+,z_1^-),\ldots,(z_k^+,z_k^-)\big] \longrightarrow u(z_i^+).$$

For $c_1,\ldots,c_k\in\mathbb{Z}^+$, we define

$$\langle c_1,\ldots,c_k\rangle_d^{\phi} = \int_{\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi}} \mathrm{ev}_1^*H^{c_1}\,\ldots\,\mathrm{ev}_k^*H^{c_k} \in \mathbb{Z}\,, \tag{1.4}$$

where $H\in H^2(\mathbb{P}^{2n-1})$ is the hyperplane class. For dimensional reasons,

$$\langle c_1,\ldots,c_k\rangle_d^{\phi} \neq 0 \qquad\Longrightarrow\qquad c_1+\ldots+c_k = n(d+1)-2+k\,. \tag{1.5}$$

Similarly to [27, Lemma 10.1], the numbers (1.4) are enumerative counts of real curves in $\mathbb{P}^{2n-1}$, i.e. of curves preserved by $\phi$, but now with some sign. They are invariant under the permutations of the insertions and satisfy the usual divisor relation,

$$\langle c_1,\ldots,c_k,1\rangle_d^{\phi} = d\,\langle c_1,\ldots,c_k\rangle_d^{\phi}\,. \tag{1.6}$$

The latter holds because the fiber of the forgetful morphism

$$\overline{\mathfrak{M}}_{0,k+1}(\mathbb{P}^{2n-1},d)^{\phi} \longrightarrow \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)^{\phi}$$

is oriented by $z_{k+1}^+$ and every degree $d$ curve in $\mathbb{P}^{2n-1}$ meets a generic hyperplane in $d$ points.

By [5, Theorem 1.10] and [5, Remark 1.11], the numbers (1.4) with $\phi = \tau_{2n}, \eta_{2n}$ vanish if either $d$ or any $c_i$ is even; see also Corollary 2.6(1). By [5, Remark 1.11] and Corollary 2.6(2),

$$\langle c_1, \ldots, c_k \rangle_d^{\tau_{2n}} = \pm \langle c_1, \ldots, c_k \rangle_d^{\eta_{2n}}; \tag{1.7}$$

the sign depends on the orientations of $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1}, d)^{\tau_{2n}}$ and $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1}, d)^{\eta_{2n}}$. Systems of such orientations, compatible with the recursion of Theorem 1.2 for $\mathbb{P}^{2n-1}$ and the WDVV-type relation of Theorem 2.1 for more general real symplectic manifolds, are described in Section 2. They ensure a fixed sign in (1.7) and can be specified by choosing the sign of the $d=1$ numbers in (1.4).

*Remark* 1.1. The orientations for the $\tau_{4n}$ and $\eta_{4n}$ moduli spaces are determined by a spin structure on $\mathbb{RP}^{4n-1}$ and a real square root of the canonical line bundle $\mathcal{K}_{\mathbb{P}^{4n-1}}$ of $\mathbb{P}^{4n-1}$, respectively. On the other hand, $\mathbb{RP}^{4n+1}$ does not admit a spin structure, while $\mathcal{K}_{\mathbb{P}^{4n+1}}$ does not admit a real square root. A relatively spin structure on $\mathbb{RP}^{4n+1}$ does not provide a system of orientations compatible with the recursion of Theorem 1.2, because such a system is not compatible with smoothing a conjugate pair of nodes, as needed for the statement of Lemma 5.2; see Remark 2.7 for more details.

For any $d, c_1, \ldots, c_k \in \mathbb{Z}^+$, let

$$\big\langle c_1, \ldots, c_k \big\rangle_d^{\mathbb{P}^{2n-1}} = \int_{\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1},d)} \mathrm{ev}_1^* H^{c_1} \ldots \mathrm{ev}_k^* H^{c_k} \in \mathbb{Z}^{\geqslant 0},$$

where $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{2n-1}, d)$ is the usual moduli space of stable (complex) $k$-marked genus 0 degree $d$ holomorphic maps to $\mathbb{P}^{2n-1}$, denote the (complex) genus 0 Gromov-Witten invariants of $\mathbb{P}^{2n-1}$; they are computed in [27, Theorem 10.4]. Finally, if $c_1, \ldots, c_k \in \mathbb{Z}$ and $I \subset \{1, \ldots, k\}$, let $c_I$ denote a tuple with the entries $c_i$ with $i \in I$, in some order.

**Theorem 1.2.** *Let $\phi = \tau_{2n}, \eta_{2n}$ and $d, k, n, c, c_1, \ldots, c_k \in \mathbb{Z}^+$. If $k \geqslant 2$ and $c_1, \ldots, c_k \notin 2\mathbb{Z}$,*

$$\begin{split}\big\langle c_1, c_2+2c, c_3, \ldots, c_k \big\rangle_d^{\phi} - \big\langle c_1+2c, c_2, c_3, \ldots, c_k \big\rangle_d^{\phi} = \sum_{\substack{2d_1+d_2=d\\ d_1,d_2\geqslant 1}} \sum_{I\sqcup J=\{3,\ldots,k\}} \sum_{\substack{2i+j=2n-1\\ i,j\geqslant 1}} 2^{|I|} \Big( \\ \big\langle 2c, c_1, c_I, 2i \big\rangle_{d_1}^{\mathbb{P}^{2n-1}} \big\langle c_2, c_J, j \big\rangle_{d_2}^{\phi} - \big\langle 2c, c_2, c_I, 2i \big\rangle_{d_1}^{\mathbb{P}^{2n-1}} \big\langle c_1, c_J, j \big\rangle_{d_2}^{\phi} \Big).\end{split}$$

**Corollary 1.3.** *Let $\phi = \tau_{2n}, \eta_{2n}$ and $d, k, n, c_1, \ldots, c_k \in \mathbb{Z}^+$. If $d \in 2\mathbb{Z}$ or $c_i \in 2\mathbb{Z}$ for some $i$,*

$$\big\langle c_1, c_2, \ldots, c_k \big\rangle_d^{\phi} = 0.$$

*If $k \geqslant 2$ and $c_1, \ldots, c_k \notin 2\mathbb{Z}$,*

$$\begin{split}\big\langle c_1, c_2, c_3, \ldots, c_k \big\rangle_d^{\phi} = d\big\langle c_1+c_2-1, c_3, \ldots, c_k \big\rangle_d^{\phi} + \sum_{\substack{2d_1+d_2=d\\ d_1,d_2\geqslant 1}} \sum_{I\sqcup J=\{3,\ldots,k\}} \sum_{\substack{2i+j=2n-1\\ i,j\geqslant 1}} 2^{|I|} \Big( \\ d_2\big\langle c_1-1, c_2, c_I, 2i \big\rangle_{d_1}^{\mathbb{P}^{2n-1}} \big\langle c_J, j \big\rangle_{d_2}^{\phi} - d_1\big\langle c_1-1, c_I, 2i \big\rangle_{d_1}^{\mathbb{P}^{2n-1}} \big\langle c_2, c_J, j \big\rangle_{d_2}^{\phi} \Big).\end{split}$$

**Corollary 1.4.** *Let $\phi = \tau_{2n}, \eta_{2n}$ and $d, k, n, c_1, \ldots, c_k \in \mathbb{Z}^+$ with*

$$c_1 + \ldots + c_k = n(d+1) - 2 + k \qquad \text{and} \qquad c_1, \ldots, c_k \leqslant 2n-1.$$

*(1) If $d, c_1, \ldots, c_k$ are odd, then so is $\langle c_1, c_2, \ldots, c_k \rangle_d^\phi$.*

*(2) The signed number $\langle c_1, c_2, \ldots, c_k \rangle_d^\phi$ of degree $d$ real curves in $\mathbb{P}^{2n-1}$ passing through general complex linear subspaces of codimensions $c_1, \ldots, c_k$ is zero if and only if either $d \in 2\mathbb{Z}$ or $c_i \in 2\mathbb{Z}$ for some $i$.*

The formula of Theorem 1.2, which is a special case of Theorem 2.1, can be seen as a real version of [23, Theorem 1]. Along with (1.6), it immediately implies the recursion of Corollary 1.3. The vanishing statement in this corrollary is the $\ell = 0$ case of Corollary 2.6(1). Corollary 1.3 reduces all numbers $\langle c_1, c_2, \ldots, c_k \rangle_d^\phi$, with $\phi = \tau_{2n}, \eta_{2n}$, to the single number $\big\langle 2n-1 \big\rangle_1^\phi$, i.e. the number of $\phi$-real lines through a non-real point in $\mathbb{P}^{2n-1}$. The absolute value of this number is of course 1, and we can choose a system of orientations so that $\big\langle 2n-1 \big\rangle_1^\phi = 1$. Taking $d = 1$ in Corollary 1.3, we obtain

$$\big\langle c_1, \ldots, c_k \big\rangle_1^\phi = \big\langle 2n-1 \big\rangle_1^\phi = 1$$

whenever $c_1, \ldots, c_k \in \mathbb{Z}^+$ are odd and $c_1 + \ldots + c_k = 2n-2+k$; this conclusion agrees with [5, Example 6.3]. Some other numbers obtained from Corollary 1.3 are shown in Tables 1 and 2; the degree 3 and 5 numbers in the former agree with [5].

Corollary 1.4 is deduced from Corollary 1.3 in Section 7. It can be equivalently viewed as a statement about the parity of the usual counts of genus 0 curves in $\mathbb{P}^{2n-1}$ with certain types of constraints (they must come in pairs of the same codimension). The standard WDVV recursion for counts of complex curves is not closed under the relevant restriction on the constraints. We do not see how to recover Corollary 1.4 directly from it.

In the case $\mathbb{P}^{2n-1} = \mathbb{P}^3$, the only interesting non-real constraints for the real genus 0 counts are points. Let

$$N_d^{\mathbb{R}} = (-1)^{\frac{d-1}{2}} \big\langle \underbrace{3, \ldots, 3}_{d} \big\rangle_d^{\tau_{2n}} = (-1)^{\frac{d-1}{2}} \big\langle \underbrace{3, \ldots, 3}_{d} \big\rangle_d^{\eta_{2n}} \tag{1.8}$$

be the number of degree $d$ real rational curves through $d$ non-real points in $\mathbb{P}^3$ counted with sign; by Corollary 1.4(2), $N_d^{\mathbb{R}}$ is well-defined even if $d \in 2\mathbb{Z}$. Denote by

$$N_d^{\mathbb{C}} = \big\langle \underbrace{3, \ldots, 3}_{2d} \big\rangle_d^{\mathbb{P}^{2n-1}} \qquad \text{and} \qquad \widetilde{N}_d^{\mathbb{C}} = \big\langle 2, 2, \underbrace{3, \ldots, 3}_{2d-1} \big\rangle_d^{\mathbb{P}^{2n-1}}$$

the number of degree $d$ (complex) rational curves through $2d$ points in $\mathbb{P}^3$ and the number of degree $d$ rational curves through 2 lines and $2d-1$ points in $\mathbb{P}^3$, respectively. The next corollary is also obtained in Section 7.

**Corollary 1.5.** *If $d \in \mathbb{Z}^+$ and $d \geqslant 2$, then*

$$N_d^{\mathbb{R}} = \sum_{\substack{2d_1+d_2=d \\ d_1,d_2 \geqslant 1}} (-4)^{d_1-1} d_2 \binom{d-2}{d_2-1} \widetilde{N}_{d_1}^{\mathbb{C}} N_{d_2}^{\mathbb{R}}, \qquad N_d^{\mathbb{R}} \cong_4 N_d^{\mathbb{C}} \cong_4 \begin{cases} 1, & \text{if } d \in \mathbb{Z}^+ - 2\mathbb{Z}; \\ 0, & \text{if } d \in 2\mathbb{Z}^+; \end{cases}$$

*where $\cong_4$ denotes the congruence modulo 4.*

The procedures of [32], [4, 28], and [10, 5] for determining the sign of each real curve passing through a specified real collection of constraints in $\mathbb{P}^3$ are very different and depend on some global choices. The latter affect the signs of all curves of a fixed degree in the same way, and so the real counts in each degree are determined up to an overall sign by all three procedures. In the case of conjugate pairs of point constraints and odd-degree curves (the intersection of the three settings), the three procedures yield the same count, up to a sign in each degree.

The second statement of Corollary 1.5 establishes a special case of Mikhalkin's congruence, a conjectural relation between real and complex counts of rational curves. Its analogues for counts of real rational curves with real point constraints in real del Pezzo surfaces as defined in [31] are proved in [19, 20]. By [3, Proposition 3] and [3, Theorem 2], the analogue of this statement for real point constraints in $\mathbb{P}^3$ holds with the sign modification in (1.8). This suggests that it would be natural to modify the signs of [32] as in (1.8). By [3, Theorem 2], such a modification would also ensure the positivity of counts of rational curves with real point constraints (but not with conjugate pairs of point constraints, as Table 1 shows). On the other hand, the second statement of Corollary 1.5 does not extend to more general constraints in $\mathbb{P}^3$ (it fails for $d=1$ with two conjugate pairs of line constraints) or to $\mathbb{P}^{2n-1}$ with $n\geqslant 3$ (according to Table 2).

The numbers (1.4) count real curves passing through specified constraints with signs and thus provide lower bounds for the actual numbers of such curves. There are indications that these bounds are often sharp. For example, for $d,m\in\mathbb{Z}^+$ with $d$ odd and $m=1$ if $d\geqslant 5$, there are configurations of $d-m$ conjugate pairs of points and $2m$ conjugate pairs of lines in $\mathbb{P}^3$ so that there are no real degree $d$ curves passing through them; see [21, Examples 12,17,18]. In light of the recursion of Corollary 1.5 and (7.1), [21, Proposition 3], which relates the numbers $N_d^{\mathbb{R}}$ to counts of real curves in $\mathbb{P}^1\times\mathbb{P}^1$, may be opening a way for a combinatorial proof that the numbers $N_d^{\mathbb{R}}$ provide sharp lower bounds for $d\notin 2\mathbb{Z}$ (if this is indeed the case).

The basic case (smallest $k$) of the analogue of Theorem 1.2 in complex Gromov-Witten theory is equivalent to the associativity of the quantum product on the cohomology of the manifold; see [27, Theorem 8.1]. The basic case of Theorem 2.1 is similarly equivalent to a property of the quantum product of a real symplectic manifold; see Section 7.

Theorem 1.2 is a special case of Theorem 2.1, which provides a WDVV-type relation for real Gromov-Witten invariants of real symplectic manifolds. In the next two paragraphs, we outline the two proofs of Theorem 2.1 appearing in this paper. While the first approach requires some preparation, it is more natural from the point of view of real Gromov-Witten theory. In [14], we describe a third proof of Theorem 1.2, which can be extended to some other cases of Theorem 2.1.

The WDVV relation for complex Gromov-Witten theory obtained in [22, 27] is a fairly direct consequence of a $\mathbb{C}$-codimension 1 relation on the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{0,4}$ of complex genus 0 curves with 4 marked points. According to this relation, the homology classes represented by two different nodal curves, e.g. $[1,0]$ and $[1,1]$ in Figure 1, are the same. Thus, topologically defined counts of morphisms from these two types of domains into an almost Kahler manifold are the same. As this relation simply states that two points in $\overline{\mathcal{M}}_{0,4}$ represent the same homology class, it is an immediate consequence of the connectedness of $\overline{\mathcal{M}}_{0,4}$. The WDVV-type relation of Theorem 2.1 is a fairly direct consequence of an $\mathbb{R}$-codimension 2 relation on the three-dimensional

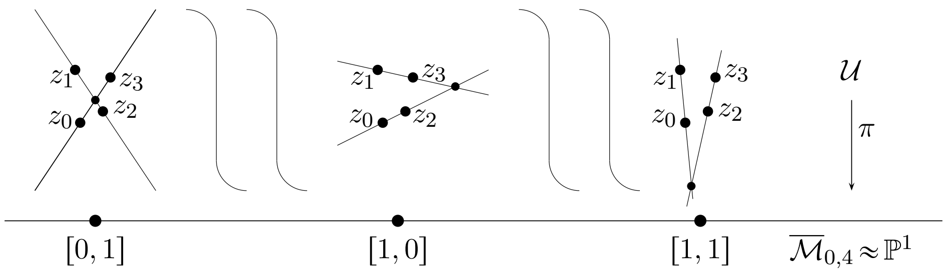


Figure 1: The universal curve $\mathcal{U} \longrightarrow \overline{\mathcal{M}}_{0,4}$.

Deligne-Mumford moduli space $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ of real genus 0 curves with 3 conjugate pairs of marked points which we establish in Section 3 through a detailed topological description of $\mathbb{R}\overline{\mathcal{M}}_{0,3}$; see Proposition 3.3 and its proof. According to this relation, the (relative) homology classes represented by two different, two-nodal degenerations of real curves are the same. Thus, topologically defined counts of morphisms from these two types of domains into a real almost Kahler manifold are the same; see Corollary 4.1. This relation, for both curves and maps, is illustrated in Figure 2, where the vertical line represents the irreducible component of the curve preserved by the involution and the two horizontal lines represent the components interchanged by the involution. In a sense, the situation with our recursion is analogous to the situation with the $\mathbb{C}$-codimension 2 recursion of [16, Lemma 1.1] on $\overline{\mathcal{M}}_{1,4}$, which had to be discovered and established before it could be applied to complex genus 1 Gromov-Witten invariants.

In Section 6, we give an alternative proof of Theorem 2.1, which bypasses Proposition 3.3. We pull back the usual relation on $\overline{\mathcal{M}}_{0,4}$ by the forgetful morphism $f_{012\bar{0}}$ which keeps the marked points $z_0^+, z_1^+, z_2^+, z_0^-$; see (6.1) and (6.2). In the proof of [27, Theorem 10.4], a nodal element of $\overline{\mathcal{M}}_{0,4}$ is a regular value of a similar map and all of its preimages are of the same type and contribute $+1$ each to the relevant count; the situation with the proof of Corollary 4.1 from Proposition 3.3 is analogous. In contrast, a nodal element of $\overline{\mathcal{M}}_{0,4}$ is not a regular value of $f_{012\bar{0}}$ and its preimages can be of four types, as indicated in Figures 6 and 7; they are morphisms from either a three-component domain or from a two-component domain. The contribution of each three-component morphism to the relevant count (6.2) is no longer necessarily $+1$; see Lemma 6.1. The stratum of two-component morphisms is not even 0-dimensional, but we show through a topological analysis that it does not contribute to the count; see Lemma 6.2.

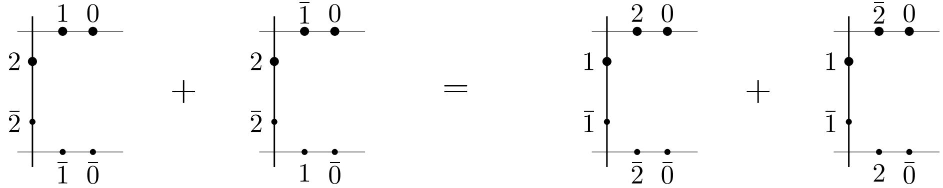

Figure 2: A relation in $H_1(\mathbb{R}\overline{\mathcal{M}}_{0,3})$; the dots labeled $i$ and $\bar{i}$ indicate the marked points $z_i^+$ and $z_i^-$, respectively.

In real Gromov-Witten theory, signs of various contributions are generally a delicate issue. It shows up explicitly in the above description of the second approach, but is hidden in the first approach. The analysis of signs for both approaches is carried out in Section 5, where different orientations of moduli spaces of constrained real morphisms are compared. This allows us to establish Propositions 4.2 and 4.3, which are used in the proofs of Theorem 2.1 in Sections 4 and 6, as well as Theorem 2.2, which provides vanishing results for real Gromov-Witten invariants of real symplectic manifolds, including in positive genera.

We would like to thank J. Morgan for his help in precisely identifying $\overline{\mathcal{M}}_{0,3}^{\mathbb{R}}$ in Remark 3.5 and E. Ionel, J. Kollár, M. Liu, N. Sheridan, J. Solomon, M. Tehrani, and G. Tian for related discussions. We are also grateful to the referees for comments on previous versions of this paper which led to significant improvements in the exposition.

## 2 Main theorems and corollaries

The formula of Theorem 1.2 is fundamentally a relation between real genus 0 GW-invariants; it is a special case of the relation of Theorem 2.1 for real symplectic manifolds. The latter implies that the real invariants of at least some real symplectic manifolds are essentially independent of the involution $\phi$; see Corollary 2.5. Likewise, the vanishing of the numbers $\langle c_1,\ldots,c_k\rangle_d^\phi$ with $c_i\in 2\mathbb{Z}$ for some $i$, established in [5, Section A.5] using the Equivariant Localization Theorem [1, (3.8)], is a special case of the general vanishing phenomenon for $\phi$-invariant insertions established in Theorem 2.2 below.

A real symplectic manifold is a triple $(X,\omega,\phi)$ consisting of a symplectic manifold $(X,\omega)$ and an involution $\phi\colon X\longrightarrow X$ such that $\phi^*\omega=-\omega$. Examples include $\mathbb{P}^{2n-1}$ with the standard Fubini-Study symplectic form $\omega_{2n}$ and the involutions (1.1) and (1.2), as well as $(\mathbb{P}^{2n},\omega_{2n+1})$ with the involution

$$\tau_{2n+1}\colon \mathbb{P}^{2n}\longrightarrow\mathbb{P}^{2n}, \qquad [X_1,\ldots,X_{2n},X_{2n+1}]\longrightarrow[\bar{X}_2,\bar{X}_1,\ldots,\bar{X}_{2n},\bar{X}_{2n-1},\bar{X}_{2n+1}],$$

which extends (1.1) to the even-dimensional projective spaces. If

$$\ell\geqslant 0, \qquad \mathbf{a}\equiv(a_1,\ldots,a_\ell)\in(\mathbb{Z}^+)^\ell, \tag{2.1}$$

and $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ is a complete intersection of multi-degree $\mathbf{a}$ preserved by $\tau_n$, $\tau_{n;\mathbf{a}}\equiv\tau_n|_{X_{n;\mathbf{a}}}$ is an anti-symplectic involution on $X_{n;\mathbf{a}}$ with respect to the symplectic form $\omega_{n;\mathbf{a}}=\omega_n|_{X_{n;\mathbf{a}}}$. Similarly, if $X_{2n;\mathbf{a}}\subset\mathbb{P}^{2n-1}$ is preserved by $\eta_{2n}$, $\eta_{2n;\mathbf{a}}\equiv\eta_{2n}|_{X_{2n;\mathbf{a}}}$ is an anti-symplectic involution on $X_{2n;\mathbf{a}}$ with respect to the symplectic form $\omega_{2n;\mathbf{a}}=\omega_{2n}|_{X_{2n;\mathbf{a}}}$.

Let $(X,\omega,\phi)$ be a real symplectic manifold. The fixed locus $X^\phi$ of $\phi$ is a Lagrangian submanifold, which may be empty. Let

$$H_2(X)_\phi=\big\{\beta\in H_2(X;\mathbb{Z})\colon\ \phi_*\beta=-\beta\big\}, \qquad H^*(X)_\pm^\phi\equiv\big\{\mu\in H^*(X)\colon\ \phi^*\mu=\pm\mu\big\}.$$

Similarly to [10, Section 1], we define

$$\mathfrak{d}\colon H_2(X)\longrightarrow H_2(X)_\phi \qquad \text{by} \qquad \mathfrak{d}(\beta)=\beta-\phi_*\beta\,. \tag{2.2}$$

A **real bundle pair** $(V,\tilde{\phi})\longrightarrow(X,\phi)$ consists of a complex vector bundle $V\longrightarrow X$ and a conjugation $\tilde{\phi}$ on $V$ lifting $\phi$, i.e. an involution restricting to an anti-complex linear homomorphism on each fiber. The fixed locus $V^{\tilde{\phi}}\longrightarrow X^{\phi}$ is then a maximal totally real subbundle of $V|_{X^{\phi}}$, i.e.

$$V|_{X^{\phi}} = V^{\tilde{\phi}} \oplus \mathfrak{i}V^{\tilde{\phi}},$$

where $\mathfrak{i}$ is the complex structure on $V$. Let

$$w_2^{\tilde{\phi}}(V) \in H^2_{\phi}(X;\mathbb{Z}_2) \equiv H^2_{\mathbb{Z}_2}(X;\mathbb{Z}_2)$$

denote the equivariant second Stiefel-Whitney class of $(V,\tilde{\phi})$; see [11, Section 2].

Let $\mathcal{J}_\omega^\phi$ be the space of $\omega$-compatible almost complex structures $J$ on $X$ such that $\phi^*J=-J$. For $J\in\mathcal{J}_\omega^\phi$, $c=\tau,\eta$, and $\beta\in H_2(X)_\phi$, denote by

$$\mathfrak{M}_{0,k}(X,\beta)^{\phi,c} \subset \overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,c} \tag{2.3}$$

the moduli space of $c$-equivalence classes of $k$-marked $J$-holomorphic $(\phi,c)$-real maps in the homology class $\beta$ and its natural compactification consisting of stable real maps from nodal domains.

By [10, Theorem 6.5], both spaces in (2.3) with $c=\tau$ are orientable in the sense of Kuranishi structures (or for a generic $J$ if $(X,\omega)$ is strongly semi-positive) if

$(\mathcal{O}_\tau)$ $X^\phi$ is orientable and there exists a real bundle pair $(E,\tilde{\phi})\longrightarrow(X,\phi)$ such that

$$w_2(TX^\phi) = w_1(E^{\tilde{\phi}})^2 \quad \text{and} \quad \frac{1}{2}\langle c_1(X),\beta'\rangle + \langle c_1(E),\beta'\rangle \in 2\mathbb{Z} \quad \forall\ \beta'\in H_2(X)_\phi\,.$$

The first requirement on $(E,\tilde{\phi})$ above implies that $TX^\phi\oplus 2E^{\tilde{\phi}}$ admits a spin structure. By [11, Theorem 1.1], both spaces in (2.3) with $c=\eta$ are orientable if

$(\mathcal{O}_\eta)$ $w_2^{\Lambda_{\mathbb{C}}^{\mathrm{top}}\mathrm{d}\phi}(\Lambda_{\mathbb{C}}^{\mathrm{top}}TX) = \kappa^2$ for some $\kappa\in H^1_\phi(X)$.

This condition implies that $(TX,\mathrm{d}\phi)$ admits a spin sub-structure, as defined above [12, Corollary 5.10]. By [11, Corollary 2.4], $(\mathcal{O}_\eta)$ holds if either $\Lambda_{\mathbb{C}}^{\mathrm{top}}(TX,\mathrm{d}\phi)$ admits a real square root, i.e. there is an isomorphism

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}(TX,\mathrm{d}\phi) \approx (L,\tilde{\phi})^{\otimes 2} \tag{2.4}$$

for a real line bundle pair $(L,\tilde{\phi})\longrightarrow(X,\phi)$, or $\pi_1(X)=0$ and $w_2(X)=0$. A fixed real square root determines a spin sub-structure on $(TX,\mathrm{d}\phi)$.

The moduli space $\overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,c}$ with $c=\tau,\eta$ has no boundary in the sense of Kuranishi structures if

$$\beta\notin \mathrm{Im}(\mathfrak{d}) \qquad \text{or} \qquad X^\phi=\varnothing. \tag{2.5}$$

Thus, it carries a virtual fundamental class if $(\mathcal{O}_c)$, with $c$ as above, and (2.5) hold. Under the above assumptions, we define

$$\langle \mu_1,\ldots,\mu_k\rangle_\beta^{\phi,c} = \int_{[\overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,c}]^{\mathrm{vir}}} \mathrm{ev}_1^*\mu_1\ \ldots\ \mathrm{ev}_k^*\mu_k \in \mathbb{Q} \tag{2.6}$$

for any $\mu_1,\ldots,\mu_k\in H^*(X)$. This number depends on the chosen orientation of the moduli space. If $c=\tau$, we orient the moduli space as in the proofs of [9, Corollary 1.8] and [10, Theorem 6.5] from any spin structure on $TX^\phi\oplus 2E^{\tilde\phi}$. If $c=\eta$, we orient the moduli space via the pinching construction of [5, Lemma 2.5] from any spin sub-structure on $(TX,\mathrm{d}\phi)$; see [12, Corollary 5.10]. In either case, we use the *same* spin structure or sub-structure for all $\beta$.

If $(\mathcal{O}_\tau)$ and $(\mathcal{O}_\eta)$ are satisfied, but not necessarily (2.5), the glued moduli space

$$\overline{\mathfrak{M}}_{0,k}(X,\beta)^\phi\equiv\overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,\tau}\cup\overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,\eta} \tag{2.7}$$

is orientable and has no boundary; see [5, Theorem 1.7]. We then define

$$\big\langle\mu_1,\ldots,\mu_k\big\rangle_\beta^\phi=\int_{[\overline{\mathfrak{M}}_{0,k}(X,\beta)^\phi]^{\mathrm{vir}}}\mathrm{ev}_1^*\mu_1\,\ldots\,\mathrm{ev}_k^*\mu_k\in\mathbb{Q} \tag{2.8}$$

for any $\mu_1,\ldots,\mu_k\in H^*(X)$. The orientations on $\overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,\tau}$ and $\overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,\eta}$ constructed as in the previous paragraph induce an orientation on $\overline{\mathfrak{M}}_{0,k}(X,\beta)^\phi$ after reversing the orientation on $\overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,\eta}$ if the chosen spin structure on $TX^\phi\oplus 2E^{\tilde\phi}$ and spin sub-structure on $(TX,\mathrm{d}\phi)$ induce the same orientation on $X^\phi$; see [5, Proposition 3.3].

Choose bases $\{\gamma_i\}_{i\leqslant\ell}$ and $\{\gamma^i\}_{i\leqslant\ell}$ for $H^*(X)$ so that

$$\gamma^i\in H^*(X)_+^\phi\cup H^*(X)_-^\phi\qquad\text{and}\qquad \mathrm{PD}_{X^2}(\Delta_X)=\sum_{i=1}^{\ell}\gamma_i\times\gamma^i\in H^*(X^2),$$

where $\Delta_X\subset X^2$ is the diagonal. If $\mu_1,\ldots,\mu_k\in H^*(X)$ and $I\subset\{1,\ldots,k\}$, let $\mu_I$ denote a tuple with the entries $\mu_i$ with $i\in I$, in some order. Let

$$\big\langle\mu_1,\ldots,\mu_k\big\rangle_\beta^X=\int_{[\overline{\mathfrak{M}}_{0,k}(X,\beta)]^{\mathrm{vir}}}\mathrm{ev}_1^*\mu_1\,\ldots\,\mathrm{ev}_k^*\mu_k\in\mathbb{Q}\,, \tag{2.9}$$

denote the (complex) genus 0 GW-invariants of $X$.

**Theorem 2.1.** *Let $(X,\omega,\phi)$ be a compact real symplectic manifold, $k\in\mathbb{Z}$ with $k\geqslant 2$,*

$$\beta\in H_2(X)_\phi-\{0\},\qquad \mu\in H^{2*}(X)_+^\phi,\quad\textit{and}\quad \mu_1,\ldots,\mu_k\in H^{2*}(X)_-^\phi.$$

*(1) If $c=\tau,\eta$ and $(\mathcal{O}_c)$ and (2.5) are satisfied, then*

$$\begin{split}\big\langle\mu_1,\mu\mu_2,\mu_3,\ldots,\mu_k\big\rangle_\beta^{\phi,c}-\big\langle\mu\mu_1,\mu_2,\mu_3,\ldots,\mu_k\big\rangle_\beta^{\phi,c}=\sum_{\substack{\mathfrak{d}(\beta_1)+\beta_2=\beta\\ \beta_1,\beta_2\in H_2(X)-\{0\}}}\;\sum_{I\sqcup J=\{3,\ldots,k\}}\;\sum_{\substack{1\leqslant i\leqslant\ell\\ \gamma^i\in H^{2*}(X)_-^\phi}}2^{|I|}\Big(\\ \big\langle\mu,\mu_1,\mu_I,\gamma_i\big\rangle_{\beta_1}^X\big\langle\mu_2,\mu_J,\gamma^i\big\rangle_{\beta_2}^{\phi,c}-\big\langle\mu,\mu_2,\mu_I,\gamma_i\big\rangle_{\beta_1}^X\big\langle\mu_1,\mu_J,\gamma^i\big\rangle_{\beta_2}^{\phi,c}\Big).\end{split}$$

*(2) If $(\mathcal{O}_\tau)$ and $(\mathcal{O}_\eta)$ are satisfied, then the above identity holds for the $\langle\ldots\rangle^\phi$ invariants.*

This theorem, established in Section 4, concerns real genus 0 GW-invariants (2.6) and (2.8) with all insertions $\mu_i$ coming from $H^{2*}(X)^{\phi}_{-}$. By the first part of Theorem 2.2 below, the invariants (2.6) and (2.8) with any insertion $\mu_i$ coming from $H^{2*}(X)^{\phi}_{+}$ vanish. The proof of Theorem 2.2 in Section 5 extends the vanishing statement of [5, Theorem 1.10] for real genus 0 invariants with even-degree insertions to all settings when the real GW-invariants are defined and the unmarked real moduli space is orientable. By [13, Theorem 1.3], this is the case in any genus under the assumptions in (2) of Theorem 2.2.

**Theorem 2.2.** *Let $(X,\omega,\phi)$ be a compact real symplectic $2n$-manifold, $\beta\in H_2(X)_{\phi}-\{0\}$, and $\mu_1,\ldots,\mu_k\in H^*(X)$ with $\mu_i\in H^*(X)^{\phi}_{+}$ for some $i$.*

*(1) Suppose $c=\tau,\eta$, $(\mathcal{O}_\tau)$ holds if $c=\tau$, and $(\mathcal{O}_\eta)$ holds if $c=\eta$. If (2.5) is satisfied, then $\langle\mu_1,\ldots,\mu_k\rangle^{\phi,c}_{\beta}=0$. If $(\mathcal{O}_\tau)$ and $(\mathcal{O}_\eta)$ are satisfied, but not necessarily (2.5), $\langle\mu_1,\ldots,\mu_k\rangle^{\phi}_{\beta}=0$.*

*(2) If $c$ is an orientation-reversing involution on a compact orientable genus $g$ surface $\Sigma_g$, $n$ is odd, $\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX,\mathrm{d}\phi)$ admits a real square root, and $X^{\phi}=\varnothing$, then real genus $g$ GW-invariants $\langle\mu_1,\ldots,\mu_k\rangle^{\phi,c}_{\beta}$ of $(X,\omega,\phi)$ vanish.*

*Remark* 2.3. Let $c_g$ be an orientation-reversing involution on $\Sigma_g$ so that $\Sigma_g^{c_g}=\varnothing$. By [26, Corollary 1.1], $c_g$ is unique up to conjugation by diffeomorphisms of $\Sigma_g$. Similarly to (2.7), the moduli spaces $\overline{\mathfrak{M}}_{g,k}(X,\beta)^{\phi,c}$ of real $J$-holomorphic maps corresponding to different topological types of involutions $c$ on $\Sigma_g$ can be glued together into a moduli space $\overline{\mathfrak{M}}_{g,k}(X,\beta)^{\phi}$ without boundary. If $X^{\phi}=\varnothing$, then

$$\overline{\mathfrak{M}}_{g,k}(X,\beta)^{\phi}=\overline{\mathfrak{M}}_{g,k}(X,\beta)^{\phi,c_g} \tag{2.10}$$

and the GW-invariants $\langle\ldots\rangle^{\phi,c_g}_{\beta}$ are the same as the combined real GW-invariants $\langle\ldots\rangle^{\phi}_{g,\beta}$ expected to arise from the left-hand side of (2.10). Since the present paper was first completed, such invariants have been defined with the condition $X^{\phi}=\varnothing$ weakened to the existence of the square root as in (2.4) such that $w_2(TX^{\phi})=w_1(L^{\tilde{\phi}})^2$; see [15, Theorems 1.3,1.4]. The proof of Theorem 2.2 applies verbatim to the real genus $g$ GW-invariants of [15, Theorems 1.4,1.5].

For a strongly semi-positive real symplectic manifold $(X,\omega,\phi)$, the real genus 0 GW-invariant through constraints $\mu_1,\ldots,\mu_k$ is of the same parity as the complex GW-invariant of the same degree through the constraints $\mu_1,\phi^*\mu_1,\ldots,\mu_k,\phi^*\mu_k$. Thus, Theorem 2.2 implies that certain complex genus 0 GW-invariants are even. For example, the GW-invariants of $\mathbb{P}^{2n-1}$ with even numbers of insertions of each codimension that include insertions of even codimensions are even. This is not the case for even-dimensional projective spaces (for which the degree $4d+1$ unmarked real moduli spaces are not orientable; see [28, Proposition 5.1]). For example, the number of lines through two points in $\mathbb{P}^n$ is 1 (these constraints are of even codimension if $n$ is even).

For a real symplectic manifold $(X,\omega,\phi)$, let $H_{\mathrm{eff}}(X)_{\phi}\subset H_2(X)_{\phi}$ denote the subset of nonzero classes that can be represented by a $J$-holomorphic map from a disjoint union of copies of $\mathbb{P}^1$ for every $J\in\mathcal{J}^{\phi}_{\omega}$.

**Corollary 2.4.** *Let $(X,\omega,\phi)$ be a compact real symplectic manifold such that every positive-degree element of $H^{2*}(X)^{\phi}_{-}$ is divisible by an element of $H^2(X)^{\phi}_{-}$ in $H^{2*}(X)$ and $\beta\in H_{\mathrm{eff}}(X)_{\phi}$. Then there exist linear maps*

$$P_{\beta';\beta}\colon\bigoplus_{k=1}^{\infty}H^{2*}(X)^{\otimes k}\longrightarrow H^{2*}(X)^{\phi}_{-},\quad \beta'\in H_{\mathrm{eff}}(X)_{\phi},\ \beta-\beta'\in\big(H_{\mathrm{eff}}(X)_{\phi}\cup\{0\}\big)\cap\mathrm{Im}(\mathfrak{d}),$$

*determined by the GW-invariants of* $(X,\omega)$ *and* $\phi_*: H_*(X)\longrightarrow H_*(X)$ *with the following properties.*

*(1) If* $c=\tau,\eta$, $(\mathcal{O}_c)$ *is satisfied, and either* $\beta\notin\mathrm{Im}(\mathfrak{d})$ *or* $X^\phi=\varnothing$, *then*

$$\big\langle\mu_1,\ldots,\mu_k\big\rangle_\beta^{\phi,c}=\sum_{\substack{\beta'\in H_{\mathrm{eff}}(X)_\phi\\ \beta-\beta'\in(H_{\mathrm{eff}}(X)_\phi\cup\{0\})\cap\mathrm{Im}(\mathfrak{d})}}\big\langle P_{\beta';\beta}(\mu_1,\ldots,\mu_k)\big\rangle_{\beta'}^{\phi,c}\quad\forall\bigotimes_{i=1}^k\mu_i\in H^{2*}(X)^{\otimes k},\ k\in\mathbb{Z}^+.\tag{2.11}$$

*(2) If* $(\mathcal{O}_\tau)$ *and* $(\mathcal{O}_\eta)$ *are satisfied, then (2.11) holds with* $\langle\cdot\rangle^{\phi,c}$ *replaced by* $\langle\cdot\rangle^\phi$.

Corollary 2.4 is deduced from Theorems 2.1 and 2.2 in Section 7. It provides the strongest results for real $\mathsf{Fano}$ symplectic manifolds, i.e. real symplectic manifolds $(X,\omega,\phi)$ such that $\langle c_1(X),\beta\rangle>0$ for all $\beta\in H_{\mathrm{eff}}(X)_\phi$. For a real Fano symplectic manifold $(X,\omega,\phi)$ and $\beta\in H_{\mathrm{eff}}(X)_\phi$, let

$$c_{\min}^\phi(\beta),c_+^\phi(\beta)\in\mathbb{Z}^+$$

denote the smallest and the second smallest values of the function

$$\big\{\beta'\in H_{\mathrm{eff}}(X)_\phi,\ \beta-\beta'\in\mathrm{Im}(\mathfrak{d})\big\}\longrightarrow\mathbb{Z}^+,\qquad \beta'\longrightarrow\langle c_1(X),\beta'\rangle;$$

if the smallest value is achieved by two different classes $\beta'$, then $c_{\min}^\phi(\beta)\equiv c_+^\phi(\beta)$. Let $\beta_\bullet^\phi\in H_{\mathrm{eff}}(X)_\phi$ be such that

$$\big\langle c_1(X),\beta_\bullet^\phi\big\rangle=c_{\min}^\phi(\beta).$$

**Corollary 2.5.** *Let* $(X,\omega,\phi)$ *and* $\beta$ *be as in Corollary 2.4. If* $X$ *is Fano and* $c_+^\phi(\beta)>(\dim X)/2+1$, *then there exists a linear map*

$$P_\beta:\bigoplus_{k=1}^\infty H^{2*}(X)^{\otimes k}\longrightarrow H^{n-1+c_{\min}(\phi)}(X)_-^\phi$$

*determined by the GW-invariants of* $(X,\omega)$ *and* $\phi_*: H_*(X)\longrightarrow H_*(X)$ *with the following properties.*

*(1) If* $c=\tau,\eta$, $(\mathcal{O}_c)$ *is satisfied, and either* $\beta\notin\mathrm{Im}(\mathfrak{d})$ *or* $X^\phi=\varnothing$, *then*

$$\big\langle\mu_1,\ldots,\mu_k\big\rangle_\beta^{\phi,c}=\big\langle P_\beta(\mu_1,\ldots,\mu_k)\big\rangle_{\beta_\bullet^\phi}^{\phi,c}\quad\forall\,\mu_1,\ldots,\mu_k\in H^{2*}(X)^{\otimes k},\ k\in\mathbb{Z}^+.\tag{2.12}$$

*(2) If* $(\mathcal{O}_\tau)$ *and* $(\mathcal{O}_\eta)$ *are satisfied, then (2.12) holds with* $\langle\cdot\rangle^{\phi,c}$ *replaced by* $\langle\cdot\rangle^\phi$.

*If* $X$ *is Fano and* $c_{\min}^\phi(\beta)>(\dim X)/2+1$, *then the invariants* $\langle\cdot\rangle_\beta^{\phi,c}$ *and* $\langle\cdot\rangle_\beta^\phi$ *with insertions from* $H^{2*}(X)$ *vanish under the assumptions in (1) and (2), respectively.*

The virtual dimensions of the moduli spaces in (2.7) are

$$\dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{0,k}(X,\beta)^\phi=\dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,c}=\big\langle c_1(X),\beta\big\rangle+(n-3)+2k\,,\tag{2.13}$$

where $n=(\dim X)/2$. In particular, the real genus 0 one-insertion GW-invariants $\langle\mu\rangle_\beta^{\phi,c}$ and $\langle\mu\rangle_\beta^\phi$ (whenever they are defined) vanish if $\langle c_1(X),\beta\rangle>n+1$. Thus, Corollary 2.5 is an immediate consequence of Corollary 2.4.

For $n\in\mathbb{Z}$ and $\ell, \mathbf{a}$ as in (2.1), let

$$|\mathbf{a}| = a_1+\ldots+a_\ell, \qquad \langle \mathbf{a}\rangle_n = 2n-2-|\mathbf{a}|-\ell.$$

If $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ is a complete intersection as before, then

$$\Lambda_{\mathbb{C}}^{\mathrm{top}} TX_{n;\mathbf{a}} \approx \mathcal{O}_{\mathbb{P}^{n-1}}\big(n-|\mathbf{a}|\big)\big|_{X_{n;\mathbf{a}}}.$$

By the Lefschetz Theorem on Hyperplane Sections [17, p156], $\pi_1(X_{n;\mathbf{a}})=0$ if the complex dimension of $X_{n;\mathbf{a}}$ is at least 2. If $n\in 2\mathbb{Z}$ and $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ is $\eta_n$-invariant, then $w_2(X_{n;\mathbf{a}})=0$ and $X_{n;\mathbf{a}}^{\eta_{n;\mathbf{a}}}=\varnothing$. By [11, Corollary 2.4], an $\eta_n$-invariant complete intersection $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ thus satisfies $(\mathcal{O}_\eta)$ if its complex dimension is at least 2 or $X_{n;\mathbf{a}}\approx\mathbb{P}^1$. If the complex dimension of $X_{n;\mathbf{a}}$ is 1 and $X_{n;\mathbf{a}}\not\approx\mathbb{P}^1$, then the moduli spaces in (2.7) with $X=X_{n;\mathbf{a}}$ are empty. A $\tau_n$-invariant complete intersection $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ satisfies $(\mathcal{O}_\tau)$ if

$$n-|\mathbf{a}|\in 2\mathbb{Z} \qquad \text{and} \qquad a_1^2+\ldots+a_\ell^2-|\mathbf{a}|\in 4\mathbb{Z}; \tag{2.14}$$

see the proof of [10, Corollary 6.8]. If the first condition in (2.14) is satisfied, then $(X_{n;\mathbf{a}},\tau_{n;\mathbf{a}})$ satisfies $(\mathcal{O}_\eta)$.

For $d\in\mathbb{Z}$, let $\langle d\rangle\subset H_2(X_{n;\mathbf{a}})$ denote the subset of classes $\beta$ whose image in $\mathbb{P}^{n-1}$ is $d$ times the homology class of a line $\mathbb{P}^1\subset\mathbb{P}^{n-1}$. If in addition $\phi$ is an involution on $X_{n;\mathbf{a}}$, let

$$\langle d\rangle_\phi = \langle d\rangle \cap H_2\big(X_{n;\mathbf{a}}\big)_\phi\,.$$

If the complex dimension of $X_{n;\mathbf{a}}$ is at least 3, $\langle d\rangle$ consists of a single element. In all cases, $\beta\in\langle d\rangle_\phi$ satisfies the first condition in (2.5) if $d\notin 2\mathbb{Z}$. We denote by $H\in H^2(X_{n;\mathbf{a}})$ the restriction of the hyperplane class.

Suppose $X=X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ is a complete intersection of multi-degree $\mathbf{a}$ invariant under $\phi_{\mathbb{P}^{n-1}}\equiv\eta_n$ or $\phi_{\mathbb{P}^{n-1}}\equiv\tau_n$, $\phi=\phi_{\mathbb{P}^{n-1}}|_X$, $c=\eta,\tau$, and $d\in\mathbb{Z}$. We denote by

- $\langle\ldots\rangle_d^\phi$ the sum of the numbers (2.8) over $\beta\in\langle d\rangle_\phi$ if $\phi_{\mathbb{P}^{n-1}}=\eta_n$ or (2.14) is satisfied;
- $\langle\ldots\rangle_d^{\phi,c}$ the sum of the numbers (2.6) over $\beta\in\langle d\rangle_\phi$ if $\phi_{\mathbb{P}^{n-1}}=\eta_n$, or

$(\tau_n\eta)$ $d\notin 2\mathbb{Z}$ , $c=\eta$, and the first condition in (2.14) is satisfied, or

$(\tau_n\tau)$ $d\notin 2\mathbb{Z}$ , $c=\tau$, and both conditions in (2.14) are satisfied.

The next corollary is also proved in Section 7.

**Corollary 2.6.** *Suppose $n\in\mathbb{Z}^+$, $\ell\in\mathbb{Z}^{\geqslant 0}$, $\mathbf{a}\in(\mathbb{Z}^+)^\ell$, $X=X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ is a complete intersection of multi-degree $\mathbf{a}$ invariant under $\phi_{\mathbb{P}^{n-1}}\equiv\eta_n$ or $\phi_{\mathbb{P}^{n-1}}\equiv\tau_n$, and $\phi=\phi_{\mathbb{P}^{n-1}}|_X$.*

*(1) Let $c=\eta,\tau$ and $\mu_1,\ldots,\mu_k\in H^{2*}(X)$. If $\phi_{\mathbb{P}^{n-1}}=\eta_n$ and $c=\tau$, then $\langle\mu_1,\ldots,\mu_k\rangle_d^{\phi,c}=0$. The same conclusion holds if either $(\tau_n\eta)$ holds or*

- *$\phi_{\mathbb{P}^{n-1}}=\eta_n$ or $(\tau_n\tau)$ is satisfied and*
- *$a_i\in 2\mathbb{Z}$ for some $i$, or $\mu_j\in H^{4*}(X)$ for some $j$, or $d\in 2\mathbb{Z}$.*

*If the last bullet condition holds and either* $\phi_{\mathbb{P}^{n-1}}=\eta_n$ *or (2.14) is satisfied, then* $\langle\mu_1,\ldots,\mu_k\rangle_d^\phi=0$.

(2) *Suppose* $3|\mathbf{a}|-\ell<2n$ *and* $d\in\mathbb{Z}$. *Then there exists a linear map*

$$C_d\colon \bigoplus_{k=1}^{\infty} H^{2*}(X)^{\otimes k} \longrightarrow \mathbb{Z}$$

*determined by the GW-invariants of* $(X,\omega_n|_X)$ *and* $\phi_*: H_*(X)\longrightarrow H_*(X)$ *such that for all* $\mu_1,\ldots,\mu_k\in H^{2*}(X)$

(2a) $\langle\mu_1,\ldots,\mu_k\rangle_d^\phi = C_d(\mu_1,\ldots,\mu_k)\langle H^{\langle\mathbf{a}\rangle_n}\rangle_1^\phi$ *if* $\phi_{\mathbb{P}^{n-1}}=\eta_n$ *or (2.14) is satisfied;*

(2b) $\langle\mu_1,\ldots,\mu_k\rangle_d^{\phi,c} = C_d(\mu_1,\ldots,\mu_k)\langle H^{\langle\mathbf{a}\rangle_n}\rangle_1^{\phi,c}$ *if* $c=\eta,\tau$ *and either* $(\tau_n\eta)$ *or the first bullet condition in (1) holds.*

For example, the genus 0 real GW-invariants (1.4) of $(\mathbb{P}^{2n-1},\phi)$ with $\phi=\eta_{2n},\tau_{2n}$ satisfy

$$\langle c_1,\ldots,c_k\rangle_d^\phi = C_d(c_1,\ldots,c_k)\langle 2n-1\rangle_1^\phi$$

for some $C_d(c_1,\ldots,c_k)\in\mathbb{Z}$ independent of the choice of $\phi$. This implies (1.7). Corollary 2.6(2) extends (1.7) to Fano complete intersections $X_{n;\mathbf{a}}\subset\mathbb{P}^{2n-1}$ with $n\in 2\mathbb{Z}^+$ that are preserved by both $\tau_n$ and $\eta_n$. The approach to (1.7) in [5] extends to $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ without a restriction on $\mathbf{a}$, but is generally limited to complete intersections in real symplectic manifolds with large torus actions and insertions coming from the ambient manifolds. Corollary 2.6(1) extends the vanishing statement of [5, Theorem 1.10] to complete intersection using completely different reasoning. More generally, physical considerations in [30] suggest that the real genus 0 GW-invariants vanish whenever (2.5) does not hold, but $(\mathcal{O}_\tau)$ and $(\mathcal{O}_\eta)$ are satisfied, i.e. when the gluing of the two parts of the moduli space as in (2.7) is necessary and possible.

If $X^\phi$ is orientable or $\overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,\eta}\neq\varnothing$, then $\langle c_1(X),\beta\rangle\in 2\mathbb{Z}$ and

$$\dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{0,k}(X,\beta)^\phi = \dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,c} \cong n-3 \mod 2; \tag{2.15}$$

see (2.13). Thus, the real genus 0 GW-invariants (2.6) and (2.8) with all insertions $\mu_i\in H^{2*}(X)$ vanish if $n\in 2\mathbb{Z}$. In this case, Theorem 2.1 and Corollary 2.4 are inutile. Theorem 2.1 can be extended to odd-degree cohomology insertions at the cost of adding signs for each summand depending on the permutation of the odd-degree insertions. In particular, the formula of Theorem 2.1 is valid without any changes if there is only one odd-degree insertion, $\mu$ or $\mu_i$. Corollary 2.4 extends to odd-degree insertions as a reduction to invariants with at most one even-degree insertion which does not increase the number of odd insertions.

Theorem 2.1 can be extended to the real GW-invariants with real marked points defined in [10]. These invariants are defined by intersecting with the pull-back of a homology class $\Gamma$ from the corresponding Deligne-Mumford space of real curves by the forgetful map; see [10, Section 1]. Since the proof of Theorem 2.1 is essentially intersection theory, it readily fits with the definition of the invariants in [10]. The analogue of the right-hand side of the formula in Theorem 2.1 would then involve Kunneth-style splitting of $\Gamma$ between the real and complex GW-invariants represented by the diagrams in Figures 2, 6, and 7 and all splittings of $\{3,\ldots,k\}$ into three subsets $I^+,I^-,J$.

It would no longer be possible to merge $I^+$ and $I^-$ into a single subset, as done in the proof of Theorem 2.1. For a related reason, Theorem 2.2 does not extend to GW-invariants with real marked points.

*Remark* 2.7. The homomorphism (2.2) factors through a similar doubling homomorphism

$$\mathfrak{d}\colon H_2(X,X^\phi;\mathbb{Z})\longrightarrow H_2(X)_\phi;$$

see [10, Section 1]. Let

$$\partial\colon H_2(X,X^\phi;\mathbb{Z})\longrightarrow H_1(X^\phi;\mathbb{Z})$$

denote the boundary homomorphism. By the proof of [10, Theorem 6.5] and [12, Corollary 5.9], $\overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,\tau}$ is orientable if

($\mathcal{O}'_\tau$) $X^\phi$ is orientable and there exist $\varpi\in H^2(X;\mathbb{Z}_2)$ and $\kappa\in H^1(X^\phi;\mathbb{Z}_2)$ such that

$$\begin{gathered}w_2(TX^\phi)=\kappa^2+\varpi|_{X^\phi}\qquad\text{and}\\ \frac{1}{2}\langle c_1(X),\mathfrak{d}(\beta')\rangle+\langle\varpi,\mathfrak{d}(\beta')\rangle+\langle\kappa,\partial\beta'\rangle\in 2\mathbb{Z}\quad\forall\ \beta'\in H_2(X,X_\phi;\mathbb{Z})\,.\end{gathered}\tag{2.16}$$

The two requirements on $(\varpi,\kappa)$ imply that $(X,X^\phi)$ admits a relatively spin sub-structure in the sense of [12, Definition 5.5] and that it can be chosen so that the $(\phi,\tau)$-moduli space is orientable, respectively. The relatively spin condition of [7, Theorem 8.1.1] is the $\kappa=0$ case of the first requirement; ($\mathcal{O}_\tau$) is effectively the $\kappa=0$, $\varpi=w_2(E)$ case of ($\mathcal{O}'_\tau$). Theorems 2.1 and 2.2 and Corollaries 2.4 and 2.5 can be extended with the assumption ($\mathcal{O}_\tau$) relaxed to ($\mathcal{O}'_\tau$). The key difference is that this would introduce the sign $(-1)^{\langle\varpi,\beta_1\rangle}$ over $\mathcal{N}_{\beta_1,\beta_2}$ as happens in [14]. This sign can be absorbed into the numbers (2.9) whenever (2.5) is satisfied, but this would result in a different dependence on the complex GW-invariants for the $\tau$- and $\eta$-invariants. It can be absorbed into the numbers (2.6) with $c=\tau$, similarly to (1.4), if $\phi^*\varpi=\varpi$; this is the case for $\tau$-relatively spin structures in the sense of [8, Definition 3.11]. We avoid such a sign modification in (2.6) by treating $(\mathbb{P}^{2n-1},\tau_{2n})$ as a special case of ($\mathcal{O}_\tau$).

Throughout this section and Sections 4-6, the moduli spaces $\overline{\mathfrak{M}}_{0,k}(X,\beta)^{\phi,c}$ and their glued, constrained, and complex versions refer to regularizations of these spaces. If $(X,\omega,\phi)$ is semi-positive in the sense of [33, Definition 1.2], e.g. $\mathbb{P}^{2n-1}$, the latter are obtained by choosing a generic $J\in\mathcal{J}_\omega^\phi$ and the invariants are defined through pairing with the pseudocycles determined by the moduli spaces. Both proofs of Theorem 2.1, outlined at the end of Section 1, are completely geometric in this case and have no relation to virtual fundamental class (VFC) constructions. The situation with the first proof in the general case is analogous to that with the WDVV and Getzler's relations in complex GW-theory: as the relation of Proposition 3.3 is universal (induced from the moduli of domains), its validity is independent of the choice of VFC construction and depends only on properties of GW-invariants any such construction must yield to be relevant. The relevant properties are the $g=0$ case of Kontsevich-Manin's axioms 2.2.0 (*Effectivity*), 2.2.1 (*$S_n$-invariance*), 2.2.2 (*Grading*), 2.2.4 (*Divisor*), and 2.2.6 (*Splitting*) in [22] and their real analogues (*Splitting* at interior nodes only). Suitable adaptations to the real case of the usual VFC constructions of [24, 6] are carried out in [28, Section 7], [8, Section 7], and [5, Section 2.3]. The invariants arising from these adaptations satisfy the real analogues of the first three axioms above for trivial reasons. The proofs of the last two axioms in the complex case readily extend to the real case.

## 3 A homology relation for $\mathbb{R}\overline{\mathcal{M}}_{0,3}$

In this section, we formulate and prove a codimension 2 relation on the Deligne-Mumford moduli space $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ of real genus 0 curves with 3 pairs of marked points; see Proposition 3.3. Its proof involves a detailed topological description of $\mathbb{R}\overline{\mathcal{M}}_{0,3}$.

For $c=\tau,\eta$ and $k\in\mathbb{Z}^+$, denote by $\overline{\mathcal{M}}^c_{0,k+1}$ the moduli space of $c$-real rational curves with $k+1$ conjugate pairs of marked points. As it is convenient to designate one of the pairs as principal, we index the pairs by the set $\{0,1,\ldots,k\}$ and view 0 as the principal index. Thus, the main stratum of $\overline{\mathcal{M}}^c_{0,k+1}$ is the quotient of

$$\big\{\big((z_0^+,z_0^-),(z_1^+,z_1^-),\ldots,(z_k^+,z_k^-)\big):\ z_i^\pm\in\mathbb{P}^1,\ z^+=c(z_i^-),\ z_i^+\neq z_j^+,z_j^-\ \forall\,i\neq j,\ z_i^+\neq z_i^-\big\}$$

by the natural action of the subgroup $\mathrm{PSL}_2^c\mathbb{C}\subset\mathrm{PSL}_2\mathbb{C}$ of automorphisms of $\mathbb{P}^1$ commuting with $c$. Many notions concerning $\overline{\mathcal{M}}^c_{0,k+1}$ are defined below with respect to the index 0, which in these cases is implicitly understood.

The moduli spaces $\overline{\mathcal{M}}^\eta_{0,k+1}$ and $\overline{\mathcal{M}}^\tau_{0,k+1}$ are $(2k-1)$-dimensional manifolds with the same boundary,

$$\partial\overline{\mathcal{M}}^\tau_{0,k+1}=\partial\overline{\mathcal{M}}^\eta_{0,k+1}\,.$$

The latter consists of the curves with no irreducible component fixed by the involution; the strata of $\overline{\mathcal{M}}^\tau_{0,k+1}$ with two invariant bubbles attached at a real node are of codimension 1, but not a boundary for this space. Gluing along the common boundary, we obtain the moduli space

$$\overline{\mathcal{M}}^{\mathbb{R}}_{0,k+1}\equiv\overline{\mathcal{M}}^\tau_{0,k+1}\cup\overline{\mathcal{M}}^\eta_{0,k+1}\,;$$

it is a $(2k-1)$-dimensional manifold without boundary. We will use $\mathbb{R}\overline{\mathcal{M}}_{0,k+1}$ to refer to any one of these three moduli spaces and $(z_i^+,z_i^-)$ to denote the $i$-th conjugate pair of marked points.

If $c=\tau,\eta$, $\overline{\mathcal{M}}^c_{0,2}$ is a compact connected one-dimensional manifold with boundary and is therefore an interval. It has a canonical orientation induced by requiring the boundary point corresponding to the two-component curve with the marked points $z_0^+$ and $z_1^-$ on the same component to be the initial point of the interval. An explicit orientation-preserving isomorphism is given by the cross-ratio

$$\overline{\mathcal{M}}^c_{0,2}\longrightarrow\mathbb{I}\equiv[0,\infty],\qquad\big[(z_0^+,z_0^-),(z_1^+,z_1^-)\big]\longrightarrow(-1)^c\,\frac{z_1^+-z_0^+}{z_1^--z_0^+}:\frac{z_1^+-z_0^-}{z_1^--z_0^-}\,,\tag{3.1}$$
$$\text{where}\qquad(-1)^c=\begin{cases}1,&\text{if }c=\tau;\\-1,&\text{if }c=\eta;\end{cases}$$

with $z_0^+=0$, the above element of $\overline{\mathcal{M}}^c_{0,2}$ is sent to $|z_1^+|^2$. For $k\geqslant2$, $\overline{\mathcal{M}}^c_{0,k+1}$ is oriented using the first element in each conjugate pair $(z_i^+,z_i^-)$ with $i\geqslant2$ to orient the general fiber of the forgetful morphism $\overline{\mathcal{M}}^c_{0,k+1}\longrightarrow\overline{\mathcal{M}}^c_{0,2}$. Since the boundaries of $\overline{\mathcal{M}}^\eta_{0,k+1}$ and $\overline{\mathcal{M}}^\tau_{0,k+1}$ are oriented in the same way, we obtain an orientation on $\overline{\mathcal{M}}^{\mathbb{R}}_{0,k+1}$ by reversing the orientation on $\overline{\mathcal{M}}^\eta_{0,k+1}$. An explicit orientation-preserving isomorphism of $\overline{\mathcal{M}}^{\mathbb{R}}_{0,2}$ with $S^1\equiv\mathbb{R}\sqcup\{\infty\}$ is given by the map in (3.1) with $(-1)^c$ dropped. The general fibers of the forgetful morphism $\overline{\mathcal{M}}^{\mathbb{R}}_{0,k+1}\longrightarrow\overline{\mathcal{M}}^{\mathbb{R}}_{0,2}$ are again oriented using the first element in each conjugate pair $(z_i^+,z_i^-)$ with $i\geqslant2$.

**Lemma 3.1.** *Let $k\in\mathbb{Z}^+$ and let $\mathbb{R}\overline{\mathcal{M}}_{0,k+1}$ denote $\overline{\mathcal{M}}_{0,k+1}^{\tau}$, $\overline{\mathcal{M}}_{0,k+1}^{\eta}$, or $\overline{\mathcal{M}}_{0,k+1}^{\mathbb{R}}$.*

*(1) For every $i=0,1,\ldots,k$, the automorphism of $\mathbb{R}\overline{\mathcal{M}}_{0,k+1}$ interchanging the marked points in the $i$-th conjugate pair is orientation-reversing.*

*(2) For all $i,j=0,1,\ldots,k$, the automorphism of $\mathbb{R}\overline{\mathcal{M}}_{0,k+1}$ interchanging the $i$-th and $j$-th conjugate pairs of marked points is orientation-preserving.*

*Proof.* (1) For $i=0,1$, this automorphism interchanges the two boundary points of $\overline{\mathcal{M}}_{0,2}^{c}$ with $c=\tau,\eta$. Thus, it is orientation-reversing on the base of the forgetful morphism

$$\mathbb{R}\overline{\mathcal{M}}_{0,k+1}\longrightarrow\mathbb{R}\overline{\mathcal{M}}_{0,2} \tag{3.2}$$

for every $k\geqslant 1$. Since this automorphism takes a general fiber of (3.2) to another general fiber in an orientation-preserving way, it is orientation-reversing on $\mathbb{R}\overline{\mathcal{M}}_{0,k+1}$. For $i\geqslant 2$, the automorphism of $\mathbb{R}\overline{\mathcal{M}}_{0,k+1}$ interchanging the marked points in the $i$-th conjugate pair takes a general fiber of (3.2) to itself in an orientation-reversing way. Thus, it is again orientation-reversing on $\mathbb{R}\overline{\mathcal{M}}_{0,k+1}$.

(2) If $i,j\leqslant 1$ or $i,j\geqslant 2$, the automorphism of $\mathbb{R}\overline{\mathcal{M}}_{0,k+1}$ interchanging $i$-th and $j$-th conjugate pairs of marked points takes a general fiber of (3.2) to itself in an orientation-preserving way and so is orientation-preserving on $\mathbb{R}\overline{\mathcal{M}}_{0,k+1}$. Thus, it remains to consider the case $i=1$ and $j=2$. Since the corresponding automorphism of $\mathbb{R}\overline{\mathcal{M}}_{0,k+1}$, with $k\geqslant 2$, takes a general fiber of the forgetful morphism $\mathbb{R}\overline{\mathcal{M}}_{0,k+1}\longrightarrow\mathbb{R}\overline{\mathcal{M}}_{0,3}$ to itself in an orientation-preserving way, it is sufficient to check that it is orientation-preserving for $k=2$. The latter is the case if and only if the forgetful morphisms

$$f_1,f_2\colon\mathbb{R}\overline{\mathcal{M}}_{0,3}\longrightarrow\mathbb{R}\overline{\mathcal{M}}_{0,2}\,,\qquad \begin{aligned} f_1\big([(z_0^+,z_0^-),(z_1^+,z_1^-),(z_2^+,z_2^-)]\big)&=\big[(z_0^+,z_0^-),(z_1^+,z_1^-)\big],\\ f_2\big([(z_0^+,z_0^-),(z_1^+,z_1^-),(z_2^+,z_2^-)]\big)&=\big[(z_0^+,z_0^-),(z_2^+,z_2^-)\big],\end{aligned} \tag{3.3}$$

induce the same orientation on $\mathbb{R}\overline{\mathcal{M}}_{0,3}$.

It is enough to check that $f_1$ and $f_2$ induce the same orientation on the tangent space at a three-component curve $\mathcal{C}$ with $z_1^+$ and $z_2^+$ on the same bubble component $\mathcal{C}^{\mathbb{C}}$, i.e. as in the first diagram in Figure 2, but with the label 0 interchanged with 2 and the label $\bar{0}$ interchanged with $\bar{2}$. The restrictions of $f_1$ and $f_2$ to the space $\Gamma$ of such curves are the same and take $\Gamma$ isomorphically onto $\mathbb{R}\overline{\mathcal{M}}_{0,2}$; thus, $f_1$ and $f_2$ induce the same orientations on $T_{\mathcal{C}}\Gamma$. The vertical tangent bundles of $f_1$ and $f_2$ along $\mathcal{C}$ are canonically isomorphic to the normal bundle of $\Gamma$ in $\mathbb{R}\overline{\mathcal{M}}_{0,3}$. The orientation of the fiber of the vertical tangent bundle of $f_1$ at $\mathcal{C}$ given by varying $z_2^+$ is the complex orientation of the tangent node of the real component $\mathcal{C}^{\mathbb{R}}$ of $\mathcal{C}$ at the node separating $\mathcal{C}^{\mathbb{R}}$ from $\mathcal{C}^{\mathbb{C}}$. The same is the case for the orientation of the vertical tangent bundle of $f_2$ at $\mathcal{C}$ given by varying $z_1^+$. Thus, the orientations of the normal bundle of $\Gamma$ in $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ with respect to the orientations induced by $f_1$ and $f_2$ are the same. This implies that the orientations induced by $f_1$ and $f_2$ on $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ are the same as well. $\square$

For $i=1,2$, let $\Gamma_i\subset\mathbb{R}\overline{\mathcal{M}}_{0,3}$ denote the closure of the subset $\mathring{\Gamma}_i$ consisting of the three-component real curves $(\mathcal{C},c)$ such that the marked point $z_i^+$ lies on the same component as the marked point $z_0^+$. Let $\Gamma_{\bar{i}}\subset\mathbb{R}\overline{\mathcal{M}}_{0,3}$ denote the closure of the subset $\mathring{\Gamma}_{\bar{i}}$ consisting of the three-component real curves $(\mathcal{C},c)$ such that the marked point $z_i^-$ lies on the same component as the marked point $z_0^+$. The stability condition implies that such a three-component curve has

($\mathbb{R}$) a component $\mathcal{C}^{\mathbb{R}}$ preserved by $c$ and containing the conjugate pair $(z_{3-i}^+, z_{3-i}^-)$ of marked points and a conjugate pair $(z_\bullet^+, z_\bullet^-)$ of nodes, and

($\mathbb{C}$) a pair of conjugate components, with the component $\mathcal{C}^{\mathbb{C}}$ containing the marked point $z_0^+$ also carrying the marked point $z_i^+$ in the case of $\Gamma_i$ and $z_i^-$ in the case of $\Gamma_{\bar{i}}$;

see Figures 3-5. We will take $z_\bullet^+ \in \mathcal{C}^{\mathbb{R}}$ to be the node identified with a point $z^{\mathbb{C}} \in \mathcal{C}^{\mathbb{C}}$. Thus, there are canonical isomorphisms

$$\Gamma_i \approx \mathbb{R}\overline{\mathcal{M}}_{0,2} \times \overline{\mathcal{M}}_{0,3} \qquad \text{and} \qquad \Gamma_{\bar{i}} \approx \mathbb{R}\overline{\mathcal{M}}_{0,2} \times \overline{\mathcal{M}}_{0,3}^- , \tag{3.4}$$

where the superscript $-$ indicates that one of the marked points (the one corresponding to $z_i^-$) is decorated with the minus sign. Following the principle introduced in [10], we define the canonical orientation of $\overline{\mathcal{M}}_{0,3}^-$ to be the opposite of the canonical (complex) orientation of $\overline{\mathcal{M}}_{0,3}$ and then use (3.4) to orient $\Gamma_i$ and $\Gamma_{\bar{i}}$. Thus, the orientation on $\Gamma_i$ is the same as the one induced by the natural isomorphism $\Gamma_i \approx \mathbb{R}\overline{\mathcal{M}}_{0,2}$, while the orientation on $\Gamma_{\bar{i}}$ is the opposite of the one induced by the natural isomorphism $\Gamma_{\bar{i}} \approx \mathbb{R}\overline{\mathcal{M}}_{0,2}$. The canonical orientation of $\mathbb{R}\overline{\mathcal{M}}_{0,2}$ is defined above, but this choice of the orientation does not affect the validity of Lemma 3.2 or Proposition 3.3. Whenever $\mathbb{R}\overline{\mathcal{M}}_{0,3} = \overline{\mathcal{M}}_{0,3}^c$ for a specific $c = \tau, \eta, \mathbb{R}$, we will write $\Gamma_*^c$, where $* = i, \bar{i}$ with $i = 1, 2$, for $\Gamma_*$.

With $\Gamma = \Gamma_i, \Gamma_{\bar{i}}$, $i = 1, 2$, as in (3.4), let

$$L_\Gamma^{\mathbb{R}} \longrightarrow \mathbb{R}\overline{\mathcal{M}}_{0,2} \qquad \text{and} \qquad L_\Gamma^{\mathbb{C}} \longrightarrow \overline{\mathcal{M}}_{0,3}, \overline{\mathcal{M}}_{0,3}^-$$

be the universal tangent line bundles at the marked points $z_\bullet^+$ and $z^{\mathbb{C}}$, respectively, and

$$L_\Gamma = \pi_1^* L_\Gamma^{\mathbb{R}} \otimes_{\mathbb{C}} \pi_2^* L_\Gamma^{\mathbb{C}} \longrightarrow \Gamma ,$$

where $\pi_1, \pi_2$ are the component projection maps.

**Lemma 3.2.** *Let $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ denote $\overline{\mathcal{M}}_{0,3}^\tau$, $\overline{\mathcal{M}}_{0,3}^\eta$, or $\overline{\mathcal{M}}_{0,3}^{\mathbb{R}}$.*

*(1) For $i = 1, 2$, the automorphism of $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ interchanging the marked points in the $i$-th conjugate pair restricts to an orientation-reversing isomorphism from $\Gamma_i$ to $\Gamma_{\bar{i}}$ and canonically lifts to a $\mathbb{C}$-linear isomorphism from $L_{\Gamma_i}$ to $L_{\Gamma_{\bar{i}}}$.*

*(2) The automorphism of $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ interchanging the 1st and 2nd conjugate pairs of marked points restricts to an orientation-preserving isomorphism from $\Gamma_1$ to $\Gamma_2$ and canonically lifts to a $\mathbb{C}$-linear isomorphism from $L_{\Gamma_1}$ to $L_{\Gamma_2}$.*

*(3) For $i = 1, 2$, the oriented normal bundle of $\mathring{\Gamma} = \mathring{\Gamma}_i, \mathring{\Gamma}_{\bar{i}}$ in $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ is isomorphic to $L_\Gamma$ with its canonical complex orientation.*

*Proof.* (1,2) It is immediate that the automorphism in (1) interchanges $\Gamma_i$ and $\Gamma_{\bar{i}}$ and the automorphism in (2) interchanges $\Gamma_1$ and $\Gamma_2$. These restrictions respect the component moduli spaces in (3.4) and induce the identity on the first component (the second component is a point). Given our choice of orientations, the domain and target orientations of the automorphism in (1) are opposite, while the domain and target orientations of the automorphism in (2) are the same. This implies the first parts of the first two statements in the lemma. Since these automorphisms respect the component moduli spaces in (3.4), they canonically lift to all universal tangent line bundles for these moduli spaces and thus to $L_{\Gamma_i}$. They act by the identity on the tangent spaces at $z_\bullet^+$ and $z^{\mathbb{C}}$ and thus $\mathbb{C}$-linearly on $L_\Gamma$.

(3) The restriction of the forgetful morphism $f_1$ in (3.3) to $\Gamma_2$ is an orientation-preserving isomorphism. By the definition of the orientation on $\mathbb{R}\overline{\mathcal{M}}_{0,3}$, the vertical tangent bundle along $\mathring{\Gamma}_2$ is thus oriented by the complex orientation of $L^{\mathbb{C}}_{\Gamma_2} \approx L_{\Gamma_2}$. Since the vertical tangent bundle of $f_1$ along $\mathring{\Gamma}_2$ is canonically isomorphic to the normal bundle of $\Gamma_2$ in $\mathbb{R}\overline{\mathcal{M}}_{0,3}$, this implies the last statement of the lemma in the $\mathring{\Gamma} = \mathring{\Gamma}_2$ case. The remaining three cases follow from this case, the first two statements of the lemma, and the $k\!=\!2$ case of Lemma 3.1. $\square$

**Proposition 3.3.** *Let $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ denote $\overline{\mathcal{M}}^\tau_{0,3}$, $\overline{\mathcal{M}}^\eta_{0,3}$, or $\overline{\mathcal{M}}^{\mathbb{R}}_{0,3}$. The submanifolds $\Gamma_1, \Gamma_{\bar{1}}, \Gamma_2, \Gamma_{\bar{2}}$ of $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ determine relative cycles in $(\mathbb{R}\overline{\mathcal{M}}_{0,3}, \partial\mathbb{R}\overline{\mathcal{M}}_{0,3})$ and*

$$[\Gamma_1] + [\Gamma_{\bar{1}}] = [\Gamma_2] + [\Gamma_{\bar{2}}] \in H_1\big(\mathbb{R}\overline{\mathcal{M}}_{0,3}, \partial\mathbb{R}\overline{\mathcal{M}}_{0,3}; \mathbb{Q}\big). \tag{3.5}$$

Since $\partial\Gamma_i$ and $\partial\Gamma_{\bar{i}}$ are contained in $\partial\mathbb{R}\overline{\mathcal{M}}_{0,3}$, only the second statement of this proposition remains to be established. The relation (3.5) in fact holds over $\mathbb{Z}$; though we do not need this stronger statement, we give two separate reasons for it in Remarks 3.4 and 3.5.

***Proof for $\overline{\mathcal{M}}^\eta_{0,3}$.*** The boundary of $\overline{\mathcal{M}}^\eta_{0,3}$ has four components, which we denote by $S_{12}$, $S_{1\bar{2}}$, $S_{\bar{1}2}$, and $S_{\bar{1}\bar{2}}$, which contain the two-component curves with the points $\{z_1^+, z_2^+\}$, $\{z_1^+, z_2^-\}$, $\{z_1^-, z_2^+\}$, and $\{z_1^-, z_2^-\}$, respectively, on the same component as the base point $z_0^+$; each of them is isomorphic to $S^2$. The forgetful morphism

$$\overline{\mathcal{M}}^\eta_{0,3} \longrightarrow \overline{\mathcal{M}}^\eta_{0,2} \approx \mathbb{I} \equiv [0, \infty] \tag{3.6}$$

is a singular fibration; see Figure 3. The fiber over every interior point is a sphere with four special points corresponding to the strata where $z_2^+$ collides with $z_0^+$, $z_0^-$, $z_1^+$, or $z_1^-$. The fiber over the boundary point $0\!\in\!\mathbb{I}$ consists of the spheres $S_{12}$ and $S_{1\bar{2}}$ joined together by the interval $\Gamma^\eta_1$ defined above. The fiber over the boundary point $\infty\!\in\!\mathbb{I}$ consists of the spheres $S_{\bar{1}2}$ and $S_{\bar{1}\bar{2}}$ joined together by the interval $\Gamma^\eta_{\bar{1}}$. The lines $\Gamma^\eta_2$ and $\Gamma^\eta_{\bar{2}}$ connect the boundary spheres in the two fibers: $S_{12}$ with $S_{\bar{1}2}$ and $S_{1\bar{2}}$ with $S_{\bar{1}\bar{2}}$, respectively.

Let

$$H_1(\overline{\mathcal{M}}^\eta_{0,3}) \xrightarrow{j_*} H_1(\overline{\mathcal{M}}^\eta_{0,3}, \partial\overline{\mathcal{M}}^\eta_{0,3}) \xrightarrow{\partial} H_0(\partial\overline{\mathcal{M}}^\eta_{0,3})$$

denote the homomorphisms in the homology long exact sequence for the pair $(\overline{\mathcal{M}}^\eta_{0,3}, \partial\overline{\mathcal{M}}^\eta_{0,3})$. With the canonical orientations on $\Gamma^\eta_i$ and $\Gamma^\eta_{\bar{i}}$ described above

$$\partial\big[\Gamma^\eta_1 - \Gamma^\eta_{\bar{2}} + \Gamma^\eta_{\bar{1}} - \Gamma^\eta_2\big] = 0;$$

see Figure 3. Thus, $[\Gamma^\eta_1 - \Gamma^\eta_{\bar{2}} + \Gamma^\eta_{\bar{1}} - \Gamma^\eta_2]$ is the image of an element of $H_1(\overline{\mathcal{M}}^\eta_{0,3})$ under $j_*$. A representative $\Gamma^\eta$ for this class is obtained by connecting the end points of the line segments inside each boundary sphere. This loop can be homotoped away from the fibers over $0, \infty \in \mathbb{I}$ by smoothing out the nodes. The resulting loop in $S^2\!\times\!\mathbb{R}^+$ is therefore contractible and hence is trivial in $H_1(\overline{\mathcal{M}}^\eta_{0,3})$. This implies (3.5) in the $\eta$ case. $\square$

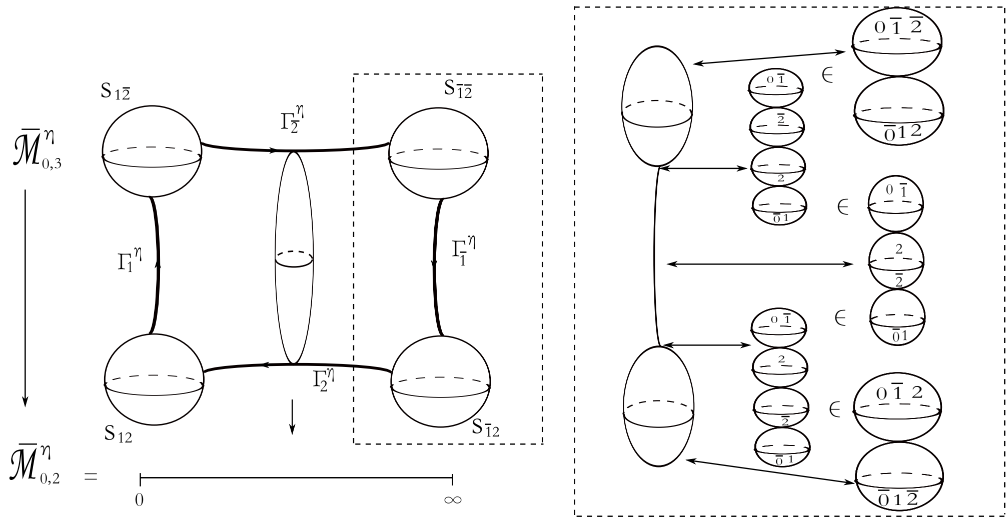


Figure 3: The moduli space $\overline{\mathcal{M}}^{\eta}_{0,3}$ as a fibration over $\overline{\mathcal{M}}^{\eta}_{0,2}$; the labels $i$ and $\bar{i}$ indicate the marked points $z_i^+$ and $z_i^-$, respectively.

***Proof for $\overline{\mathcal{M}}^{\tau}_{0,3}$.*** In comparison with (3.6), the forgetful morphism

$$\overline{\mathcal{M}}^{\tau}_{0,3} \longrightarrow \overline{\mathcal{M}}^{\tau}_{0,2} \approx \mathbb{I} \equiv [0,\infty]$$

has an additional singular value: the point $1\in\mathbb{I}$ corresponding to the two-component curve with $z_0^+$ and $z_1^+$ on separate invariant bubbles; see Figure 4. The fiber $F_1$ over this point consists of two copies of $\mathbb{RP}^2$ joined along a non-contractible circle in each copy or equivalently the quotient of $S^2$ by the action of the antipodal map on the equator only. The complement of the common circle in one copy of $\mathbb{RP}^2$ consists of the two-component curves, with each component fixed by the involution, with $z_2^+$ on the same component as $z_0^+$; the complement in the other copy consists of the two-component curves, with each component fixed by the involution, with $z_2^+$ on the same component as $z_1^+$. The circle corresponds to the three-component curves with each component fixed by the involution and $z_2^+$ on the middle component; see Figure 4.

By the same reasoning as in the $\eta$ case, the class $[\Gamma^{\tau}_1 - \Gamma^{\tau}_{\bar{2}} + \Gamma^{\tau}_{\bar{1}} - \Gamma^{\tau}_2]$ is in the image of an element in $H_1(\overline{\mathcal{M}}^{\tau}_{0,3})$ which can be represented by a loop in $\overline{\mathcal{M}}^{\tau}_{0,3}$ away from the fibers over $0,\infty\in\mathbb{I}$. This loop can be homotoped to a loop in the special fiber $F_1$. Since $\pi_1(F_1)\approx\mathbb{Z}_2$, it still represents the zero class in $H_1(\overline{\mathcal{M}}^{\tau}_{0,3})$ with $\mathbb{Q}$-coefficients. $\square$

***Proof for $\overline{\mathcal{M}}^{\mathbb{R}}_{0,3}$.*** The fibers of the forgetful morphism

$$\overline{\mathcal{M}}^{\mathbb{R}}_{0,3} \longrightarrow \overline{\mathcal{M}}^{\mathbb{R}}_{0,2} \approx S^1 \tag{3.7}$$

away from the identification points of $\overline{\mathcal{M}}^{\eta}_{0,2}$ and $\overline{\mathcal{M}}^{\tau}_{0,2}$ are as described in the $\eta,\tau$ cases; see Figure 5. A fiber over either of the two identification points, $0,\infty\in\mathbb{I}$, consists of two spheres joined by a circle. The submanifolds $\Gamma^{\mathbb{R}}_*$ with $*=1,\bar{1},2,\bar{2}$ form 4 loops in $\overline{\mathcal{M}}^{\mathbb{R}}_{0,3}$:

$$\Gamma^{\mathbb{R}}_1 = \Gamma^{\tau}_1 - \Gamma^{\eta}_1, \qquad \Gamma^{\mathbb{R}}_{\bar{1}} = \Gamma^{\tau}_{\bar{1}} - \Gamma^{\eta}_{\bar{1}}, \qquad \Gamma^{\mathbb{R}}_2 = \Gamma^{\tau}_2 - \Gamma^{\eta}_2, \qquad \Gamma^{\mathbb{R}}_{\bar{2}} = \Gamma^{\tau}_{\bar{2}} - \Gamma^{\eta}_{\bar{2}}.$$

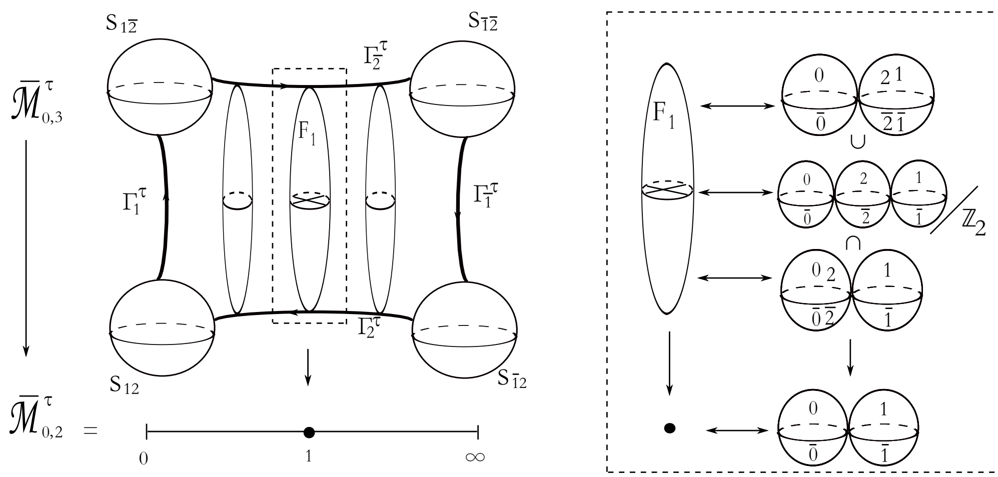


Figure 4: The moduli space $\overline{\mathcal{M}}^{\tau}_{0,3}$ as a fibration over $\overline{\mathcal{M}}^{\tau}_{0,2}$; the labels $i$ and $\bar{i}$ indicate the marked points $z_i^+$ and $z_i^-$, respectively.

Connecting the points of these loops on each of the four spheres by paths as before, we obtain the loops $\Gamma^{\eta}\subset\overline{\mathcal{M}}^{\eta}_{0,3}$ and $\Gamma^{\tau}\subset\overline{\mathcal{M}}^{\tau}_{0,3}$ as in the $\eta,\tau$ cases above so that

$$\big[\Gamma^{\mathbb{R}}_1-\Gamma^{\mathbb{R}}_{\bar{2}}+\Gamma^{\mathbb{R}}_{\bar{1}}-\Gamma^{\mathbb{R}}_2\big]=[\Gamma^{\tau}]-[\Gamma^{\eta}].$$

By the $\eta,\tau$ cases above, $[\Gamma^{\eta}]$ and $[\Gamma^{\tau}]$ are zero in $H_1(\overline{\mathcal{M}}^{\eta}_{0,3})$ and $H_1(\overline{\mathcal{M}}^{\tau}_{0,3})$, respectively. $\square$

*Remark* 3.4. The same argument can be used to obtain 3-term relations in $H_1(\mathbb{R}\overline{\mathcal{M}}_{0,3},\partial\mathbb{R}\overline{\mathcal{M}}_{0,3})$ by going diagonally in Figures 3-5. While these relations are nominally stronger than (3.5), we do not see any applications for them at this point and they have a less appealing appearance than (3.5). On the other hand, they can be used to conclude that (3.5) holds over $\mathbb{Z}$ as follows. Let $\alpha$ denote a nontrivial loop in the fiber $F_1$ in the proof of the $\tau$ case of Proposition 3.3. The loops formed by the upper left and lower right triangles equal to $\varepsilon_l\alpha$ and $\varepsilon_r\alpha$ in homology, for some $\varepsilon_l,\varepsilon_r\in\{0,1\}$. Pulling back the loops to $\overline{\mathfrak{M}}^{\tau}_{0,3}(\mathbb{P}^2,1)$ by (4.1), evaluating on 3 conjugate pairs of lines over $\mathbb{Z}_2$, and using Proposition 4.2, we find that each of the three segments in each of the triangles contributes $1\in\mathbb{Z}_2$ (the number of real lines through a non-real point) to the total count for the triangle. Thus, $\varepsilon_l,\varepsilon_r=1$ (the preimage of the loop $\alpha$ in fact corresponds to the number of real lines through 2 real points in $\mathbb{P}^2$). This implies that the loop $\Gamma^{\tau}_1-\Gamma^{\tau}_{\bar{2}}+\Gamma^{\tau}_{\bar{1}}-\Gamma^{\tau}_2$, which is the sum of the two triangular loops, is contractible. Thus, (3.5) holds over $\mathbb{Z}$.

*Remark* 3.5. A local model for (3.7) near the intersection point of an $S^2$ and $S^1$ in the same fiber is given by

$$\mathbb{R}\times\mathbb{C}\longrightarrow\mathbb{R},\qquad (t,z)\longrightarrow t|z|^2\,. \tag{3.8}$$

Local models for (3.7) around $S^2\equiv\mathbb{C}\sqcup\{\infty\}$ and $S^1\equiv\mathbb{R}\sqcup\{\infty\}$ are given by

$$S^2\times\mathbb{R}\longrightarrow\mathbb{R},\quad (z,t)\longrightarrow\frac{2t}{1+|z|^2}\,,\qquad S^1\times\mathbb{C}\longrightarrow\mathbb{R},\quad (t,z)\longrightarrow\frac{2|z|^2}{t+t^{-1}}\,. \tag{3.9}$$

The remaining singular fiber of (3.7) is obtained by blowing up a point of another compact orientable 3-manifold $\widetilde{\mathcal{M}}^{\mathbb{R}}_{0,3}$. The latter is isomorphic to the orientable "double connect-sum" of two copies of

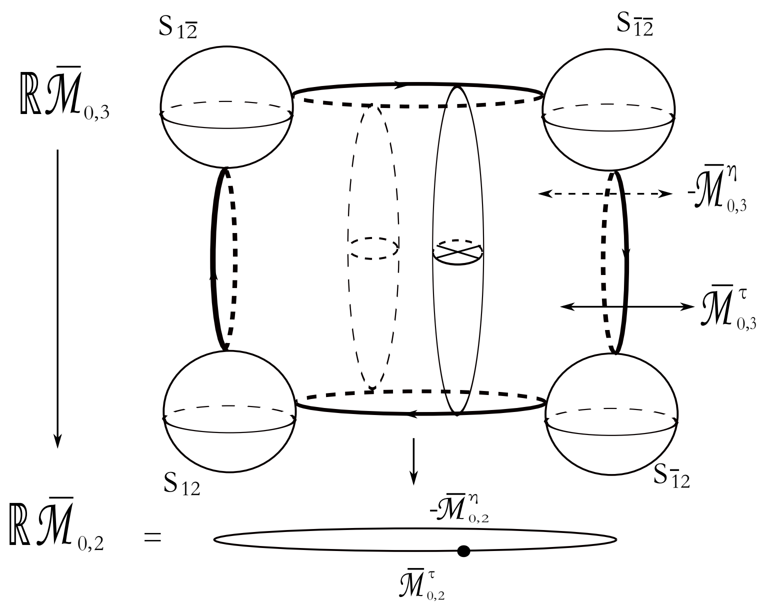


Figure 5: The moduli space $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ as a fibration over $\mathbb{R}\overline{\mathcal{M}}_{0,2}$; the labels $i$ and $\bar{i}$ indicate the marked points $z_i^+$ and $z_i^-$, respectively.

$S^1\times S^2$, i.e. the manifold obtained by removing two disjoint three-balls from each copy of $S^1\times S^2$ and gluing the two copies together along the common boundary so that the glued manifold is orientable. The manifold $\widetilde{\mathcal{M}}_{0,3}^{\mathbb{R}}$ can be obtained by contracting the second copy of $\mathbb{RP}^2$ described in the proof of the $\tau$ case of Proposition 3.3; the loop $\Gamma_1^\tau-\Gamma_2^\tau+\Gamma_{\bar{1}}^\tau-\Gamma_{\bar{2}}^\tau$ then arises from a contractible loop in the complement of the blowup point in $\widetilde{\mathcal{M}}_{0,3}^{\mathbb{R}}$ and thus is contractible in $\overline{\mathcal{M}}_{0,3}^{\tau}$. This implies that (3.5) holds over $\mathbb{Z}$.

# 4 Proof of Theorem 2.1

The relation on $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ of Proposition 3.3 induces relations between counts of real maps from nodal domains into a real symplectic manifold $(X,\omega,\phi)$; see Corollary 4.1. Proposition 4.2, which is proved in Section 5, expresses these counts in terms of real GW-invariants and a decorated version of complex GW-invariants via the Kunneth splitting of the diagonal $\Delta_X$ in $X^2$. Proposition 4.3, which is also proved in Section 5, relates the decorated invariants to the usual complex GW-invariants. We conclude this section by deducing Theorem 2.1 from Corollary 4.1 and Propositions 4.2 and 4.3.

Let $(X,\omega,\phi)$ be a compact real symplectic $2n$-manifold, $\beta\in H_2(X)_\phi$, and $k\in\mathbb{Z}$ with $k\geqslant 2$. Let

$$f_{012}^c\colon \overline{\mathfrak{M}}_{0,k+1}(X,\beta)^{\phi,c}\longrightarrow \overline{\mathcal{M}}_{0,3}^c,\quad c=\tau,\eta,\qquad f_{012}^{\mathbb{R}}\colon \overline{\mathfrak{M}}_{0,k+1}(X,\beta)^{\phi}\longrightarrow \overline{\mathcal{M}}_{0,3}^{\mathbb{R}},\tag{4.1}$$

be the forgetful morphisms keeping the first three conjugate pairs of marked points only (i.e. those indexed by 0,1,2). If $c=\tau,\eta$ and $(\mathcal{O}_c)$ and (2.5) in Section 2 are satisfied, we set

$$\mathbb{R}f_{012}=f_{012}^c,\qquad \mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta)=\overline{\mathfrak{M}}_{0,k+1}(X,\beta)^{\phi,c},\qquad \mathbb{R}\overline{\mathcal{M}}_{0,3}=\overline{\mathcal{M}}_{0,3}^c\,.$$

If $(\mathcal{O}_\tau)$ and $(\mathcal{O}_\eta)$ are satisfied, but not (2.5), we set

$$\mathbb{R}f_{012} = f_{012}^{\mathbb{R}}, \qquad \mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta) = \overline{\mathfrak{M}}_{0,k+1}(X,\beta)^{\phi}, \qquad \mathbb{R}\overline{\mathcal{M}}_{0,3} = \overline{\mathcal{M}}_{0,3}^{\mathbb{R}}.$$

In all cases, we index the conjugate pairs of marked points of elements of $\mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta)$ by the set $\{0,1,\ldots,k\}$. For any relative cycle $\Gamma$ in $(\mathbb{R}\overline{\mathcal{M}}_{0,3}, \partial\mathbb{R}\overline{\mathcal{M}}_{0,3})$ and $\mu_0,\ldots,\mu_k \in H^*(X)$, we define

$$\langle \mu_0,\ldots,\mu_k \rangle_\beta^\Gamma = \int_{[\mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta)]^{\mathrm{vir}}} \mathbb{R}f_{012}^*\mathrm{PD}([\Gamma])\,\mathrm{ev}_0^*\mu_0 \,\ldots\, \mathrm{ev}_k^*\mu_k. \tag{4.2}$$

This number counts degree $\beta$ real morphisms into $(X,\phi)$ from domains that stabilize to elements of $\Gamma$ after dropping the conjugate pairs labeled by the set $\{3,\ldots,k\}$. From Proposition 3.3, we immediately obtain the following corollary.

**Corollary 4.1** (of Proposition 3.3)**.** *Let $(X,\omega,\phi)$ be a compact real symplectic manifold, $\beta \in H_2(X)_\phi$, and $\mu_0,\ldots,\mu_k \in H^*(X)$ for some $k \geqslant 2$. If $c=\tau,\eta$ and the conditions $(\mathcal{O}_c)$ and (2.5) in Section 2 are satisfied, then*

$$\langle \mu_0,\ldots,\mu_k \rangle_\beta^{\Gamma_1^c} + \langle \mu_0,\ldots,\mu_k \rangle_\beta^{\Gamma_{\bar{1}}^c} = \langle \mu_0,\ldots,\mu_k \rangle_\beta^{\Gamma_2^c} + \langle \mu_0,\ldots,\mu_k \rangle_\beta^{\Gamma_{\bar{2}}^c}, \tag{4.3}$$

*where $\Gamma_1^c, \Gamma_{\bar{1}}^c, \Gamma_2^c, \Gamma_{\bar{2}}^c$ are the relative cycles in $(\mathbb{R}\overline{\mathcal{M}}_{0,3}^c, \partial\mathbb{R}\overline{\mathcal{M}}_{0,3}^c)$ defined in Section 3 and represented by the diagrams in Figures 3 and 4. If the conditions $(\mathcal{O}_\tau)$ and $(\mathcal{O}_\eta)$ are satisfied, but not necessarily (2.5), then (4.3) holds with $c=\mathbb{R}$.*

We next express the numbers appearing in Corollary 4.1 in terms of complex and real GW-invariants. Let $(X,\omega,\phi)$ be a compact real symplectic $2n$-manifold, $\beta \in H_2(X)_\phi$, and $k \in \mathbb{Z}^{\geqslant 0}$. We denote by

$$\overline{\mathfrak{M}}_{k+1}(\beta) = \overline{\mathfrak{M}}_{0,k+1}(X,\beta)$$

the moduli space of stable genus 0 degree $\beta$ maps with marked points indexed by the set $\{0,1,\ldots,k\}$. For any $I \subset \{1,\ldots,k\}$, let $\overline{\mathfrak{M}}_{k+1}^I(\beta)$ be the space $\overline{\mathfrak{M}}_{k+1}(\beta)$ with the reverse orientation if $|I|$ is odd and let

$$\mathrm{ev}^I : \overline{\mathfrak{M}}_{k+1}^I(\beta) \longrightarrow X^{k+1}$$

be the modification of the total evaluation map

$$\mathrm{ev} \equiv \mathrm{ev}_0 \times \mathrm{ev}_1 \times \ldots \times \mathrm{ev}_k : \overline{\mathfrak{M}}_{k+1}(\beta) \longrightarrow X^{k+1} \tag{4.4}$$

obtained by replacing $\mathrm{ev}_i$ with $\phi \circ \mathrm{ev}_i$ whenever $i \in I$. For any $\mu_0,\ldots,\mu_k \in H^*(X)$, define

$$\langle \mu_0,\ldots,\mu_k \rangle_\beta^I = \int_{[\overline{\mathfrak{M}}_{k+1}^I(\beta)]^{\mathrm{vir}}} \mathrm{ev}^{I*}(\mu_0 \times \ldots \times \mu_k).$$

This setup is motivated by the introduction of sign decorations for disk maps in [10]. The next two propositions are established in Section 5. As before, if $\mu_1,\ldots,\mu_k \in H^*(X)$ and $I \subset \{1,\ldots,k\}$, let $\mu_I$ denote a tuple with the entries $\mu_i$ with $i \in I$, in some order.

**Proposition 4.2.** *Let $(X,\omega,\phi)$ be a compact real symplectic manifold, $\beta \in H_2(X)_\phi$, $\Gamma_1^c, \Gamma_{\bar{1}}^c, \Gamma_2^c, \Gamma_{\bar{2}}^c$ be the relative cycles in $(\mathbb{R}\overline{\mathcal{M}}_{0,3}, \partial\mathbb{R}\overline{\mathcal{M}}_{0,3})$ defined in Section 3 and represented by the diagrams in*

*Figure 2, and $\{\gamma_i\}_{i\leqslant\ell}$ and $\{\gamma^i\}_{i\leqslant\ell}$ be dual bases for $H^*(X)$. If $c=\tau,\eta$ and the conditions $(\mathcal{O}_c)$ and (2.5) in Section 2 are satisfied, then*

$$\begin{aligned}\big\langle\mu_0,\ldots,\mu_k\big\rangle_\beta^{\Gamma_j^c} =&\big\langle\mu_0\mu_j,\mu_{3-j},\mu_3,\ldots,\mu_k\big\rangle_\beta^{\phi,c}\\ &+\sum_{\substack{\mathfrak{d}(\beta_1)+\beta_2=\beta\\ \beta_1,\beta_2\in H_2(X)-\{0\}}}\;\sum_{\substack{I^+\sqcup J\sqcup I^-=\{1,\ldots,k\}\\ j\in I^+,\,3-j\in J}}\;\sum_{\substack{1\leqslant i\leqslant\ell\\ \gamma^i\in H^{2*}(X)_-^\phi}}\big\langle\mu_0,\mu_{I^+\sqcup I^-},\gamma_i\big\rangle_{\beta_1}^{I^-}\big\langle\mu_J,\gamma^i\big\rangle_{\beta_2}^{\phi,c}\end{aligned}\qquad(4.5)$$

*for all $j=1,2$, $k\geqslant 2$, and $\mu_0,\ldots,\mu_k\in H^{2*}(X)$. The same identity also holds with $(\Gamma_j,\mu_0\mu_j,j\in I^+)$ replaced by $(\Gamma_{\bar{j}},-\mu_0\phi^*\mu_j,j\in I^-)$. If the conditions $(\mathcal{O}_\tau)$ and $(\mathcal{O}_\eta)$ are satisfied, but not necessarily (2.5), then the four identities hold with $\Gamma_*^c=\Gamma_*^{\mathbb{R}}$ and $\langle\ldots\rangle^{\phi,c}=\langle\ldots\rangle^\phi$.*

**Proposition 4.3.** *Let $(X,\omega,\phi)$ be a compact real symplectic $2n$-manifold, $\beta\in H_2(X)_\phi$, $k\in\mathbb{Z}^{\geqslant 0}$, and $I\subset\{0,1,\ldots,k\}$. For all $\mu_0,\ldots,\mu_k\in H^*(X)_-^\phi\cup H^*(X)_+^\phi$,*

$$\langle\mu_0,\ldots,\mu_k\rangle_\beta^I=(-1)^{\varepsilon_I(\mu)}\langle\mu_0,\ldots,\mu_k\rangle_\beta^X\,,$$

*where $\varepsilon_I(\mu)=|\{i\in I:\ \mu_i\in H^*(X)_+^\phi\}|$.*

***Proof of Theorem 2.1.*** We apply Corollary 4.1 with $\mu_0\equiv\mu,\mu_1,\ldots,\mu_k$ as in the statement of Theorem 2.1. Since $\mu_i\in H^*(X)_-^\phi$ for all $i=1,\ldots,k$,

$$\big\langle\mu_0,\mu_{I^+\sqcup I^-},\gamma_i\big\rangle_{\beta_1}^{I^-}=\big\langle\mu_0,\mu_{I^+\sqcup I^-},\gamma_i\big\rangle_{\beta_1}^{X}$$

for all decompositions $I^+\sqcup J\sqcup I^-=\{1,\ldots,k\}$ and for all four terms in (4.3); see Proposition 4.3. If $c=\tau,\eta$ and the conditions $(\mathcal{O}_c)$ and (2.5) are satisfied, Proposition 4.2 thus reduces the left-hand side of (4.3) to

$$2\Bigg(\big\langle\mu_0\mu_1,\mu_2,\mu_3,\ldots,\mu_k\big\rangle_\beta^{\phi,c}+\sum_{\substack{\mathfrak{d}(\beta_1)+\beta_2=\beta\\ \beta_1,\beta_2\in H_2(X)-\{0\}}}\;\sum_{I\sqcup J=\{3,\ldots,k\}}\;\sum_{\substack{1\leqslant i\leqslant\ell\\ \gamma^i\in H^{2*}(X)_-^\phi}}2^{|I|}\big\langle\mu_0,\mu_1,\mu_I,\gamma_i\big\rangle_{\beta_1}^X\big\langle\mu_2,\mu_J,\gamma^i\big\rangle_{\beta_2}^{\phi,c}\Bigg)$$

and the right-hand side of (4.3) to

$$2\Bigg(\big\langle\mu_1,\mu_0\mu_2,\mu_3,\ldots,\mu_k\big\rangle_\beta^{\phi,c}+\sum_{\substack{\mathfrak{d}(\beta_1)+\beta_2=\beta\\ \beta_1,\beta_2\in H_2(X)-\{0\}}}\;\sum_{I\sqcup J=\{3,\ldots,k\}}\;\sum_{\substack{1\leqslant i\leqslant\ell\\ \gamma^i\in H^{2*}(X)_-^\phi}}2^{|I|}\big\langle\mu_0,\mu_2,\mu_I,\gamma_i\big\rangle_{\beta_1}^X\big\langle\mu_1,\mu_J,\gamma^i\big\rangle_{\beta_2}^{\phi,c}\Bigg)\,.$$

Setting the two expressions equal, we obtain the formula in Theorem 2.1. If the conditions $(\mathcal{O}_\tau)$ and $(\mathcal{O}_\eta)$ are satisfied, but not necessarily (2.5), the same argument applies with $\langle\ldots\rangle^{\phi,c}$ replaced by $\langle\ldots\rangle^\phi$. $\square$

# 5 Orientations and signs

In this section, we analyze and compare orientations of various moduli spaces of complex and real maps. We use these comparisons to establish Proposition 4.3, Theorem 2.2, and Proposition 4.2.

***Proof of Proposition 4.3.*** For each cycle $h: Y \longrightarrow X$ representing the Poincare dual of an element of $H^*(X)_{\pm}^{\phi}$, let $\varepsilon(h)=\pm 1$, respectively. Define an involution $\Theta^I: X^{k+1} \longrightarrow X^{k+1}$ by

$$\Theta^I\big(x_0,\ldots,x_k\big) \longrightarrow \big(\Theta^I_0(x_0),\ldots,\Theta^I_k(x_k)\big), \qquad \text{where} \qquad \Theta^I_i(x) = \begin{cases} x, & \text{if } i\notin I; \\ \phi(x), & \text{if } i\in I. \end{cases}$$

We can assume that the cohomology degrees of $\mu_0,\mu_1,\ldots,\mu_k$ satisfy

$$\deg\mu_0 + \ldots + \deg\mu_k = \dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{k+1}(\beta) = 2\big(\big\langle c_1(X),\beta\big\rangle + n-2+k\big)\,,$$

where $2n=\dim X$. Choose a generic collection of representatives $h_i: Y_i \longrightarrow X$ for the Poincare duals of $\mu_0,\ldots,\mu_k$, respectively. The Poincare dual of $\phi^*\mu_i$ is then represented by the cycle

$$\overline{h_i} \equiv \phi\circ h_i: \bar{Y}_i \equiv (-1)^n Y_i \longrightarrow X, \tag{5.1}$$

with $-Y_i$ denoting $Y_i$ with the opposite orientation. Let

$$\langle\mathbf{h}\rangle = h_0\times\ldots\times h_k: \mathbf{Y} \equiv Y_0\times\ldots\times Y_k \longrightarrow X^{k+1}.$$

We denote by $\mathbf{Y}^I$ the modification of $\mathbf{Y}$ with the $i$-th factor replaced by $\varepsilon(h_i)\bar{Y}_i$ and by

$$\langle\mathbf{h}\rangle^I: \mathbf{Y}^I \longrightarrow X^{k+1}$$

the modification of $\langle\mathbf{h}\rangle$ with the $i$-th factor map replaced by $\overline{h_i}$ whenever $i\in I$. Thus, $\langle\mathbf{h}\rangle^I=\Theta^I\circ\langle\mathbf{h}\rangle$.

We set

$$\overline{\mathfrak{M}}^I_{\mathbf{h}}(\beta) = \big\{(u,\mathbf{y})\in\overline{\mathfrak{M}}^I_{k+1}(\beta)\times\mathbf{Y}^I:\ \mathrm{ev}^I(u)=\langle\mathbf{h}\rangle^I(\mathbf{y})\big\}\,, \qquad \overline{\mathfrak{M}}_{\mathbf{h}}(\beta) \equiv \overline{\mathfrak{M}}^{\varnothing}_{\mathbf{h}}(\beta)\,.$$

As sets, these two objects are the same. For a generic tuple $\mathbf{h}$, the restriction of the total evaluation map (4.4) to every stratum of $\overline{\mathfrak{M}}_{k+1}(\beta)$ is transverse to $\langle\mathbf{h}\rangle$ in $X^{k+1}$ and thus $\overline{\mathfrak{M}}_{\mathbf{h}}(\beta)$ is a finite collection of signed weighted points contained in the main stratum of the moduli space. Since $\mathbf{h}$ and $\mathbf{h}^I$ represent the Poincare duals of $\mu_0\times\ldots\times\mu_k$, the signed weighted cardinalities of $\overline{\mathfrak{M}}_{\mathbf{h}}(\beta)$ and $\overline{\mathfrak{M}}^I_{\mathbf{h}}(\beta)$ are the numbers $\langle\mu_0,\ldots,\mu_k\rangle^X_\beta$ and $\langle\mu_0,\ldots,\mu_k\rangle^I_\beta$, respectively.

The sign of each element $(u,\mathbf{y})$ of $\overline{\mathfrak{M}}^I_{\mathbf{h}}(\beta)$ is determined by the orientations of $\overline{\mathfrak{M}}^I_{k+1}(\beta)$, $\mathbf{Y}^I$, and $X^{k+1}$ via the maps $\mathrm{ev}^I$ and $\langle\mathbf{h}\rangle^I$. It is the sign of the isomorphism

$$\mathrm{d}\big\{\mathrm{ev}^I\times\langle\mathbf{h}\rangle^I\big\}: T(\overline{\mathfrak{M}}^I_{k+1}(\beta)\times\mathbf{Y}^I)|_{(u,\mathbf{y})} \longrightarrow \frac{T(X^{k+1}\times X^{k+1})|_{\Delta_{X^{k+1}}}}{T(\Delta_{X^{k+1}})}\bigg|_{(\mathrm{ev}^I(u),\langle\mathbf{h}\rangle^I(\mathbf{y}))}\,, \tag{5.2}$$

where $\Delta_{X^{k+1}}\subset X^{k+1}\times X^{k+1}$ is the diagonal. By the chain rule,

$$\mathrm{d}\big\{\mathrm{ev}^I\times\langle\mathbf{h}\rangle^I\big\} = \mathrm{d}\big\{\Theta^I\times\Theta^I\big\}\circ\mathrm{d}\big\{\mathrm{ev}\times\langle\mathbf{h}\rangle\big\}\,.$$

The sign of the isomorphism

$$\mathrm{d}\big\{\Theta^I\times\Theta^I\big\}: \frac{T(X^{k+1}\times X^{k+1})|_{\Delta_{X^{k+1}}}}{T(\Delta_{X^{k+1}})}\bigg|_{(\mathrm{ev}(u),\langle\mathbf{h}\rangle(\mathbf{y}))} \longrightarrow \frac{T(X^{k+1}\times X^{k+1})|_{\Delta_{X^{k+1}}}}{T(\Delta_{X^{k+1}})}\bigg|_{(\mathrm{ev}^I(u),\langle\mathbf{h}\rangle^I(\mathbf{y}))}$$

is $(-1)^{n|I|}$. The orientations of $\overline{\mathfrak{M}}^I_{k+1}(\beta)$ and $\overline{\mathfrak{M}}_{k+1}(\beta)$ differ by $(-1)^{|I|}$, while the orientations of $\mathbf{Y}^I$ and $\mathbf{Y}$ differ by

$$(-1)^{n|I|+|\{i\in I:\,\mu_i\in H^*(X)^{\phi}_{-}\}|}\,.$$

Thus, the signed weighted cardinalities of $\overline{\mathfrak{M}}^I_{\mathbf{h}}(\beta)$ and $\overline{\mathfrak{M}}_{\mathbf{h}}(\beta)$ differ by the sign $(-1)^{\varepsilon_I(\mu)}$. $\square$

We next recall how the main stratum $\mathfrak{M}_{k+1}^{\phi,c}(\beta)$ of the moduli space $\overline{\mathfrak{M}}_{k+1}(X,\beta)^{\phi,c}$ with $c=\tau,\eta$ is oriented if the condition $(\mathcal{O}_c)$ in Section 2 is satisfied. We begin with the case $k=-1$. By definition,

$$\mathfrak{M}_0^{\phi,c}(\beta)=\mathcal{P}_0^{\phi,c}(\beta)/G_c\,, \qquad g\cdot u=u\circ g,$$

where $\mathcal{P}_0^{\phi,c}(\beta)$ is the space of parametrized $(\phi,c)$-real degree $\beta$ $J$-holomorphic maps $\mathbb{P}^1\longrightarrow X$ and $G_c\subset\mathrm{PSL}_2\mathbb{C}$ is the subgroup of automorphisms of $\mathbb{P}^1$ commuting with $c$. The latter is oriented by the short exact sequence

$$0\longrightarrow T_{\mathrm{id}}S^1\longrightarrow T_{\mathrm{id}}G_c\longrightarrow\mathbb{C}\longrightarrow 0,$$

where $\mathbb{C}=T_0\mathbb{C}$ corresponds to shifting the origin and $S^1\subset G_c$ is the subgroup of standard rotations of $\mathbb{C}$, which we identify with $S^1\subset\mathbb{C}^*$. The (virtual) tangent space of $\mathcal{P}_0^{\phi,c}(\beta)$ at a point $u\in\mathcal{P}_0^{\phi,c}(\beta)$ is the index of the linearization $D_u^c$ of the $\bar\partial$-operator at $u$. If $c=\tau$, we orient this index as in the proofs of [9, Corollary 1.8] and [10, Lemma 7.3] from a fixed spin structure on $TX^\phi\oplus 2E^{\tilde\phi}$, with $E$ as in $(\mathcal{O}_\tau)$. If $c=\eta$, we orient the index via the pinching construction of [5, Lemma 2.5] from a fixed spin sub-structure on $(TX,\mathrm{d}\phi)$; see [12, Corollary 5.10]. The orientation of $\mathfrak{M}_0^{\phi,c}(\beta)$ at $[u]$ is then specified by

$$\mathrm{ind}\,D_u^c\approx T_{[u]}\mathfrak{M}_0^{\phi,c}(\beta)\oplus T_{\mathrm{id}}G_c\,.$$

The order of the factors on the right-hand side above is motivated by the choice of the orientation on $\overline{\mathcal{M}}_{0,2}^c$ in Section 3. For $k\geqslant 0$, $\overline{\mathfrak{M}}_{k+1}(X,\beta)^{\phi,c}$ is oriented using the first element in each conjugate pair $(z_i,z_{\bar i})$ to orient the general fibers of the forgetful morphism

$$\overline{\mathfrak{M}}_{k+1}(X,\beta)^{\phi,c}\longrightarrow\overline{\mathfrak{M}}_0(X,\beta)^{\phi,c} \tag{5.3}$$

obtained by forgetting the $k$ pairs of conjugate marked points.

In this paper, we use a different natural construction of orientation on $\overline{\mathfrak{M}}_{k+1}(X,\beta)^{\phi,c}$ in the stable range, i.e. $k\geqslant 1$; in Lemma 5.1, we show that the two orientations coincide. It is obtained using the forgetful morphism

$$f\colon\overline{\mathfrak{M}}_{k+1}(X,\beta)^{\phi,c}\longrightarrow\overline{\mathcal{M}}_{0,k+1}^c$$

and the orientation on $\overline{\mathcal{M}}_{0,k+1}^c$ defined in Section 3. For a general $[u]\in\overline{\mathfrak{M}}_{k+1}(X,\beta)^{\phi,c}$ in this case, the domain $\Sigma_u$ of $u$ with its marked points is stable and thus $\mathcal{C}=[\Sigma_u]$ is the image of $[u]$ in $\overline{\mathcal{M}}_{0,k+1}^c$. The (virtual) vertical tangent bundle of $f$ at such $[u]$ is the index of $D_u^c$. The orientation of $\overline{\mathfrak{M}}_{k+1}(X,\beta)^{\phi,c}$ is then specified by

$$T_{[u]}\overline{\mathfrak{M}}_{k+1}(X,\beta)^{\phi,c}\approx\mathrm{ind}\,D_u^c\oplus T_{[\Sigma_u]}\overline{\mathcal{M}}_{0,k+1}^c\,, \tag{5.4}$$

with $\mathrm{ind}\,D_u^c$ oriented as in the previous paragraph.

**Lemma 5.1.** *Let $c=\tau,\eta$, $(X,\omega,\phi)$ be a compact real symplectic manifold satisfying the condition $(\mathcal{O}_c)$ in Section 2, $k\in\mathbb{Z}^{\geqslant 0}$, and $\beta\in H_2(X)_\phi$.*

*(1) For every $i=0,1,\ldots,k$, the automorphism of $\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi,c}$ interchanging the marked points in the $i$-th conjugate pair is orientation-reversing.*

*(2) For all $i,j=0,1,\ldots,k$, the automorphism of $\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi,c}$ interchanging the $i$-th and $j$-th conjugate pairs of marked points is orientation-preserving.*

*(3) If $k \geqslant 1$, the two orientations on $\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi,c}$ described above are the same.*

*If the conditions $(\mathcal{O}_\tau)$ and $(\mathcal{O}_\eta)$ are satisfied, the three statements also apply with $\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi,c}$ replaced by $\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi}$.*

*Proof.* (1,2) Both automorphisms take a fiber of (5.3) to the same fiber. The restriction of the automorphism in (1) to a fiber of (5.3) is orientation-reversing, while the restriction of the automorphism in (2) to a fiber of (5.3) is orientation-preserving. This implies the first two statements of the lemma.

(3) Let $u$ be an element of $\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi,c}$ at a point $u$ with smooth domain $\Sigma_u$ and $u_0$ be its image of $[u]$ under (5.3). The first orientation of $\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi,c}$ described above satisfies

$$\begin{aligned} T_{[u]}\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi,c} \oplus T_{\mathrm{id}}G_c &\approx T_{[u_0]}\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi,c} \oplus \bigoplus_{i=0}^{k} T_{z_i^+}\Sigma_u \oplus T_{\mathrm{id}}G_c \\ &\approx T_{[u_0]}\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi,c} \oplus T_{\mathrm{id}}G_c \oplus \bigoplus_{i=0}^{k} T_{z_i^+}\Sigma_u \approx \mathrm{ind}\, D^c_{u_0} \oplus \bigoplus_{i=0}^{k} T_{z_i^+}\Sigma_u\,. \end{aligned}$$

The orientation of $\overline{\mathcal{M}}^c_{0,k+1}$ chosen in Section 3 at a smooth curve $\mathcal{C} = [(z_0^+, z_0^-), \ldots, (z_k^+, z_k^-)]$ is described by

$$T_{z_0^+}\mathcal{C} \oplus \ldots \oplus T_{z_k^+}\mathcal{C} \approx T_{\mathcal{C}}\overline{\mathcal{M}}^c_{0,k+1} \oplus T_{\mathrm{id}}G_c\,.$$

Thus, the second orientation of $\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi,c}$ described above satisfies

$$T_{[u]}\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi,c} \oplus T_{\mathrm{id}}G_c \approx \mathrm{ind}\, D^c_{u_0} \oplus T_{[\Sigma_0]}\overline{\mathcal{M}}^c_{0,k+1} \oplus T_{\mathrm{id}}G_c \approx \mathrm{ind}\, D^c_{u_0} \oplus \bigoplus_{i=0}^{k} T_{z_i^+}\Sigma_u\,.$$

Thus, the two orientations of $T_{[u]}\overline{\mathfrak{M}}_{0,k+1}(\beta)^{\phi,c}$ are the same. □

If $c$ is an orientation-reversing involution on a compact orientable surface $\Sigma$ of genus $g$ and $(X,\omega,\phi)$ is a compact real symplectic $2n$-manifold such that $n$ is odd, $X^\phi = \varnothing$, and $\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX, \mathrm{d}\phi)$ admits a real square root, then the moduli spaces $\overline{\mathfrak{M}}_{g,k+1}(X,\beta)^{\phi,c}$ are oriented via the analogue of the morphism (5.3). Thus, the first two statements of Lemma 5.1 also hold if $\overline{\mathfrak{M}}_{0,k+1}(X,\beta)^{\phi,c}$ is replaced by $\overline{\mathfrak{M}}_{g,k+1}(X,\beta)^{\phi,c}$.

***Proof of Theorem 2.2.*** We denote by $\mathbb{R}\overline{\mathfrak{M}}_{g,k}(\beta)$ the appropriate moduli space of real morphisms, as determined by the case of Theorem 2.2 under consideration. We can assume that the cohomology degrees of $\mu_1, \ldots, \mu_k$ satisfy

$$\deg \mu_1 + \ldots + \deg \mu_k = \dim^{\mathrm{vir}} \mathbb{R}\overline{\mathfrak{M}}_{g,k}(\beta) = \big\langle c_1(X), \beta\big\rangle + (n-3)(1-g) + 2k\,.$$

Choose $h_i\colon Y_i \longrightarrow X$ as in the proof of Proposition 4.3 and define $\langle \mathbf{h}\rangle$, $\langle \mathbf{h}\rangle^I$, $\mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}}(\beta)$, and $\mathbb{R}\overline{\mathfrak{M}}^I_{\mathbf{h}}(\beta)$, for any subset $I \subset \{1, \ldots, k\}$, as before, but starting with the moduli space $\mathbb{R}\overline{\mathfrak{M}}_{g,k}(\beta)$ in the last two cases. By exactly the same argument as in the proof of Proposition 4.3, the signed weighted cardinalities of $\mathbb{R}\overline{\mathfrak{M}}^I_{\mathbf{h}}(\beta)$ and $\mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}}(\beta)$ differ by the sign $(-1)^{\varepsilon_I(\mu)}$.

If $\mu_{i^*} \in H^*(X)^\phi_+$, we apply the above conclusion with $I = \{i^*\}$. The signed weighted cardinalities of $\mathbb{R}\overline{\mathfrak{M}}^I_{\mathbf{h}}(\beta)$ and $\mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}}(\beta)$ are then opposite. Interchanging the points in the $i^*$-th conjugate pair induces an orientation-preserving isomorphism from $\mathbb{R}\overline{\mathfrak{M}}^I_{\mathbf{h}}(\beta)$ to $\mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}'}(\beta)$, where $\mathbf{h}'$ is the tuple obtained by replacing $h_{i*}$ with

$$\overline{h_{i*}} = \phi\circ h_{i*} : (-1)^n Y_{i*} \longrightarrow X\,;$$

this cycle represents the Poincare dual of $\phi^*\mu_{i*} = \mu_{i*}$. Thus, the signed weighted cardinalities of $\mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}}(\beta)$ and $\mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}'}(\beta)$ are opposite. Since both of them are equal to the real invariant $\langle\mu_1,\ldots,\mu_k\rangle^\phi_\beta$ in question, the latter vanishes. $\square$

In the remainder of this section, we establish Proposition 4.2. The key point in its proof is that all orientations are chosen compatibly; in particular, the oriented normal bundle of $\Gamma_*$ in $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ and the oriented normal bundle of its preimage in $\mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta)$ are given by the complex line bundle of smoothings of the node on the bubble containing $z_0$. We proceed with the notation and assumptions as in the statement of Proposition 4.2. We will also use the same notation for the uncompactified moduli spaces (maps only from smooth domains) as we have introduced for the compactified moduli spaces.

For $\beta\in H_2(X)_\phi$, denote by $\mathcal{N}_\beta\subset\mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta)$ the sub-orbifold of maps from domains consisting of precisely three components with one invariant bubble and two conjugate bubbles with the marked point $z_0^+$ on one of the conjugate bubbles. For $u\in\mathcal{N}_\beta$, denote by $u^{\mathbb{C}}$ the restriction of $u$ to the component containing $z_0^+$ and by $z^{\mathbb{C}}$ the marked point corresponding to the node on this component; denote by $u^{\mathbb{R}}$ the restriction of $u$ to the invariant component and by $z_\bullet^+$ the marked point on this component corresponding to the same node as $z^{\mathbb{C}}$. If $\beta=\mathfrak{d}(\beta_1)+\beta_2$ and $\{1,\ldots,k\}=I^+\sqcup J\sqcup I^-$, let

$$\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}\subset\mathcal{N}_\beta$$

be the subspace of the maps $u$ so that the degrees of $u^{\mathbb{C}}$ and $u^{\mathbb{R}}$ are $\beta_1$ and $\beta_2$, respectively, and the rest of the marked points carried by the component containing $z_0^+$ are the first elements in the pairs of conjugate points indexed by $I^+$ and the second elements in the pairs indexed by $I^-$. If

$$(\beta_1,I^+,I^-)=(0,\varnothing,\varnothing)\qquad\text{or}\qquad(\beta_2,J)=(0,\varnothing),\tag{5.5}$$

$\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}=\varnothing$ for stability reasons.

The restrictions $u^{\mathbb{C}}$ and $u^{\mathbb{R}}$ determine an isomorphism

$$\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}\approx\big\{(u^{\mathbb{C}},u^{\mathbb{R}})\in\mathfrak{M}^{I^-}_{|I^+|+|I^-|+2}(\beta_1)\times\mathbb{R}\mathfrak{M}_{|J|+1}(\beta_2)\colon u^{\mathbb{C}}(z^{\mathbb{C}})=u^{\mathbb{R}}(z_\bullet^+)\big\},\tag{5.6}$$

with the marked points of the elements of $\mathfrak{M}^{I^-}_{|I^+|+|I^-|+2}(\beta_1)$ indexed by 0, the elements of $I^+\sqcup I^-$, and the superscript $\mathbb{C}$; under either of the conditions (5.5), one of the moduli spaces on the right-hand side of (5.6) is empty for stability reasons. The inverse map is obtained by identifying the marked point $z^{\mathbb{C}}$ of the domain of $u^{\mathbb{C}}$ with the marked point $z_\bullet^+$ of the domain of $u^{\mathbb{R}}$ and the marked point $c(z^{\mathbb{C}})$ of the map $\phi\circ u^{\mathbb{C}}\circ c$ with $z_\bullet^-=c(z_\bullet^+)$; the marked points of $u^{\mathbb{C}}$ indexed by $I^+$ become the first points in the corresponding pair of the nodal map, while those indexed by $I^-$ become the second. As in Section 4, $\mathfrak{M}^{I^-}_{|I^+|+|I^-|+2}(\beta_1)$ is oriented by twisting the canonical complex orientation of $\mathfrak{M}_{|I^+|+|I^-|+2}(\beta_1)$ by $(-1)^{|I^-|}$. The canonical orientation of $X$ and the chosen orientations of $\mathfrak{M}^{I^-}_{|I^+|+|I^-|+2}(\beta_1)$ and $\mathbb{R}\mathfrak{M}_{|J|+1}(\beta_2)$ induce an orientation on each component of $\mathcal{N}_\beta$ via the isomorphism (5.5).

Let $L^{\mathbb{C}} \longrightarrow \mathfrak{M}^{I^-}_{|I^+|+|I^-|+2}(\beta_1)$ and $L^{\mathbb{R}} \longrightarrow \mathbb{R}\mathfrak{M}_{|J|+1}(\beta_2)$ be the universal tangent line bundles at the marked points $z^{\mathbb{C}}$ and $z^+_\bullet$, respectively, and

$$L = \pi_1^* L^{\mathbb{C}} \otimes_{\mathbb{C}} \pi_2^* L^{\mathbb{R}} \longrightarrow \mathcal{N}_\beta \,,$$

where $\pi_1, \pi_2$ are the component projection maps. The line bundle $L \longrightarrow \mathcal{N}_\beta$ is the normal bundle of $\mathcal{N}_\beta$ in $\mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta)$. There is a gluing map

$$\Phi\colon U \longrightarrow \mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta), \tag{5.7}$$

where $U \subset L$ is a neighborhood of the zero set in $L$; it is obtained via a $(\phi, c)$-equivariant version of a standard gluing construction, such as in [24, Section 3].

If $k \geqslant 2$ and $|J \cap \{1,2\}| = 1$, $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}$ is a topological component of the pre-image of $\mathring{\Gamma}$ under the forgetful morphism $\mathbb{R}f_{012}$ in (4.1) for some $\Gamma = \Gamma_i, \Gamma_{\bar{i}}$, with $i = 1, 2$. In this case, the restriction of $L$ to $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}$ equals $\mathbb{R}f_{012}^* L_\Gamma$, where $L_\Gamma \longrightarrow \Gamma$ is the complex line bundle defined in Section 3. The gluing map $\Phi$ in (5.7) can be chosen so that its restriction to each such component $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}$ lifts any pre-specified gluing map on $L_{\Gamma;\mathbf{h}}$ over $\mathbb{R}f_{012}$.

**Lemma 5.2.** *If $k \geqslant 2$ and $|J \cap \{1,2\}| = 1$, the restriction of the gluing map (5.7) to a neighborhood of $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}$ in $L$ is orientation-preserving with respect to the complex orientation on $L$ and the orientation on the base described above.*

*Proof.* This follows readily from the definitions of the three orientations above; we follow the second construction, which is described just before Lemma 5.1. Let $\Gamma$ be as in the preceding paragraph. If $k = 2$, $\Phi$ can be chosen so that there is a commutative diagram

$$\begin{tikzcd}
U \arrow[r, "\Phi"] \arrow[d, "f_{012}"'] & \mathbb{R}\overline{\mathfrak{M}}_3(\beta) \arrow[d, "f_{012}"] \\
U_\Gamma \arrow[r, "\Phi_\Gamma"] & \mathbb{R}\overline{\mathcal{M}}_{0,3}
\end{tikzcd}$$

with the bottom arrow being some gluing map on a neighborhood of $\mathring{\Gamma}$ in $L_\Gamma$. By Lemma 3.2, $\Phi_\Gamma$ is orientation-preserving. Since all domains are stable in this case, the vertical tangent spaces of the vertical arrows in the diagram are oriented by orienting the indices of the linearized $\bar{\partial}$-operators; see [11, Section 6].

The index for the complex moduli space has a canonical orientation; see [25, p51]. The indices for the two real moduli spaces are oriented from either the same trivialization of $TX^\phi \oplus 2E^{\tilde{\phi}}$ over a loop in $X^\phi$ or from the same trivialization of $(TX, \mathrm{d}\phi)$ over a $\mathbb{Z}_2$-invariant loop in $X$ by pinching off the relevant vector bundle onto a conjugate pair of sphere bubbles, as in the proofs of [10, Lemma 7.3] and in [11, Theorem 1.1]; the index over the first of these bubbles, $B$, has a canonical complex orientation. Thus, the index of an element of $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}$ is oriented by introducing an extra pinching in $B$ as compared to what is used to orient nearby real maps from $\mathbb{P}^1$. This pinching, which is given by the inverse of $\Phi$, induces the same canonical orientation over $B$. Thus, the orientation of the index for a map from $\mathbb{P}^1$ is equivalent to the orientation obtained from the orientation of an element of $\mathcal{N}_\beta$ by smoothing the node.

If $k \geqslant 3$, $\Phi$ can be chosen so that there is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\Phi"] \arrow[d] & \mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta) \arrow[d] \\ U' \arrow[r, "\Phi'"] & \mathbb{R}\overline{\mathfrak{M}}_3(\beta) \end{tikzcd} \tag{5.8}$$

with the vertical arrows being forgetful maps again. Since the fibers of the right arrow are oriented by the first points in each conjugate pair and the orientation of the fibers of the left arrow is based on the number of conjugate pairs with the second point carried by $u^{\mathbb{C}}$, Lemma 3.1 implies that $\Phi$ is again orientation-preserving between the fibers and thus between the spaces on the first line of (5.8). $\square$

*Remark* 5.3. The assumption that $k \geqslant 2$ and $|J \cap \{1,2\}| = 1$ in Lemma 5.2 is not necessary, but it simplifies the argument. The general case is not needed for our purposes.

***Proof of Proposition 4.2.*** We can assume that the cohomology degrees of $\mu_0, \ldots, \mu_k$ satisfy

$$\deg \mu_0 + \ldots + \deg \mu_k = \dim^{\mathrm{vir}} \mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta) - 2 = \langle c_1(X), \beta \rangle + n - 3 + 2k, \tag{5.9}$$

where $2n = \dim X$. Choose a generic collection of representatives $h_i \colon Y_i \longrightarrow X$ for the Poincare duals of $\mu_0, \ldots, \mu_k$, respectively, and define

$$\mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}}(\beta) = \big\{(u, \mathbf{y}) \in \mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta) \times Y_0 \times \ldots \times Y_k \colon \mathrm{ev}_i(u) = h_i(y_i) \ \forall\, i = 0, \ldots, k\big\}.$$

If $\beta = \mathfrak{d}(\beta_1) + \beta_2$ and $\{1, \ldots, k\} = I^+ \sqcup J \sqcup I^-$, let

$$\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}(\mathbf{h}) = \mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-} \cap \mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}}(\beta). \tag{5.10}$$

If the representatives $h_i$ for $\mu_i$ are generic, each set $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}(\mathbf{h})$ is a compact zero-dimensional suborbifold of the oriented orbifold $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}$ and thus has a well-defined signed weighted cardinality. The latter is computed by the usual Kunneth decomposition, with respect to the specified orientations of $\mathfrak{M}^{I^-}_{|I^+|+|I^-|+2}(\beta_1)$ and $\mathbb{R}\mathfrak{M}_{|J|+1}(\beta_2)$; this gives the last sum in (4.5) if $\beta_1, \beta_2 \neq \varnothing$, but without the restriction $\gamma^i \in H^{2*}(X)^{\phi}_{-}$. Since $\mu_i \in H^{2*}(X)$ for all $i$, the complex GW-invariant in (4.5) with $\gamma_i \in H^{2*-1}(X)$ vanishes for dimensional reasons; the real GW-invariant in (4.5) with $\gamma^i \in H^*(X)^{\phi}_{+}$ vanishes by Theorem 2.2. If $\beta_1 = 0$ and $|I^+ \sqcup I^-| \geqslant 2$ or $\beta_2 = 0$ and $|J| \geqslant 1$, $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}(\mathbf{h}) = \varnothing$; otherwise, the marked points on $u^{\mathbb{C}}$ (in the first case) or on $u^{\mathbb{R}}$ (in the second case) could vary while staying inside of the zero-dimensional $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}(\mathbf{h})$. The case $\beta_1 = 0$ and $|I^+ \sqcup I^-| = 1$ reduces as usual to an invariant like the first term on the right-hand side of (4.5); as described below, there is only one decomposition $\{1, \ldots, k\} = I^+ \sqcup J \sqcup I^-$ with $|I^+ \sqcup I^-| = 1$ relevant to each of the four cases of Proposition 4.2.

For any $\Gamma \subset \mathbb{R}\overline{\mathcal{M}}_{0,3}$, define

$$\begin{aligned} Z_\Gamma &= \big\{\big(u, y_0, \ldots, y_k\big) \in \mathbb{R}f^{-1}_{012}(\Gamma) \times Y_0 \times \ldots \times Y_k \colon \mathrm{ev}_i(u) = h_i(y_i) \ \forall\, i = 0, \ldots, k\big\} \\ &= \big(\mathbb{R}f^{-1}_{012}(\Gamma) \times Y_0 \times \ldots \times Y_k\big) \cap \mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}}(\beta)\,. \end{aligned}$$

For $j=1,2$ and generically chosen constraints $h_i$,

$$Z_{\Gamma_j} = \bigsqcup_{\substack{\mathfrak{d}(\beta_1)+\beta_2=\beta\\ \beta_1,\beta_2\in H_2(X),\,\beta_2\neq 0}} \bigsqcup_{\substack{I^+\sqcup J\sqcup I^-=\{1,\ldots,k\}\\ j\in I^+,\,3-j\in J}} \mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}(\mathbf{h}); \tag{5.11}$$

this decomposition corresponds to the first two sums in (4.5) and the first term on the right-hand side of (4.5). It also holds with $(\Gamma_j, j\in I^+)$ replaced by $(\Gamma_{\bar{j}}, j\in I^-)$.

Let $L_{\Gamma;\mathbf{h}}\longrightarrow Z_\Gamma$ denote the restriction of

$$\pi_1^*L = L\times Y_0\times\ldots\times Y_k \longrightarrow \mathcal{N}_\beta\times Y_0\times\ldots\times Y_k\,. \tag{5.12}$$

As in complex GW-theory, a small modification of the gluing map (5.7) gives rise to a gluing map

$$\Phi_{\Gamma;\mathbf{h}}\colon U_{\Gamma;\mathbf{h}} \longrightarrow \mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}}(\beta),$$

where $U_{\Gamma;\mathbf{h}}\subset L_{\Gamma;\mathbf{h}}$ is a neighborhood of the zero section in $L_{\Gamma;\mathbf{h}}$ (a finite collection of disks in this case). Such a modification can be chosen to be of the form

$$\Phi_{\Gamma;\mathbf{h}}(u,\upsilon) = \Phi\big(\psi(u,\upsilon),\upsilon\big) \qquad \forall\ (u,\upsilon)\in U_{\Gamma;\mathbf{h}},$$

for some smooth function $\psi$ on $U_{\Gamma;\mathbf{h}}$ sending $(u,0)$ to $u$. Thus, the induced map

$$\mathrm{d}(\mathbb{R}f_{012}\circ\Phi_{\Gamma;\mathbf{h}})\colon \pi_1^*\,\mathbb{R}f_{012}^*L_\Gamma \longrightarrow L_\Gamma$$

between the normal bundle of $Z_\Gamma$ in $\mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}}(\beta)$ and of $\Gamma$ in $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ is the identity. Since $\Phi_{\Gamma;\mathbf{h}}$ is orientation-preserving by Lemma 5.2, it follows that every signed weighted element of $Z_\Gamma$ contributes $+1$ to the number (4.2). By the last two paragraphs, the signed weighted cardinality of $Z_\Gamma$ is given by the right-hand side of (4.5). $\square$

# 6 Alternative proof of Theorem 2.1

In this section, we give a proof of Theorem 2.1 (in effect of a combination of Corollary 4.1 and Proposition 4.2) which bypasses the real Deligne-Mumford moduli space $\mathbb{R}\overline{\mathcal{M}}_{0,3}$ of Section 3. We instead pull back the standard relation on $\overline{\mathcal{M}}_{0,4}$ by the forgetful morphism

$$\begin{gathered} f_{012\bar{0}}\colon \mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta) \longrightarrow \overline{\mathcal{M}}_{0,4}\,,\\ \big[u,(z_0^+,z_0^-),\ldots,(z_k^+,z_k^-)\big] \longrightarrow [z_0^+,z_1^+,z_2^+,z_0^-], \end{gathered} \tag{6.1}$$

preserving the marked points $z_0^+,z_1^+,z_2^+,z_0^-$ only (and stabilizing the domain if necessary).

As in Section 4, we either fix $c=\tau,\eta$ and assume that the conditions $(\mathcal{O}_c)$ and (2.5) in Section 2 are satisfied or assume that the conditions $(\mathcal{O}_\tau)$ and $(\mathcal{O}_\eta)$, but not necessarily (2.5), are satisfied. In both cases, we continue with the abbreviations for moduli spaces of maps introduced in Section 4 (before Corollary 4.1 for the $\mathbb{R}$-spaces and before Proposition 4.2 for the $\mathbb{C}$-spaces). We can again assume that (5.9) holds and choose generic representative $h_i\colon Y_i\longrightarrow X$ for the Poincare duals of $\mu_0\equiv\mu,\mu_1,\ldots,\mu_k$.

Let $\Omega_{0,4}\in H^2(\overline{\mathcal{M}}_{0,4})$ be the Poincare dual of the point class and

$$\widetilde{N}^{\mathbb{R}}_{\beta}(\mu_0,\ldots,\mu_k)=\int_{[\mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta)]^{\mathrm{vir}}}f^*_{012\bar{0}}\Omega_{0,4}\,\mathrm{ev}_0^*\mu_0\,\ldots\,\mathrm{ev}_k^*\mu_k\,. \tag{6.2}$$

For any $\lambda\in\mathcal{M}_{0,4}$, define

$$\begin{aligned}Z_\lambda&=\big\{(u,y_0,\ldots,y_k)\in f^{-1}_{012\bar{0}}(\lambda)\times Y_0\times\ldots\times Y_k\colon \mathrm{ev}_i(u)=h_i(y_i)\ \forall\, i=0,\ldots,k\big\}\\&\subset\mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta)\times Y_0\times\ldots\times Y_k\,.\end{aligned}$$

This subset is a compact oriented 0-dimensional suborbifold, i.e. a finite set of weighted points, if $\lambda$ is generic. The number (6.2) is the signed weighted cardinality $|Z_\lambda|^\pm$ of this set.

We prove Theorem 2.1 by explicitly describing the elements of $Z_{[1,1]}$ and $Z_{[1,0]}$, with notation as in Figure 1, and determining their contribution to the number (6.2). The domain $\Sigma_u$ of each element $[u]$ of $Z_{[1,1]}$ and $Z_{[1,0]}$ consists of at least two irreducible components. If (2.5) holds, $\Sigma_u$ has an odd number of irreducible components; the involution $c_u$ associated with $u$ restricts to $c$ on one of the components and interchanges the others in pairs. For dimensional reasons, the number of irreducible components of $\Sigma_u$ cannot be greater than 3 and thus must be either 2 or 3. Each map $u$ with its marked points is completely determined by its restriction $u^{\mathbb{R}}$ to the component $\Sigma_u^{\mathbb{R}}$ of $\Sigma_u$ preserved by $c_u$ (if the number of irreducible components is odd) and its restriction $u^{\mathbb{C}}$ to either of the other components.

We depict all possibilities for the elements of $Z_{[1,1]}$ and $Z_{[1,0]}$ in Figures 6 and 7, respectively. In each of the first three diagrams in these figures, the vertical line represents the irreducible component $\Sigma_u^{\mathbb{R}}$ of $\Sigma_u$ preserved by $c_u$, while the two horizontal lines represent the components of $\Sigma_u$ interchanged by $c_u$; in the last diagram in each figure, the two lines represent the components of $\Sigma_u$ interchanged by $c_u$. The homology classes next to the lines specify the degrees of $u$ on the corresponding components. The larger dots on the three lines indicate the locations of the marked points $z_0^+,z_1^+,z_2^+$; we label them by the constraints they map to, i.e. $\mu,\mu_1,\mu_2$, in order to make the connection with the expression in Theorem 2.1 more apparent. If a marked point $z_i^+$ lies on the bottom component, its conjugate lies on the top component. In such a case, we indicate the conjugate point by a small dot on the upper component and label it with $\bar{\mu}_i$; the restriction of $u$ to the upper component maps this point to the image of $\phi\circ h_i$. By the definition of $Z_{[1,1]}$, each diagram in Figure 6 contains a node separating the marked points $z_0^+,z_1^+$ (i.e. the larger dots labeled by $\mu,\mu_1$) from the marked points $z_2^+,z_0^-$ (i.e. dots labeled by $\mu_2,\bar{\mu}$). Similarly, each diagram in Figure 7 contains a node separating the marked points $z_0^+,z_2^+$ from the marked points $z_1^+,z_0^-$. We arrange the diagrams in both cases so that the pair of marked points containing $z_0^+$ lies above the other pair. The remaining marked points, $z_3^\pm,\ldots,z_k^\pm$, are distributed between the components in some way. In the case of the first three diagrams in each figure, such a distribution is described by a partition of $\{1,\ldots,k\}$ into subsets $I^+,J,I^-$ of plus-decorated marked points on the top, middle, and bottom components, respectively.

Each element $u$ of $Z_{[1,1]}$ and $Z_{[1,0]}$ described by the first three diagrams in Figures 6 and 7, respectively, is an element of the subspace

$$\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}(\mathbf{h})\subset\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}\times Y_0\times\ldots\times Y_k\subset\mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta)\times Y_0\times\ldots\times Y_k \tag{6.3}$$

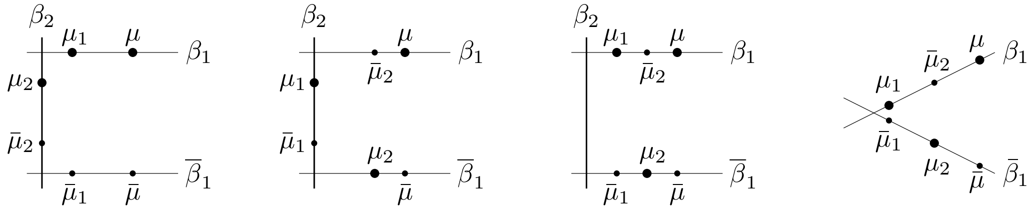

Figure 6: Domains of elements of $Z_{[1,1]}$

defined in (5.10) for some $\beta_1,\beta_2$ and $I^+,J,I^-$ with $\beta=\mathfrak{d}(\beta_1)+\beta_2$ and $\{1,\ldots,k\}=I^+\sqcup J\sqcup I^-$. An element $u$ of the first space in (6.3) has a well-defined nonzero weight $w(u)$ with respect to the orientation of $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}$ described below (5.6). The sum of these weights over all elements $u$ represented by a fixed diagram with fixed $(\beta_1,\beta_2)$ and $(I^+,J,I^-)$ is the signed weighted cardinality of $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}(\mathbf{h})$ computed via the usual Kunneth decomposition; see the first paragraph in the proof of Proposition 4.2. As an isolated element of $Z_{[1,1]}$ or $Z_{[1,0]}$, $u$ has a well-defined contribution $\varepsilon(u)w(u)$ to the number (6.2), i.e. the signed number of nearby elements of $Z_\lambda$, with $\lambda\in\mathcal{M}_{0,4}$ close to $[1,1]$ or $[1,0]$. By Lemma 6.1 below, $\varepsilon(u)=1$ for all elements $u$ represented by the first diagrams in Figures 6 and 7, $\varepsilon(u)=-1$ for the second diagrams in these figures, and $\varepsilon(u)=0$ for the third diagrams. Even if the contributions from the third diagrams were nonzero, they would have been the same for $Z_{[1,1]}$ and $Z_{[1,0]}$ by symmetry and so would have had no effect on the recursion of Theorem 2.1. The reason behind Lemma 6.1 is that the oriented normal bundle of $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}$ inside $\mathbb{R}\overline{\mathfrak{M}}_{k+1}(\beta)$ is given by the complex line bundle of smoothings of the top node in the first three diagrams, which is conjugate to the complex line bundle of smoothings of the bottom node, while the complex tangent bundle of $[1,1]$ or $[1,0]$ in $\overline{\mathcal{M}}_{0,4}$ corresponds to the smoothings of the node separating $\{z_0^+,z_1^+\}$ from $\{z_2^+,z_0^-\}$ in the case of $[1,1]$ and $\{z_0^+,z_2^+\}$ from $\{z_1^+,z_0^-\}$ in the case of $[1,0]$.

The remaining elements of $Z_{[1,1]}$ and $Z_{[1,0]}$, i.e. those described by the last diagrams in Figures 6 and 7, respectively, form one-dimensional subspaces $Z'_{[1,1]}\subset Z_{[1,1]}$ and $Z'_{[1,0]}\subset Z_{[1,0]}$; these diagrams appear only if (2.5) is not satisfied. By Lemma 6.2 below, no topological component of $Z'_{[1,1]}$ or $Z'_{[1,0]}$ contributes to the number (6.2).

**Lemma 6.1.** *Suppose $u\in Z_{[1,1]}$ and the domain of $u$ contains an irreducible component $\Sigma_u^{\mathbb{R}}$ fixed by the involution $c_u$.*

*(1) If $\Sigma_u^{\mathbb{R}}$ contains the marked point $z_2^+$, $\varepsilon(u)=1$.*

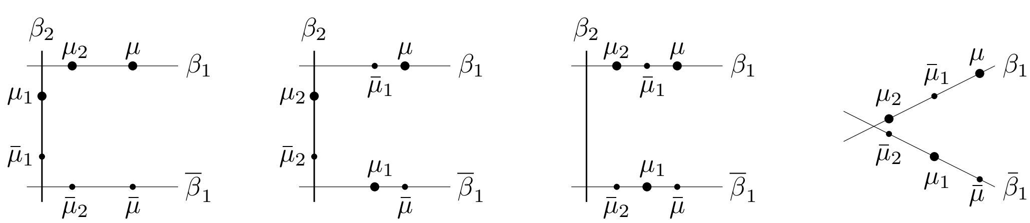

Figure 7: Domains of elements of $Z_{[1,0]}$

*(2) If $\Sigma_u^{\mathbb{R}}$ contains the marked point $z_1^+$, $\varepsilon(u)=-1$.*

*(3) If $\Sigma_u^{\mathbb{R}}$ contains neither of the marked points $z_1^+, z_2^+$, $\varepsilon(u)=0$.*

*The same statements with 1 and 2 interchanged hold for $u\in Z_{[1,0]}$.*

*Proof.* Let $L_{\mathbf{h}} \longrightarrow Z_{[1,1]}-Z'_{[1,1]}, Z_{[1,0]}-Z'_{[1,0]}$ be the restriction of the line bundle $\pi_1^*L$ defined in (5.12). As in complex GW-theory, a small modification of the gluing map (5.7) gives rise to a gluing map

$$\Phi_{\mathbf{h}}: U_{\mathbf{h}} \longrightarrow \mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}}(\beta),$$

where $U_{\mathbf{h}} \subset L_{\mathbf{h}}$ is a neighborhood of the zero section in $L_{\mathbf{h}}$, which lifts any pre-specified family of smoothings of the domain. Over the subsets $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}(\mathbf{h})$ corresponding to the first two diagrams in Figures 6 and 7, $\Phi_{\mathbf{h}}$ is orientation-preserving by Lemma 5.2. The differential

$$\mathrm{d}\big\{f_{012\bar{0}}\circ\Phi_{\mathbf{h}}\big\}: L \longrightarrow \big\{f_{012\bar{0}}\circ\Phi_{\mathbf{h}}\big\}^*T\mathcal{M}_{0,4} \tag{6.4}$$

is the composition of the differential for smoothing the nodes in $\overline{\mathfrak{M}}_{k+2}(\beta)$,

$$\mathrm{d}(f_{012\bar{0}}\circ\Phi^{\mathbb{C}}): L\oplus L' \longrightarrow \big\{f_{012\bar{0}}\circ\Phi^{\mathbb{C}}\big\}^*T\mathcal{M}_{0,4}\,,$$

where $L'$ is the analogue of $L$ for the second node, with the embedding

$$L \longrightarrow L\oplus L', \qquad \upsilon \longrightarrow \big(\upsilon, \mathrm{d}c(\upsilon)\big).$$

The restriction of the latter differential to the component, $L$ or $L'$, corresponding to the node separating off two of the marked points $\{z_0^+, z_1^+, z_2^+, z_0^-\}$ is a $\mathbb{C}$-linear isomorphism, while the restriction to the other component is trivial. Over the subsets $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}(\mathbf{h})$ corresponding to the first diagrams in Figures 6 and 7, the former component is $L$ and (6.4) is an orientation-preserving map. Over the subsets $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}(\mathbf{h})$ corresponding to the second diagrams in Figures 6 and 7, the former component is $L'$ and (6.4) is an orientation-reversing map. This establishes the first two statements of Lemma 6.1.

Near the spaces $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}$ corresponding to the second-to-last diagrams in Figures 6 and 7, the morphism

$$f_{012\bar{0}}: \overline{\mathfrak{M}}_{k+2}(\beta) \longrightarrow \overline{\mathcal{M}}_{0,4}$$

is locally of the form

$$L\oplus L' \longrightarrow \overline{\mathcal{M}}_{0,4}, \qquad (\upsilon,\upsilon') \longrightarrow a\upsilon\upsilon',$$

for some $a$ dependent only on $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}$. Thus, the restriction of $f_{012\bar{0}}$ to $\mathbb{R}\overline{\mathfrak{M}}_{\mathbf{h}}(\beta)$ is locally of the form

$$L \longrightarrow \overline{\mathcal{M}}_{0,4}, \qquad \upsilon \longrightarrow a\upsilon\bar{\upsilon}\,.$$

The image of this maps is one-dimensional, which implies the third claim of Lemma 6.1. □

**Lemma 6.2.** *The contribution of every topological component of $Z'_{[1,1]}$ and $Z'_{[1,0]}$ to the number (6.2) is 0.*

*Proof.* If $(X,\omega)$ is strongly semi-positive, each topological component $C$ of $Z'_{[1,1]}$ and $Z'_{[1,0]}$ is a circle. In general, $C$ is obtained by gluing several circles along some intervals as specified by branching of the multi-section $\mathfrak{s}$ used to regularize the moduli space. Along $C$, $\mathfrak{s}$ can be represented by several single-valued sections obtained by gluing together local representatives as in [6, Section 3]. Each such section determines disjoint circles in $Z'_{[1,1]}$ or $Z'_{[1,0]}$. For the purposes of studying the nearby elements of $Z_\lambda$ that lie in the zero set of each of these sections, it is sufficient to assume that each topological component $C$ of $Z'_{[1,1]}$ and $Z'_{[1,0]}$ is the circle $S^1$.

There is a gluing map

$$\Phi\colon C\times(-\delta,\delta)\longrightarrow \bigcup_{\lambda\in\overline{\mathcal{M}}_{0,4}} Z_\lambda \tag{6.5}$$

for $\delta\in\mathbb{R}^+$ sufficiently small, which restricts to the identity along $C\times\{0\}$; it is obtained via a $(\phi,c)$-equivariant version of a standard gluing construction, such as in [24, Section 3], with $c=\tau,\eta$. In particular, we can normalize the elements of $C$ by setting the marked point $z_0^+=0$ and the node to $\infty$ on one of the components of the domain and setting $z_2^+=1$, $z_0^-=\infty$, and the node to 0 on the other component. For each $t\in\mathbb{R}^*$ sufficiently small, we can define a marked pregluing map $u_t\colon\mathbb{P}^1\longrightarrow X$ with the same values at the marked points as $u$ and with the cross-ratio $f_{012\bar{0}}$ given by

$$\lambda = f_{012\bar{0}}(u_t) = t\,z_1^+(u)\in\mathbb{C}^*\subset\mathcal{M}_{0,4}$$

in some chart on $\overline{\mathcal{M}}_{0,4}$. This map can then be deformed to an element $\tilde{u}_t$ of $Z_\lambda$, with the same $\lambda\in\mathcal{M}_{0,4}$. Since $C$ consists of two-bubble maps (no additional bubbling), the gluing construction can be carried out on the entire space $C$ in this case.

Let $\bar{\mathbb{R}}^+=\mathbb{R}^{\geqslant0}$ and $\bar{\mathbb{R}}^-=\mathbb{R}^{\leqslant0}$. The restriction of $f_{012\bar{0}}\circ\Phi$ to $C\times((-\delta,\delta)\cap\bar{\mathbb{R}}^\pm)$ is the composition of the maps

$$\begin{aligned} C\times((-\delta,\delta)\cap\bar{\mathbb{R}}^\pm)&\longrightarrow\big\{z\in\mathbb{C}\colon\,|z|<\delta\big\}, &\qquad (e^{\mathrm{i}\theta},t)&\longrightarrow|t|e^{\mathrm{i}\theta},\\ \big\{z\in\mathbb{C}\colon\,|z|<\delta\big\}&\longrightarrow\mathbb{C}, & re^{\mathrm{i}\theta}&\longrightarrow\pm rz_1^+\big(e^{\mathrm{i}\theta}\big). \end{aligned}$$

The two maps, for $\bar{\mathbb{R}}^+$ and $\bar{\mathbb{R}}^-$, described by the first line above have opposite local degrees, while the two maps described by the second map have the same local degrees. Thus, the local degree of the map

$$f_{012\bar{0}}\circ\Phi : C\times(-\delta,\delta)\longrightarrow\mathbb{C}, \qquad (u,t)\longrightarrow f_{012\bar{0}}\big(\Phi(u,t)\big)=t\,z_1^+(u),$$

is zero. This implies the claim. $\square$

***Proof of Theorem 2.1.*** We compute the number (6.2) by adding up the contributions from the elements represented by the diagrams in Figure 6. We then compute it from the diagrams in Figure 7 and compare the two expressions for the number (6.2).

By Lemmas 6.1 and 6.2, only the first two diagrams in Figure 6 and 7 contribute. By the Kunneth decomposition, as in the first part of the proof of Proposition 4.2, and by Proposition 4.3, the signed cardinality of $\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}(\mathbf{h})$ is given by

$$\big|\mathcal{N}_{\beta_1,\beta_2;I^+,J,I^-}(\mathbf{h})\big|^\pm = \sum_{\substack{1\leqslant i\leqslant\ell\\ \gamma^i\in H^{2*}(X)^\phi_-}}\big\langle\mu_0,\mu_{I^+\sqcup I^-},\gamma_i\big\rangle^X_{\beta_1}\big\langle\mu_J,\gamma^i\big\rangle^{\mathbb{R}}_{\beta_2}, \tag{6.6}$$

where $\langle\ldots\rangle^{\mathbb{R}}$ denotes $\langle\ldots\rangle^{\phi,c}$ if (2.5) holds and $\langle\ldots\rangle^{\phi}$ otherwise. If $\beta_1=0$ and the complex invariant in (6.6) is nonzero, then $|I^+\sqcup I^-|=1$ for dimensional reasons.

By Lemma 6.1(1), the contribution to the number (6.2) from the first diagram in Figure 6 equals the sum of (6.6) over all admissible $(\beta_1,\beta_2)$ and $(I,J)$ with $1\in I$ and $2\in J$ and all partitions of $I-\{1\}$ into two subsets $I^+$ and $I^-$. By Lemma 6.1(2), the contribution to the number (6.2) from the second diagram in Figure 6 equals the negative of the sum of (6.6) over all admissible $(\beta_1,\beta_2)$ and $(I,J)$ with $2\in I$ and $1\in J$ and all partitions of $I-\{2\}$ into two subsets $I^+$ and $I^-$. Thus, the number (6.2) equals

$$\big\langle\mu\mu_1,\mu_2,\mu_3,\ldots,\mu_k\big\rangle_{\beta}^{\mathbb{R}}-\big\langle\mu_1,\mu\mu_2,\mu_3,\ldots,\mu_k\big\rangle_{\beta}^{\mathbb{R}}+\sum_{\substack{\mathfrak{d}(\beta_1)+\beta_2=\beta\\ \beta_1,\beta_2\in H_2(X)-\{0\}}}\sum_{I\sqcup J=\{3,\ldots,k\}}\sum_{\substack{1\leqslant i\leqslant\ell\\ \gamma^i\in H^{2*}(X)_-^{\phi}}}2^{|I|}\bigg(\big\langle\mu,\mu_1,\mu_I,\gamma_i\big\rangle_{\beta_1}^{X}\big\langle\mu_2,\mu_J,\gamma^i\big\rangle_{\beta_2}^{\mathbb{R}}-\big\langle\mu,\mu_2,\mu_I,\gamma_i\big\rangle_{\beta_1}^{X}\big\langle\mu_1,\mu_J,\gamma^i\big\rangle_{\beta_2}^{\mathbb{R}}\bigg).$$

Considering the first two diagrams in Figure 7, we similarly find that the number (6.2) equals

$$\big\langle\mu_1,\mu\mu_2,\mu_3,\ldots,\mu_k\big\rangle_{\beta}^{\mathbb{R}}-\big\langle\mu\mu_1,\mu_2,\mu_3,\ldots,\mu_k\big\rangle_{\beta}^{\mathbb{R}}+\sum_{\substack{\mathfrak{d}(\beta_1)+\beta_2=\beta\\ \beta_1,\beta_2\in H_2(X)-\{0\}}}\sum_{I\sqcup J=\{3,\ldots,k\}}\sum_{\substack{1\leqslant i\leqslant\ell\\ \gamma^i\in H^{2*}(X)_-^{\phi}}}2^{|I|}\bigg(\big\langle\mu,\mu_2,\mu_I,\gamma_i\big\rangle_{\beta_1}^{X}\big\langle\mu_1,\mu_J,\gamma^i\big\rangle_{\beta_2}^{\mathbb{R}}-\big\langle\mu,\mu_1,\mu_I,\gamma^i\big\rangle_{\beta_1}^{X}\big\langle\mu_2,\mu_J,\gamma^i\big\rangle_{\beta_2}^{\mathbb{R}}\bigg).$$

Setting the two expressions equal, we obtain the formula in Theorem 2.1. □

# 7 Miscellaneous odds and ends

We begin this section by deducing Corollaries 1.4 and 1.5 from Corollary 1.3. We then deduce Corollaries 2.4 and 2.6 from Theorems 2.1 and 2.2 and relate the formula of Theorem 2.1 to the quantum product on the cohomology of the symplectic manifold $(X,\omega)$. We conclude with tables of counts of real curves in $\mathbb{P}^3$, $\mathbb{P}^5$, and $\mathbb{P}^7$ and a discussion of their compatibility.

***Proof of Corollary 1.4.*** (1) The claim holds for $d,k=1$, since there is a unique $\phi$-real line through any point in $\mathbb{P}^{2n-1}$. Modulo 2, the recursion of Corollary 1.3 becomes

$$\big\langle c_1,c_2,c_3,\ldots,c_k\big\rangle_d^{\phi}\cong\big\langle c_1+c_2-1,c_3,\ldots,c_k\big\rangle_d^{\phi}+\sum_{\substack{2d_1+d_2=d\\ d_1,d_2\geqslant1}}\sum_{\substack{2i+j=2n-1\\ i,j\geqslant1}}\bigg(\big\langle c_1-1,c_2,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_3,\ldots,c_k,j\big\rangle_{d_2}^{\phi}+d_1\big\langle c_1-1,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}\big\langle c_2,\ldots,c_k,j\big\rangle_{d_2}^{\phi}\bigg).$$

For dimensional reasons,

$$\big\langle c_1-1,c_2,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}=0\quad\forall\ d_1\geqslant2,\qquad\big\langle c_1-1,2i\big\rangle_{d_1}^{\mathbb{P}^{2n-1}}=0\quad\forall\ d_1\geqslant1.$$

Thus, the mod 2 recursion reduces to

$$\big\langle c_1, c_2, c_3, \ldots, c_k \big\rangle_d^\phi \cong \big\langle c_1+c_2-1, c_3, \ldots, c_k \big\rangle_d^\phi + \sum_{\substack{2i+j=2n-1\\ i,j\geqslant 1}} \big\langle c_1-1, c_2, 2i \big\rangle_1^{\mathbb{P}^{2n-1}} \big\langle c_3, \ldots, c_k, j \big\rangle_{d-2}^\phi.$$

If $c_1+c_2-1 \leqslant 2n-1$, $\langle c_1+c_2-1, c_3, \ldots, c_k \rangle_d^\phi$ is odd by induction on $k$ for a fixed $d \geqslant 1$ odd and $\langle c_1-1, c_2, 2i \rangle_1^{\mathbb{P}^{2n-1}} = 0$ for dimensional reasons. It follows that

$$\big\langle c_1, c_2, c_3, \ldots, c_k \big\rangle_d^\phi \cong 1 \mod 2$$

in this case. If $c_1+c_2-1 > 2n-1$, then

$$d \geqslant 3, \qquad \langle c_1+c_2-1, c_3, \ldots, c_k \rangle_d^\phi = 0,$$

and $\langle c_1-1, c_2, 2i \rangle_1^{\mathbb{P}^{2n-1}} = 0$ if $c_1+c_2+2i \neq 4n$. Since the linear span of general $\mathbb{P}^{2n-1-(c_1-1)}$ and $\mathbb{P}^{2n-1-c_2}$, with

$$1 \leqslant c_1, c_2 \leqslant 2n-1 \qquad \text{and} \qquad c_1+c_2 > 2n,$$

in $\mathbb{P}^{2n-1}$ is a $\mathbb{P}^{4n-c_1-c_2}$, it intersects a general $\mathbb{P}^{2n-1-(4n-c_1-c_2)}$ in a single point. This point lies on the unique line passing through linear subspaces of $\mathbb{P}^{2n-1}$ of codimensions $c_1, c_2, 2i$ whenever $c_1+c_2+2i = 4n$. Thus,

$$\big\langle c_1, c_2, c_3, \ldots, c_k \big\rangle_d^\phi \cong \big\langle c_3, \ldots, c_k, 2n-1-(4n-c_1-c_2) \big\rangle_{d-2}^\phi \mod 2$$

in this case; the last number is odd by the induction on $d$.

(2) The second claim of Corollary 1.4 follows from the first and [5, Theorem 1.8]; the latter is contained in Corollary 2.6 and Theorem 2.2. □

***Proof of Corollary 1.5.*** The first statement follows immediately from Corollary 1.3 and implies that

$$N_d^{\mathbb{R}} \cong_4 \begin{cases} 1, & \text{if } d \in \mathbb{Z}^+ - 2\mathbb{Z}; \\ 0, & \text{if } d \in 2\mathbb{Z}^+ . \end{cases}$$

We use simultaneous induction on the degree $d$ to show that

$$N_d^{\mathbb{C}} \cong_4 \begin{cases} 1, & \text{if } d \in \mathbb{Z}^+ - 2\mathbb{Z}; \\ 0, & \text{if } d \in 2\mathbb{Z}^+ ; \end{cases} \qquad \text{and} \qquad \widetilde{N}_d^{\mathbb{C}} \cong_4 \begin{cases} 1, & \text{if } d \in \mathbb{Z}^+ - 2\mathbb{Z} \text{ or } d = 2; \\ 2, & \text{if } d = 4; \\ 0, & \text{if } d \in 2\mathbb{Z}^+ - \{2,4\} ; \end{cases}$$

from the base case $N_1^{\mathbb{C}} = 1$ (the number of lines through 2 points in $\mathbb{P}^3$). By [27, Theorem 10.4],

$$\begin{aligned} N_d^{\mathbb{C}} &= \sum_{\substack{d_1+d_2=d\\ d_1,d_2\geqslant 1}} \left( d_2^2 \binom{2d-3}{2d_1-2} - d_1 d_2 \binom{2d-3}{2d_1-1} \right) \widetilde{N}_{d_1}^{\mathbb{C}} N_{d_2}^{\mathbb{C}} , \\ \widetilde{N}_d^{\mathbb{C}} &= d N_d^{\mathbb{C}} + \sum_{\substack{d_1+d_2=d\\ d_1,d_2\geqslant 1}} \left( d_1 d_2^2 \binom{2d-2}{2d_1-1} - d_2^3 \binom{2d-2}{2d_1-2} \right) \widetilde{N}_{d_1}^{\mathbb{C}} N_{d_2}^{\mathbb{C}} . \end{aligned} \tag{7.1}$$

By (7.1), $\widetilde{N}_1^{\mathbb{C}}, N_3^{\mathbb{C}} = 1$ and $\widetilde{N}_3^{\mathbb{C}} = 5$. Modulo 4, the summands in (7.1) with $d_2$ even vanish (by Corollary 1.4, $N_{d_2}^{\mathbb{C}} \in 2\mathbb{Z}$ if $d_2 \in 2\mathbb{Z}$). Thus, by the inductive assumption only the summands with $d_1 = 2, 4$ may be nonzero in either sum in (7.1) with $d \geqslant 5$ odd. These two summands contribute $d$–2 and $d$–3, respectively, i.e. 1 together, to the first sum. They contribute $d$–1 and 0, respectively, i.e. again 1 together with the term $dN_d^{\mathbb{C}}$, to the second sum.

For $d\in 2\mathbb{Z}^+$, (7.1) and the inductive assumptions give

$$\begin{aligned} N_d^{\mathbb{C}} &\cong_4 \sum_{\substack{d_1+d_2=d\\ d_1,d_2\geqslant 1 \text{ odd}}} \left(\binom{2d-3}{2d_1-2} - (d-1)\binom{2d-3}{2d_1-1}\right) = (2-d) \sum_{\substack{d_1+d_2=d\\ d_1,d_2\geqslant 1 \text{ odd}}} \binom{2d-3}{2d_1-1}, \\ \widetilde{N}_d^{\mathbb{C}} &\cong_4 \sum_{\substack{d_1+d_2=d\\ d_1,d_2\geqslant 1 \text{ odd}}} \left(d_1\binom{2d-2}{2d_1-1} - d_2\binom{2d-2}{2d_1-2}\right) = \frac{1}{2} \sum_{\substack{d_1+d_2=d\\ d_1,d_2\geqslant 1 \text{ odd}}} \binom{2d-2}{2d_1-1}. \end{aligned} \tag{7.2}$$

By symmetry, the last expression on the first line above equals

$$\frac{2-d}{2} \sum_{\substack{d_1+d_2=d\\ d_1,d_2\geqslant 1 \text{ odd}}} \left(\binom{2d-3}{2d_1-1} + \binom{2d-3}{2d_2-1}\right) = \frac{2-d}{2} \sum_{\substack{d_1+d_2=d\\ d_1,d_2\geqslant 1 \text{ odd}}} \binom{2d-2}{2d_1-1}.$$

Each of the last binomial coefficients is even. If in addition $d\in 4\mathbb{Z}$, these coefficients come in pairs: the one for $d_1$ and $d-d_1$ are the same. This shows that $N_d^{\mathbb{C}}\in 4\mathbb{Z}$ if $d\in 2\mathbb{Z}$.

By (7.2), $\widetilde{N}_2^{\mathbb{C}} \cong_4 1$ and $\widetilde{N}_4^{\mathbb{C}} \cong_4 2$. Suppose $d = 2(d'+1)$ with $d' \geqslant 2$, $d_1 = 2d_1'+1$, and $d_2 = 2d_2'+1$ (so that $d_1' + d_2' = d'$). By Kummer's Theorem, the highest power of 2 that divides half of the last binomial coefficient in (7.2) is the number $c_2(d_1', d_2')$ of carries in the addition of $d_1'$ and $d_2'$ modulo 2. The pairs $(d_1', d_2')$ for which $c_2(d_1', d_2') = 0$ are obtained from $d'$ by distributing the 1's in the binary representation of $d'$ between $d_1'$ and $d_2'$. Thus, the number of such pairs $(d_1', d_2')$ is $2^{\#}$, where $\#$ is the number of 1's in the binary representation of $d'$. Since $d_1' \neq d_2'$ for such pairs, the contribution from $(d_1', d_2')$ and $(d_2', d_1')$ to the last expression in (7.2), including the half factor, is 2 modulo 4. Thus, the contribution from all such pairs to the last expression in (7.2) is $2^{\#}$. The contribution from any other pair $(d_1', d_2')$ is divisible by 2, since $c_2(d_1', d_2') \geqslant 1$, and such pairs come in pairs giving the same contribution to (7.2), unless $d_1' = d_2'$. If $d_1' = d_2'$ and thus $d' \in 2\mathbb{Z}^+$, $c_2(d_1', d_2') = 1$ if and only if $\# = 1$. Thus, if $d' \in 2\mathbb{Z}^+$, the total contribution to (7.2) from the terms with $c_2(d_1', d_2') = 0$ and the term with $d_1' = d_2'$ is 0 modulo 4. If $d' \notin 2\mathbb{Z}^+$, $\# \geqslant 2$, since $d' \geqslant 2$, and so this contribution is still 0 modulo 4. This shows that $\widetilde{N}_d^{\mathbb{C}} \in 4\mathbb{Z}$ if $d\in 2\mathbb{Z}$ and $d\geqslant 6$. □

***Proof of Corollary 2.4.*** We can assume that $X$ is connected and thus $H^0(X)_-^{\phi} = \{0\}$. By Gromov's compactness theorem, we can rescale $\omega$ so that

$$\inf\left\{\omega(\beta) \colon \beta\in H_{\mathrm{eff}}(X)_\phi\right\} = 1.$$

We prove the claim by induction on the number

$$\langle\beta\rangle_k \equiv \omega(\beta)+k \in (1,\infty) \qquad \forall\, k\in\mathbb{Z}^+,\ \beta\in H_{\mathrm{eff}}(X)_\phi\,.$$

Let $\{\gamma_i\}_{i\leqslant\ell}$ and $\{\gamma^i\}_{i\leqslant\ell}$ be as in Theorem 2.1. By the divisor relation,

$$\begin{aligned} \big\langle\mu_1,\ldots,\mu_k\big\rangle_\beta^{\phi,c} &= \langle\mu_2,\beta\rangle\big\langle\mu_1,\mu_3,\ldots,\mu_k\big\rangle_\beta^{\phi,c},\\ \big\langle\mu_1,\ldots,\mu_k\big\rangle_\beta^{\phi} &= \langle\mu_2,\beta\rangle\big\langle\mu_1,\mu_3,\ldots,\mu_k\big\rangle_\beta^{\phi}\end{aligned}\qquad \forall\,\mu_2\in H^2(X), \tag{7.3}$$

whenever these invariants are defined, $\beta\neq0$, and $k\geqslant2$.

By the linearity of real genus 0 GW-invariants and Theorem 2.2(1), it is sufficient to construct the maps $P_{\beta;\beta'}$ on the direct sum of $(H^{2*}(X)_-^\phi)^{\otimes k}$ so that these maps satisfy (2.11) with $H^{2*}(X)$ replaced by $H^{2*}(X)_-^\phi$. For $\beta,\beta'\in H_{\mathrm{eff}}(X)_\phi$, define

$$\begin{gathered} P_{\beta';\beta}\colon H^{2*}(X)_-^\phi\longrightarrow H^{2*}(X)_-^\phi,\qquad P_{\beta';\beta}(\mu)=\begin{cases}\mu, & \text{if } \beta'=\beta;\\ 0, & \text{if } \beta'\neq\beta;\end{cases}\\ P_{\beta';\beta}=0\colon \bigoplus_{k=1}^{\infty}\big(H^{2*}(X)_-^\phi\big)^{\otimes k}\longrightarrow H^{2*}(X)_-^\phi \qquad \text{if}\quad \beta-\beta'\notin H_{\mathrm{eff}}(X)_\phi\cup\{0\}\,. \end{gathered}$$

These linear maps satisfy the $k=1$ case of (2.11) under the assumptions in (1) and of (2.11) with $\langle\cdot\rangle^{\phi,c}$ replaced by $\langle\cdot\rangle^{\phi}$ under the assumptions in (2).

Suppose $M\in\mathbb{Z}^+$ and for every pair $(k,\beta)$ in $\mathbb{Z}^+\times H_{\mathrm{eff}}(X)_\phi$ with $\langle\beta\rangle_k<M$ there exist linear maps

$$P_{\beta';\beta}\colon \big(H^{2*}(X)_-^\phi\big)^{\otimes k}\longrightarrow H^{2*}(X)_-^\phi \qquad \text{with}\quad \beta'\in H_{\mathrm{eff}}(X)_\phi \tag{7.4}$$

that satisfy (2.11) under the assumptions in (1) and (2.11) with $\langle\cdot\rangle^{\phi,c}$ replaced by $\langle\cdot\rangle^{\phi}$ under the assumptions in (2). Let $(k,\beta)$ be a pair in $\mathbb{Z}^+\times H_{\mathrm{eff}}(X)_\phi$ such that

$$k>1\qquad\text{and}\qquad M\leqslant\langle\beta\rangle_k<M+1.$$

Choose a basis $\mathcal{B}_k$ for $(H^{2*}(X)_-^\phi)^{\otimes k}$ consisting of products of homogeneous elements (each factor $\mu_i$ lies in $H^c(X)_-^\phi$ for some $c\in2\mathbb{Z}^+$).

Let $\mu_1\otimes\ldots\otimes\mu_k\in\mathcal{B}_k$. By the divisibility assumption, there exist $\mu\in H^{2*}(X)_+^\phi$ and $\mu_2'\in H^2(X)_-^\phi$ such that $\mu_2=\mu\mu_2'$. For each $\beta'\in H_{\mathrm{eff}}(X)_\phi$ such that $\beta-\beta'\in H_{\mathrm{eff}}(X)_\phi\cup\{0\}$, define

$$\begin{split} P_{\beta';\beta}(\mu_1,\ldots,\mu_k)=\langle\mu_2',\beta\rangle P_{\beta';\beta}\big(\mu\mu_1,\mu_3,\ldots,\mu_k\big)+\sum_{\substack{\mathfrak{d}(\beta_1)+\beta_2=\beta\\ \mathfrak{d}(\beta_1),\beta_2\in H_{\mathrm{eff}}(X)_\phi}}\ \sum_{I\sqcup J=\{3,\ldots,k\}}\ \sum_{\substack{1\leqslant i\leqslant\ell\\ \gamma^i\in H^{2*}(X)_-^\phi}}2^{|I|}\Bigg(\\ \big\langle\mu,\mu_1,\mu_I,\gamma_i\big\rangle_{\beta_1}^X P_{\beta';\beta_2}\big(\mu_2',\mu_J,\gamma^i\big)-\big\langle\mu,\mu_2',\mu_I,\gamma_i\big\rangle_{\beta_1}^X P_{\beta';\beta_2}\big(\mu_1,\mu_J,\gamma^i\big)\Bigg). \end{split}$$

The values of $P_{\beta';\beta}$ and $P_{\beta';\beta_2}$ above are well-defined because

$$\langle\beta\rangle_{k-1}=\langle\beta\rangle_k-1<M,\quad \langle\beta_2\rangle_{|J|+2}=\langle\beta\rangle_k-\omega\big(\mathfrak{d}(\beta_1)\big)-|I|\leqslant\langle\beta\rangle_k-1<M\,.$$

By (7.3), the equation in Theorem 2.1 with $\mu_2$ replaced by $\mu_2'$ is equivalent to

$$\big\langle\mu_1,\mu_2,\mu_3,\ldots,\mu_k\big\rangle_\beta^{\phi,c}=\langle\mu_2',\beta\rangle\big\langle\mu\mu_1,\mu_3,\ldots,\mu_k\big\rangle_\beta^{\phi,c}+\sum_{\substack{\mathfrak d(\beta_1)+\beta_2=\beta\\ \mathfrak d(\beta_1),\beta_2\in H_{\mathrm{eff}}(X)_\phi}}\sum_{I\sqcup J=\{3,\ldots,k\}}\sum_{\substack{1\leqslant i\leqslant\ell\\ \gamma^i\in H^{2*}(X)^\phi_-}}2^{|I|}\bigg(\big\langle\mu,\mu_1,\mu_I,\gamma_i\big\rangle_{\beta_1}^X\big\langle\mu_2',\mu_J,\gamma^i\big\rangle_{\beta_2}^{\phi,c}-\big\langle\mu,\mu_2',\mu_I,\gamma_i\big\rangle_{\beta_1}^X\big\langle\mu_1,\mu_J,\gamma^i\big\rangle_{\beta_2}^{\phi,c}\bigg)$$

under the assumptions in (1) and with $\langle\cdot\rangle^{\phi,c}$ replaced by $\langle\cdot\rangle^\phi$ under the assumptions in (2). Along with the assumption on (7.4), this implies that the elements $P_{\beta';\beta}(\mu_1,\ldots,\mu_k)$ of $H^{2*}(X)^\phi_-$ satisfy (2.11) under the assumptions in (1) and (2.11) with $\langle\cdot\rangle^{\phi,c}$ replaced by $\langle\cdot\rangle^\phi$ under the assumptions in (2). This completes the inductive step of the proof. $\square$

***Proof of Corollary 2.6.*** By the discussion above the statement of this corollary and immediately after $(\mathcal O_\tau)$ and $(\mathcal O_\eta)$ in Section 2, the real genus 0 GW-invariants $\langle\ldots\rangle_d^\phi$ and $\langle\ldots\rangle_d^{\phi,c}$ of $(X,\phi)$ are defined in all cases considered in the statement of Corollary 2.6. By the sentence below (2.15), we can assume that the complex dimension of $X$ is odd. By the Lefschetz Theorem on Hyperplane Sections [17, p156] and Poincare Duality, $H^{2*}(X)$ is then generated by $H^2$ over $\mathbb Q$ and $H^{4*}(X)=H^{4*}(X)^\phi_+$. In light of Theorem 2.2, this implies that the real genus 0 degree $d$ GW-invariants of $(X,\omega_n|_X,\phi)$ with any insertion $\mu_j\in H^{4*}(X)$ vanish.

If $\phi_{\mathbb P^{n-1}}=\eta_n$ and $c=\tau$, the moduli spaces in (2.3) are empty for any $J\in\mathcal J_\omega^\phi$ because $X^\phi=\varnothing$. The same is the case if $(\tau_n\eta)$ holds because the involution $\tau_n$ lifts to an involution $\widetilde\tau_n$ on the line bundle $\mathcal O_{\mathbb P^{n-1}}(1)$, a real degree $d$ map from $(\mathbb P^1,\eta)$ to $(\mathbb P^{n-1},\tau_n)$ pulls back $(\mathcal O_{\mathbb P^{n-1}}(1),\widetilde\tau_n)$ to a degree $d$ line bundle over $\mathbb P^1$ with an involution lifting $\eta$, and only even-degree line bundles over $\mathbb P^1$ admit such lifts. This establishes the vanishing claim if either $\phi_{\mathbb P^{n-1}}=\eta_n$ and $c=\tau$ or $(\tau_n\eta)$ holds. The assumption that the degrees of $\mu_j$ are even is not necessary in these cases.

By the real version of Quantum Lefschetz Hyperplane Theorem (as in [10, Proposition 7.7]), the real genus 0 degree $d$ GW-invariants of $(X,\omega_n|_X,\phi)$ with insertions $\mu_j=H^{c_j}$ for some $c_j\in\mathbb Z^{\geqslant0}$ are equal to the real genus 0 GW-invariants of $(\mathbb P^{n-1},\phi_{\mathbb P^{n-1}})$ twisted by the Euler class of a vector bundle. If either $a_i\in2\mathbb Z$ for some $i$ or $d\in2\mathbb Z$ and $\ell\in\mathbb Z^+$, then this bundle contains a subbundle of odd rank and the invariants of $(X,\omega_n|_X,\phi)$ vanish.

Suppose $d\in2\mathbb Z$ and $\ell=0$, i.e. $X_{n;\mathbf a}=\mathbb P^{n-1}$. If $[\mathbb P^1]$ is the generator of $H_2(\mathbb P^{n-1})$, then

$$c^\phi_{\min}\big(d[\mathbb P^1]\big)=2n>(\dim_{\mathbb R}X)/2+1=n.$$

Thus, the real genus 0 degree $d$ GW-invariants of $(\mathbb P^{n-1},\omega_n,\phi_{\mathbb P^{n-1}})$ with $d\in2\mathbb Z$ vanish by the last statement of Corollary 2.5. This concludes the proof of Corollary 2.6(1).

It remains to establish Corollary 2.6(2). We assume that

$$\dim_{\mathbb C}X=n-1-\ell\geqslant0.$$

Along with the assumption on $|\mathbf a|$, this implies that $n>|\mathbf a|$. By Corollary 2.6(1) and the reasoning above, we can also assume that $a_i\notin2\mathbb Z$ for every $i$, $d\notin2\mathbb Z$, and $X$ is odd-dimensional. The last assumption implies that $H^{2*}(X)$ is generated by $H^2(X)^\phi_-$ as an algebra over $\mathbb{Q}$, $H_2(X)=H_2(X)_\phi$ is one-dimensional, and $(X,\omega_n|_X,\phi)$ is a real Fano symplectic manifold. Along with the middle assumption, it implies that

$$\begin{aligned} c^\phi_{\min}\big(d[\mathbb{P}^1]\big) &= n-|\mathbf{a}| = \langle\mathbf{a}\rangle_n - \big((\dim_{\mathbb{R}}X)/2-1\big),\\ c^\phi_+\big(d[\mathbb{P}^1]\big) &= 3\big(n-|\mathbf{a}|\big) > n-\ell = (\dim_{\mathbb{R}}X)/2+1. \end{aligned}$$

Corollary 2.6(2) thus follows from the first statement of Corollary 2.5 and the linearity of real genus 0 GW-invariants. □

Analogously to the situation in complex GW-theory, Theorem 2.1 is related to the quantum cohomology of $(X,\omega)$. Let $(X,\omega,\phi)$ be a real symplectic manifold. Suppose that either

(C1) the conditions $(\mathcal{O}_\tau)$ and $(\mathcal{O}_\eta)$ in Section 2 hold or

(C2) $c\in\{\tau,\eta\}$ is fixed and $(\mathcal{O}_c)$ holds.

In the first case, let

$$H_2(X)^\star_\phi = H_2(X)_\phi-\{0\}\,.$$

In the second case, let $H_2(X)^\star_\phi$ be as above if $X^\phi=\varnothing$ and $H_2(X)_\phi-\mathrm{Im}(\mathfrak{d})$ if $X^\phi\neq\varnothing$. For each $\beta\in H_2(X)^\star_\phi$, denote by $\langle\ldots\rangle^\phi_\beta$ the real invariant (2.8) in the case (C1) and the real invariant (2.6) in the case (C2).

Choose bases $\{\gamma_i\}_{i\leqslant\ell}$ and $\{\gamma^i\}_{i\leqslant\ell}$ for $H^*(X)$ so that

$$\mathrm{PD}_{X^2}(\Delta_X) = \sum_{i=1}^{\ell}\gamma_i\times\gamma^i \in H^*(X^2),$$

as before. Let $q$ denote the formal variable in the Novikov ring $\Lambda$ on $H_2(X;\mathbb{Z})$ and set

$$\widetilde{\Lambda}=\Lambda[q^{1/2}],\quad \widetilde{Q}H^*(X)=H^*(X)\otimes\widetilde{\Lambda},\quad \widetilde{Q}H^*(X)^\phi_\pm = H^*(X)^\phi_\pm\otimes\widetilde{\Lambda} = QH(X)^\phi_\pm[q^{1/2}]\,;$$

see [25, Section 11.1]. We define a homomorphism of modules over $\tilde{\Lambda}$ by

$$\mathfrak{R}_\phi\colon \widetilde{Q}H^*(X)\longrightarrow \widetilde{Q}H^*(X)^\phi_{(-1)^{n+1}},\quad \mathfrak{R}_\phi\mu = \sum_{\beta\in H_2(X)^\star_\phi}\sum_{i=1}^{\ell}\big\langle\mu,\gamma_i\big\rangle^\phi_\beta\gamma^i q^{\beta/2}\quad\forall\ \mu\in H^*(X)\,,$$

where $2n=\dim X$. By Theorems 2.2 and 2.1,

$$\mathfrak{R}_\phi\mu=0\quad\forall\ \mu\in\widetilde{Q}H^*(X)^\phi_+\qquad\text{and}\qquad \mathfrak{R}_\phi\mu_1*\mu_2=\mu_1*\mathfrak{R}_\phi\mu_2\quad\forall\ \mu_1,\mu_2\in\widetilde{Q}H^*(X)^\phi_-\,,$$

respectively, where $*$ is the quantum product. If in addition $\langle c_1(X),\beta\rangle\in 2\mathbb{Z}$ for all $\beta\in H_2(X)$ that can be represented by $J$-holomorphic spheres for a generic $J\in\mathcal{J}^\phi_\omega$, then

$$\mathfrak{R}_\phi\mu_-*\mu_+=\mathfrak{R}_\phi\big(\mu_-*\mu_+\big)\qquad\forall\ \mu_-\in\widetilde{Q}H^*(X)^\phi_-\,,\mu_+\in\widetilde{Q}H^*(X)^\phi_+\,;$$

this can be seen by an argument similar to the proof of Proposition 4.3.

We conclude with some counts of real curves in $\mathbb{P}^3$, $\mathbb{P}^5$, and $\mathbb{P}^7$; see Tables 1 and 2. These numbers are consistent with basic algebro-geometric considerations [17, p177].

| $d$ | $N_d^{\mathbb{R}}$ |
|---|---|
| 1 | 1 |
| 3 | 1 |
| 5 | 5 |
| 7 | 85 |
| 9 | 1993 |
| 11 | 136457 |
| 13 | 3991693 |
| 15 | 1580831965 |
| 17 | -129358296175 |
| 19 | 106335656443537 |
| 21 | -39705915765949931 |
| 23 | 27364388694945255653 |
| 25 | -19263282511829476981415 |
| 27 | 17458116427845844069499545 |
| 29 | -18101279473337469331178336611 |
| 31 | 22138019795038729862257691515501 |

Table 1: The number $N_d^{\mathbb{R}}$ of degree $d$ real rational curves through $d$ non-real points in $\mathbb{P}^3$.

(1) Every degree 1 curve lies in a $\mathbb{P}^1$, every non-real point $p$ in $\mathbb{P}^{2n-1}$ determines a real $\mathbb{P}^1 \subset \mathbb{P}^{2n-1}$, and a real line passing through $p$ lies in this $\mathbb{P}^1$. Thus, $N_1^{\mathbb{R}}$, $\langle 5^1 3^0 \rangle_1^{\tau_5}$, and $\langle 7^1 5^0 3^0 \rangle_1^{\tau_7}$ should equal 1, at least in the absolute value.

(2) Every degree 3 curve lies in a $\mathbb{P}^3$, every two general non-real points $p_1$ and $p_2$ in $\mathbb{P}^{2n-1}$ determine a real $\mathbb{P}^3 \subset \mathbb{P}^{2n-1}$, for $n \geqslant 2$, and a real degree 3 curve passing through $p_1$ and $p_2$ lies in this $\mathbb{P}^3$. Thus, a real degree 3 curve in $\mathbb{P}^5$ passing through two general points $p_1$ and $p_2$ and a general plane $\pi$ lies in the real $\mathbb{P}^3$ determined by these two points and passes through the point $\pi \cap \mathbb{P}^3$; so the number $\langle 5^2 3^1 \rangle_3^{\tau_5}$ should equal $N_3^{\mathbb{R}}$, at least in the absolute value. By the same reasoning, the number $\langle 7^2 5^0 3^1 \rangle_3^{\tau_7}$ should also equal $N_3^{\mathbb{R}}$.

(3) Every degree 5 curve lies in a $\mathbb{P}^5$, every three non-real points $p_1$, $p_2$, and $p_3$ in $\mathbb{P}^7$ determine a real $\mathbb{P}^5$, and a real degree 5 curve passing through $p_1$, $p_2$, and $p_3$ lies in this $\mathbb{P}^5$. Thus, the numbers $\langle 7^3 5^1 3^0 \rangle_5^{\tau_7}$ and $\langle 7^3 5^0 3^2 \rangle_5^{\tau_7}$ should equal $\langle 5^4 3^0 \rangle_5^{\tau_5}$ and $\langle 5^3 3^2 \rangle_5^{\tau_5}$, respectively.

| $d$ | cond | $\langle 5^a 3^b \rangle_d^{\tau_5}$ |
|---|---|---|
| 1 | $5^1 3^0$ | 1 |
| 1 | $5^0 3^2$ | 1 |
| 3 | $5^2 3^1$ | -1 |
| 3 | $5^1 3^3$ | -3 |
| 3 | $5^0 3^5$ | -5 |
| 5 | $5^4 3^0$ | 1 |
| 5 | $5^3 3^2$ | 1 |
| 5 | $5^2 3^4$ | -7 |
| 5 | $5^1 3^6$ | 93 |
| 5 | $5^0 3^8$ | 12417 |
| 7 | $5^5 3^1$ | -23 |
| 7 | $5^4 3^3$ | -213 |
| 7 | $5^3 3^5$ | -2679 |
| 7 | $5^2 3^7$ | -23001 |
| 7 | $5^1 3^9$ | 874089 |
| 7 | $5^0 3^{11}$ | 90271011 |
| 9 | $5^7 3^0$ | 21 |
| 9 | $5^6 3^2$ | -503 |
| 9 | $5^5 3^4$ | -16399 |
| 9 | $5^4 3^6$ | -394863 |
| 9 | $5^3 3^8$ | -6924579 |
| 9 | $5^2 3^{10}$ | 69060873 |
| 9 | $5^1 3^{12}$ | 19824606009 |
| 9 | $5^0 3^{14}$ | 1811570349393 |

| $d$ | cond | $\langle 7^a 5^b 3^c \rangle_d^{\tau_7}$ |
|---|---|---|
| 1 | $7^1 5^0 3^0$ | 1 |
| 1 | $7^0 5^1 3^1$ | 1 |
| 1 | $7^0 5^0 3^3$ | 1 |
| 3 | $7^2 5^0 3^1$ | -1 |
| 3 | $7^1 5^2 3^0$ | -1 |
| 3 | $7^1 5^1 3^2$ | -3 |
| 3 | $7^1 5^0 3^4$ | -5 |
| 3 | $7^0 5^3 3^1$ | -3 |
| 3 | $7^0 5^2 3^3$ | -1 |
| 3 | $7^0 5^1 3^5$ | 89 |
| 3 | $7^0 5^0 3^7$ | 1155 |
| 5 | $7^3 5^1 3^0$ | 1 |
| 5 | $7^3 5^0 3^2$ | 1 |
| 5 | $7^2 5^2 3^1$ | -3 |
| 5 | $7^2 5^1 3^3$ | -27 |
| 5 | $7^2 5^0 3^5$ | -175 |
| 5 | $7^1 5^4 3^0$ | -11 |
| 5 | $7^1 5^3 3^2$ | -71 |
| 5 | $7^1 5^2 3^4$ | -239 |
| 5 | $7^1 5^1 3^6$ | 2181 |
| 5 | $7^1 5^0 3^8$ | 75405 |
| 5 | $7^0 5^5 3^1$ | -55 |
| 5 | $7^0 5^4 3^3$ | 349 |
| 5 | $7^0 5^3 3^5$ | 20589 |
| 5 | $7^0 5^2 3^7$ | 438481 |
| 5 | $7^0 5^1 3^9$ | 7937169 |
| 5 | $7^0 5^0 3^{11}$ | 139758309 |

Table 2: The numbers $\langle 5^a 3^b \rangle_d^{\tau_5}$ and $\langle 7^a 5^b 3^c \rangle_d^{\tau_7}$ of degree $d$ real rational curves through $a$ non-real points and $b$ non-real planes in $\mathbb{P}^5$ and through $a$ non-real points, $b$ non-real planes, and $c$ non-real linear $\mathbb{P}^4$'s in $\mathbb{P}^7$, respectively.

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