Some properties of hypergeometric series associated with mirror symmetry

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Abstract

We show that certain hypergeometric series used to formulate mirror symmetry for Calabi-Yau hypersurfaces, in string theory and algebraic geometry, satisfy a number of interesting properties. Many of these properties are used in separate papers to verify the BCOV prediction for the genus one Gromov-Witten invariants of a quintic threefold and more generally to compute the genus one Gromov-Witten invariants of any Calabi-Yau projective hypersurface.

1. Introduction

An astounding prediction for the genus zero Gromov-Witten invariants of (counts of rational curves in) a quintic threefold was made in [CaDGP]. It was formulated in terms of the function $\mathcal{F}$ defined in (1) below and related objects. This 1991 mirror symmetry prediction was mathematically verified about five years later. The 1993 mirror symmetry prediction of [BCOV] for the genus one Gromov-Witten invariants of a quintic threefold was verified in [Z1], and a generalization to degree n hypersurfaces in $\mathbb{C}P^{n-1}$ for arbitrary n was formulated and proved in [Z2]. The proofs in these two papers make use of the properties of $\mathcal{F}$ described by Theorems 1–3 below. Theorem 4 explores related properties of $\mathcal{F}$ which appear to be of interest in their own right and may also be of use in computation of higher genus Gromov-Witten invariants. Some further conjectural properties are stated in Section 3.

We denote by

$$\mathcal{P} \subset 1 + x\mathbb{Q}(w)[[x]]$$

the subgroup of power series in x with constant term 1 whose coefficients are rational functions in w which are holomorphic at $w=0$. Thus, the evaluation map

$$\mathcal{P} \to 1 + x\mathbb{Q}[[x]], \qquad F(w,x) \mapsto F(0,x)\,,$$

is well-defined. We define a map $\mathbf{M}: \mathcal{P} \to \mathcal{P}$ by

$$\mathbf{M}F(w,x) \;=\; \left\{1 + \frac{x}{w}\frac{\partial}{\partial x}\right\}\frac{F(w,x)}{F(0,x)}\;.$$

Our first result says that the hypergeometric functions arising in the mirror symmetry predictions are periodic fixed points of the map $\mathbf{M}$.

Theorem 1. *Let n be a positive integer and $\mathcal{F} \in \mathcal{P}$ the hypergeometric series*

$$\mathcal{F}(w,x) \;=\; \sum_{d=0}^{\infty} x^d \frac{\prod_{r=1}^{r=nd}(nw+r)}{\prod_{r=1}^{r=d}((w+r)^n-w^n)}\,. \tag{1}$$

Then $\mathbf{M}^n\mathcal{F} = \mathcal{F}$.

Note that we consider n as fixed and therefore omit it from the notations.

If we now define further power series $\mathcal{F}_p \in \mathcal{P}$ and $I_p \in 1 + x\mathbb{Q}[[x]]$ for all $p \geq 0$ by

$$\mathcal{F}_p(w,x) \;=\; \mathbf{M}^p\mathcal{F}(w,x)\,, \qquad I_p(x) \;=\; \mathcal{F}_p(0,x)\,,$$

so that $\mathcal{F}_{p+1} = (1 + w^{-1}x\,d/dx)(\mathcal{F}_p/I_p)$, then Theorem 1 says that $\mathcal{F}_{n+p} = \mathcal{F}_p$ and consequently $I_{n+p} = I_p$ for all $p \geq 0$. The next result gives further properties of the functions $\{I_p\}_{p\in\mathbb{Z}/n\mathbb{Z}}$.

Theorem 2. *The power series $I_p(x)$, $0 \leq p \leq n-1$, satisfy*

$$I_0(x)\,I_1(x)\,\cdots\,I_{n-1}(x) \;=\; (1-n^nx)^{-1}, \tag{2}$$

$$I_0(x)^{n-1}I_1(x)^{n-2}\cdots I_{n-1}(x)^0 \;=\; (1-n^nx)^{-(n-1)/2}\,, \tag{3}$$

$$I_p(x) \;=\; I_{n-1-p}(x) \qquad (0 \leq p \leq n-1)\,. \tag{4}$$

We note that (2) and the symmetry property (4) imply (3). However, (3) is simpler to prove directly than (4) and will be verified together with (2) before we give the proof of (4).

The power series I_p describe the structure of $\mathcal{F}$ at $w=0$. We will also give information about its structure at $w=\infty$. We begin with the following observation, which will be proved in Subsection 2.3.

Lemma 1. *If $F \in \mathcal{P}$ and $\mathbf{M}^kF = F$ for some $k > 0$, then every coefficient of the power series $\log F(w,x) \in \mathbb{Q}(w)[[x]]$ is $\mathrm{O}(w)$ as $w \to \infty$.*

Applying this lemma to $F = \mathcal{F}$, which satisfies its hypothesis by Theorem 1, we find that $\log\mathcal{F}(w,x)$ has an asymptotic expansion $\sum_{j=-1}^{\infty}\mu_j(x)w^{-j}$ with $\mu_j(x) \in x\mathbb{Q}[[x]]$ for all $j \geq -1$ or equivalently, that $\mathcal{F}(w,x)$ itself has an asymptotic expansion

$$\mathcal{F}(w,x) \;\sim\; e^{\mu(x)w}\sum_{s=0}^{\infty}\Phi_s(x)\,w^{-s} \qquad (w \to \infty) \tag{5}$$

for some power series $\mu = \mu_{-1}$, $\Phi_0 = e^{\mu_0}$, $\Phi_1 = \Phi_0\mu_1$, $\ldots$ in $\mathbb{Q}[[x]]$.

Theorem 3. *The first three coefficients $\mu(x)$, $\Phi_0(x)$, and $\Phi_1(x)$ in the expansion* (5) *are given by*

$$\mu(x) = \int_0^x \frac{L(u)-1}{u}\,du\,, \quad \Phi_0(x) = L(x)\,, \quad \Phi_1(x) = \frac{(n-2)(n+1)}{24n}\bigl(L(x)-L(x)^n\bigr)\,, \tag{6}$$

where $L(x)$ denotes the power series $(1-n^nx)^{-1/n} \in \mathbb{Z}[[x]]$.

The proof of this theorem in Subsection 2.3 can be systematized and streamlined to obtain an algorithm for computing every Φ_s by a differential recursion, which we now state. For integers $m \geq j \geq 0$ (and for our fixed integer n) we define $\mathcal{H}_{m,j} = \mathcal{H}_{m,j}(X) \in \mathbb{Q}[X]$ inductively by

$$\mathcal{H}_{0,j} = \delta_{0,j}, \qquad \mathcal{H}_{m,j} = \mathcal{H}_{m-1,j} + (X-1)\Bigl(X\frac{d}{dX} + \frac{m-j}{n}\Bigr)\mathcal{H}_{m-1,j-1} \quad \text{for } m \geq 1 \tag{7}$$

(with the convention that $\mathcal{H}_{m-1,j-1} = 0$ if $j = 0$). For example, for $0 \leq j \leq 2$ we find

$$\begin{aligned} \mathcal{H}_{m,0}(X) = 1, \qquad \mathcal{H}_{m,1}(X) \ &= \ \frac{1}{n}\binom{m}{2}(X-1), \\ \mathcal{H}_{m,2}(X) \ &= \ \frac{1}{n^2}\binom{m}{3}\bigl((n+1)X-1\bigr)(X-1) \ + \ \frac{3}{n^2}\binom{m}{4}(X-1)^2\,; \end{aligned} \tag{8}$$

more generally, $\mathcal{H}_{m,j}$ for fixed $j \geq 1$ and varying m has the form $\sum_{k=1}^{j}\binom{m}{j+k}Q_{j,k}(X)$ with $Q_{j,k} \in \mathbb{Z}[n^{-1}, X]$ defined inductively by

$$Q_{0,k} = \delta_{0,k}, \qquad Q_{j,k} = (X-1)\bigl(XQ'_{j-1,k} + (k\,Q_{j-1,k} + (k+j-1)\,Q_{j-1,k-1})/n\bigr) \quad \text{for } j \geq 1.$$

We now define differential operators $\mathfrak{L}_k$ $(0 \leq k \leq n)$ on $\mathbb{Q}[[x]]$ by

$$\mathfrak{L}_k \ = \ \sum_{i=0}^{k}\Biggl(\binom{n}{i}\mathcal{H}_{n-i,k-i}(L^n) - (L^n-1)\sum_{r=1}^{k-i}\binom{n-r}{i}\frac{S_r(n)}{n^r}\,\mathcal{H}_{n-i-r,k-i-r}(L^n)\Biggr)D^i\,, \tag{9}$$

where $D = x\,d/dx$ and $S_r(n)$ denotes the rth elementary symmetric function of $1, 2, \ldots, n$ (a Stirling number of the first kind). Using (8), we find that the first two of these operators are

$$\mathfrak{L}_1 \ = \ nD - (L^n - 1) = nLDL^{-1}\,, \tag{10}$$

$$\mathfrak{L}_2 \ = \ \binom{n}{2}D^2 - \frac{3(n-1)}{2}(L^n-1)D \ + \ \frac{n-1}{n}\Bigl(\frac{(n-2)(n-11)}{24}L^n - 1\Bigr)(L^n-1)\,. \tag{11}$$

Theorem 4. *(i) The power series $\Phi_s \in \mathbb{Q}[[x]]$, $s \geq 0$, are determined by the first-order ODEs*

$$\mathfrak{L}_1(\Phi_s) \ + \ \frac{1}{L}\,\mathfrak{L}_2(\Phi_{s-1}) \ + \ \frac{1}{L^2}\mathfrak{L}_3(\Phi_{s-2}) \ + \ \cdots \ + \ \frac{1}{L^{n-1}}\,\mathfrak{L}_n(\Phi_{s+1-n}) \ = \ 0, \quad s \geq 0, \tag{12}$$

(with the convention $\Phi_r = 0$ for $r < 0$) together with the initial condition $\Phi_s(0) = \delta_{0,s}$.
(ii) For fixed s and n, $\Phi_s(x)$ belongs to $L\,\mathbb{Q}[L]$.
(iii) For fixed s, $\Phi_s(x)$ belongs to $\mathbb{Q}(n)[L, L^{-1}, L^n]$.

The meaning of part *(iii)* in Theorem 4 is that for each $s \geq 0$ there exists

$$\Psi_s \equiv \Psi_s(a, X, Y, Z) \in \mathbb{Q}(a)[X, Y, Z]$$

such that the function $\Phi_s(x)$ defined by (1) and (5) is given by

$$\Phi_s(x) = \Psi_s\bigl(n, L(x), L(x)^{-1}, L(x)^n\bigr).$$

In particular, *(iii)* neither implies nor is implied by *(ii)*.

For example, from (12) for $s=0$ and $s=1$ together with equations (10) and (11) one finds the second and third identity in (6), and continuing the same way one obtains

$$\Phi_2 = \frac{(n+1)^2(n-2)^2}{2\,(24n)^2}(L-2L^n+L^{2n-1}) = \Phi_1^2/2L\Phi_0\,,$$
$$\begin{aligned}\Phi_3 = \frac{(n+1)(n-2)}{30\,(24n)^3}\Big\{&(1003n^4-2366n^3+3759n^2-1676n-164)\,L^{3n-2}\\ &-\,72\,(n-1)(3n-1)(7n^2-9n+14)\,L^{2n-2}\\ &+\,15\,(n+1)^2(n-2)^2\big(L^{2n-1}-L^n\big)\\ &+\,72\,(n-1)(7n^3-17n^2+22n-24)\,L^{n-2}\\ &+\,(5n^4+134n^3-447n^2+308n-556)\,L\Big\}\,,\end{aligned}$$

illustrating parts (ii) and (iii) of the theorem. These expressions, and the similar formulas obtained for $s\le 7$, suggest that in fact Φ_s for s fixed and n varying is an element of $\mathbb{Q}[n,n^{-1},L,L^{-1},L^n]$, sharpening statement (iii), but we do not know how to prove this. Some further data and a further conjecture concerning the functions $\Phi_s(x)$ are given in Section 3.

Remark. Although the calculations in this paper are completely independent of the theory of Gromov-Witten invariants, this theory does form the principal motivation, so we give here a brief indication of how the function $\mathcal{F}$ defined in (1) is used in GW-theory. Mathematical formulations of mirror symmetry ordinarily involve the coefficients of $w^0,\ldots,w^{n-1}$ of the Taylor series expansion of the function $\widetilde{\mathcal{F}}_0$ in equation (25) at $w=0$. These coefficients are of course the same as the corresponding coefficients of $\mathcal{F}$. The power series $\widetilde{\mathcal{F}}_0$ and $\mathcal{F}$ are the evaluations at $\alpha_1=\ldots=\alpha_n=0$ and $\hbar=1$ of the functions

$$\widetilde{\mathcal{Y}}_0 \equiv \sum_{d=0}^{\infty} x^d \frac{\prod_{r=1}^{r=nd}(nw+r\hbar)}{\prod_{r=1}^{r=d}\prod_{k=1}^{k=n}(w-\alpha_k+r\hbar)} \qquad \text{and}$$
$$\mathcal{Y} \equiv \sum_{d=0}^{\infty} x^d \frac{\prod_{r=1}^{r=nd}(nw+r\hbar)}{\prod_{r=1}^{r=d}\big(\prod_{k=1}^{k=n}(w-\alpha_k+r\hbar)-\prod_{k=1}^{k=n}(w-\alpha_k)\big)}\,,$$

which appear prominently in the proofs of genus 0 mirror symmetry in [Gi] and of genus 1 mirror symmetry in [Z1], respectively; both proofs involve a localization computation on $\mathbb{P}^{n-1}$. As elements of

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1};\mathbb{Q})\big[\big[\hbar^{-1},x\big]\big] \approx \Big(\mathbb{Q}[\alpha_1,\ldots,\alpha_n]/\prod_{k=1}^{k=n}(w-\alpha_k)\Big)\big[\big[\hbar^{-1},x\big]\big],$$

where $\mathbb{T}$ is the n-torus acting in the usual way on $\mathbb{P}^{n-1}$, $\widetilde{\mathcal{Y}}_0$ and $\mathcal{Y}$ are the same. On the other hand,

$$\mathrm{Res}_{\hbar=0}\big\{\mathcal{Y}(w=\alpha_i)\big\} = \big\{\mathrm{Res}_{\hbar=0}\mathcal{Y}(w)\big\}\big|_{w=\alpha_i}\,.$$

This property, which clearly does not hold for $\widetilde{\mathcal{Y}}_0$ (viewed as a power series in rational functions), is used in an essential way in Subsection 3.4 in [Z1] and eventually reduces the problem of computing genus 1 GW-invariants of X_n to determining the first three coefficients in the expansion (5) of $\mathcal{F}$.

2. Proofs

2.1. Preliminaries

It will be convenient to introduce notations D and D_w for the first order differential operators $D = x\,\frac{d}{dx}$ and $D_w = D + w$ on $\mathbb{Q}(w)[[x]]$. (Here we think of w as a parameter rather than a variable and write simply $\frac{d}{dx}$ instead of $\frac{\partial}{\partial x}$.) The effect of D_w on a power series $\sum c_d(w)x^d \in \mathbb{Q}(w)[[x]]$ is to multiply each $c_d(w)$ by $w+d$, so D_w has an inverse operator D_w^{-1} which replaces each $c_d(w)$ by $(w+d)^{-1}c_d(w)$. The operator $\mathbf{M}$ defined above can be written in terms of D_w as $F(w,x) \mapsto w^{-1}D_w\big[F(w,x)/F(0,x)\big]$.

We remark that instead of working with the functions $\mathcal{F}_p(w,x)$, we could have worked with the functions $R_p(w,t) = e^{wt}\mathcal{F}_p(w,e^t)$, which are the objects that actually arise in the analysis of the mirror symmetry predictions for Gromov-Witten invariants. If we had done that, then the differential operator $D_w = w + x\,d/dx$ would have been replaced by the simpler differential operator d/dt, explaining why this operator plays such a ubiquitous role in our analysis. But it is easier, both in the calculations and for purposes of exposition, to work with power series over $\mathbb{Q}(w)$ in a single variable x rather than with objects in the less familiar space $e^{wt}\mathbb{Q}(w)[[e^t]]$.

The following lemma and its corollary are key to the proofs of the four theorems stated above.

Lemma 2. *Suppose $c_0,\,\dots,\,c_m,\ f,\ g,\ a$ are functions of t (with f not identically 0) satisfying*

$$\begin{aligned} c_m\, f^{(m)} \;+\; c_{m-1}\, f^{(m-1)} \;+\; \dots \;+\; c_0\, f \;&=\; 0\,,\\ c_m\, g^{(m)} \;+\; c_{m-1}\, g^{(m-1)} \;+\; \dots \;+\; c_0\, g \;&=\; a\,, \end{aligned} \tag{13}$$

where $f^{(k)} = d^kf/dt^k$. Then the function $h := (g/f)'$ satisfies

$$\widetilde{c}_{m-1}\, h^{(m-1)} \;+\; \widetilde{c}_{m-2}\, h^{(m-2)} \;+\; \dots \;+\; \widetilde{c}_0\, h \;=\; a, \tag{14}$$

where $\widetilde{c}_s(t) = \sum_{r=s+1}^m \binom{r}{s+1}\, c_r(t)\, f^{(r-1-s)}(t)$.

Proof: Using Leibnitz's rule and (13), we find

$$a \;=\; \sum_{r=0}^m c_r\,\big(f\cdot g/f\big)^{(r)} \;=\; \sum_{r=0}^m c_r\left(f^{(r)}g/f \;+\; \sum_{s=0}^{r-1}\binom{r}{s+1}f^{(r-1-s)}h^{(s)}\right) \;=\; \sum_{s=0}^{m-1}\widetilde{c}_s\, h^{(s)}\,.$$

Corollary 1. *Suppose $F(w,x) \in \mathcal{P}$ satisfies*

$$\left(\sum_{r=0}^m C_r(x)\, D_w^r\right) F(w,x) \;=\; A(w,x) \tag{15}$$

for some power series $C_0(x),\,\dots,\,C_m(x) \in \mathbb{Q}[[x]]$ and $A(w,x) \in \mathbb{Q}(w)[[x]]$ with $A(0,x) \equiv 0$. Then

$$\left(\sum_{s=0}^{m-1} \widetilde{C}_s(x)\, D_w^s\right)\mathbf{M}F(w,x) \;=\; \frac{1}{w}\, A(w,x)\,, \tag{16}$$

where $\widetilde{C}_s(x) := \sum_{r=s+1}^m \binom{r}{s+1}\, C_r(x)\, D^{r-1-s}F(0,x)$.

Proof: Apply the lemma with $c_r(t) = C_r(e^t),\ f(t) = F(0,e^t),\ g(t) = e^{wt}F(w,e^t),\ a(t) = e^{wt}A(w,e^t)$, noting that then $h(t) = we^{wt}\mathbf{M}F(w,e^t)$.

2.2. Proof of Theorems 1 and 2

For the proof of (2) and (3), it is convenient to define $\mathcal{F}_p(w,x)$ also for $p=-1$. Set

$$\mathcal{F}_{-1}(w,x) \;=\; w\,D_w^{-1}\mathcal{F}(w,x) \;=\; \sum_{d=0}^{\infty} x^d \frac{\prod_{r=0}^{r=nd-1}(nw+r)}{\prod_{r=1}^{r=d}\big((w+r)^n-w^n\big)} \;\in\; \mathcal{P}\,. \tag{17}$$

We have $\mathcal{F}_{-1}(0,x)=1$ and $w^{-1}D_w\mathcal{F}_{-1}=\mathcal{F}$, so $\mathcal{F}_p=\mathbf{M}^{p+1}\mathcal{F}_{-1}$ for all $p\ge 0$, justifying the notation. It is straightforward to check that $\mathcal{F}_{-1}$ is a solution of the differential equation

$$\Big(D_w^n \,-\, x\prod_{j=0}^{n-1}\big(nD_w+j\big)\Big)\,\mathcal{F}_{-1} \;=\; w^n\,\mathcal{F}_{-1}\,. \tag{18}$$

This has the form of (15) with $F=\mathcal{F}_{-1}$, $A=w^n\mathcal{F}_{-1}$, $m=n$, and

$$C_n(x) \;=\; 1-n^n x\,,\quad C_r(x) \;=\; -n^r\,S_{n-r}(n-1)\,x \qquad (0<r<n), \qquad C_0(x) \;=\; 0\,, \tag{19}$$

where $S_{n-r}(n-1)$ as before denotes the $(n-r)$-th elementary symmetric function of $1,2,\dots,n-1$. Applying Corollary 1 repeatedly, we obtain

$$\sum_{s=0}^{n-1-p} C_s^{(p)}(x)\,D_w^s\mathcal{F}_p(w,x) \;=\; w^{n-p-1}\,\mathcal{F}_{-1}(w,x) \qquad (0\le p\le n-1), \tag{20}$$

where $C_s^{(0)}(x)=C_{s+1}(x)$ with $C_r(x)$ as in (19) and $C_s^{(p)}$ for $p>0$ is given inductively by

$$C_s^{(p)} \;=\; \sum_{r=s+1}^{n-p}\binom{r}{s+1}\,C_r^{(p-1)}(x)\,D^{r-1-s}I_{p-1}(x)\,. \tag{21}$$

In particular, by induction on p we find that the first two coefficients in (20) are given by

$$C_{n-1-p}^{(p)} \;=\; (1-n^n x)\prod_{r=0}^{p-1} I_r(x)\,, \tag{22}$$

$$C_{n-2-p}^{(p)} \;=\; \Big(-\frac{n^n(n-1)}{2}\,x \,+\, (1-n^n x)\sum_{r=0}^{p-1}(n-r-1)\,\frac{I_r'(x)}{I_r(x)}\Big)\prod_{r=0}^{p-1} I_r(x)\,. \tag{23}$$

Equations (20) and (22) for $p=n-1$ give

$$(1-n^n x)\prod_{r=0}^{n-2} I_r(x)\,\mathcal{F}_{n-1}(w,x) \;=\; \mathcal{F}_{-1}(w,x)\,. \tag{24}$$

Setting $w=0$ in this relation and using $\mathcal{F}_{-1}(0,x)=1$ gives equation (2). Then substituting (2) back into (24) gives $\mathcal{F}_{n-1}/I_{n-1}=\mathcal{F}_{-1}$ and hence, applying $w^{-1}D_w$ to both sides, $\mathcal{F}_n=\mathcal{F}$, proving Theorem 1. Similarly, taking $p=n-2$ in equations (20), (22), and (23) and then setting $w=0$ gives

$$\sum_{r=0}^{n-2}(n-r-1)\,\frac{I_r'(x)}{I_r(x)} \;=\; \frac{n-1}{2}\,\frac{n^n\,x}{1-n^n x}\,,$$

and integrating this and exponentiating gives (3).

Finally, we must prove the reflection symmetry (4). For this purpose, it is useful to construct the power series I_p in another way. Define a function $\widetilde{\mathcal{F}}_0 \in \mathcal{P}$ by

$$\widetilde{\mathcal{F}}_0(w,x) \;=\; \sum_{d=0}^{\infty} x^d \, \frac{\prod_{r=1}^{r=nd}(nw+r)}{\prod_{r=1}^{r=d}(w+r)^n} \tag{25}$$

and set $\widetilde{\mathcal{F}}_p(w,x) \;=\; \mathbf{M}^p \widetilde{\mathcal{F}}_0(w,x)$ for all $p \geq 0$. Since $\widetilde{\mathcal{F}}_0(w,x)$ is congruent to $\mathcal{F}(w,x)$ modulo w^n, we find by induction on p that $\widetilde{\mathcal{F}}_p(w,x)$ is congruent to $\mathcal{F}_p(w,x)$ modulo w^{n-p} for all $0 \leq p \leq n-1$ and hence that $I_p(x) = \widetilde{\mathcal{F}}_p(0,x)$ in this range. We now argue as above, using $\widetilde{\mathcal{F}}_0$ instead of $\mathcal{F}_{-1}$. This function satisfies the differential equation

$$\Bigl(D_w^{n-1} \;-\; nx \prod_{j=1}^{n-1}\bigl(nD_w + j\bigr)\Bigr)\, \widetilde{\mathcal{F}}_0 \;=\; w^{n-1}\,.$$

Applying Corollary 1 repeatedly, we obtain

$$\sum_{s=0}^{n-1-p} \widetilde{C}_s^{(p)}(x)\, D_w^s \widetilde{\mathcal{F}}_p(w,x) \;=\; w^{n-p-1}$$

for $0 \leq p \leq n-1$, where the coefficients $\widetilde{C}_s^{(p)}(x) \in \mathbb{Q}[[x]]$ can be calculated recursively, the top one being given by

$$\widetilde{C}_{n-1-p}^{(p)}(x) = (1 - n^n x) I_0(x) \cdots I_{p-1}(x).$$

Specializing to $p = n-1$ and using (2), we find that $\widetilde{\mathcal{F}}_{n-1}(w,x) = I_{n-1}(x)$ is independent of w. Now by downwards induction on p, using the equation $\widetilde{\mathcal{F}}_p = I_p w D_w^{-1} \widetilde{\mathcal{F}}_{p+1}$, we can "reconstruct" all of the power series $\widetilde{\mathcal{F}}_p(w,x)$ $(n-1 \geq p \geq 0)$ from their special values $I_p(x) = \widetilde{\mathcal{F}}_p(0,x)$ at $w = 0$, obtaining in particular the formula

$$w^{1-n}\, \widetilde{\mathcal{F}}_0(w,x) \;=\; I_0\, D_w^{-1}\, I_1\, D_w^{-1} \,\cdots I_{n-2}\, D_w^{-1}\, I_{n-1}$$

for the initial series $\widetilde{\mathcal{F}}_0$. Comparing the coefficients of x^d on both sides of this equation, we find

$$\frac{n^{-1}\prod_{r=0}^{nd}(nw+r)}{[w(w+1)\cdots(w+d)]^n} \;=\; \sum_{\substack{d_0,\dots,d_{n-1}\geq 0\\ d_0+\dots+d_{n-1}=d}} \frac{c_0(d_0)\,\cdots\,c_{n-1}(d_{n-1})}{(w+d_1+\dots+d_{n-1})(w+d_2+\dots+d_{n-1})\cdots(w+d_{n-1})}$$

for all $d \geq 0$, where $c_p(d)$ denotes the coefficient of x^d in $I_p(x)$. Splitting up the sum on the right into the subsum over n-tuples $(d_0,\dots,d_{n-1})$ with $\max\{d_r\} \leq d-1$ and the sum over the n-tuples which are permutations of $(d,0,\dots,0)$, and using that $c_p(0) = 1$ for all p, we can rewrite this equation as

$$\sum_{p=0}^{n-1} \frac{c_p(d)}{w^{n-p-1}(w+d)^p} \;=\; \frac{\prod_{r=0}^{nd}(nw+r)}{n\prod_{r=0}^{d}(w+r)^n} \;-\; \sum_{\substack{0\leq d_0,\dots,d_{n-1}<d\\ d_0+\dots+d_{n-1}=d}} \frac{c_0(d_0)\,\cdots\,c_{n-1}(d_{n-1})}{(w+d_1+\dots+d_{n-1})\cdots(w+d_{n-1})}$$

Now suppose by induction that $c_p(d') = c_{n-p-1}(d')$ for all $d' < d$ and all $0 \leq p \leq n-1$. (Notice that this is true for $d' = 0$ because $c_p(0) = I_p(0) = 1$ for all p, providing the starting point for the induction.) Then both terms on the right are $(-1)^{n-1}$-invariant under the map $w \to -w-d$, as one sees for the second term by making the renumbering $d_r \to d_{n-1-r}$. It follows that the left-hand side has the same invariance and hence that $c_p(d) = c_{n-1-p}(d)$ for all $0 \leq p \leq n-1$, completing the inductive proof of the desired symmetry $I_{n-1-p} = I_p$.

2.3. Proof of Theorem 3

We now turn to the expansion of $\mathcal{F}(w,x)$ near $w=\infty$. We first prove Lemma 1, which said that any periodic fixed point of the map $\mathbf{M}:\mathcal{P}\to\mathcal{P}$ has a logarithm which belongs to $w\,\mathbb{Q}[[x,w^{-1}]]$.

Proof of Lemma 1: The effect of $\mathbf{M}$ on logarithms is given by $\mathbf{M}\big(e^{H(w,x)}\big)=e^{H^*(w,x)}$, where

$$H^*(w,x) \;=\; H(w,x)-H(0,x)+\log\biggl(1+\frac{DH(w,x)-DH(0,x)}{w}\biggr); \tag{26}$$

here, as before, D denotes $x\frac{\partial}{\partial x}$. Suppose that $H(w,x):=\log F(w,x)$ is not $\mathrm{O}(w)$, and let e be the smallest integer such that the coefficient of x^e in $H(w,x)$ is not $\mathrm{O}(w)$ as $w\to\infty$. Then

$$H(w,x) \;=\; Cx^e w^N + x\mathrm{O}_w(w) + x^e\mathrm{O}_w(w^{N-1}) + \mathrm{O}(x^{e+1}) \qquad (w\to\infty) \tag{27}$$

for some $C\neq 0$ and $N\geq 2$, where $\mathrm{O}_w(w^\nu)$ denotes a polynomial in x with coefficients that grow at most like w^ν as $w\to\infty$ and $\mathrm{O}(x^{e+1})$ denotes an element of $x^{e+1}\,\mathbb{Q}(w)[[x]]$. From (26) and (27),

$$H^*(w,x) \;=\; H(w,x) + Cex^e w^{N-1} + x\mathrm{O}_w(1) + x^e\mathrm{O}_w(w^{N-2}) + \mathrm{O}(x^{e+1}).$$

This has the same form as (27) with the same C, e, and N. Iterating, we find that

$$\log\big(\mathbf{M}^k F(w,x)\big) \;=\; H(w,x) + kCex^e w^{N-1} + x\mathrm{O}_w(1) + x^e\mathrm{O}_w(w^{N-2}) + \mathrm{O}(x^{e+1}),$$

and this contradicts the assumption that $\mathbf{M}^k F = F$, since $C\neq 0$ and $N\geq 2$.

As already mentioned in the introduction, Lemma 1 together with Theorem 1 implies that $\mathcal{F}(w,x)$ has an asymptotic expansion of the form (5). From the proof of the lemma, we see that each $\mathcal{F}_p(w,x)=\mathbf{M}^p\mathcal{F}(w,x)$ has an asymptotic expansion

$$\mathcal{F}_p(w,x) \;\sim\; e^{\mu(x)w}\sum_{s=0}^{\infty}\Phi_{p,s}(x)\,w^{-s} \qquad (w\to\infty) \tag{28}$$

of the same form, with the same function $\mu(x)$ in the exponent. The equation $\mathcal{F}_{p+1}=\mathbf{M}\mathcal{F}_p$ gives

$$\Phi_{0,s}=\Phi_s, \quad \Phi_{p+1,s}=\frac{1+\mu'}{I_p}\,\Phi_{p,s} \;+\; \begin{cases}\big(\Phi_{p,s-1}/I_p\big)' & \text{if } s\geq 1,\\ \qquad 0 & \text{otherwise,}\end{cases} \tag{29}$$

where f' denotes $Df=x\,df/dx$. We want to solve these equations by induction on p for small s.

Before doing this, we begin with the following observation. Let $L(x)=(1-n^n x)^{-1/n}$ as in Theorem 3. Then (2) says that the product of the functions $I_p(x)/L(x)$ ($p\in\mathbb{Z}/n\mathbb{Z}$) equals 1, so if we define

$$H_p(x) \;=\; \frac{L(x)^p}{I_0(x)\cdots I_{p-1}(x)} \qquad (p\geq 0), \tag{30}$$

then we have the properties

$$H_0=1, \quad H_p/H_{p+1}=I_p/L, \quad H_1H_2\cdots H_n=1, \quad H_{p+n}=H_p, \quad H_{n-p}=H_p^{-1}, \tag{31}$$

where the last equality is originally true for $0\leq p\leq n$ but then, in view of the periodicity of $\{H_p\}$, holds for any $p\in\mathbb{Z}/n\mathbb{Z}$. A number of identities below are simpler to state in terms of the functions $H_p(x)$ than in terms of the original functions $I_p(x)$.

The case $s=0$ of (29) gives by induction the formula $\Phi_{p,0} = (1+\mu')^p/I_0\cdots I_{p-1}$. Combining this with the formulas $\mathcal{F}_n = \mathcal{F}$ and (2), we obtain $(1+\mu')^n = L^n$, from which the first equation in (6) follows since $\mu(x)$ is a power series in x with no constant term. This also gives us the formula

$$\Phi_{p,0}(x) \;=\; H_p(x)\,\Phi_0(x) \qquad \text{for all } p\ge 0,$$

with H_p as in (30). Now substituting this into the case $s=1$ of (29) we find inductively

$$\Phi_{p,1}(x) \;=\; H_p(x)\left(\Phi_1(x) \,+\, p\,\frac{\Phi_0'-L'}{L} \,+\, \frac{\Phi_0}{L}\sum_{r=1}^{p}\frac{H_r'}{H_r}\right) \qquad \text{for all } p\ge 0.$$

Setting $p=n$ in this relation and using the third and fourth of equations (31) and $\mathcal{F}_n = \mathcal{F}$, we deduce that $\Phi_0 = L$, which is the second assertion of Theorem 3. At the same time we can refine the last two equations to

$$\Phi_{p,0} \;=\; H_p\,L, \qquad \Phi_{p,1} \;=\; H_p\left(\Phi_1 \,+\, \sum_{r=1}^{p}\frac{H_r'}{H_r}\right) \qquad (p\ge 0). \tag{32}$$

The proof of the third identity in (6) is similar, but the calculations are more complicated. The case $s=2$ of (29) gives by induction the formula

$$\Phi_{p,2} \;=\; H_p\left(\Phi_2 \,+\, p\Bigl(\frac{\Phi_1}{L}\Bigr)' \,+\, \Bigl(\sum_{r=1}^{p}\frac{H_r'}{H_r}\Bigr)\frac{\Phi_1}{L} \,+\, \frac{1}{L}\sum_{s=2}^{p}\sum_{r=1}^{s-1}\frac{H_r'}{H_r}\frac{H_s'}{H_s} \,+\, \Bigl(\frac{1}{L}\sum_{r=1}^{p-1}(p-r)\frac{H_r'}{H_r}\Bigr)'\right)$$

for all $p\ge 0$. Taking $p=n$, observing that

$$\sum_{s=2}^{n}\sum_{r=1}^{s-1}\frac{H_r'}{H_r}\frac{H_s'}{H_s} \equiv \frac{1}{2}\left(\Bigl(\sum_{p=1}^{n}\frac{H_p'}{H_p}\Bigr)^2 - \sum_{p=1}^{n}\Bigl(\frac{H_p'}{H_p}\Bigr)^2\right) = -\frac{1}{2}\sum_{p=1}^{n}\Bigl(\frac{H_p'}{H_p}\Bigr)^2$$

by the third equation in (31), and using $\mathcal{F}_n = \mathcal{F}$, we find that

$$n\Bigl(\frac{\Phi_1}{L}\Bigr)' \;=\; \frac{1}{2L}\sum_{p=1}^{n}\Bigl(\frac{H_p'}{H_p}\Bigr)^2 \,+\, \Bigl(\frac{1}{L}\sum_{p=0}^{n-1}p\,\frac{H_p'}{H_p}\Bigr)' \;=\; -\frac{(n+1)(n-2)}{24}\,\bigl(L^{n-1}\bigr)',$$

the last equation being Lemma 3 below. Integrating and using $\Phi_1(0)=0$ gives the last identity in (6).

Lemma 3. *The functions* $\{H_p(x)\}_{p\in\mathbb{Z}/n\mathbb{Z}}$ *satisfy*

$$\frac{1}{2L}\sum_{p \pmod n}\Bigl(\frac{H_p'}{H_p}\Bigr)^2 \;=\; -\Bigl(\frac{(n+1)(n-2)}{24}\,L^{n-1} \,+\, \frac{1}{L}\sum_{p=0}^{n-1}p\,\frac{H_p'}{H_p}\Bigr)'\,. \tag{33}$$

The proof consists of expressing the left-hand side of (33) in terms of the functions $I_0, I_1, \ldots, I_{n-1}$ and their derivatives, getting rid of all square terms via the product rule, and then eliminating I_{n-1}, I_{n-2}, and I_{n-3}. The last elimination is achieved by computing the coefficients $C_p^{(n-3-p)}$ inductively by (20), starting with

$$C_{n-3}^{(0)} \;=\; -n^{n-2}\,S_2(n-1)\,x = -\frac{(n-1)(n-2)(3n-1)}{24}L'/L^{n+1}\,,$$

and then setting $p = n-3$, exactly as we did with $C_p^{(n-1-p)}$ and $C_p^{(n-2-p)}$ in Subsection 2.2 to prove eqs. (2) and (3). At this stage, all terms involving products of two functions I_p cancel, and the resulting expression can be integrated. We omit the details, which are somewhat tedious, since the last identity in (6) also follows easily from Theorem 4.

2.4. Proof of Theorem 4

We set $X = L^n$ and $Y = (L^n - 1)/n$. Note that

$$D(\mu) = L - 1, \qquad D(L) = LY, \qquad D(X) = X^2 - X, \qquad D(Y) = XY. \tag{34}$$

The first identity implies that $D_w\, e^{\mu w} = e^{\mu w}\, \widetilde{D}_w$, where $\widetilde{D}_w = D + Lw$. By induction on k, the powers of the differential operator $\widetilde{D}_w$ are given by

$$\begin{aligned} \widetilde{D}_w^k \;&=\; \sum_{m=0}^{k} \binom{k}{m} \widetilde{D}_w^m(1)\, D^{k-m} \\ &= D^k + k\, Lw\, D^{k-1} + \frac{k(k-1)}{2} \big((Lw)^2 + Y(Lw)\big)\, D^{k-2} + \dots . \end{aligned} \tag{35}$$

A second induction gives the formula

$$\widetilde{D}_w^m(1) \;=\; \sum_{j=0}^{m} \mathcal{H}_{m,j}(X)\, (Lw)^{m-j}, \tag{36}$$

with $\mathcal{H}_{m,j} \in \mathbb{Z}[X, Y] \subseteq \mathbb{Q}[X]$ given by (7).

The function $\mathcal{F}(w, x)$ satisfies the ODE

$$\Big(D_w^n \;-\; w^n \;-\; x \prod_{j=1}^{n} \big(nD_w + j\big)\Big)\mathcal{F} \;=\; 0\,.$$

Since $D_w\, e^{\mu w} = e^{\mu w}\, \widetilde{D}_w$, the function $\widetilde{\mathcal{F}}(w, x) = e^{-\mu(x) w}\mathcal{F}(w, x)$ satisfies the differential equation $\mathfrak{L}\widetilde{\mathcal{F}} = 0$, where $\mathfrak{L}$ is the differential operator

$$\begin{aligned} \mathfrak{L} \;&=\; L^n \Big(\widetilde{D}_w^n \;-\; w^n \;-\; x \prod_{j=1}^{n} \big(n\widetilde{D}_w + j\big)\Big) \\ &=\; \widetilde{D}_w^n \;-\; (Lw)^n \;-\; (L^n - 1) \sum_{r=1}^{n} \frac{S_r(n)}{n^r}\, \widetilde{D}_w^{n-r}\,. \end{aligned}$$

Using (35) and (36), we can expand $\mathfrak{L}$ as $\mathfrak{L} = \sum_{k=1}^{n} (Lw)^{n-k} \mathfrak{L}_k$, with $\mathfrak{L}_k$ defined by (9). Combining the differential equation $\mathfrak{L}\widetilde{\mathcal{F}} = 0$ with the asymptotic expansion $\widetilde{\mathcal{F}}(w, x) \sim \sum_{s \ge 0} \Phi_s(x) w^{-s}$ for large w, we obtain (12).

We will next use (12) to prove by induction that Φ_s belongs to $L\mathbb{Q}[L]$. Since $\mathfrak{L}_1(L\mathbb{Q}[L]) = L^2 Y \mathbb{Q}[L]$, it suffices to show that

$$\mathfrak{L}_k(L\mathbb{Q}[L]) \subseteq L^{k+1} Y\, \mathbb{Q}[L] \qquad (2 \le k \le n). \tag{37}$$

Let $\mathcal{I} \subset \mathbb{Q}[L]$ be the ideal generated by XY. Since D and Y commute modulo $\mathcal{I}$ by (34) and since $(D - rY)L^r = 0$, we have

$$(D-Y)(D-2Y)\dots(D-kY)L^r \;\in\; \begin{cases} L^r\,\mathcal{I} & \text{if } 1 \le r \le k\,, \\ L^r Y\,\mathbb{Q}[L] & \text{if } r \ge k+1\,. \end{cases}$$

Therefore (37) is a consequence of the following lemma.

Lemma 4. *For all $k > 1$,*

$$\mathfrak{L}_k \;\equiv\; \binom{n}{k}(D-Y)(D-2Y)\cdots(D-kY) \pmod{\mathcal{I}}\,.$$

Proof: The recursion (7) for $H_{m,j}$ shows that $H_{m,j} \equiv h_{m,j}\,Y^j \pmod{\mathcal{I}}$, where $h_{m,j} \in \mathbb{Z}$ is given recursively by

$$h_{0,j} = \delta_{0,j}, \qquad h_{m,j} = h_{m-1,j} + (m-j)h_{m-1,j-1} \quad \forall\; m \ge 1 \tag{38}$$

(with $h_{m-1,j-1} = 0$ for $j = 0$). Thus $h_{m,j} = \mathfrak{S}_m^{(m-j)}$, where $\mathfrak{S}_m^{(k)}$ denotes a Stirling number of the second kind (the number of ways of partitioning a set of m elements into k non-empty subsets). We also note $(1 - L^n)n^{-r} \equiv (-1)^r Y^r \pmod{\mathcal{I}}$ for all $r \ge 1$. Combining these facts with (9), we find that

$$\mathfrak{L}_k \;\equiv\; \sum_{i=0}^{k}\Biggl(\sum_{r=0}^{k-i}(-1)^r\binom{n-r}{i}\,S_r(n)\,\mathfrak{S}_{n-r-i}^{(n-k)}\Biggr)\,Y^{k-i}D^i \pmod{\mathcal{I}}\,.$$

The desired congruence for $\mathfrak{L}_k$ now follows from the generating series calculation

$$\begin{aligned}
&\sum_{i=0}^{k}\Biggl(\sum_{r=0}^{k-i}(-1)^r\binom{n-r}{i}\,S_r(n)\,\mathfrak{S}_{n-r-i}^{(n-k)}\Biggr)\,t^i \\
&= \sum_{i=0}^{n}\Biggl(\sum_{r=0}^{n-i}(-1)^r\binom{n-r}{i}\,S_r(n)\,\Biggl[\frac{1}{(n-k)!}\sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n-k}{j}j^{n-r-i}\Biggr]\Biggr)\,t^i \\
&= \frac{1}{(n-k)!}\sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n-k}{j}\sum_{r=0}^{n}(-1)^r S_r(n)\sum_{i=0}^{n-r}\binom{n-r}{i}j^{n-r-i}t^i \\
&= \frac{1}{(n-k)!}\sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n-k}{j}\sum_{r=0}^{n}(-1)^r S_r(n)(j+t)^{n-r} \\
&= \frac{n!}{(n-k)!}\sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n-k}{j}\binom{j+t-1}{n} \\
&= \frac{n!}{(n-k)!}\binom{t-1}{k} \;=\; \binom{n}{k}\,(t-1)(t-2)\cdots(t-k)\,,
\end{aligned}$$

where the first equality follows from the well-known fact that the expression in square brackets equals $\mathfrak{S}_{n-r-i}^{(n-k)}$ if $i + r \le k$ and 0 for $i + r > k$ and the second-to-last equality is obtained by expanding $(1+u)^{t-1}((1+u)-1)^{t-1}$ by the binomial theorem and equating coefficients of t^n.

This completes the proof of part (ii) of Theorem 4. Part (iii) of Theorem 4 follows from the differential equation (12) by induction on s.

3. Further discussion of the large w expansion of $\mathcal{F}(w,x)$

In this final section we give some further information and conjectures about the power series $\Phi_s(x)$ defined by equation (5). We begin by giving the numerical values for $n \le 5$ and $s \le 4$. For this purpose it is convenient to divide Φ_s/L by $((n-2)(n+1)/24n)^s/s!$ and write the result as the sum of $(1-L^{n-1})^s$ and a correction term, because the formulas then become much simpler than without this renormalization:

$$\begin{array}{lll}
n=3: & s=1: & 1-L^2 \\
 & s=2: & (1-L^2)^2 \\
 & s=3: & (1-L^2)^3 + 144\,(1-5L^3+4L^6) \\
 & s=4: & (1-L^2)^4 + 576\,(1-94L^2-5L^3+245L^5+4L^6-151L^8) \\
 & & \\
n=4: & s=1: & 1-L^3 \\
 & s=2: & (1-L^3)^2 \\
 & s=3: & (1-L^3)^3 + \frac{36}{25}\,(4+72L-297L^5+221L^9) \\
 & n=4 & (1-L^3)^4 + \frac{144}{125}\,(884+360L-20L^3-19584L^4-1485L^5 \\
 & & \qquad\qquad +44253L^8+1105L^9-25513L^{12}) \\
 & & \\
n=5: & s=1: & 1-L^4 \\
 & s=2 & (1-L^4)^2 \\
 & s=3: & (1-L^4)^3 + \frac{32}{45}\,(7+134L^2-504L^7+363L^{12}) \\
 & s=4: & (1-L^4)^4 + \frac{16}{135}\,(168+8576L+3216L^2-168L^4-127568L^6 \\
 & & \qquad -12096L^7+270144L^{11}+8712L^{12}-150984L^{16})
\end{array}$$

Table: List of values of $s!\left(\frac{24n}{(n-2)(n+1)}\right)^s \Phi_s/L$ for $s = 1, 2, 3, 4$ and $n = 3, 4, 5$

This suggests that the series $\sum_s (\Phi_s/L)\, w^{-s}$ is given to a first approximation by a pure exponential $\exp\left(\frac{(n-2)(n+1)}{24n}(1-L^{n-1})/w\right)$ and hence that the formulas for the coefficients of the expansion (5) may become simpler if we take the logarithm. Doing this, we find an expansion which begins

$$\log\mathcal{F}(w,x) \;=\; \mu(x)w \,+\, \log L(x) \,+\, \frac{(n-2)(n+1)(1-L(x)^{n-1})}{24n}\,w^{-1} \,+\, 0\,w^{-2} \,+\, \cdots$$

and in which, at least experimentally, the coefficient of w^{-j} for $j \ge 1$ is the sum of a term independent of x and a term of the form L^{-j} times a polynomial (without constant term) in L^n. By applying the operator $w^{-1}D$ and adding 1, this can be stated more elegantly as follows.

Conjecture: If $\mathcal{F}$ is given by (1), then

$$1 \,+\, \frac{x}{w}\,\frac{\partial}{\partial x}\,\log\mathcal{F}(w,x) \;\stackrel{?}{=}\; L\sum_{k=0}^{\infty}\frac{P_k(n,L^n)}{(nLw)^k}\,, \tag{39}$$

where $P_k(n,X)$ is a polynomial in X of degree k with coefficients in $\mathbb{Q}[n]$.

We have verified this conjecture up to order $\mathrm{O}\big(w^{-6}\big)$, with the values of the corresponding coefficients P_k being given by

$$\begin{aligned}
P_0(n,X) &= 1\,, \\
P_1(n,X) &= X-1\,, \\
P_2(n,X) &= -\frac{(n+1)(n-1)(n-2)}{24}(X-1)X\,, \\
P_3(n,X) &= 0\,, \\
P_4(n,X) &= \frac{(n+1)(n-1)(n-2)}{5760}(X-1)(A_3X^3+A_2X^2+A_1X)\,, \\
P_5(n,X) &= -\frac{(n+1)(n-1)(n-2)}{5760}(X-1)(B_4X^4+B_3X^3+B_2X^2+B_1X),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= (n-3)(7n^3-17n^2+22n-24)\,, \\
A_2 &= -(2n-3)(3n-1)(7n^2-9n+14)\,, \\
A_3 &= 3\,(14n^4-33n^3+52n^2-23n-2)\,, \\
B_1 &= -(n-3)(n-4)(7n^3-17n^2+22n-24)\,, \\
B_2 &= 2\,(n-1)(n-2)(49n^3-115n^2+152n-124)\,, \\
B_3 &= -4\,(n-1)(3n-1)(3n-4)(7n^2-9n+14)\,, \\
B_4 &= 8\,(n-1)(3n-2)(7n^3-11n^2+17n-1)\,.
\end{aligned}$$

The coefficients of the polynomials P_k follow no apparent pattern apart from the divisibility by $(n+1)(n-1)(n-2)X(X-1)$: the common factors of A_1 and B_1 and of A_2 and B_3 are striking, but nothing similar occurs for the next two polynomials. On the other hand, there is a simple formula for the leading coefficient of $P_k(n,X)$ with respect to n, namely (at least up to $k=7$)

$$P_k(n,X) \;=\; \begin{cases} \alpha_j\, e_k(X)\, n^{4j-1} \;+\; \mathrm{O}(n^{4j-2}) & \text{if } k=2j>0, \\ (j-1)\alpha_j\, e_k(X)\, n^{4j} \;+\; \mathrm{O}(n^{4j-1}) & \text{if } k=2j+1, \end{cases}$$

where α_j denotes the coefficient of u^{2j} in $\frac{u/2}{\sinh u/2}$ ($\alpha_0=1$, $\alpha_1=-\frac{1}{24}$, $\alpha_2=\frac{7}{5760}$, $\alpha_3=-\frac{31}{967680}$, $\dots$) and where $e_1=X-1$, $e_2=X^2-X$, $e_3=2X^3-3X^2+X$, $\dots$ are the polynomials defined by

$$e_k(X) \;=\; \sum_{l=1}^{k}(-1)^{k-l}\,(l-1)!\,\mathfrak{S}_k^{(l)}\,X^l \quad\in\quad \mathbb{Z}[X]$$

with $\mathfrak{S}_k^{(l)}$ as before a Stirling number of the second kind. This is interesting because the argument $X=L^n$ of $P_k(n,X)$ in equation (39) is in fact $(1-n^nx)^{-1}$ and the functions $e_k((1-x)^{-1})$ have the basic property

$$e_k\Big(\frac{1}{1-x}\Big) \;=\; \sum_{d=1}^{\infty} d^{k-1}x^d \quad\in\; x\,\mathbb{Z}[[x]] \qquad (k\ge 1)\,.$$

There is also a possible intriguing connection with modular and elliptic functions since, for example, the power series in two variables $\sum \alpha_j e_{2j}\big(\frac{1}{1-x}\big)\,u^{2j-1}$ is closely related to the expansion of the Weierstrass $\wp$-function and related Jacobi forms. This suggests possible hidden modularity properties of the original function $\mathcal{F}(w,x)$.

As a final remark, we observe that (39), if it is true, defines the power series $\mathcal{F}(w,x)$ even for non-integral values of n and shows that this function is analytic in n as well as in w and x. This seems surprising since $\mathcal{F}$ is defined as a hypergeometric function of order n and we would usually not expect such series to have a reasonable interpolation with respect to the order of the differential equation which they satisfy.

References

[BCOV] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Holomorphic Anomalies in Topological Field Theories*, Nucl. Phys. B405 (1993), 279–304.

[CaDGP] P. Candelas, X. de la Ossa, P. Green, L. Parkes, *A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory*, Nuclear Phys. B359 (1991), 21–74.

[Gi] A. Givental, *The Mirror Formula for Quintic Threefolds*, Amer. Math. Soc. Transl. Ser. 2, 196 (1999).

[Z1] A. Zinger, *The Reduced Genus-One Gromov-Witten Invariants of Calabi-Yau Hypersurfaces*, math/0705.2397.

[Z2] A. Zinger, *Standard vs. Reduced Genus-One Gromov-Witten Invariants*, math/0706.0715.

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