On the Refined Symplectic Sum Formula for Gromov-Witten Invariants

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January 21, 2020

Abstract

We describe the extent to which Ionel-Parker's proposed refinement of the standard relative Gromov-Witten invariants sharpens the usual symplectic sum formula. The key product operation on the target spaces for the refined invariants is specified in terms of abelian covers of symplectic divisors, making it suitable for studying from a topological perspective. We give several qualitative applications of this refinement, which include vanishing results for Gromov-Witten invariants.

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*Partially supported by NSF grant 0846978

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1 Introduction

Gromov-Witten invariants of symplectic manifolds, which include nonsingular projective varieties, are certain counts of pseudo-holomorphic curves that play prominent roles in symplectic topology, algebraic geometry, and string theory. The decomposition formulas, known as symplectic sum formulas in symplectic topology and degeneration formulas in algebraic geometry, are one of the main tools used to compute Gromov-Witten invariants; they relate Gromov-Witten invariants of one symplectic manifold to Gromov-Witten invariants of two simpler symplectic manifolds. Unfortunately, the formulas of [15, 16] do not completely determine the former in terms of the latter in many cases because of the so-called vanishing cycles: second homology classes in the first manifold which vanish when projected to the union of the other two manifolds; see (1.12). A refinement to the usual relative Gromov-Witten invariants of [14, 16] is sketched in [11]; the aim of this refinement is to resolve the unfortunate deficiency of the formulas of [15, 16] in [12]. In [5], we formally constructed the refinement to relative invariants suggested in [11] and discussed the invariance and computability aspects of the resulting curve counts. In this paper, we describe the extent to which it sharpens the usual symplectic sum formula and obtain some qualitative applications.

1.1 Relative GW-invariants

Let (X,ω) be a compact symplectic manifold and J be an ω-tame almost complex structure on X. For $g,k\in\mathbb{Z}^{\geq 0}$ and $A\in H_2(X;\mathbb{Z})$, we denote by $\overline{\mathfrak{M}}_{g,k}(X,A)$ the moduli space of stable J-holomorphic k-marked degree A maps from connected nodal curves of genus g. By [18, 6, 2], this moduli space carries a virtual class, which is independent of J and of representative ω in a deformation equivalence class of symplectic forms on X. If $V\subset X$ is a compact symplectic divisor (symplectic submanifold of real codimension 2), $\ell\in\mathbb{Z}^{\geq 0}$, $\mathbf{s}\equiv(s_1,\ldots,s_\ell)$ is an ℓ-tuple of positive integers such that

$$s_1+\ldots+s_\ell = A\cdot V, \tag{1.1}$$

and J restricts to an almost complex structure on V, let $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$ denote the moduli space of stable J-holomorphic $(k+\ell)$-marked maps from connected nodal curves of genus g that have contact with V at the last ℓ marked points of orders $s_1,\ldots,s_\ell$. According to [16, 14], this moduli space carries a virtual class, which is independent of J and of representative ω in a deformation equivalence class of symplectic forms on (X,V).

There are natural evaluation morphisms

$$\mathrm{ev}_X \equiv \mathrm{ev}_1 \times \ldots \times \mathrm{ev}_k : \overline{\mathfrak{M}}_{g,k}(X,A), \overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A) \longrightarrow X^k, \tag{1.2}$$

$$\mathrm{ev}^V_X \equiv \mathrm{ev}_{k+1} \times \ldots \times \mathrm{ev}_{k+\ell} : \overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A) \longrightarrow V_{\mathbf{s}} \equiv V^\ell, \tag{1.3}$$

sending each stable map to its values at the marked points. The (absolute) GW-invariants of (X,ω) are obtained by pulling back elements of $H^*(X^k;\mathbb{Q})$ by the morphism (1.2) and integrating them and other natural classes on $\overline{\mathfrak{M}}_{g,k}(X,A)$ against the virtual class of $\overline{\mathfrak{M}}_{g,k}(X,A)$. The (relative) GW-invariants of (X,V,ω) are obtained by pulling back elements of $H^*(X^k;\mathbb{Q})$ and $H^*(V_{\mathbf{s}};\mathbb{Q})$ by the morphisms (1.2) and (1.3), and integrating them and other natural classes on $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$ against the virtual class of $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$.

As emphasized in [11, Section 5], two preimages of the same point in $V_{\mathbf{s}}$ under (1.3) determine an element of

$$\mathcal{R}^V_X \equiv \ker\big\{\iota^X_{X-V*}\colon H_2(X-V;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})\big\}, \tag{1.4}$$

where $\iota^X_{X-V}\colon X-V \longrightarrow X$ is the inclusion; see [5, Section 2.1]. The elements of $\mathcal{R}^V_X$, called rim tori in [11], can be represented by circle bundles over loops γ in V; see [5, Section 3.1]. By standard topological considerations,

$$\mathcal{R}^V_X \approx H_1(V;\mathbb{Z})_X \equiv \frac{H_1(V;\mathbb{Z})}{H^V_X}, \qquad \text{where} \quad H^V_X \equiv \big\{A\cap V\colon A\in H_3(X;\mathbb{Z})\big\}\,; \tag{1.5}$$

see [5, Corollary 3.2].

The main claim of [11, Section 5] is that the above observations can be used to lift (1.3) over some regular (Galois), possibly disconnected (unramified) covering

$$\pi^V_{X;\mathbf{s}}\colon \widehat{V}_{X;\mathbf{s}} \longrightarrow V_{\mathbf{s}}; \tag{1.6}$$

the topology of this cover is specified in [5, Section 6.1]. Its group of deck transformations is

$$\mathrm{Deck}\big(\pi^V_{X;\mathbf{s}}\big) = \frac{\mathcal{R}^V_X}{\mathcal{R}'^V_{X;\mathbf{s}}} \times \mathcal{R}'^V_{X;\mathbf{s}} \tag{1.7}$$

for a certain submodule $\mathcal{R}'^V_{X;\mathbf{s}}$ of $\mathcal{R}^V_X$. For example,

$$\mathcal{R}'^V_{X;\mathbf{s}} = \begin{cases} \{0\}, & \text{if } \ell=0; \\ \gcd(\mathbf{s})\mathcal{R}^V_X, & \text{if } |\pi_0(V)|=1. \end{cases}$$

As discussed in [5, Section 1.1], the topology of the covering (1.6) is usually very complicated. Since

$$\mathrm{ev}^V_X = \pi^V_{X;\mathbf{s}} \circ \widetilde{\mathrm{ev}}^V_X\colon \overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A) \longrightarrow V_{\mathbf{s}} \tag{1.8}$$

for some morphism

$$\widetilde{\mathrm{ev}}^V_X\colon \overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A) \longrightarrow \widehat{V}_{X;\mathbf{s}}\,, \tag{1.9}$$

the numbers obtained by pulling back elements of $H^*(\widehat{V}_{X;\mathbf{s}};\mathbb{Q})$ by (1.9), instead of elements of $H^*(V_{\mathbf{s}};\mathbb{Q})$ by (1.3), and integrating them and other natural classes on $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$ against the virtual class of $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$ refine the usual GW-invariants of (X,V,ω). We will call these numbers the IP-counts for (X,V,ω). These numbers generally depend on the choice of the lift (1.9).

The construction of the coverings (1.6) is recalled in Section 2.2. The lifts (1.9) can be chosen systematically in a manner suitable for use in the symplectic sum context; see Proposition 2.2. In [5], we deduced vanishing results for the standard GW-invariants of (X,V,ω) from the existence of the lifts (1.9). We use a very basic case of these vanishing results in Section 6 to streamline the proof of [12, (15.4)], after correcting its statement; this formula computes the GW-invariants of the blowup $\widehat{\mathbb{P}}^2_9$ of $\mathbb{P}^2$ at 9 points.

1.2 Symplectic sum formulas

Let (X,ω_X) and (Y,ω_Y) be compact symplectic manifolds with a common compact symplectic divisor $V\subset X,Y$. We note that

$$e(\mathcal{N}_X V) = -e(\mathcal{N}_Y V) \in H^2(V;\mathbb{Z}) \tag{1.10}$$

if and only if there exists an isomorphism

$$\Phi\colon \mathcal{N}_X V \otimes \mathcal{N}_Y V \approx V\times\mathbb{C} \tag{1.11}$$

of complex line bundles. A symplectic sum of symplectic manifolds (X,ω_X) and (Y,ω_Y) with a common symplectic divisor V such that (1.10) holds is a symplectic manifold $(Z,\omega_Z)=(X\#_V Y,\omega_\#)$ obtained from X and Y by gluing the complements of tubular neighborhoods of V in X and Y along their common boundary as directed by Φ. In fact, the symplectic sum construction of [8, 24] produces a symplectic fibration $\pi\colon \mathcal{Z}\longrightarrow\Delta$ with central fiber $\mathcal{Z}_0=X\cup_V Y$, where $\Delta\subset\mathbb{C}$ is a disk centered at the origin and $\mathcal{Z}$ is a symplectic manifold with symplectic form $\omega_{\mathcal{Z}}$ such that

- π is surjective and is a submersion outside of $V\subset\mathcal{Z}_0$,
- the restriction ω_λ of $\omega_{\mathcal{Z}}$ to $\mathcal{Z}_\lambda\equiv\pi^{-1}(\lambda)$ is nondegenerate for every $\lambda\in\Delta^*$,
- $\omega_{\mathcal{Z}}|_X=\omega_X$, $\omega_{\mathcal{Z}}|_Y=\omega_Y$.

The symplectic manifolds $(\mathcal{Z}_\lambda,\omega_\lambda)$ with $\lambda\in\Delta^*$ are then symplectically deformation equivalent to each other and denoted $(X\#_V Y,\omega_\#)$. However, different homotopy classes of the isomorphisms (1.11) give rise to generally different topological manifolds; see [7]. There is also a retraction $q\colon \mathcal{Z}\longrightarrow\mathcal{Z}_0$ such that $q_\lambda\equiv q|_{\mathcal{Z}_\lambda}$ restricts to a diffeomorphism

$$\mathcal{Z}_\lambda - q_\lambda^{-1}(V)\longrightarrow \mathcal{Z}_0 - V$$

and to an S^1-fiber bundle $q_\lambda^{-1}(V)\longrightarrow V$, whenever $\lambda\in\Delta^*$. We denote by $q_\#\colon X\#_V Y\longrightarrow X\cup_V Y$ a typical collapsing map q_λ.

The symplectic sum formulas of [16, 15] for GW-invariants relate the absolute GW-invariants of a smooth fiber $\mathcal{Z}_\lambda=X\#_V Y$ to the GW-invariants of a singular fiber $\mathcal{Z}_0=X\cup_V Y$ and to the relative GW-invariants of the pairs (X,V) and (Y,V). The first relation is often called a degeneration

relation for GWs	ST name	AG name
$X\#_V Y$ vs. $X\cup_V Y$	degeneration formula	invariance property
$X\#_V Y$ vs. (X,V) and (Y,V)	decomposition formula	degeneration formula

Table 1: ST and AG terminology describing the two types of symplectic sum formulas for GW-invariants; our terminology is in sans-serif.

formula for GW-invariants in symplectic topology and an invariance property of GW-invariants in algebraic geometry; the formula [16, (5.7)] and the second formula at the bottom of [15, p201] fall under this category. The second relation is often called a decomposition formula for GW-invariants in symplectic topology and a degeneration formula for GW-invariants in algebraic geometry; the three formulas [16, (5.4),(5.7),(5.8)] together and the first formula at the bottom of [15, p201] fall under this category. In order to reduce confusion, we will call the first relation an invariance property and the second a decomposition formula; see Table 1. As indicated below, a decomposition formula for GW-invariants is an immediate consequence of an invariance property in the basic symplectic sum settings of [16, 15]. However, the difference between the two formulas turns out to be insurmountable in the refined setting of [12] and substantial in the (unrefined) multifold degeneration settings of [25, 10, 1].

With $V\subset X,Y$ as above, let

$$\begin{aligned} \mathcal{R}_{X,Y}^V &= \ker\big\{q_{\#*}\colon H_2(X\#_V Y;\mathbb{Z})\longrightarrow H_2(X\cup_V Y;\mathbb{Z})\big\}, \\ H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z}) &= \big\{(A_X,A_Y)\in H_2(X;\mathbb{Z})\times H_2(Y;\mathbb{Z})\colon\ A_X\cdot_X V = A_Y\cdot_Y V\big\}. \end{aligned} \tag{1.12}$$

As recalled in [5, Section 2.2], there is a natural homomorphism

$$H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z})\longrightarrow H_2(X\#_V Y;\mathbb{Z})/\mathcal{R}_{X,Y}^V, \quad (A_X,A_Y)\longrightarrow A_X\#_V A_Y. \tag{1.13}$$

It is obtained by representing A_X and A_Y by cycles in X and Y with the same contacts with V and smoothing out the nodes of the resulting cycle into $X\cup_V Y\subset\mathcal{Z}$. Let $\eta\in H_2(X\#_V Y;\mathbb{Z})/\mathcal{R}_{X,Y}^V$ be an $\mathcal{R}_{X,Y}^V$-coset of $H_2(X\#_V Y;\mathbb{Z})$ and $g\in\mathbb{Z}^{\ge 0}$. According to the invariance formulas of [16, 15], the sum of the genus g GW-invariants of $X\#_V Y$ with degrees $A\in\eta$ is the same as the sum of the genus g GW-invariants of $X\cup_V Y$ of degrees

$$(A_X,A_Y)\in H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z}) \qquad \text{s.t.} \quad A_X\#_V A_Y=\eta$$

with the same cohomological insertions. The allowed cohomological insertions consist of intrinsic classes on moduli spaces of stable maps, such as ψ- and λ-classes, and pullbacks of cohomology classes on $\mathcal{Z}$ by the evaluation morphism (1.2); we will call such insertions Φ-admissible inputs. Two characterizations of cohomology classes on a smooth fiber $\mathcal{Z}_\lambda=X\#_V Y$ that are restrictions of cohomology classes on $\mathcal{Z}$ are provided in [5, Section 4.4]. By Gromov's Compactness Theorem for J-holomorphic curves, both sums have only finitely many possibly nonzero terms for each fixed g and η (independently of the cohomological insertions).

The GW-invariants of $X\cup_V Y$ count curves that lie in X and Y and meet the divisor V at the same points of V and with the same order of contact; see Figure 1. The contacts of such a curve with V

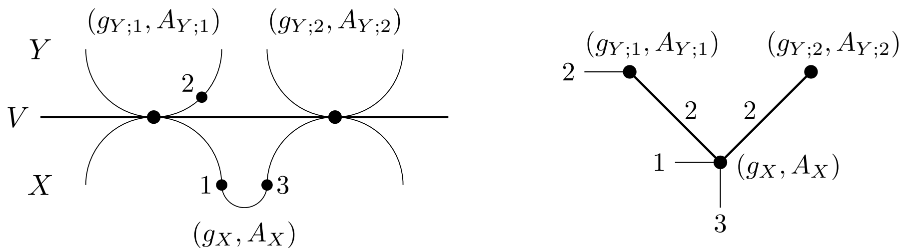

Figure 1: A curve in $X \cup_V Y$ possibly contributing to the genus $g_X + g_{Y;1} + g_{Y;2}$ degree $(A_X, A_{Y;1} + A_{Y;2})$ GW-invariant of $X \cup_V Y$ and the corresponding bipartite graph.

can be described by a tuple $\mathbf{s} \in \mathbb{Z}_+^\ell$, where $\ell \in \mathbb{Z}^{\geq 0}$ is the number of nodes on V. Its contribution to the corresponding GW-invariant of $X \cup_V Y$ is the product of the orders of contacts,

$$\langle \mathbf{s} \rangle \equiv s_1 \cdot \ldots \cdot s_\ell \,.$$

The combinatorial structure of such a curve is described by a bipartite graph Γ. The vertices of Γ specify the genus and degree of each maximal connected curve mapped into X and Y; its edges determine the meeting pattern between the components and the degrees of contacts with V. In the terminology of [15], Γ corresponds to a triple (Γ_1, Γ_2, I), with Γ_1 and Γ_2 specifying the components mapped into X and Y, respectively, and I determining the distribution of the marked points between the components. Each graph Γ and an ordering of $\ell = \ell(\Gamma)$ relative contact points determine moduli spaces $\overline{\mathfrak{M}}_\Gamma^V(X)$ and $\overline{\mathfrak{M}}_\Gamma^V(Y)$ of relative stable maps into (X, V) and (Y, V) from disconnected domains with the same relative contact vector $\mathbf{s} = \mathbf{s}(\Gamma)$. These moduli spaces are quotients of products of moduli spaces of stable maps into (X, V) and (Y, V) from connected domains by $\mathrm{Aut}(\Gamma_1, I|_{\Gamma_1})$ and $\mathrm{Aut}(\Gamma_2, I|_{\Gamma_2})$, respectively.

The morphisms (1.2) and (1.3) induce morphisms on $\overline{\mathfrak{M}}_\Gamma^V(X)$ and $\overline{\mathfrak{M}}_\Gamma^V(Y)$. Let

$$\overline{\mathfrak{M}}_\Gamma^V(X \cup_V Y) \subset \overline{\mathfrak{M}}_\Gamma^V(X) \times \overline{\mathfrak{M}}_\Gamma^V(Y)$$

denote the preimage of the diagonal $\Delta_\mathbf{s}^V \subset V_\mathbf{s}^2$ under the product evaluation morphism

$$\mathrm{ev}_X^V \times \mathrm{ev}_Y^V \colon \overline{\mathfrak{M}}_\Gamma^V(X) \times \overline{\mathfrak{M}}_\Gamma^V(Y) \longrightarrow V_\mathbf{s}^2 \,. \tag{1.14}$$

The moduli space of curves into $X \cup_V Y$ of type Γ is the quotient of $\overline{\mathfrak{M}}_\Gamma^V(X \cup_V Y)$ by the action of the symmetric group $\mathbb{S}_\ell$ permuting the relative marked points. The number of such maps is computed by pulling back the Poincare dual $\mathrm{PD}_\mathbf{s}^V \Delta$ of $\Delta_\mathbf{s}^V$ by (1.14) and integrating it over the product of the moduli spaces along with the original cohomology insertions. By the Kunneth formula for cohomology [23, Theorem 60.6],

$$\mathrm{PD}_\mathbf{s}^V \Delta = \sum_{i=1}^N \kappa_{X;i} \otimes \kappa_{Y;i} \in H^{(n-1)\ell}(V_\mathbf{s}^2; \mathbb{Q}) \tag{1.15}$$

for some $\kappa_{X;i}, \kappa_{Y;i} \in H^*(V_\mathbf{s}; \mathbb{Q})$. Thus, the contribution to the GW-invariant of $X \cup_V Y$ from maps of type Γ is the sum of N products of relative invariants of (X, V) and (Y, V) with relative insertions $\kappa_{X;i}$ and $\kappa_{Y;i}$, times $\langle \mathbf{s}(\Gamma) \rangle / \ell(\Gamma)!$. So, in this case, a decomposition formula for GW-invariants is a direct consequence of an invariance property for GW-invariants.

An obvious deficiency of the decomposition formulas of [16, 15] is that they express sums of GW-invariants of $X\#_V Y$ over degrees differing by elements of $\mathcal{R}_{X,Y}^V$ in terms of relative GW-invariants of (X,V) and (Y,V); it would of course be preferable to express individual GW-invariants of $X\#_V Y$ in terms of relative GW-invariants of (X,V) and (Y,V). The rim tori refinement of relative invariants is introduced in [11] with the aim of resolving this deficiency in [12].

For $\ell\in\mathbb{Z}^{\geq 0}$ and $\mathbf{s}\in\mathbb{Z}_+^{\ell}$, let

$$\widehat{V}_{X,Y;\mathbf{s}} = \widehat{V}_{X;\mathbf{s}} \times_{V_{\mathbf{s}}} \widehat{V}_{Y;\mathbf{s}} \equiv \big\{\pi_{X;\mathbf{s}}^V \times \pi_{Y;\mathbf{s}}^V\big\}^{-1}\big(\Delta_{\mathbf{s}}^V\big). \tag{1.16}$$

The idea of [12] is that there is a continuous map

$$g_{A_X,A_Y}\colon \widehat{V}_{X,Y;\mathbf{s}} \longrightarrow A_X\#_V A_Y \subset H_2(X\#_V Y;\mathbb{Z}) \tag{1.17}$$

such that its composition with the restriction of

$$\widetilde{\mathrm{ev}}_X^V \times \widetilde{\mathrm{ev}}_Y^V \colon \overline{\mathfrak{M}}_\Gamma^V(X) \times \overline{\mathfrak{M}}_\Gamma^V(Y) \longrightarrow \widehat{V}_{X;\mathbf{s}} \times \widehat{V}_{Y;\mathbf{s}} \tag{1.18}$$

to $\overline{\mathfrak{M}}_\Gamma^V(X\cup_V Y)$ is the homology degree of the glued map into $X\#_V Y$; see Proposition 4.2 and Figure 2. Thus, the space of maps into $X\cup_V Y$ contributing to the GW-invariant of $X\#_V Y$ of a degree $A\in A_X\#_V A_Y$ is the preimage of

$$\widehat{V}_{X,Y;\mathbf{s}}^A \equiv g_{A_X,A_Y}^{-1}(A) \tag{1.19}$$

under the morphism (1.18).

Each $\widehat{V}_{X,Y;\mathbf{s}}^A$ is a closed oriented submanifold of the oriented manifold $\widehat{V}_{X;\mathbf{s}} \times \widehat{V}_{Y;\mathbf{s}}$. As suggested by [12, Definition 10.2], it thus determines a $\mathbb{Z}$-cohomology class on $\widehat{V}_{X;\mathbf{s}} \times \widehat{V}_{Y;\mathbf{s}}$; see the last paragraph on page 118 in [22]. For our purposes, it is sufficient to consider its image

$$\mathrm{PD}_{X,Y;\mathbf{s}}^{V,A}\Delta \in H^*\big(\widehat{V}_{X;\mathbf{s}} \times \widehat{V}_{Y;\mathbf{s}};\mathbb{Q}\big) \tag{1.20}$$

in the $\mathbb{Q}$-cohomology cohomology of $\widehat{V}_{X;\mathbf{s}} \times \widehat{V}_{Y;\mathbf{s}}$. In Section 3.1, we describe the class (1.20) as a homomorphism on the $\mathbb{Q}$-homology of $\widehat{V}_{X;\mathbf{s}} \times \widehat{V}_{Y;\mathbf{s}}$ obtained by intersecting cycles with $\widehat{V}_{X,Y;\mathbf{s}}^A$. The contribution to the IP-count of $X\cup_V Y$ from maps of type Γ is computed by pulling back this class by the morphism (1.18). Thus, the approach of [12] expresses GW-invariants of $X\#_V Y$ of *each* degree $A\in H_2(X\#_V Y;\mathbb{Z})$ in terms of IP-counts of $X\cup_V Y$, i.e. provides a refined *invariance property* for GW-invariants. It can be summarized as follows.

Theorem (Refined Invariance Property for GW-Invariants)**.** *Let (X,ω_X) and (Y,ω_Y) be compact symplectic manifolds with a common compact symplectic divisor $V\subset X,Y$ and Φ be an isomorphism of complex line bundles as in (1.11). For all $g\in\mathbb{Z}^{\geq 0}$, $A\in H_2(X\#_V Y;\mathbb{Z})$, and Φ-admissible insertions κ,*

$$\mathrm{GW}_{g,A}^{X\#_V Y}(\kappa) = \sum_{\substack{(A_X,A_Y)\in H_2(X;\mathbb{Z})\times_V H_2(Y;\mathbb{Z})\\ A\in A_X\#_V A_Y}} \sum_{\substack{\mathbf{s}\in\mathbb{Z}_+^\ell\\ \ell\geq 0}} \widetilde{\mathrm{GW}}_{g,(A_X,A_Y);\mathbf{s}}^{X\cup_V Y}\big(\kappa;\mathrm{PD}_{X,Y;\mathbf{s}}^{V,A}\Delta\big), \tag{1.21}$$

where $\widetilde{\mathrm{GW}}$ denotes an IP-count for $X\cup_V Y$.

If the $\mathbb{Q}$-homology of either $\widehat{V}_{X;\mathbf{s}}$ or $\widehat{V}_{Y;\mathbf{s}}$ is finitely generated, then

$$\mathrm{PD}^{V,A}_{X,Y;\mathbf{s}}\Delta = \sum_{i=1}^{N} \widetilde{\kappa}_{X;i}\otimes\widetilde{\kappa}_{Y;i} \in H^{(n-1)\ell}(\widehat{V}_{X;\mathbf{s}}\times\widehat{V}_{Y;\mathbf{s}};\mathbb{Q}) \tag{1.22}$$

for some $\widetilde{\kappa}_{X;i}\in H^*(\widehat{V}_{X;\mathbf{s}};\mathbb{Q})$ and $\widetilde{\kappa}_{Y;i}\in H^*(\widehat{V}_{Y;\mathbf{s}};\mathbb{Q})$. This is also the case if the submodule $\mathcal{R}^V_{X,Y}$ of $H_1(X\#_V Y;\mathbb{Z})$ is finite; see Corollary 4.3. In such cases, the approach of [12] provides a refined *decomposition formula* for GW-invariants of $X\#_V Y$ in terms of IP-counts for (X,V) and (Y,V). However, in general the homologies of $\widehat{V}_{X;\mathbf{s}}$ and $\widehat{V}_{Y;\mathbf{s}}$ are not finitely generated and a Kunneth decomposition (1.22) need not exist; see Example 3.7. In these cases, the approach of [12] does not provide a decomposition formula for GW-invariants of $X\#_V Y$ in terms of any kind of invariants of (X,V) and (Y,V).

1.3 Vanishing applications

Even in cases when a Kunneth decomposition (1.22) exists (and $\mathcal{R}^V_{X;Y}\neq\{0\}$, $|\mathbf{s}|\neq 0$), the use of the refined decomposition formula of [12] for quantitative applications does not appear practical outside of rare cases, in part because of the dependence of the IP-counts for (X,V) and (Y,V) on the lifts (1.9). However, we are able to extract some qualitative applications from the approach of [11, 12].

Let (X,ω) be a symplectic manifold and $V\subset X$ be a common symplectic divisor with connected components $V_1,\ldots,V_N$. Denote by

$$\widehat{V}_X \longrightarrow V_1\times\ldots\times V_N$$

the covering projection corresponding to the preimage of H^V_X under the natural homomorphism

$$\pi_1(V_1\times\ldots\times V_N) \longrightarrow H_1(V_1\times\ldots\times V_N;\mathbb{Z}) = \bigoplus_{r=1}^{N} H_1(V_r;\mathbb{Z}) = H_1(V;\mathbb{Z}),$$

i.e. $\widehat{V}_{X;(1)^N}$ in the notation of Section 2.2. We will call $V\subset X$ virtually connected if the cokernel of the composition homomorphism

$$H_1(V_r;\mathbb{Z}) \longrightarrow H_1(V;\mathbb{Z}) \longrightarrow H_1(V;\mathbb{Z})_X \tag{1.23}$$

is finite for every component $V_r\subset V$. For example, this is the case if V is connected or $X=\mathbb{P}^1\times F$ and $V=\{0,\infty\}\times F$ for some connected symplectic manifold F; the homomorphism (1.23) is surjective in both cases. We will call a class $A\in H_2(X-V;\mathbb{Z})$ ω-effective if for every ω-tame almost complex structure J on (X,V) there exists a J-holomorphic map $u\colon\Sigma\longrightarrow X-V$ from a compact Riemann surface such that

$$u_*[\Sigma] = A \in H_2(X-V;\mathbb{Z}).$$

We denote by $\mathrm{Eff}_\omega(X,V)\subset H_2(X-V;\mathbb{Z})$ the subset of effective classes.

Theorem 1.1. *Let (X,ω_X) and (Y,ω_Y) be compact symplectic manifolds with a common compact symplectic divisor $V\subset X,Y$ and Φ be an isomorphism of complex line bundles as in (1.11). Suppose $\mathcal{R}^V_{X,Y}$ is infinite, $V\subset X$ is virtually connected, and $H_*(\widehat{V}_X;\mathbb{Q})$ is finitely generated. If*

$$A \in H_2(X\#_V Y;\mathbb{Z}) - \iota^{X\#_V Y}_{X-V*}\big(\mathrm{Eff}_{\omega_X}(X,V)\big) - \iota^{X\#_V Y}_{Y-V*}\big(\mathrm{Eff}_{\omega_Y}(Y,V)\big), \tag{1.24}$$

then all degree A GW-invariants of $X\#_V Y$ with Φ-admissible inputs vanish.

By [5, Corollary 4.2(1)], $\mathcal{R}^V_{X,Y}$ is infinite if either of the homomorphisms

$$\iota^X_{V*}\colon H_2(V;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z}) \qquad \text{or} \qquad \iota^Y_{V*}\colon H_2(V;\mathbb{Z}) \longrightarrow H_2(Y;\mathbb{Z})$$

is injective and

$$H_1(V;\mathbb{Q}) \neq \big\{B\cap_X V\colon\, B\in H_3(X;\mathbb{Q})\big\} + \big\{B\cap_Y V\colon\, B\in H_3(Y;\mathbb{Q})\big\}\,.$$

In light of [20, Assertion 6] in the case $H_1(V;\mathbb{Z})$ is of rank 1 and its extension [21], the finite generation condition can be satisfied only if $\chi(V)=0$ (assuming $\mathcal{R}^V_{X,Y}$ is infinite). All covers of the fibrations in [5, Theorem 1.1] have finitely generated homology. A characterization of abelian covers with finitely generated homology for an arbitrary compact base is provided by [4, Theorem 1], though it appears difficult to use in practice.

In a typical situation, most classes $A\in H_2(X\#_V Y;\mathbb{Z})$ with potentially nonzero GW-invariants satisfy (1.24). For example, if X and Y are Kahler, $V\subset X$ is ample, and $V\subset Y$ is anti-ample, then $X-V$ and $Y-V$ contain no curves and the restriction (1.24) is no longer necessary. The same is the case if $\mathrm{PD}_X V$ and $\mathrm{PD}_Y V$ are nonzero multiples of the cohomology classes represented by ω_X and ω_Y, respectively. Remark 4.8 describes a generalization of Theorem 1.1.

If $X\longrightarrow\Sigma_1$ and $Y\longrightarrow\Sigma_2$ are possibly singular fibrations over curves with the same smooth fiber F and $V\subset X,Y$ is a union of finitely many fibers, all elements A in the image of $H_2(F;\mathbb{Z})$ in $H_2(X\#_V Y;\mathbb{Z})$ fail the condition (1.24). For example, $\mathbb{T}^{2n}$ with $n>1$ is the symplectic sum of $\mathbb{P}^1\times\mathbb{T}^{2n-2}$ with itself along $V=\{0,\infty\}\times\mathbb{T}^{2n-2}$ with respect to the canonical isomorphism (1.11). By Theorem 1.1, the GW-invariants of $\mathbb{T}^{2n}$ in classes A not contained in a fiber vanish. By changing the projection, we recover the vanishing of all GW-invariants of $\mathbb{T}^{2n}$ with $n>1$, except in degree 0.

The $K3$ surface $\mathbb{K}_3$ is the symplectic sum of the blowup $\widehat{\mathbb{P}}^2_9$ of $\mathbb{P}^2$ at 9 points with itself along a smooth fiber $V=F$ of the fibration $\widehat{\mathbb{P}}^2_9\longrightarrow\mathbb{P}^1$ with respect to the canonical isomorphism (1.11). In this case, Theorem 1.1 recovers the vanishing of the GW-invariants of $\mathbb{K}_3$ except in degrees that are multiples of a fiber of the fibration $\mathbb{K}_3\longrightarrow\mathbb{P}^1$ (all GW-invariants of $\mathbb{K}_3$ are known to vanish).

The symplectic sum of $\mathbb{P}^1\times F$ with itself along $V=\{0,\infty\}\times F$ with respect to the canonical isomorphism (1.11) is $\mathbb{T}^2\times F$. Applying Theorem 1.1 in this case to the genus 1 GW-invariants in the section class $A=[\mathbb{T}^2\times\mathrm{pt}]$, we obtain the following statement about the maximal abelian cover $\widehat{F}\longrightarrow F$, i.e. the covering projection corresponding to the commutator subgroup of $\pi_1(F)$.

Corollary 1.2. *Let (F,ω) be a compact connected symplectic manifold. If $H_1(F;\mathbb{Q})\neq\{0\}$ and $\chi(F)\neq0$, then $H_*(\widehat{F};\mathbb{Q})$ is not finitely generated over $\mathbb{Q}$.*

By [20, Assertion 6] in the case $H_1(F;\mathbb{Z})$ is of rank 1 and its extension [21], the conclusion of this corollary holds for all finite simplicial complexes F (not just compact symplectic manifolds) and for all infinite abelian covers (not just the maximal one). Thus, its conclusion is not surprising. What perhaps is surprising is that this purely topological property is detected by the refined invariance property for GW-invariants arising from [12].

1.4 GW-invariants and the flux group

We next describe qualitative applications of the refined decomposition formula of [12] that equate GW-invariants of the symplectic sum in classes differing by some rim tori. In certain cases, these observations suffice to express individual GW-invariants of the symplectic sum in terms of the usual relative GW-invariants of the two pieces.

If V is any topological space, a loop of homeomorphisms

$$\Psi_t\colon V\longrightarrow V, \qquad t\in[0,1], \quad \Psi_0=\Psi_1,$$

and a point $x\in V$ determines a loop $t\longrightarrow\Psi_t(x)$ in V and thus an element of $H_1(V;\mathbb{Z})$. The latter is independent of the choice of $x\in V$. We denote the set of all elements of $H_1(V;\mathbb{Z})$ obtained in this way by $\mathrm{Flux}(V)$. It is a subgroup of $H_1(V;\mathbb{Z})$, usually called the flux subgroup (or group). If in addition $V\subset X$ is a compact oriented submanifold of a compact oriented manifold and $H_1(V;\mathbb{Z})_X$ is as in (1.5), let

$$\mathrm{Flux}(V)_X\subset H_1(V;\mathbb{Z})_X$$

denote the image of $\mathrm{Flux}(V)$ under the quotient projection.

Let $V\subset X$ be a symplectic divisor with topological components $V_1,\ldots,V_N$. For each $r=1,\ldots,N$, denote by $\mathfrak{f}_{X,V_r}\in H_1(S_XV_r;\mathbb{Z})$ the homology class of a fiber of the circle bundle in $\mathcal{N}_XV_r\longrightarrow V_r$. If Y is another symplectic manifold containing V and Φ is an isomorphism of complex line bundles as in (1.11), let

$$\delta_\Phi\colon H_2(X\#_VY;\mathbb{Z})\longrightarrow H_1(S_XV;\mathbb{Z})$$

be the connecting homomorphism of the Mayer-Vietoris sequence for $X\#_VY=(X-V)\cup(Y-V)$, i.e. as in the first exact sequence in the proof of [5, Lemma 4.1] with $m=\mathfrak{c}=2$. For each $A\in A_X\#_VA_Y$,

$$\delta_\Phi(A)=\sum_{r=1}^{N}|A|_{V_r}\mathfrak{f}_{X,V_r}\in H_1(S_XV;\mathbb{Z}), \qquad \text{where} \quad |A|_{V_r}\equiv A_X\cdot_X V_r\in\mathbb{Z}\,. \tag{1.25}$$

If $\mathcal{N}_XV_r\approx V_r\times\mathbb{C}$, $\mathfrak{f}_{X,V_r}\neq0$ and so the number $|A|_{V_r}$ depends only on $A_X\#_VA_Y\subset H_2(X\#_VY;\mathbb{Z})$.

Proposition 1.3. *Let (X,ω_X) and (Y,ω_Y) be compact symplectic manifolds, $V\subset X,Y$ be a common compact connected symplectic divisor such that $\mathcal{N}_XV\approx V\times\mathbb{C}$ and $\mathrm{Flux}(V)_X=H_1(V;\mathbb{Z})_X$, and Φ be an isomorphism of complex line bundles as in (1.11). If*

$$A_1,A_2\in H_2(X\#_VY;\mathbb{Z}) \qquad \textit{and} \qquad A_1-A_2\in|A_1|_V\mathcal{R}_{X,Y}^V, \tag{1.26}$$

then the GW-invariants of $X\#_VY$ of degrees A_1 and A_2 with Φ-admissible inputs are the same.

Proposition 1.4. *Let (X,ω_X) and (Y,ω_Y) be compact symplectic manifolds, $V\subset X,Y$ be a common compact symplectic divisor with topological components $V_1,\ldots,V_N$ such that $\mathcal{N}_XV\approx V\times\mathbb{C}$ and $\mathrm{Flux}(V)_X=H_1(V;\mathbb{Z})_X$, and Φ be an isomorphism of complex line bundles as in (1.11). If*

$$A_1,A_2\in H_2(X\#_VY;\mathbb{Z}), \qquad A_1-A_2\in\mathcal{R}_{X,Y}^V, \tag{1.27}$$

and $|A_1|_{V_r}$ is prime relative to $|\mathcal{R}_{X,Y}^V|$ for every r, then the GW-invariants of $X\#_VY$ of degrees A_1 and A_2 with Φ-admissible inputs are the same.

If the vanishing cycles module $\mathcal{R}^V_{X,Y}$ is infinite, we call the numbers $|A_1|_{V_r}$ and $|\mathcal{R}^V_{X,Y}|$ relatively prime if $|A_1|_{V_r}=\pm 1$. Propositions 1.3 and 1.4 are special cases of Theorem 4.9. The latter also leads to a vanishing result for the GW-invariants of $X\#_V Y$ in the spirit of Theorem 1.1, but with different assumptions; see Corollary 4.10.

The flux condition in the above propositions is automatically satisfied if V is a union of the tori $\mathbb{T}^{2n-2}$. The relevant covers $\widehat{V}_{X;\mathbf{s}}$ and $\widehat{V}_{Y;\mathbf{s}}$ are then of the form $\mathbb{R}^m\times\mathbb{T}^{m'}$; the groups of deck transformations act trivially on them and thus on the Poincare duals (1.20) of the diagonals components (1.19). This approach provides a direct justification of Propositions 1.3 and 1.4 from the refined invariance property for GW-invariance arising from [12] when V is a union of tori.

1.5 Miscellaneous considerations

The algebraic approach of [15] considers only Kahler fibrations $\pi: \mathcal{Z}\longrightarrow\Delta$ which come with an ample line bundle $\mathcal{L}\longrightarrow\mathcal{Z}$. Since every element of $\mathcal{R}^V_{X,Y}$ in $\mathcal{Z}_\lambda$ can be represented by a totally real submanifold, its homology intersection with every complex hyperplane in $\mathcal{Z}_\lambda$ is zero. Thus, by the Lefschetz Theorem on $(1,1)$-classes [9, p163], an element of $\mathcal{R}^V_{X,Y}$ in $\mathcal{Z}_\lambda$ determines a class in $H^{n-2,n}(\mathcal{Z}_\lambda)\oplus H^{n,n-2}(\mathcal{Z}_\lambda)$, where n is the complex dimension of $\mathcal{Z}_\lambda$, X, and Y. In particular, curve classes in $H_2(\mathcal{Z}_\lambda;\mathbb{Z})$ differing by an element of $\mathcal{R}^V_{X,Y}$ differ by a torsion class. This observation and Theorem 1.1 suggest the following conjecture about non-Kahler symplectic sums.

Conjecture 1.5. Let (X,ω_X) and (Y,ω_Y) be compact symplectic manifolds with a common compact symplectic divisor $V\subset X,Y$ and Φ be an isomorphism of complex line bundles as in (1.11). If $A_1,A_2\in H_2(X\#_V Y;\mathbb{Z})$, $A_1-A_2\in\mathcal{R}^V_{X,Y}$, and some GW-invariants of $X\#_V Y$ of degrees A_1 and A_2 are nonzero, then A_1-A_2 is a torsion class.

Remark 1.6. The reasoning above Corollary 1.5 does not apply outside of the Kahler setting. For example, let $X=\mathbb{P}^1\times\mathbb{T}^2$, $V=\{0,\infty\}\times\mathbb{T}^2$,

$$f_1,f_2\colon \mathbb{T}^2\longrightarrow X-V,\qquad f_1(\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2})=(2,\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2}),\quad f_2(\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2})=(\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_1},\mathrm{e}^{\mathrm{i}\theta_2}).$$

Since the images of the embeddings f_1 and f_2 are disjoint symplectic submanifolds of X, we can choose an almost complex structure J_X on X which is standard around V and makes these images J_X-holomorphic. The two maps f_1 and f_2 differ by a rim torus. Since both maps miss V, they induce J-holomorphic maps into $Z=X\#_V X$, which differ by a non-trivial element of $\mathcal{R}^V_{X,X}\approx\mathbb{Z}^2$.

1.6 Outline of the paper

We review the notation for abelian covers from [5] in Section 2.1 and the definition of the coverings (1.6) in Section 2.2. In Section 4.1, we define the map (1.17) as a special case of the map (3.4) for arbitrary abelian covers constructed in Section 3.1. In Section 4.2, the map (3.4) is used to construct and study the diagonal components (3.6) and their cohomology classes (3.9), which specialize to (1.19) an (1.20), respectively, in the symplectic sum context. Theorem 1.1 and Corollary 1.2 are established in Section 4.3. A rim tori refinement of (1.17) is defined in Section 5.2 as a special case of the convolution product on abelian covers defined in Section 5.1; see (5.24) and (5.5). Section 6 streamlines the computation of some GW-invariants of $\widehat{\mathbb{P}}^2_9$ in [12] by making use of a vanishing result for relative GW-invariants, which is an immediate consequence of the existence of IP-counts, and corrects some statements in [12] concerning these counts.

The purpose of this paper is to investigate the topological aspects of the rim tori refinement to the standard symplectic sum formula. We pre-suppose that the latter has been established and describe the necessary steps to implement the suggestion of [12] as an enhancement on an existing analytic proof. We deduce some qualitative applications arising from this refinement and discuss its usability for quantitative purposes. A significant number of examples are included in Section 4 for illustrative purposes; some of them are also used in Section 6.

The authors would like to thank E. Ionel, D. McDuff, M. McLean, J. Milnor, and J. Starr for enlightening discussions.

2 Review of preliminaries

The coverings (1.6) are special cases of the abelian covers described in Section 2.1. The former are constructed in [5, Section 6.1] and reviewed in Section 2.2. We also recall a crucial statement concerning choices of the lifts (1.9) established in [5]; see Proposition 2.2.

2.1 Abelian covers of topological spaces

Let $\mathbb{Z}_\pm \subset \mathbb{Z}$ denote the nonzero integers. For a tuple $\mathbf{s}=(s_1,\ldots,s_\ell)\in\mathbb{Z}_\pm^\ell$ with $\ell\in\mathbb{Z}^{\ge 0}$, we denote by $\gcd(\mathbf{s})$ the greatest common divisor of $s_1,\ldots,s_\ell$; if $\ell=0$, we set $\gcd(\mathbf{s})=0$.

Let V be a topological space. For any submodule $H\subset H_1(V;\mathbb{Z})$, let

$$q_H\colon H_1(V;\mathbb{Z}) \longrightarrow \mathcal{R}_H \equiv \frac{H_1(V;\mathbb{Z})}{H} \tag{2.1}$$

be the projection to the corresponding quotient module. If $V_1,\ldots,V_N$ are the topological components of V, $\ell_1,\ldots,\ell_N\in\mathbb{Z}^{\ge 0}$, and $\mathbf{s}_1\in\mathbb{Z}_\pm^{\ell_1},\ldots,\mathbf{s}_N\in\mathbb{Z}_\pm^{\ell_N}$, then the topological space

$$V_{\mathbf{s}_1\ldots\mathbf{s}_N} \equiv V_1^{\ell_1}\times\ldots\times V_N^{\ell_N}$$

is connected.

With V and $\mathbf{s}_1,\ldots,\mathbf{s}_N$ as above, define

$$\begin{gathered}\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}\colon H_1\big(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z}\big) = \bigoplus_{r=1}^{N} H_1(V_r;\mathbb{Z})^{\oplus\ell_r} \longrightarrow H_1(V;\mathbb{Z}), \qquad (2.2)\\ \Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}\big((\gamma_{r;i})_{i\le\ell_r,r\le N}\big) = \sum_{r=1}^{N}\sum_{i=1}^{\ell_r} s_{r;i}\gamma_{r;i}\,.\end{gathered}$$

For any submodule $H\subset H_1(V;\mathbb{Z})$, let

$$H_{\mathbf{s}_1\ldots\mathbf{s}_N} = \Phi^{-1}_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(H) \subset H_1\big(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z}\big), \tag{2.3}$$

$$\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N} = \mathrm{Im}\big\{q_H\circ\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}\big\} \subset \mathcal{R}_H, \qquad \mathcal{R}_{H;\mathbf{s}_1\ldots\mathbf{s}_N} = \frac{\mathcal{R}_H}{\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}}\times\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\,. \tag{2.4}$$

If $\gcd(\mathbf{s}_r)=1$ for every $r=1,\ldots,N$, then

$$\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}=\mathcal{R}_{H;\mathbf{s}_1\ldots\mathbf{s}_N}=\mathcal{R}_H\,.$$

If V is connected, then $\mathcal{R}'_{H;\mathbf{s}}=\gcd(\mathbf{s})\mathcal{R}_H$ for any $\mathbf{s}\in\mathbb{Z}^\ell_\pm$ and $H_{(1)}=H$.

For each $r=1,\ldots,N$, let $\widehat{V}_r\longrightarrow V_r$ be the maximal abelian cover of V_r, i.e. the covering projection corresponding to the commutator subgroup of $\pi_1(V)$. The group of deck transformations of this regular covering is $H_1(V_r;\mathbb{Z})$. The maximal abelian cover of $V_{\mathbf{s}_1\ldots\mathbf{s}_N}$ is given by

$$\widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}\equiv\prod_{r=1}^N\widehat{V}_r^{\ell_r}\longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N}\,; \tag{2.5}$$

there is a natural action of $H_1(V_{\mathbf{s}};\mathbb{Z})$ on this space. For any submodule $H\subset H_1(V;\mathbb{Z})$, let

$$\begin{aligned}\pi'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\colon\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}&\equiv\widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}/H_{\mathbf{s}_1\ldots\mathbf{s}_N}\longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N}\,,\\ \pi_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\colon\widehat{V}_{H;\mathbf{s}_1\ldots\mathbf{s}_N}&\equiv\frac{\mathcal{R}_H}{\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}}\times\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N}\,.\end{aligned} \tag{2.6}$$

The groups of deck transformations of these regular coverings are

$$\mathrm{Deck}\big(\pi'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\big)=\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\qquad\text{and}\qquad\mathrm{Deck}\big(\pi_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\big)=\mathcal{R}_{H;\mathbf{s}_1\ldots\mathbf{s}_N}, \tag{2.7}$$

respectively. We will write elements of the second covering in (2.6) as

$$\big([\gamma]_{H;\mathbf{s}_1\ldots\mathbf{s}_N},[\widehat{x}]_H\big)\in\frac{\mathcal{R}_H}{\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}}\times\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}, \tag{2.8}$$

with the first component denoting the image of $\gamma\in H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z})$ under the homomorphism

$$H_1(V;\mathbb{Z})\longrightarrow\mathcal{R}_H\longrightarrow\frac{\mathcal{R}_H}{\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}}$$

and the second component denoting the image of $\widehat{x}\in\widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}$.

A collection $\{\gamma_j\}\subset H_1(V;\mathbb{Z})$ of representatives for the elements of $\mathcal{R}_H/\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$ induces a homomorphism

$$H_1(V;\mathbb{Z})\longrightarrow\mathrm{Deck}\big(\pi_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\big),\qquad\eta\longrightarrow\Theta_\eta\,,$$

as follows. For every $\eta\in H_1(V;\mathbb{Z})$ and a coset representative γ_j, let $\gamma_j(\eta)$ be the unique coset representative from the chosen collection such that

$$\gamma_j+\eta-\gamma_j(\eta)-\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\eta_j)\in H \tag{2.9}$$

for some $\eta_j\in H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z})$. Define

$$\Theta_\eta\colon\widehat{V}_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\longrightarrow\widehat{V}_{H;\mathbf{s}_1\ldots\mathbf{s}_N},\quad\Theta_\eta\big([\gamma_j]_{H;\mathbf{s}_1\ldots\mathbf{s}_N},[\widehat{x}]_H\big)=\big([\gamma_j(\eta)]_{H;\mathbf{s}_1\ldots\mathbf{s}_N},[\eta_j\cdot\widehat{x}]_H\big). \tag{2.10}$$

Since (2.9) determines η_j up to an element of $H_{\mathbf{s}_1\ldots\mathbf{s}_N}$, the last component of Θ_η is well-defined.

The coverings $V_{\mathbf{s}_1\ldots\mathbf{s}_N}$ often do not have finitely generated homology groups. Some cases when their homology groups are finitely generated are described in [5, Section 5.2].

The next example underlines most examples in Section 4.

Example 2.1 ([5, Example 5.1])**.** If $V=\mathbb{T}^2$, $\ell\in\mathbb{Z}^+$, and $H=\{0\}$, then

$$\widehat{V}_{H;\mathbf{s}}=\mathbb{C}\times\mathbb{T}_{\mathbf{s}}^{2(\ell-1)}, \quad \text{where} \quad \mathbb{T}_{\mathbf{s}}^{2(\ell-1)}=\big\{(z_i)_{i\le\ell}\in\mathbb{C}^\ell\colon \sum_{i=1}^{\ell}s_iz_i\in\mathbb{Z}\oplus\mathfrak{i}\mathbb{Z}\big\}/\mathbb{Z}^{2\ell}\subset\mathbb{T}^{2\ell}=V_{\mathbf{s}}.$$

The second covering in (2.6) can be written as

$$\mathbb{C}\times\mathbb{T}_{\mathbf{s}}^{2(\ell-1)}\longrightarrow\mathbb{T}^{2\ell}, \qquad \big(z,[z_i]_{i\le\ell}\big)\longrightarrow\Big[z_i-\frac{z}{s_i}\Big]_{i\le\ell}. \tag{2.11}$$

Under the standard identification of $H_1(\mathbb{T}^2;\mathbb{Z})$ with $\mathbb{Z}\oplus\mathfrak{i}\mathbb{Z}$, the action of $H_1(\mathbb{T}^2;\mathbb{Z})^{\oplus\ell}$ on this cover is given by

$$(\gamma_{i'})_{i'\le\ell}\cdot\big(z,[z_i]_{i\le\ell}\big)=\bigg(z+\frac{1}{\ell}\sum_{i'=1}^{\ell}s_{i'}\gamma_{i'},\,\Big[z_i+\frac{1}{\ell s_i}\sum_{i'=1}^{\ell}s_{i'}\gamma_{i'}\Big]_{i\le\ell}\bigg). \tag{2.12}$$

The group of deck transformations of this cover is $\mathbb{Z}^2_{\gcd(\mathbf{s})}\oplus\gcd(\mathbf{s})\mathbb{Z}^2$. The action of the second component is induced by the action of $H_1(\mathbb{T}^2;\mathbb{Z})^{\oplus\ell}$ via the surjective homomorphism

$$H_1(\mathbb{T}^2;\mathbb{Z})^{\oplus\ell}\longrightarrow\gcd(\mathbf{s})H_1(\mathbb{T}^2;\mathbb{Z}), \qquad (\gamma_{i'})_{i'\le\ell}\longrightarrow\sum_{i'=1}^{\ell}s_{i'}\gamma_{i'}\,. \tag{2.13}$$

2.2 The rim tori covers and lifts

The coverings (1.6) are defined below as special cases of the abelian covers (2.6). The relative evaluation morphisms (1.3) lift over these coverings, though not uniquely. Fixing lifts of these morphisms (as is essentially done in [11]) corresponds to choosing base points in certain spaces; see [5, Remark 6.7]. These choices can be made in a systematic manner; see Proposition 2.2.

Let X be a manifold and $V\subset X$ be a closed submanifold of codimension $\mathfrak{c}$. Denote by $S_XV\subset\mathcal{N}_XV$ the sphere subbundle of the normal bundle of V in X, which we will view as a hypersurface in X, and let

$$\mathcal{R}_X^V\equiv\ker\big\{\iota_{X-V*}^X\colon H_{\mathfrak{c}}(X-V;\mathbb{Z})\longrightarrow H_{\mathfrak{c}}(X;\mathbb{Z})\big\}. \tag{2.14}$$

If in addition X and V are compact and oriented, we define

$$\begin{aligned}\cap V\colon H_*(X;\mathbb{Z})&\longrightarrow H_{*-\mathfrak{c}}(V;\mathbb{Z}), & A\cap V&=\mathrm{PD}_V\big((\mathrm{PD}_XA)|_V\big),\\ \Delta_X^V\colon H_m(V;\mathbb{Z})&\longrightarrow H_{m+\mathfrak{c}-1}(S_XV;\mathbb{Z}), & \Delta_X^V(\gamma)&=\mathrm{PD}_{S_XV}\big(q_X^{V\,*}(\mathrm{PD}_V\gamma)\big),\end{aligned}$$

where $q_X^V\colon S_XV\longrightarrow V$ is the projection map. Let

$$H_X^V=\big\{A\cap V\colon A\in H_{\mathfrak{c}+1}(X;\mathbb{Z})\big\}\subset H_1(V;\mathbb{Z}), \quad H_1(V;\mathbb{Z})_X=\frac{H_1(V;\mathbb{Z})}{H_X^V}. \tag{2.15}$$

By [5, Corollary 3.2], the homomorphism

$$\iota_{S_XV*}^{X-V}\circ\Delta_X^V\colon H_1(V;\mathbb{Z})_X\longrightarrow\mathcal{R}_X^V\subset H_{\mathfrak{c}}(X-V;\mathbb{Z}) \tag{2.16}$$

is well-defined and is an isomorphism.

With notation as in (2.1),

$$\mathcal{R}_{H_X^V} \equiv H_1(V;\mathbb{Z})_X \approx \mathcal{R}_X^V .$$

Let $V_1,\ldots,V_N$ be the topological components of V. For $\ell_1,\ldots,\ell_N\in\mathbb{Z}^{\ge0}$ and $\mathbf{s}_1\in\mathbb{Z}_{\pm}^{\ell_1},\ldots,\mathbf{s}_N\in\mathbb{Z}_{\pm}^{\ell_N}$, define

$$H_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V = \big(H_X^V\big)_{\mathbf{s}_1\ldots\mathbf{s}_N} \subset H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z}),$$
$$\mathcal{R}'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N} = \mathcal{R}'_{H_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V} \subset H_1(V;\mathbb{Z})_X\,, \qquad \mathcal{R}^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N} = \mathcal{R}_{H_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V}\,.$$

The rim tori covers (1.6) are the abelian covers

$$\begin{aligned}\pi'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \pi'^V_{H^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}} &: \widehat{V}'_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \widehat{V}'_{H^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}} \longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N}\,,\\ \pi^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \pi^V_{H^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}} &: \widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \widehat{V}_{H^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}} \longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N}\,.\end{aligned} \tag{2.17}$$

By (2.7), the groups of deck transformations of these regular coverings are

$$\mathrm{Deck}\big(\pi'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\big) = \mathcal{R}'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \qquad \text{and} \qquad \mathrm{Deck}\big(\pi^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\big) = \mathcal{R}^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\,, \tag{2.18}$$

respectively. We will write elements of the second covering in (2.17) as

$$\big([\gamma]_{X;\mathbf{s}_1\ldots\mathbf{s}_N}, [\widehat{x}]_X\big) \in \frac{\mathcal{R}_X^V}{\mathcal{R}'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}} \times \widehat{V}'_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\,, \tag{2.19}$$

with notation as in (2.8) for $H=H_X^V$.

If Σ is a compact oriented m-dimensional manifold, $A\in H_m(X;\mathbb{Z})$, $k\in\mathbb{Z}^{\ge0}$, and $p>m$, let $\mathfrak{X}_{\Sigma,k}(X,A)$ be the space of tuples $(z_1,\ldots,z_k,f)$ such that $f\in L_1^p(\Sigma;X)$, $f_*[\Sigma]=A$, and $z_1,\ldots,z_k\in\Sigma$ are distinct points. If $m=\mathfrak{c}$ and $r=1,\ldots,N$, each isolated point $z\in f^{-1}(V_r)$ has well-defined order of contact with V_r, $\mathrm{ord}_z^{V_r}f\in\mathbb{Z}$; see the beginning of [5, Section 2.1]. If in addition $\mathbf{s}_1,\ldots,\mathbf{s}_N$ are as before and (1.1) holds for each $(V,\mathbf{s})=(V_r,\mathbf{s}_r)$, let

$$\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A) \subset \mathfrak{X}_{\Sigma,k+\ell_1+\ldots+\ell_N}(X,A)$$

be the subspace of tuples $(z_1,\ldots,z_{k+\ell_1+\ldots+\ell_N},f)$ such that

$$\begin{aligned} f^{-1}(V_r) &= \big\{z_{k+\ell_1+\ldots+\ell_{r-1}+1},\ldots,z_{k+\ell_1+\ldots+\ell_r}\big\} && \forall\ r=1,\ldots,N,\\ \mathrm{ord}^{V_r}_{z_{k+\ell_1+\ldots+\ell_{r-1}+i}}f &= s_{r;i} && \forall\ i=1,2,\ldots,\ell_r,\ r=1,\ldots,N. \end{aligned}$$

We denote by

$$\mathrm{ev}_X^V = \mathrm{ev}_{k+1}\times\ldots\times\mathrm{ev}_{k+\ell_1+\ldots+\ell_N} : \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A) \longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N} \tag{2.20}$$

the total relative evaluation morphism. Any pair $\mathbf{f},\mathbf{f}'\in\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A)$ with

$$\mathrm{ev}_X^V\big(\mathbf{f}\big) = \mathrm{ev}_X^V(\mathbf{f}') \in V_{\mathbf{s}}$$

determines an element $[f\#(-f')]$ of $\mathcal{R}_X^V\subset H_{\mathfrak{c}}(X-V;\mathbb{Z})$; see [5, Section 2.1].

By [5, Lemma 6.3], the morphism (2.20) lifts over the coverings (2.17). Such lifts can be chosen so that they are compatible with the morphisms between the configuration spaces of maps obtained by dropping some of the components of V and the corresponding relative contact points; see [5, Figure 1]. They can also be chosen compatibly with the isomorphism (2.16), in the sense described below.

Proposition 2.2 ([5, Theorem 6.5]). *Suppose X is a compact oriented manifold, $V\subset X$ is a compact oriented submanifold of codimension $\mathfrak{c}$ with connected components $V_1,\ldots,V_N$, $A\in H_{\mathfrak{c}}(X;\mathbb{Z})$, and $\mathbf{s}_r\in\mathbb{Z}_{\pm}^{\ell_r}$ for $r=1,\ldots,N$. Let $\{\gamma_j\}\subset H_1(V;\mathbb{Z})$ be a collection of representatives for the elements of $\mathcal{R}_X^V/\mathcal{R}'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$. If Σ is a compact oriented $\mathfrak{c}$-dimensional manifold and $k\in\mathbb{Z}^{\geq 0}$, there exists a lift*

$$\widetilde{\mathrm{ev}}_X^V\colon \mathfrak{X}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}^{V_1,\ldots,V_N}(X,A)\longrightarrow \widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \tag{2.21}$$

of the morphism ev_X^V in (2.20) over the covering $\pi^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$ in (2.17) with the following property. For any $\mathbf{f},\mathbf{f}'\in\mathfrak{X}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}^{V_1,\ldots,V_N}(X,A)$ with

$$\widetilde{\mathrm{ev}}_X^V(\mathbf{f})=\big([\gamma_j]_{X;\mathbf{s}_1\ldots\mathbf{s}_N},[\gamma\cdot\widehat{x}]_X\big)\quad \textit{and}\quad \widetilde{\mathrm{ev}}_X^V(\mathbf{f}')=\big([\gamma_{j'}]_{X;\mathbf{s}_1\ldots\mathbf{s}_N},[\widehat{x}]_X\big) \tag{2.22}$$

for some $\widehat{x}\in\widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}$, $\gamma\in H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z})$, and j,j' indexing the coset representatives, the map components of $\mathbf{f}$ and $\mathbf{f}'$ satisfy

$$\big[f\#(-f')\big]=\iota_{S_XV*}^{X-V}\big(\Delta_X^V\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma)+\gamma_j-\gamma_{j'}\big)\big)\in H_{\mathfrak{c}}(X-V;\mathbb{Z}). \tag{2.23}$$

Furthermore, $\widetilde{\mathrm{ev}}_X^V(\mathbf{f}')$ is the unique point in $\pi^{V\ -1}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}(\mathrm{ev}_X^V(\mathbf{f}'))$ so that (2.23) holds for a given value of $\widetilde{\mathrm{ev}}_X^V(\mathbf{f})$.

3 Diagonal components for abelian covers

In Section 3.1, we define diagonal components for arbitrary abelian covers that specialize to (1.19) in the symplectic sum setting. Each diagonal component $\widehat{V}^{\eta}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}$ in (3.6) determines an intersection homomorphism (3.7) on the product of two abelian covers of $V_{\mathbf{s}_1\ldots\mathbf{s}_N}$ and thus a cohomology class (3.9), as suggested by [12, Definition 10.2]; in the symplectic sum context, this class specializes to (1.20). In Section 3.2, we describe cases when these classes split into products of cohomology classes from the two factors and when these classes are the same for different choices of η; see Lemmas 3.4 and 3.8.

3.1 Notation and examples

We continue with the notation for the abelian covers of a topological space V with connected components $V_1,\ldots,V_N$ introduced in Section 2.1. For $\mathbf{s}_1\in\mathbb{Z}_{\pm}^{\ell_1},\ldots,\mathbf{s}_N\in\mathbb{Z}_{\pm}^{\ell_N}$, denote by

$$\Delta^V_{\mathbf{s}_1\ldots\mathbf{s}_N}\subset V^2_{\mathbf{s}_1\ldots\mathbf{s}_N}\equiv V_{\mathbf{s}_1\ldots\mathbf{s}_N}\times V_{\mathbf{s}_1\ldots\mathbf{s}_N}$$

the diagonal. For any submodules $H_1,H_2\subset H_1(V;\mathbb{Z})$, define

$$\begin{aligned}\widehat{V}'_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}&=\widehat{V}'_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\times_{V_{\mathbf{s}_1\ldots\mathbf{s}_N}}\widehat{V}'_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\equiv\big\{\pi'_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\times\pi'_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\big\}^{-1}\big(\Delta^V_{\mathbf{s}_1\ldots\mathbf{s}_N}\big),\\ \widehat{V}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}&=\widehat{V}_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\times_{V_{\mathbf{s}_1\ldots\mathbf{s}_N}}\widehat{V}_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\equiv\big\{\pi_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\times\pi_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\big\}^{-1}\big(\Delta^V_{\mathbf{s}_1\ldots\mathbf{s}_N}\big).\end{aligned} \tag{3.1}$$

Thus,

$$\widehat{V}'_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N} = \bigcup_{\gamma\in H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z})} \big\{\big([\gamma\cdot\widehat{x}]_{H_1},[\widehat{x}]_{H_2}\big)\colon\, \widehat{x}\in\widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}\big\},$$

$$\widehat{V}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N} = \frac{\mathcal{R}_{H_1}}{\mathcal{R}'_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}}\times\frac{\mathcal{R}_{H_2}}{\mathcal{R}'_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}}\times\widehat{V}'_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\,.$$

If $H_{12}\subset H_1(V;\mathbb{Z})$ is a module containing H_1 and H_2, define

$$\Psi'^{H_{12}}_{H_1,H_2}\colon \widehat{V}'_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\longrightarrow \mathcal{R}'_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}\subset\mathcal{R}_{H_{12}}\,,\quad \Psi'^{H_{12}}_{H_1,H_2}\big([\gamma\cdot\widehat{x}]_{H_1},[\widehat{x}]_{H_2}\big)=\big[\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma)\big]_{H_{12}}\,,$$

where $[\gamma']_{H_{12}}$ denotes the coset of $\gamma'\in H_1(V;\mathbb{Z})$ modulo H_{12}. Since γ above is well-defined up to an element of

$$(H_1)_{\mathbf{s}_1\ldots\mathbf{s}_N}+(H_2)_{\mathbf{s}_1\ldots\mathbf{s}_N}\subset(H_{12})_{\mathbf{s}_1\ldots\mathbf{s}_N}\,,$$

the map $\Psi'^{H_{12}}_{H_1,H_2}$ is well-defined. Combining $\Psi'^{H_{12}}_{H_1,H_2}$ with the homomorphism,

$$\begin{gathered}\frac{\mathcal{R}_{H_1}}{\mathcal{R}'_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}}\times\frac{\mathcal{R}_{H_2}}{\mathcal{R}'_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}}\longrightarrow\frac{\mathcal{R}_{H_{12}}}{\mathcal{R}'_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}},\\ \big([\gamma_1]_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N},[\gamma_2]_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\big)\longrightarrow[\gamma_1-\gamma_2]_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N},\end{gathered}\tag{3.2}$$

we obtain a continuous map

$$\Psi^{H_{12}}_{H_1,H_2}\colon \widehat{V}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\longrightarrow\mathcal{R}_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}\,;$$

its target is a discrete set.

The map (3.2) can be lifted to a map (not a homomorphism) to $\mathcal{R}_{H_{12}}$ by choosing collections

$$\{\gamma_{1;j_1}\},\{\gamma_{2;j_2}\}\subset H_1(V;\mathbb{Z})\tag{3.3}$$

of representatives for the elements of $\mathcal{R}_{H_1}/\mathcal{R}'_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}$ and $\mathcal{R}_{H_2}/\mathcal{R}'_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}$, respectively:

$$\big([\gamma_{1;j_1}]_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N},[\gamma_{2;j_2}]_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\big)\longrightarrow[\gamma_{1;j_1}-\gamma_{2;j_2}]_{H_{12}}.$$

We can then "lift" $\Psi^{H_{12}}_{H_1,H_2}$ to a continuous map

$$\begin{gathered}\widetilde{\Psi}^{H_{12}}_{H_1,H_2}\colon \widehat{V}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\longrightarrow\mathcal{R}_{H_{12}}\,,\\ \widetilde{\Psi}^{H_{12}}_{H_1,H_2}\big(\big([\gamma_{1;j_1}]_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N},[\gamma\cdot\widehat{x}]_{H_1}\big),\big([\gamma_{2;j_2}]_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N},[\widehat{x}]_{H_2}\big)\big)=\big[\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma)+\gamma_{1;j_1}-\gamma_{2;j_2}\big]_{H_{12}}\,.\end{gathered}\tag{3.4}$$

The latter gives rise to the refined gluing degree map (1.17) in the symplectic sum context; see (4.10). This map is compatible with the deck transformations (2.10) associated with the collections (3.3), i.e.

$$\begin{aligned}\widetilde{\Psi}^{H_{12}}_{H_1,H_2}\big(\Theta_\eta(\widetilde{x}_1),\widetilde{x}_2\big)&=\widetilde{\Psi}^{H_{12}}_{H_1,H_2}\big(\widetilde{x}_1,\widetilde{x}_2\big)+[\eta]_{H_{12}},\\ \widetilde{\Psi}^{H_{12}}_{H_1,H_2}\big(\widetilde{x}_1,\Theta_\eta(\widetilde{x}_2)\big)&=\widetilde{\Psi}^{H_{12}}_{H_1,H_2}\big(\widetilde{x}_1,\widetilde{x}_2\big)-[\eta]_{H_{12}}\end{aligned}\tag{3.5}$$

for all $(\widetilde{x}_1,\widetilde{x}_2)\in\widehat{V}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}$ and $\eta\in H_1(V;\mathbb{Z})$.

For $\eta\in\mathcal{R}_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}$ and $\widetilde{\eta}\in\mathcal{R}_{H_{12}}$, let

$$\widehat{V}^{\eta}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}=\big\{\Psi^{H_{12}}_{H_1,H_2}\big\}^{-1}(\eta),\qquad \widehat{V}^{\widetilde{\eta}}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}=\big\{\widetilde{\Psi}^{H_{12}}_{H_1,H_2}\big\}^{-1}(\widetilde{\eta}). \tag{3.6}$$

These subsets of $\widehat{V}_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}$ are closed.

If in addition V is an oriented manifold, then so are the subsets (3.6). Thus, they define intersection homomorphisms

$$H_*\big(\widehat{V}_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q}\big)\longrightarrow\mathbb{Q},\quad h\longrightarrow h\cdot\widehat{V}^{\eta}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N},\, h\cdot\widehat{V}^{\widetilde{\eta}}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}, \tag{3.7}$$

as follows. On the homology of dimension different from half the dimension, we take these homomorphisms to be zero. A (rational multiple of a) homology class of half the dimension can be represented by a smooth map

$$h\colon Z\longrightarrow\widehat{V}_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N} \tag{3.8}$$

from a compact oriented manifold which intersects the submanifolds (3.6) transversely. Thus, their preimages in Z are closed zero-dimensional submanifolds of Z. We take the intersection numbers in (3.7) to be the signed cardinalities of these finite sets (divided by the appropriate multiple if necessary). A cobordism between two representatives for the same homology class provides cobordisms between the preimages of the submanifolds (3.6) with respect to the two representatives; thus, the intersection homomorphisms (3.7) are well-defined. By the Universal Coefficient Theorem [23, Theorem 53.5], these homomorphisms correspond to some elements

$$\mathrm{PD}^{\eta}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta,\mathrm{PD}^{\widetilde{\eta}}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta\in H^*\big(\widehat{V}_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q}\big), \tag{3.9}$$

respectively. Since the homologies of $\widehat{V}_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}$ and $\widehat{V}_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}$ may not be finitely generated, these classes need not admit finite Kunneth decompositions as in (1.22). By Examples 3.1-3.3 and Lemma 3.4 below, they do admit such decompositions in some interesting cases.

Example 3.1. Let $V=\mathbb{T}^2$, $\ell\in\mathbb{Z}^+$, and $\mathbf{s}\in\mathbb{Z}^{\ell}_{\pm}$. We identify $H_1(V;\mathbb{Z})$ with $\mathbb{Z}\oplus\mathfrak{i}\mathbb{Z}$ and denote by $0\subset H_1(V;\mathbb{Z})$ the zero submodule. By Example 2.1,

$$\widehat{V}_{0,0;\mathbf{s}}=\big\{\big(x,[z_i+s_i^{-1}x]_{i\le\ell},y,[z_i+s_i^{-1}y]_{i\le\ell}\big)\in\mathbb{C}\times\mathbb{T}^{2(\ell-1)}_{\mathbf{s}}\times\mathbb{C}\times\mathbb{T}^{2(\ell-1)}_{\mathbf{s}}\big\}.$$

Let $\{\gamma_j\}\subset\mathbb{Z}\oplus\mathfrak{i}\mathbb{Z}$ be a collection of representatives for the elements of $\mathbb{Z}_{\gcd(\mathbf{s})}\oplus\mathfrak{i}\mathbb{Z}_{\gcd(\mathbf{s})}$. It is convenient to identify base points for the components of $\widehat{V}_{0;\mathbf{s}}$ as

$$\big([\gamma_j]_{0;\mathbf{s}},[\widehat{x}_{\mathbf{s}}]_0\big)=\big(\ell^{-1}\gamma_j,[\ell^{-1}s_i^{-1}\gamma_j]_{i\le\ell}\big)\in\widehat{V}_{0;\mathbf{s}}=\mathbb{C}\times\mathbb{T}^{2(\ell-1)}_{\mathbf{s}}.$$

The map (3.4) with $H_1,H_2,H_{12}=0$ is then given by

$$\widetilde{\Psi}^0_{0,0;\mathbf{s}}\colon\widehat{V}_{0,0;\mathbf{s}}\longrightarrow\mathbb{Z}\oplus\mathfrak{i}\mathbb{Z},\qquad \widetilde{\Psi}^0_{0,0;\mathbf{s}}\big(x,[\mathbf{z}],y,[\mathbf{z}']\big)=\ell\,(x-y);$$

see (2.12) and (2.13). The diagonal components $\widehat{V}^{\widetilde{\eta}}_{0,0;\mathbf{s}}$ are naturally indexed by $\mathbb{Z}\oplus\mathfrak{i}\mathbb{Z}$ and

$$\widehat{V}^{\widetilde{\eta}}_{0,0;\mathbf{s}}=\Delta^{\widetilde{\eta}}_{\mathbb{C}}\times\Delta^{\widetilde{\eta}}_{\mathbb{T}^{2\ell}}\cap\mathbb{C}\times\mathbb{T}^{2(\ell-1)}_{\mathbf{s}}\times\mathbb{C}\times\mathbb{T}^{2(\ell-1)}_{\mathbf{s}}\subset\mathbb{C}\times\mathbb{T}^{2\ell}\times\mathbb{C}\times\mathbb{T}^{2\ell}, \tag{3.10}$$

where
$$\Delta_{\mathbb{C}}^{\widetilde{\eta}}=\{(x+\widetilde{\eta},x)\in\mathbb{C}^2\}\quad\text{and}\quad\Delta_{\mathbb{T}^{2\ell}}^{\widetilde{\eta}}=\big\{\big([z_i+s_i^{-1}\widetilde{\eta}]_{i\le\ell},[z_i]_{i\le\ell}\big)\in(\mathbb{T}_{\mathbf{s}}^{2(\ell-1)})^2\big\}$$
are translates of the diagonals. In particular,

$$h\cdot\big(\Delta_{\mathbb{C}}^{\widetilde{\eta}}\times\Delta_{\mathbb{T}^{2\ell}}^{\widetilde{\eta}}\big)=0\qquad\forall\,h\in H_*\big(\mathbb{C}\times\mathbb{T}^{2\ell}\times\mathbb{C}\times\mathbb{T}^{2\ell};\mathbb{Q}\big),\ \widetilde{\eta}\in\mathbb{Z}\oplus\mathfrak{i}\mathbb{Z}.$$

Combining the last observation with (3.10),

$$\mathrm{PD}_{0,0;\mathbf{s}}^{\widetilde{\eta}}\Delta=0\in H^*\big(\mathbb{C}\times\mathbb{T}_{\mathbf{s}}^{2(\ell-1)}\times\mathbb{C}\times\mathbb{T}_{\mathbf{s}}^{2(\ell-1)};\mathbb{Q}\big)\qquad\forall\,\widetilde{\eta}\in\mathbb{Z}\oplus\mathfrak{i}\mathbb{Z}.\tag{3.11}$$

This is consistent with the vanishing of the GW-invariants of $\mathbb{K}_3$; see Example 4.4.

Example 3.2. For $V=\mathbb{T}^2$, $\ell\in\mathbb{Z}^+$, $\mathbf{s}\in\mathbb{Z}_{\pm}^{\ell}$, and $H=H_1(V;\mathbb{Z})$,

$$\widetilde{\Psi}_{0,H;\mathbf{s}}^{H}\colon\widehat{V}_{0,H;\mathbf{s}}=\big\{\big(z,[z_i]_{i\le\ell},[z_i-s_i^{-1}z]_{i\le\ell}\big)\in\mathbb{C}\times\mathbb{T}_{\mathbf{s}}^{2(\ell-1)}\times\mathbb{T}^{2\ell}\big\}\longrightarrow\mathcal{R}_H=\{0\}.$$

The preimage of $\widehat{V}_{0,H;\mathbf{s}}=\widehat{V}_{0,H;\mathbf{s}}^0$ under the automorphism

$$\Theta_{\mathbf{s}}\colon\mathbb{C}\times\mathbb{T}_{\mathbf{s}}^{2(\ell-1)}\times\mathbb{T}^{2\ell}\longrightarrow\mathbb{C}\times\mathbb{T}_{\mathbf{s}}^{2(\ell-1)}\times\mathbb{T}^{2\ell},\quad\Theta_{\mathbf{s}}\big(z,[z_i]_{i\le\ell},[z_i']_{i\le\ell}\big)=\big(z,[z_i]_{i\le\ell},[z_i'-s_i^{-1}z]_{i\le\ell}\big),$$

is the intersection
$$\mathbb{C}\times\Delta_{\mathbb{T}^{2\ell}}\cap\mathbb{C}\times\mathbb{T}_{\mathbf{s}}^{2(\ell-1)}\times\mathbb{T}^{2\ell}\subset\mathbb{C}\times\mathbb{T}^{2\ell}\times\mathbb{T}^{2\ell}.$$
Since $\Theta_{\mathbf{s}}$ induces the identity on the cohomology, it follows that

$$\mathrm{PD}_{0,H;\mathbf{s}}^{0}\Delta=1\times\big(\mathrm{PD}_{(\mathbb{T}^{2\ell})^2}\Delta_{\mathbb{T}^{2\ell}}\big)\big|_{\mathbb{T}_{\mathbf{s}}^{2(\ell-1)}\times\mathbb{T}^{2\ell}}\in H^*\big(\mathbb{C}\times\mathbb{T}_{\mathbf{s}}^{2(\ell-1)}\times\mathbb{T}^{2\ell};\mathbb{Q}\big).\tag{3.12}$$

In the $\ell=1$ case, $\mathbb{T}_{\mathbf{s}}^{2(\ell-1)}$ consists of $|\mathbf{s}|^2$ points and (3.12) reduces to

$$\mathrm{PD}_{0,H;(s)}^{0}\Delta=1\times\mathrm{PD}_{\mathbb{T}^2}(\mathrm{pt})\in H^*\big(\{1,\ldots,s^2\}\times\mathbb{C}\times\mathbb{T}^2;\mathbb{Q}\big).\tag{3.13}$$

In the symplectic sum decomposition for $\widehat{\mathbb{P}}_9^2=\widehat{\mathbb{P}}_9^2\#_V(\mathbb{P}^1\times V)$, (3.13) corresponds to putting the whole fiber on the X-side and a point on the Y-side; see Example 4.6.

Example 3.3. Let F be a compact connected oriented manifold, $V=\{0,\infty\}\times F$, $\ell_1,\ell_2\in\mathbb{Z}^{\ge0}$, $\mathbf{s}_1\in\mathbb{Z}_{\pm}^{\ell_1}$, and $\mathbf{s}_2\in\mathbb{Z}_{\pm}^{\ell_2}$. We take

$$H_1=H_2=H_{12}=H_\Delta\subset H_1(V;\mathbb{Z})=H_1(F;\mathbb{Z})\oplus H_1(F;\mathbb{Z})$$

to be the diagonal subgroup. By [5, Example 6.2], $\widehat{V}_{H_\Delta;\mathbf{s}_1\mathbf{s}_2}=\widehat{F}_{0;\mathbf{s}}$, where $\mathbf{s}$ is the merged tuple of $\mathbf{s}_1$ and $-\mathbf{s}_2$. With the identifications of [5, Examples 3.6,4.7], the map (3.4) becomes

$$\begin{gathered}\frac{H_1(F;\mathbb{Z})}{\gcd(\mathbf{s})H_1(F;\mathbb{Z})}\times\frac{H_1(F;\mathbb{Z})}{\gcd(\mathbf{s})H_1(F;\mathbb{Z})}\times\widehat{F}_{0;\mathbf{s}}'\times_{F_{\mathbf{s}}}\widehat{F}_{0;\mathbf{s}}'\longrightarrow H_1(F;\mathbb{Z}),\\ \widetilde{\Psi}_{H_\Delta,H_\Delta;\mathbf{s}_1\mathbf{s}_2}^{H_\Delta}\big([\gamma_{j_1}]_{0;\mathbf{s}},[\gamma_{j_2}]_{0;\mathbf{s}},[\gamma\cdot\widehat{x}]_0,[\widehat{x}]_0\big)=\Phi_{F;\mathbf{s}}(\gamma)+\gamma_{j_1}-\gamma_{j_2}.\end{gathered}$$

If $F=\mathbb{T}^2$ and $\ell\equiv\ell_1+\ell_2>0$, we associate representatives γ_j for the elements of $\mathbb{Z}_{\gcd(\mathbf{s})}\oplus\mathfrak{i}\mathbb{Z}_{\gcd(\mathbf{s})}$ with base points for the connected components of $\mathbb{C}\times\mathbb{T}_{\mathbf{s}}^{2(\ell-1)}$ as in Example 3.1. Then,

$$\begin{gathered}\widehat{V}_{H_\Delta,H_\Delta;\mathbf{s}_1\mathbf{s}_2}=\big\{\big(x,\big([z_{1;i}+s_{1;i}^{-1}x]_{i\le\ell_1},[z_{2;i}-s_{2;i}^{-1}x]_{i\le\ell_2}\big),y,\big([z_{1;i}+s_{1;i}^{-1}y]_{i\le\ell_1},[z_{2;i}-s_{2;i}^{-1}y]_{i\le\ell_2}\big)\big):\\ x,y\in\mathbb{C},\big([z_{1;i}+s_{1;i}^{-1}x]_{i\le\ell_1},[z_{2;i}-s_{2;i}^{-1}x]_{i\le\ell_2}\big),\big([z_{1;i}+s_{1;i}^{-1}y]_{i\le\ell_1},[z_{2;i}-s_{2;i}^{-1}y]_{i\le\ell_2}\big)\in\mathbb{T}_{\mathbf{s}}^{2(\ell-1)}\big\},\\ \widetilde{\Psi}_{H_\Delta,H_\Delta;\mathbf{s}_1\mathbf{s}_2}^{H_\Delta}\colon\widehat{V}_{H_\Delta,H_\Delta;\mathbf{s}_1\mathbf{s}_2}\longrightarrow\mathbb{Z}\oplus\mathfrak{i}\mathbb{Z},\qquad \widetilde{\Psi}_{H_\Delta,H_\Delta;\mathbf{s}_1\mathbf{s}_2}^{H_\Delta}\big(x,[\mathbf{z}],y,[\mathbf{z}']\big)=\ell\,(x-y).\end{gathered}$$

The diagonal components $\widehat{V}^{\widetilde{\eta}}_{H_\Delta,H_\Delta;\mathbf{s}_1\mathbf{s}_2}$ are again indexed by $\mathbb{Z}\oplus\mathfrak{i}\mathbb{Z}$ and have the same structure as in Example 3.1. Thus,

$$\mathrm{PD}^{\widetilde{\eta}}_{H_\Delta,H_\Delta;\mathbf{s}_1\mathbf{s}_2}\Delta=0\in H^*\big(\widehat{V}_{H_\Delta;\mathbf{s}_1\mathbf{s}_2}\times\widehat{V}_{H_\Delta;\mathbf{s}_1\mathbf{s}_2};\mathbb{Q}\big)\qquad\forall\,\widetilde{\eta}\in\mathbb{Z}\oplus\mathfrak{i}\mathbb{Z}. \tag{3.14}$$

This is consistent with the vanishing of the GW-invariants of $\mathbb{T}^2\times\mathbb{T}^2$; see Example 4.5.

3.2 Some properties

We begin this section by describing cases when the diagonal classes (3.9) split into products of cohomology classes from the two factors. We then show in certain cases these classes are the same for different choices of η.

Lemma 3.4. *Suppose V is a compact oriented manifold with topological components $V_1,\ldots,V_N$,*

$$H_1,H_2\subset H_{12}\subset H_1(V;\mathbb{Z})$$

are submodules, $\mathbf{s}_r\in\mathbb{Z}_{\pm}^{\ell_r}$, $\eta\in\mathcal{R}_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}$, and $\widetilde{\eta}\in\mathcal{R}_{H_{12}}$. The classes (3.9) admit finite Kunneth decompositions as in (1.22) if either $\mathcal{R}_{H_{12}}$ is finite or V is connected and $H_*(\widehat{V}_{H_{12}};\mathbb{Q})$ is finitely generated.

Proof. The inclusions $H_i\subset H_{12}$ induce projections

$$\Theta'_{H_{12},H_i}\colon\widehat{V}'_{H_i;\mathbf{s}_1\ldots\mathbf{s}_N}\equiv\widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}/(H_i)_{\mathbf{s}_1\ldots\mathbf{s}_N}\longrightarrow\widehat{V}'_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}\equiv\widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}/(H_{12})_{\mathbf{s}_1\ldots\mathbf{s}_N}\,.$$

Combining them with the restrictions of the homomorphism (3.2) to each component of the domain, we obtain a covering projection

$$\Theta\colon\widehat{V}_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\longrightarrow\widehat{V}_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}$$

such that

$$\begin{gathered}\pi_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\times\pi_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}=\pi_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}\times\pi_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}\circ\Theta,\\ \Psi_{H_1,H_2}^{H_{12}}=\Psi_{H_{12},H_{12}}^{H_{12}}\circ\Theta|_{\widehat{V}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}}.\end{gathered} \tag{3.15}$$

Thus,

$$\mathrm{PD}^{\eta}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta=\Theta^*\left(\mathrm{PD}^{\eta}_{H_{12},H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}\Delta\right) \tag{3.16}$$

for every $\eta\in\mathcal{R}_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}$.

If $\mathcal{R}_{H_{12}}$ is finite, then $\widehat{V}_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}$ is a finite cover of a compact manifold and so its homology is finitely generated. By the Kunneth formula for cohomology [23, Corollary 60.7],

$$H^*\big(\widehat{V}_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}\!\times\!\widehat{V}_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q}\big)\approx H^*\big(\widehat{V}_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q}\big)\otimes H^*\big(\widehat{V}_{H_{12};\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q}\big).$$

Thus, $\mathrm{PD}^{\eta}_{H_{12},H_{12};\mathbf{s}_1\ldots\mathbf{s}_N}$ admits a Kunneth decomposition in this case. By (3.16), so does the class $\mathrm{PD}^{\eta}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta$. In light of [5, Lemma 5.2], the same reasoning applies if V is connected and $H_*(\widehat{V}_{H_{12}};\mathbb{Q})$ is finitely generated.

The identity (3.16) holds with Ψ and η replaced by $\widetilde{\Psi}$ and $\widetilde{\eta}$ if the relevant collections

$$\{\gamma_{1;j_1}\},\{\gamma_{2;j_2}\},\{\gamma_{12;j}\}\subset H_1(V;\mathbb{Z})$$

are compatible, i.e.

$$\big\{[\gamma_{1;j_1}]_{H_{12}}\big\},\big\{[\gamma_{2;j_2}]_{H_{12}}\big\}=\big\{[\gamma_{12;j}]_{H_{12}}\big\}.$$

Changing the first or second collection would change a Kunneth decomposition of the cohomology class $\mathrm{PD}^{\widetilde{\eta}}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta$ by pulling back the cohomology classes involved by diffeomorphisms of the topological components of the factors of the domain. Thus, the existence of a Kunneth decomposition for this class is independent of the three collections. We can thus assume that these collections are compatible and (3.16) holds. The reasoning in the previous paragraph then also applies to $\mathrm{PD}^{\widetilde{\eta}}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta$. $\square$

Remark 3.5. The statement of Lemma 3.4 for a connected V and its proof can be adapted to a disconnected V. For each $r=1,\ldots,N$, let

$$\mathcal{R}_{H;r}=q_H\big(H_1(V_r;\mathbb{Z})\big)\subset\mathcal{R}_H\,;$$

these modules span $\mathcal{R}_H$. The first factor in the definition of $\widehat{V}_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$ in (2.6) is finite if and only if the submodule

$$\breve{\mathcal{R}}_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\equiv\sum_{\substack{1\le r\le N\\ \ell_r\neq0}}\mathcal{R}_{H;r}\subset\mathcal{R}_H$$

has finite index. This index is finite if $\ell_r\neq0$ whenever $H_1(V_r;\mathbb{Q})\neq\{0\}$ or if $V=\{0,\infty\}\times F$ for some connected F and

$$H=H_\Delta\subset H_1(V;\mathbb{Z})=H_1(F;\mathbb{Z})\oplus H_1(F;\mathbb{Z})$$

is the diagonal. If this index is finite for $H=H_{12}$, the argument in the proof of Lemma 3.4 concerning connected V applies via [5, Remark 5.3], which extends [5, Lemma 5.2] to this setting.

Remark 3.6. By [4, Theorem 1], $H_*(\widehat{V}_{H_{12}};\mathbb{Q})$ is finitely generated if $H_*(\widehat{V}_H;\mathbb{Q})$ is finitely generated for some submodule $H\subset H_{12}$. Thus, the last condition in Lemma 3.4 could instead require that $H_*(\widehat{V}_H;\mathbb{Q})$ be finitely generated for some submodule $H\subset H_{12}$. The approach in the proof applies directly, without use of [4, Theorem 1], if in addition H contains either H_1 or H_2.

The next example indicates that the cohomology classes (3.9) of the diagonal components (3.6) generally do not admit a Kunneth decomposition as in (1.22) if ($\mathcal{R}_{H_{12}}$ is infinite and) the homology of $\widehat{V}_{H_{12}}$ is not finitely generated. By [27, Corollary], the manifold V appearing in Example 3.7 is not symplectic. We use this particular V for the sake of simplicity. The complex blowup of $\mathbb{T}^4$ at a point could be used instead, but the notation would become a bit more involved.

Example 3.7. The maximal abelian cover $\widehat{V}$ of $V\equiv(S^1\times S^3)\#\mathbb{P}^2$ is $\mathbb{R}\times S^3$ with copies of $\mathbb{P}^2$ connected at $(i,\mathrm{pt})\in\mathbb{R}\times S^3$, with $i\in\mathbb{Z}$. If $\mathbb{P}^1_i$ is a $\mathbb{P}^1$ in the i-th $\mathbb{P}^2$, $\{\mathbb{P}^1_i: i\in\mathbb{Z}\}$ is a basis for $H_2(\widehat{V};\mathbb{Q})$. The topological components of $\widehat{V}_{0,0;(1)}\equiv\widehat{V}\times_V\widehat{V}$ are described by

$$\widehat{V}^{\gamma}_{0,0;(1)}=\big\{(\gamma\cdot\widehat{x},\widehat{x}):\ \widehat{x}\in\widehat{V}\big\},\qquad \gamma\in\mathbb{Z}.$$

For example, $\widehat{V}^0_{0,0;(1)}=\Delta_{\widehat{V}}$ is the diagonal and

$$\mathbb{P}^1_i\times\mathbb{P}^1_j\cdot\widehat{V}^0_{0,0;(1)}=\begin{cases}1, & \text{if } i=j;\\ 0, & \text{if } i\neq j.\end{cases}\tag{3.17}$$

If $e_1,\ldots,e_N$ is a basis for $\mathbb{Q}^N$, then the element

$$e_1\otimes e_1+\ldots+e_N\otimes e_N\in\mathbb{Q}^N\otimes_{\mathbb{Q}}\mathbb{Q}^N$$

cannot be written as a sum of fewer than N products $\alpha_j\otimes\beta_j$. Along with (3.17), this implies that the cohomology class on $\widehat{V}\times\widehat{V}$ determined by $\widehat{V}^0_{0,0;(1)}$ does not admit a finite Kunneth decomposition.

Let V, $V_1,\ldots,V_N$, and H_{12} be as in Lemma 3.4. For each $r=1,\ldots,N$, denote by

$$\mathrm{Flux}(V_r)_{H_{12}}\subset\mathcal{R}_{H_{12}}$$

the image of $\mathrm{Flux}(V_r)$ under the homomorphism

$$H_1(V_r;\mathbb{Z})\longrightarrow H_1(V;\mathbb{Z})\longrightarrow\frac{H_1(V;\mathbb{Z})}{H_{12}}=\mathcal{R}_{H_{12}}\,.$$

Lemma 3.8. *Suppose V, $V_1,\ldots,V_N$, H_1,H_2,H_{12}, and $\mathbf{s}_r$ are as in Lemma 3.4. If*

$$\widetilde{\eta}_1,\widetilde{\eta}_2\in\mathcal{R}_{H_{12}}\quad\textit{and}\quad\widetilde{\eta}_1-\widetilde{\eta}_2\in\bigoplus_{r=1}^{N}\gcd(\mathbf{s}_r)\mathrm{Flux}(V_r)_{H_{12}},$$

then

$$\mathrm{PD}^{\widetilde{\eta}_1}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta=\mathrm{PD}^{\widetilde{\eta}_2}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta\in H^*\big(\widehat{V}_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q}\big)\,.\tag{3.18}$$

Proof. Let

$$\gamma\equiv(\gamma_{r;i})_{i\le\ell_r,r\le N}\in\bigoplus_{r=1}^{N}\mathrm{Flux}(V_r)^{\oplus\ell_r}$$

be such that

$$\widetilde{\eta}_1-\widetilde{\eta}_2=\big[\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma)\big]_{H_{12}}\in\mathcal{R}_{H_{12}}\,.\tag{3.19}$$

For each $r=1,\ldots,N$ and $i=1,\ldots,\ell_r$, let $\Psi_{r;i;t}\colon V_r\longrightarrow V_r$ be a loop of diffeomorphisms generating $\gamma_{r;i}$ such that $\Psi_{r;i;0}=\mathrm{id}$. These loops lift to paths of diffeomorphisms

$$\begin{aligned}&\widehat{\Psi}_{r;i;t}\colon\widehat{V}_r\longrightarrow\widehat{V}_r, && t\in[0,1], \quad \widehat{\Psi}_{r;i;0}=\mathrm{id}_{\widehat{V}_r},\quad \widehat{\Psi}_{r;i;1}(\widehat{x}_{r;i})=\gamma_{r;i}\cdot\widehat{x}_{r;i},\\ &\widetilde{\Psi}_t\colon\widehat{V}_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\longrightarrow\widehat{V}_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}, && t\in[0,1],\quad \widetilde{\Psi}_t\big(\big[(\widehat{x}_{r;i})_{i\le\ell_r,r\le N}\big]_{H_{12}}\big)=\big[\big(\widehat{\Psi}_{r;i;t}(\widehat{x}_{r;i})\big)_{i\le\ell_r,r\le N}\big]_{H_{12}}.\end{aligned}$$

In particular, $\widetilde{\Psi}_1=\Theta_{\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma)}$. By (3.5) and (3.19),

$$\widehat{V}^{\widetilde{\eta}_1}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}=\big\{\widetilde{\Psi}_1\times\mathrm{id}_{\widehat{V}_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N}}\big\}\big(\widehat{V}^{\widetilde{\eta}_2}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\big).$$

Thus, the smooth proper map

$$[0,1]\times\widehat{V}^{\widetilde{\eta}_2}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}\longrightarrow\widehat{V}_{H_1;\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{H_2;\mathbf{s}_1\ldots\mathbf{s}_N},\qquad (t,\widetilde{x}_1,\widetilde{x}_2)\longrightarrow\big(\widetilde{\Psi}_t(\widetilde{x}_1),\widetilde{x}_2\big),$$

is a cobordism between $\widehat{V}^{\widetilde{\eta}_1}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}$ and $\widehat{V}^{\widetilde{\eta}_2}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}$. Its intersection with a smooth map h as in (3.8) is a compact cobordism between the intersections of h with $\widehat{V}^{\widetilde{\eta}_1}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}$ and with $\widehat{V}^{\widetilde{\eta}_2}_{H_1,H_2;\mathbf{s}_1\ldots\mathbf{s}_N}$. Thus, the intersection homomorphisms on the homology induced by the two diagonal components are the same; this establishes (3.18). $\square$

4 The refined invariance property

We now provide the details needed to refine the usual symplectic sum formula, as suggested in [12, Sections 3,10] and outlined in Section 1.2. Once the lifts (1.9) of (1.3) are chosen systematically as in Proposition 2.2, the fiber products (1.16) of the coverings (1.6) for (X,V) and (Y,V) over the diagonal in $V_{\mathbf{s}}\times V_{\mathbf{s}}$ can be split into unions of topological components (1.19), as suggested by [12, (3.10)]. We define the crucial refined degree-gluing map (1.17) in Section 4.1 as a special case of the map (3.4) for arbitrary abelian covers. It is used in Section 4.2 to define the diagonal components (1.19). Corollary 4.3 describes cases when the Poincare duals of these components split as in (1.22). The approach of [11, 12] can then be used to distinguish the GW-invariants of $X\#_V Y$ in degrees differing by elements of $\mathcal{R}^V_{X,Y}$ in terms of the IP-counts of (X,V) and (Y,V). Otherwise, they can be distinguished only in terms of the IP-counts of the singular fiber $X\cup_V Y$. The assertions made in Sections 1.3 and 1.4 are established in Section 4.3.

4.1 The refined gluing degree map

We begin this section by defining the refined gluing degree map (1.17) as a special case of the continuous map (3.4) on the diagonal fiber product of two abelian covers of the divisor V. We then show that its composition with the lifted evaluation morphisms (2.21) gives the degree of the glued map, provided the coset representatives in the constructions of (3.4) and (2.21) are chosen in the same way; see Figure 2 and Proposition 4.2.

Throughout this section, X and Y denote compact oriented manifolds, $V\subset X,Y$ is a compact oriented submanifold of codimension $\mathfrak{c}$, and $\varphi\colon S_XV\longrightarrow S_YV$ is an orientation-reversing diffeomorphism commuting with the projections to V. With

$$q_V\colon\ S_XV=S_YV\longrightarrow V$$

denoting the bundle projection map, define

$$H_{\mathfrak{c}}(SV;\mathbb{Z})_{X,Y}=\big\{A_{SV}\in H_{\mathfrak{c}}(SV;\mathbb{Z})\colon\ q_{V*}(A_{SV})\in\ker\iota^X_{V*}\cap\ker\iota^Y_{V*}\big\}.\tag{4.1}$$

Let

$$\begin{aligned}H_{\mathfrak{c}}(X;\mathbb{Z})\times_V H_{\mathfrak{c}}(Y;\mathbb{Z})=\big\{(A_X,A_Y)\in H_{\mathfrak{c}}(X;\mathbb{Z})\times H_{\mathfrak{c}}(Y;\mathbb{Z})\colon\\ A_X\cdot_X V_r=A_Y\cdot_Y V_r\ \ \forall\,r=1,\ldots,N\big\},\end{aligned}\tag{4.2}$$

where $\cdot_X$ and $\cdot_Y$ are the homology intersection pairings in X and Y, respectively.

Let $X\#_\varphi Y$ be the manifold obtained by gluing the complements of tubular neighborhoods of V in X and Y by φ along their common boundary. We denote by

$$q_\varphi\colon X\#_\varphi Y \longrightarrow X\cup_V Y$$

a continuous map which restricts to the identity outside of a tubular neighborhood of $S_X V =_\varphi S_Y V$, is a diffeomorphism on the complement of $q_\varphi^{-1}(V)$, and restricts to the bundle projection $S_X V \longrightarrow V$. Let

$$\mathcal{R}^V_{X,Y} \equiv \big\{\iota^{X\#_\varphi Y}_{X-V*}(A_{X-V})+\iota^{X\#_\varphi Y}_{Y-V*}(A_{Y-V})\colon\ (A_{X-V},A_{Y-V})\in\mathcal{R}^V_X\oplus\mathcal{R}^V_Y\big\}. \tag{4.3}$$

By [5, Lemma 4.1], this definition of $\mathcal{R}^V_{X,Y}$ agrees with (1.12) in the $\mathfrak{c}=2$ case.

Define

$$\Delta^V_{X,Y}\colon H_1(V;\mathbb{Z})_X\oplus H_1(V;\mathbb{Z})_Y \longrightarrow \frac{H_1(V;\mathbb{Z})}{H^V_X+H^V_Y}\,, \qquad \big([\gamma_X]_{H^V_X},[\gamma_Y]_{H^V_Y}\big)\longrightarrow \big[\gamma_X-\gamma_Y\big]_{H^V_X+H^V_Y}.$$

Denote by $\overline{H}^V_{X,Y}$ the image of the composition

$$H_{\mathfrak{c}}(SV;\mathbb{Z})_{X,Y} \xrightarrow{(\iota^{X-V}_{SV*},-\iota^{Y-V}_{SV*})} \mathcal{R}^V_X\oplus\mathcal{R}^V_Y \xrightarrow{\approx} H_1(V;\mathbb{Z})_X\oplus H_1(V;\mathbb{Z})_Y \xrightarrow{\Delta^V_{X,Y}} \frac{H_1(V;\mathbb{Z})}{H^V_X+H^V_Y},$$

with the second arrow above given by the isomorphisms in (2.16). Let $H^V_{X,Y}\subset H_1(V;\mathbb{Z})$ be the preimage of $\overline{H}^V_{X,Y}$ under the quotient projection

$$H_1(V;\mathbb{Z})\longrightarrow \frac{H_1(V;\mathbb{Z})}{H^V_X+H^V_Y}\,.$$

By [5, Corollary 4.4(1)], there is a commutative diagram

$$\begin{array}{ccc}
\frac{H_1(V;\mathbb{Z})}{H^V_X}\oplus\frac{H_1(V;\mathbb{Z})}{H^V_Y} & \xrightarrow{\quad\overline{\Delta}^V_{X,Y}\quad} & \frac{H_1(V;\mathbb{Z})}{H^V_{X,Y}} \\
\Big\downarrow{\scriptstyle \iota^{X-V}_{S_XV*}\circ\Delta^V_X\oplus\iota^{Y-V}_{S_YV*}\circ\Delta^V_Y} & & \approx\Big\downarrow{\scriptstyle\mathfrak{R}^V_{X,Y}} \\
\mathcal{R}^V_X\oplus\mathcal{R}^V_Y & \xrightarrow{\iota^{X\#_\varphi Y}_{X-V*}+\iota^{X\#_\varphi Y}_{Y-V*}} & \mathcal{R}^V_{X,Y}
\end{array} \tag{4.4}$$

of homomorphisms, with both vertical arrows being isomorphisms.

Let $V_1,\ldots,V_N$ be the topological components of V, $\mathbf{s}_1\in\mathbb{Z}^{\ell_1}_\pm,\ldots,\mathbf{s}_N\in\mathbb{Z}^{\ell_N}_\pm$, and

$$\widehat{V}'_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}=\widehat{V}'_{H^V_X,H^V_Y;\mathbf{s}_1\ldots\mathbf{s}_N}\,,\quad \widehat{V}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}=\widehat{V}_{H^V_X,H^V_Y;\mathbf{s}_1\ldots\mathbf{s}_N};$$

see (3.1). Choose collections

$$\big\{\gamma_{X;j}\big\},\big\{\gamma_{Y;j}\big\}\subset H_1(V;\mathbb{Z}) \tag{4.5}$$

of coset representatives for the elements of $\mathcal{R}_X^V/\mathcal{R}'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$ and $\mathcal{R}_Y^V/\mathcal{R}'^V_{Y;\mathbf{s}_1\ldots\mathbf{s}_N}$. As described above (3.4), these collections determine a smooth map

$$\widetilde{\Psi}_{X,Y}^V\colon \widehat{V}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}\xrightarrow{\widetilde{\Psi}^{H_{X,Y}^V}_{H_X^V,H_Y^V}}\mathcal{R}_{H_{X,Y}^V}\xrightarrow[\approx]{\mathfrak{R}_{X,Y}^V}\mathcal{R}_{X,Y}^V\subset H_{\mathfrak{c}}(X\#_{\varphi}Y;\mathbb{Z}),$$

with $\mathfrak{R}_{X,Y}^V$ as in (4.4).

Let Σ_X,Σ_Y be compact oriented $\mathfrak{c}$-dimensional manifolds, $k_X,k_Y\in\mathbb{Z}^{\ge0}$, and (A_X,A_Y) be an element of $H_{\mathfrak{c}}(X;\mathbb{Z})\times_V H_{\mathfrak{c}}(Y;\mathbb{Z})$. Denote by

$$\widetilde{\mathrm{ev}}_X^V\colon \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_X,k_X;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A_X)\longrightarrow\widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\quad\text{and}\quad\widetilde{\mathrm{ev}}_Y^V\colon \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_Y,k_Y;\mathbf{s}_1\ldots\mathbf{s}_N}(Y,A_Y)\longrightarrow\widehat{V}_{Y;\mathbf{s}_1\ldots\mathbf{s}_N}$$

the lifted relative evaluation morphisms of Proposition 2.2 for (X,V) and (Y,V) compatible with the coset representatives (4.5). Let

$$\begin{aligned}\mathfrak{X}^{V_1,\ldots,V_N}_{(\Sigma_X,\Sigma_Y),(k_X,k_Y);\mathbf{s}_1\ldots\mathbf{s}_N}\big((X,Y),(A_X,A_Y)\big)&=\big\{\widetilde{\mathrm{ev}}_X^V\times\widetilde{\mathrm{ev}}_Y^V\big\}^{-1}(\widehat{V}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N})\\&=\big\{\mathrm{ev}_X^V\times\mathrm{ev}_Y^V\big\}^{-1}(\Delta^V_{\mathbf{s}_1\ldots\mathbf{s}_N}).\end{aligned}\tag{4.6}$$

Each element $(\mathbf{f}_X,\mathbf{f}_Y)$ of this space gives rise to a marked map

$$\mathbf{f}_X\#_{\varphi}\mathbf{f}_Y\in\bigsqcup_{A_\#\in A_X\#_{\varphi}A_Y}\mathfrak{X}_{\Sigma_X\#\Sigma_Y,k_X+k_Y}(X\#_{\varphi}Y,A_\#);\tag{4.7}$$

see [5, Section 2.2].

Fix a base point $x_{\mathbf{s}_1\ldots\mathbf{s}_N}\in V_{\mathbf{s}_1\ldots\mathbf{s}_N}$. Suppose

$$\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_X,k_X;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A_X),\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_Y,k_Y;\mathbf{s}_1\ldots\mathbf{s}_N}(Y,A_Y)\neq\emptyset.$$

Choose

$$\begin{aligned}\mathbf{f}_X^\bullet&\equiv(z^\bullet_{X;1},\ldots,z^\bullet_{X;k_X+\ell_1+\ldots+\ell_N},f_X^\bullet)\in\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_X,k_X;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A_X),\\\mathbf{f}_Y^\bullet&\equiv(z^\bullet_{Y;1},\ldots,z^\bullet_{Y;k_Y+\ell_1+\ldots+\ell_N},f_Y^\bullet)\in\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma_Y,k_Y;\mathbf{s}_1\ldots\mathbf{s}_N}(Y,A_Y)\end{aligned}\tag{4.8}$$

such that

$$\mathrm{ev}_X^V\big(\mathbf{f}_X^\bullet\big),\mathrm{ev}_Y^V\big(\mathbf{f}_Y^\bullet\big)=x_{\mathbf{s}_1\ldots\mathbf{s}_N}\in V_{\mathbf{s}_1\ldots\mathbf{s}_N}.$$

These two marked maps correspond to the initially chosen base points, denoted by $\mathbf{f}_0$, in the proof of [5, Theorem 6.5]. Let

$$A_{X,Y}=\big[f_X^\bullet\#_{\varphi}f_Y^\bullet\big]\in H_{\mathfrak{c}}(X\#_{\varphi}Y;\mathbb{Z}),\tag{4.9}$$

and define

$$\begin{aligned}&g_{A_X,A_Y}\colon\widehat{V}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}\longrightarrow H_{\mathfrak{c}}(X\#_{\varphi}Y;\mathbb{Z})\qquad\text{by}\\&g_{A_X,A_Y}(\widetilde{x},\widetilde{y})=A_{X,Y}-\widetilde{\Psi}_{X,Y}^V\big(\widetilde{\mathrm{ev}}_X^V\big(\mathbf{f}_X^\bullet\big),\widetilde{\mathrm{ev}}_Y^V\big(\mathbf{f}_Y^\bullet\big)\big)+\widetilde{\Psi}_{X,Y}^V\big(\widetilde{x},\widetilde{y}\big).\end{aligned}\tag{4.10}$$

The last map is the intended refined gluing degree map of [12, (3.10)]. By Proposition 4.2 below, it gives the degree $A_\#$ of each glued map (4.7). The statement of Proposition 4.2 is illustrated in Figure 2, where we abbreviate $\mathbf{s}_1\ldots\mathbf{s}_N$ as $\mathbf{s}$ and ignore the absolute marked points.

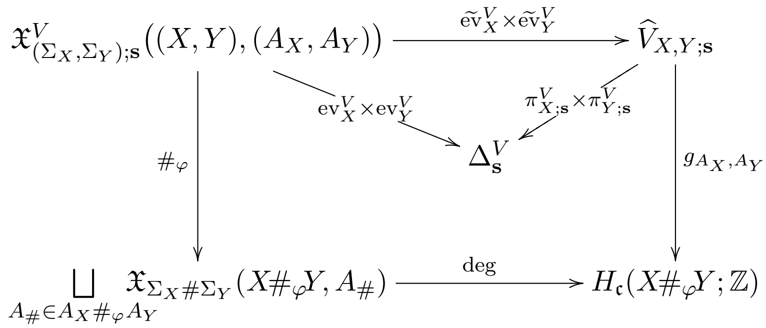

Figure 2: The evaluation and gluing maps of Proposition 4.2.

Remark 4.1. The map component $f_X\#_\varphi f_Y$ of $\mathbf{f}_X\#_\varphi\mathbf{f}_Y$ depends on the choices made in the construction of [5, Section 2.2]. However, the degree (homology class) of $f_X\#_\varphi f_Y$ does not depend on these choices.

Proposition 4.2. *Suppose X and Y are oriented manifolds, $V\subset X,Y$ is a compact oriented submanifold of codimension $\mathfrak{c}$ with connected components $V_1,\ldots,V_N$, $\varphi\colon S_XV\longrightarrow S_YV$ is an orientation-reversing diffeomorphism commuting with the projections to V, and*

$$(A_X,A_Y)\in H_{\mathfrak{c}}(X;\mathbb{Z})\times_V H_{\mathfrak{c}}(Y;\mathbb{Z}),\qquad \mathbf{s}_r\in\mathbb{Z}^{\ell_r}_{\pm}\quad\textit{with}\quad r=1,\ldots,N.$$

Then,

$$\big[\mathbf{f}_X\#_\varphi\mathbf{f}_Y\big]=g_{A_X,A_Y}\big(\widetilde{\mathrm{ev}}^V_X(\mathbf{f}_X),\widetilde{\mathrm{ev}}^V_Y(\mathbf{f}_Y)\big) \tag{4.11}$$

for all pairs $(\mathbf{f}_X,\mathbf{f}_Y)$ in $\mathfrak{X}^{V_1,\ldots,V_N}_{(\Sigma_X,\Sigma_Y),(k_X,k_Y);\mathbf{s}_1\ldots\mathbf{s}_N}\big((X,Y),(A_X,A_Y)\big)$.

Proof. Let $\widehat{x}\in\widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}$, $\gamma^\bullet\in H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z})$, and $\gamma^\bullet_X,\gamma^\bullet_Y\in H_1(V;\mathbb{Z})$ be elements of the collections in (4.5) such that

$$\widetilde{\mathrm{ev}}^V_X(\mathbf{f}^\bullet_X)=\big([\gamma^\bullet_X]_{X;\mathbf{s}_1\ldots\mathbf{s}_N},[\gamma^\bullet\cdot\widehat{x}]_X\big)\quad\text{and}\quad\widetilde{\mathrm{ev}}^V_Y(\mathbf{f}^\bullet_Y)=\big([\gamma^\bullet_Y]_{Y;\mathbf{s}_1\ldots\mathbf{s}_N},[\widehat{x}]_X\big). \tag{4.12}$$

Thus,

$$\widetilde{\Psi}^V_{X,Y}\big(\widetilde{\mathrm{ev}}^V_X(\mathbf{f}^\bullet_X),\widetilde{\mathrm{ev}}^V_Y(\mathbf{f}^\bullet_Y)\big)=\mathfrak{R}^V_{X,Y}\big(\big[\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma^\bullet)+\gamma^\bullet_X-\gamma^\bullet_Y\big]_{H^V_{X,Y}}\big)\,. \tag{4.13}$$

Since the two sides of (4.11) take discrete values and are continuous in $(\mathbf{f}_X,\mathbf{f}_Y)$, it is sufficient to verify (4.11) under the assumption that

$$\mathrm{ev}^V_X\big(\mathbf{f}_X\big),\mathrm{ev}^V_Y\big(\mathbf{f}_Y\big)=x_{\mathbf{s}_1\ldots\mathbf{s}_N}\in V_{\mathbf{s}_1\ldots\mathbf{s}_N}\,.$$

Let $\gamma,\gamma'\in H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z})$, and $\gamma_X,\gamma_Y\in H_1(V;\mathbb{Z})$ be elements of the collections in (4.5) such that

$$\widetilde{\mathrm{ev}}^V_X(\mathbf{f}_X)=\big([\gamma_X]_{X;\mathbf{s}_1\ldots\mathbf{s}_N},[\gamma\gamma'\cdot\widehat{x}]_X\big)\quad\text{and}\quad\widetilde{\mathrm{ev}}^V_Y(\mathbf{f}_Y)=\big([\gamma_Y]_{Y;\mathbf{s}_1\ldots\mathbf{s}_N},[\gamma'\cdot\widehat{x}]_X\big). \tag{4.14}$$

Thus,

$$\widetilde{\Psi}^V_{X,Y}\big(\widetilde{\mathrm{ev}}^V_X\big(\mathbf{f}_X\big),\widetilde{\mathrm{ev}}^V_Y\big(\mathbf{f}_Y\big)\big)=\mathfrak{R}^V_{X,Y}\big(\big[\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma)+\gamma_X-\gamma_Y\big]_{H^V_{X,Y}}\big)\,. \tag{4.15}$$

By (4.12), (4.14), and (2.23),

$$\begin{aligned}\big[f_X\#(-f_X^\bullet)\big] &= \iota_{S_XV*}^{X-V}\big(\Delta_X^V\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma+\gamma'-\gamma^\bullet)+\gamma_X-\gamma_X^\bullet\big)\big)\in H_{\mathfrak{c}}(X-V;\mathbb{Z}),\\ \big[f_Y\#(-f_Y^\bullet)\big] &= \iota_{S_YV*}^{Y-V}\big(\Delta_Y^V\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma')+\gamma_Y-\gamma_Y^\bullet\big)\big)\in H_{\mathfrak{c}}(Y-V;\mathbb{Z}).\end{aligned}$$

Thus,

$$\begin{aligned}\big[f_X\#_\varphi f_Y\big]-\big[f_X^\bullet\#_\varphi f_Y^\bullet\big] &= \iota_{X-V*}^{X\#_\varphi Y}\big(\big[f_X\#(-f_X^\bullet)\big]\big)+\iota_{Y-V*}^{X\#_\varphi Y}\big(\big[f_Y\#(-f_Y^\bullet)\big]\big)\\ &= \iota_{S_XV*}^{X\#_\varphi Y}\big(\Delta_X^V\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma-\gamma^\bullet)+\gamma_X-\gamma_Y-\gamma_X^\bullet+\gamma_Y^\bullet\big)\big).\end{aligned}\tag{4.16}$$

From (4.9) and (4.16), we find that

$$\begin{aligned}\big[f_X\#_\varphi f_Y\big] = A_{X,Y} &- \mathfrak{R}_{X,Y}^V\big(\big[\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma^\bullet)+\gamma_X^\bullet-\gamma_Y^\bullet\big]_{H_{X,Y}^V}\big)\\ &+\mathfrak{R}_{X,Y}^V\big(\big[\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma)+\gamma_X-\gamma_Y\big]_{H_{X,Y}^V}\big).\end{aligned}$$

Comparing this with (4.10), (4.13) and (4.15), we obtain the claim. □

4.2 Diagonal components for rim tori covers

We now use the refined gluing degree map (4.10) to split fiber products of rim tori covers into diagonal components and describe some of their properties. This completes the details needed to refine the usual symplectic sum formula, as suggested in [12, Sections 3,10] and outlined in Section 1.2. We include three examples of applying the refined invariance property for GW-invariants in simple cases when the diagonal splits and its Kunneth decomposition can be easily determined.

Similarly to (3.6), we define

$$\widehat{V}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}^A \equiv g_{A_X,A_Y}^{-1}(A)=\widehat{V}_{H_X^V,H_Y^V;\mathbf{s}_1\ldots\mathbf{s}_N}^{A-A_{X,Y}} \qquad \forall\ A\in A_X\#_\varphi A_Y\subset H_{\mathfrak{c}}(X\#_\varphi Y;\mathbb{Z}).\tag{4.17}$$

Similarly to (3.9), let

$$\mathrm{PD}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}^{V,A}\Delta\equiv \mathrm{PD}_{H_X^V,H_Y^V;\mathbf{s}_1\ldots\mathbf{s}_N}^{A-A_{X,Y}}\Delta\in H^*\big(\widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{Y;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q}\big)\tag{4.18}$$

denote the cohomology class determined by the closed submanifold (4.17). Lemma 3.4 gives the following description of some cases when this class admits a finite Kunneth decomposition.

Corollary 4.3. *Suppose X, Y, V, φ, (A_X,A_Y), and $\mathbf{s}_r\in\mathbb{Z}_{\pm}^{\ell_r}$ are as in Proposition 4.2 and $A\in A_X\#_\varphi A_Y$. The class (4.18) admits a finite Kunneth decomposition as in (1.22) if either $\mathcal{R}_{X,Y}^V$ is finite or V is connected and $H_*(\widehat{V}_{H_{X,Y}^V};\mathbb{Q})$ is finitely generated.*

Example 4.4. We take $X,Y=\widehat{\mathbb{P}}_9^2$ to be the blowup of $\mathbb{P}^2$ at 9 points and $V=F$ to be a smooth fiber of the fibration $\widehat{\mathbb{P}}_9^2\longrightarrow\mathbb{P}^1$; see the beginning of Section 6. By [5, Examples 3.5,4.6],

$$H_X^V,H_Y^V,H_{X,Y}^V=\{0\}\subset H_1(V;\mathbb{Z})\approx\mathbb{Z}^2\,.$$

Let $|A|_V \in \mathbb{Z}$ be as in (1.25). By (3.11),

$$\mathrm{PD}^{V,A}_{X,Y;\mathbf{s}}\Delta = 0 \in H^*\big(\widehat{V}_{X;\mathbf{s}}\times\widehat{V}_{Y;\mathbf{s}};\mathbb{Q}\big) \tag{4.19}$$

for all $\mathbf{s}\in\mathbb{Z}^\ell$ with $\ell>0$ and $A\in H_2(X\#_\varphi Y;\mathbb{Z})$ with $|A|_V=|\mathbf{s}|$. Since the contribution to the GW-invariant of $X\#_\varphi Y$ in a degree $A\in A_X\#_\varphi A_Y$ from its splitting as $A_X\#_\varphi A_Y$ is obtained by pulling back the classes (4.19) with $|\mathbf{s}|=|A|_V$ by $\widetilde{\mathrm{ev}}^V_X\times\widetilde{\mathrm{ev}}^V_Y$, this contribution vanishes if $|A|_V\neq 0$. If $C\subset\widehat{\mathbb{P}}^2_9$ is a holomorphic curve disjoint from V, its projection to $\mathbb{P}^1$ misses a point and thus C is a union of fibers. Therefore, the fiber class of $\widehat{\mathbb{P}}^2_9$ and its multiples are the only classes that have zero intersection with V and a priori may have nonzero GW-invariants. By (4.19), all GW-invariants of $X\#_\varphi Y$ in non-fiber classes vanish. This is a direct illustration of the vanishing statement of Theorem 1.1. For the standard identification φ, $X\#_\varphi Y=\mathbb{K}_3$. By [13, Theorem 2.4], $\mathbb{K}_3$ admits an almost complex structure J with no J-holomorphic curves. Thus, (4.19) is consistent with the vanishing of all GW-invariants of $\mathbb{K}_3$.

Example 4.5. We take $X,Y=\mathbb{P}^1\times\mathbb{T}^2$ and $V=\{0,\infty\}\times\mathbb{T}^2$. By [5, Examples 3.6,4.7],

$$H^V_X, H^V_Y, H^V_{X,Y} = H_\Delta \subset H_1(V;\mathbb{Z}) = H_1(\mathbb{T}^2;\mathbb{Z})\oplus H_1(\mathbb{T}^2;\mathbb{Z}) \approx \mathbb{Z}^2\oplus\mathbb{Z}^2\,,$$

where H_Δ is the diagonal subgroup. By (3.14),

$$\mathrm{PD}^{V,A}_{X,Y;\mathbf{s}_1\mathbf{s}_2}\Delta = 0 \in H^*\big(\widehat{V}_{X;\mathbf{s}_1\mathbf{s}_2}\times\widehat{V}_{Y;\mathbf{s}_1\mathbf{s}_2};\mathbb{Q}\big) \tag{4.20}$$

for all $\mathbf{s}_1\in\mathbb{Z}^{\ell_1},\mathbf{s}_2\in\mathbb{Z}^{\ell_2}$ with $\ell_1+\ell_2>0$ and $A\in H_2(X\#_\varphi Y;\mathbb{Z})$ with $|A|_V=|\mathbf{s}_1|+|\mathbf{s}_2|$. Similarly to Example 4.4, this implies that all GW-invariants of $X\#_\varphi Y$ in non-fiber classes vanish and provides another direct illustration of the vanishing statement of Theorem 1.1. For the standard identification φ, $X\#_\varphi Y=\mathbb{T}^2\times\mathbb{T}^2$. By [13, Theorem 2.4], $\mathbb{T}^2\times\mathbb{T}^2$ admits an almost complex structure J with no J-holomorphic curves. Thus, (4.20) is consistent with the vanishing of all GW-invariants of $\mathbb{T}^2\times\mathbb{T}^2$.

Example 4.6. We now take $X=\widehat{\mathbb{P}}^2_9$, $Y=\mathbb{P}^1\times\mathbb{T}^2$, and $V=F\subset X,Y$ to be a smooth fiber. By [5, Examples 3.5,3.6],

$$H^V_X=\{0\},\;\; H^V_Y, H^V_{X,Y} = H_1(V;\mathbb{Z})\approx\mathbb{Z}^2.$$

Since $\mathcal{R}^V_{X,Y}=H_1(V;\mathbb{Z})/H^V_{X,Y}$, there are no rim tori in $X\#_\varphi Y$ in this case. By (3.13),

$$\mathrm{PD}^{V,A}_{X,Y;(1)}\Delta = 1\times\mathrm{PD}_{\mathbb{T}^2}(\mathrm{pt}) \in H^*\big(\widehat{V}_{X;(1)}\times\widehat{V}_{Y;(1)};\mathbb{Q}\big) = H^*\big(\mathbb{C}\times\mathbb{T}^2;\mathbb{Q}\big) \tag{4.21}$$

for all $A\in H_2(X\#_\varphi Y;\mathbb{Z})$ with $|A|_V=1$. Let $\mathfrak{s}_1,\ldots,\mathfrak{s}_9,\mathfrak{f}\in H_2(X;\mathbb{Z})$ denote the homology classes of the 9 sections corresponding to the exceptional divisors and of the fiber class and $\mathfrak{s},\mathfrak{f}\in H_2(Y;\mathbb{Z})$ denote the homology classes of the section class and of the fiber class. Let

$$A_{i;d}=(\mathfrak{s}_i+d\mathfrak{f})\#_\varphi\mathfrak{s}\in H_2(X\#_\varphi Y;\mathbb{Z})\,.$$

The only decompositions $A_{i;d}=A_X\#_\varphi A_Y$ into classes A_X, A_Y with possibly nonzero GW-invariants are of the form

$$A_{i;d}=(\mathfrak{s}_i+d_1\mathfrak{f})\#_\varphi(\mathfrak{s}+d_2\mathfrak{f}) \qquad\text{with}\quad d_1,d_2\in\mathbb{Z}^{\geq 0},\;d_1+d_2=d.$$

Thus, the refined invariance property for GW-invariants of [12], the decomposition (4.21), and dimensional considerations give

$$\mathrm{GW}_{g,A_{i;d}}^{X\#_\varphi Y}\big(\mathrm{pt}^g\big) = \sum_{\substack{d_1,d_2\in\mathbb{Z}^{\ge0}\\ d_1+d_2=d}} \widetilde{\mathrm{GW}}_{g-1,\mathfrak{s}_i+d_1\mathfrak{f};(1)}^{\widehat{\mathbb{P}}^2_9,F}\big(\mathrm{pt}^{g-1};1\big)\widetilde{\mathrm{GW}}_{1,\mathfrak{s}+d_2\mathfrak{f};(1)}^{\mathbb{P}^1\times\mathbb{T}^2,F}\big(\mathrm{pt};\mathrm{PD}_{\mathbb{T}^2}(\mathrm{pt})\big), \tag{4.22}$$

where pt^g denotes g point constraints (pullback of $\mathrm{PD}_{(X\#_\varphi Y)^g}(\mathrm{pt})$ by the evaluation map at g absolute points) and $\widetilde{\mathrm{GW}}$ denotes the IP-counts with the relative constraints

$$1\in H^0\big(\widehat{V}_{X;(1)};\mathbb{Q}\big)=H^0\big(\mathbb{C};\mathbb{Q}\big) \qquad \text{and} \qquad \mathrm{PD}_{\mathbb{T}^2}(\mathrm{pt})\in H^2\big(\widehat{V}_{Y;(1)};\mathbb{Q}\big)=H^2\big(\mathbb{T}^2;\mathbb{Q}\big).$$

Since the first class above is the pullback of the cohomology class 1 on $\mathbb{T}^2$ by the covering map, (4.22) reduces to

$$\mathrm{GW}_{g,A_{i;d}}^{X\#_\varphi Y}\big(\mathrm{pt}^g\big) = \sum_{\substack{d_1,d_2\in\mathbb{Z}^{\ge0}\\ d_1+d_2=d}} \mathrm{GW}_{g-1,\mathfrak{s}_i+d_1\mathfrak{f};(1)}^{\widehat{\mathbb{P}}^2_9,F}\big(\mathrm{pt}^{g-1};1\big)\mathrm{GW}_{1,\mathfrak{s}+d_2\mathfrak{f};(1)}^{\mathbb{P}^1\times\mathbb{T}^2,F}\big(\mathrm{pt};\mathrm{PD}_{\mathbb{T}^2}(\mathrm{pt})\big), \tag{4.23}$$

with the standard relative GW-invariants on the right-hand side above. Since $\mathcal{R}_{X,Y}^V=\{0\}$, the standard symplectic sum formulas of [16, 15] leave nothing to be refined in this case. In the proof of Lemma 6.12, (4.23) is deduced from (6.15) and (6.16). The last statement is a consequence of (1.8) and $\widehat{V}_{\widehat{\mathbb{P}}^2_9;(1)}\approx\mathbb{C}$. The same conclusions are obtained in the proof of [12, Lemma 15.2] through a simultaneous triple induction with three separate applications of the standard symplectic sum formula.

We next turn to connections with the flux group defined in Section 1.4. Let X, Y, V, $V_1,\ldots,V_N$, $\mathfrak{c}$, and φ be as in Proposition 4.2. For each $r=1,\ldots,N$, define

$$\mathrm{Flux}(V_r)_{X,Y} = \mathfrak{R}_{X,Y}^V\big(\mathrm{Flux}(V_r)_{H_{X,Y}^V}\big)\subset\mathcal{R}_{X,Y}^V\subset H_{\mathfrak{c}}(X\#_\varphi Y;\mathbb{Z}). \tag{4.24}$$

In particular,

$$\mathrm{Flux}(V_r)_{X,Y} = \iota_{X-V*}^{X\#_V Y}\big(\Delta_X^V\big(\mathrm{Flux}(V_r)\big)\big) = \iota_{Y-V*}^{X\#_V Y}\big(\Delta_Y^V\big(\mathrm{Flux}(V_r)\big)\big).$$

Since $\mathfrak{R}_{X,Y}^V$ is an isomorphism, the next statement follows immediately from (4.10) and Lemma 3.8.

Corollary 4.7. *Suppose X, Y, V, φ, (A_X,A_Y), and $\mathbf{s}_r\in\mathbb{Z}_\pm^{\ell_r}$ are as in Proposition 4.2. If*

$$A_1,A_2\in A_X\#_\varphi A_Y \quad \textit{and} \quad A_1-A_2\in\bigoplus_{r=1}^N \gcd(\mathbf{s}_r)\mathrm{Flux}(V_r)_{X,Y}\subset H_{\mathfrak{c}}(X\#_\varphi Y;\mathbb{Z}), \tag{4.25}$$

then

$$\mathrm{PD}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}^{V,A_1}\Delta = \mathrm{PD}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}^{V,A_2}\Delta\in H^*\big(\widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{Y;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q}\big)\,.$$

4.3 Proofs of qualitative implications

In this section, we establish the assertions made in Sections 1.3 and 1.4. We first obtain Theorem 1.1 and deduce Corollary 1.2 from it. We then state and prove Theorem 4.9, which includes Propositions 1.3 and 1.4 as special cases, and conclude with Corollary 4.10.

Proof of Theorem 1.1. Let $V_1,\ldots,V_N$ be the connected components of V. Suppose $g\in\mathbb{Z}^{\ge0}$, A is as in (1.24), and $\kappa_\#$ is a Φ-admissible input for the GW-invariants of $X\#_V Y$ such that the corresponding GW-invariant is nonzero,

$$\mathrm{GW}^{X\#_V Y}_{g,A}(\kappa_\#)\neq 0. \tag{4.26}$$

In light of (1.21), (4.26) implies that there exist an element (A_X,A_Y) of (4.2) and relative contact vectors $\mathbf{s}_1\in\mathbb{Z}^{\ell_1},\ldots,\mathbf{s}_N\in\mathbb{Z}^{\ell_N}$ such that

$$A\in A_X\#_V A_Y, \qquad \widetilde{\mathrm{GW}}^{X\cup_V Y}_{g,(A_X,A_Y);\mathbf{s}_1\ldots\mathbf{s}_N}\big(\kappa_\#;\mathrm{PD}^{V,A}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta\big)\neq 0, \tag{4.27}$$

where $\widetilde{\mathrm{GW}}$ is an IP-count for $X\cup_V Y$ as described in Section 1.2.

By (4.27) and (1.24), $\ell_r\neq 0$ for some $r=1,\ldots,N$. Reordering the components of $V_1,\ldots,V_N$, we can assume that

$$\ell_r\neq 0 \quad \forall\, r=1,\ldots,N' \qquad\text{and}\qquad \ell_r=0 \quad \forall\, r=N'+1,\ldots,N,$$

for some $N'=1,\ldots,N$. Let W denote the union of the first N' components of V. By (2.17) and (2.6),

$$\widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{Y;\mathbf{s}_1\ldots\mathbf{s}_N}=\frac{\mathcal{R}^V_X}{\mathcal{R}'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}}\times\frac{\mathcal{R}_Y}{\mathcal{R}'^V_{Y;\mathbf{s}_1\ldots\mathbf{s}_N}}\times\widehat{W}'_{X;\mathbf{s}_1\ldots\mathbf{s}_{N'}}\times\widehat{W}'_{Y;\mathbf{s}_1\ldots\mathbf{s}_{N'}}\,. \tag{4.28}$$

Since $H_1(V_1;\mathbb{Z})$ is finitely generated, the index of $\gcd(\mathbf{s}_1)H_1(V_1;\mathbb{Z})$ in $H_1(V_1;\mathbb{Z})$ is finite. If $V\subset X$ is virtually connected, the cokernel of the homomorphism (1.23) with $r=1$ is finite and so the image of $\gcd(\mathbf{s}_1)H_1(V_1;\mathbb{Z})$ under this homomorphism is of a finite index. Thus, the first quotient on the right-hand side of (4.28) is finite if $V\subset X$ is virtually connected.

For each element $[\gamma]$ in the product of the two quotients on the right-hand side of (4.28), let

$$\mathrm{PD}^{V,A;\gamma}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta\in H^*\big(\{[\gamma]\}\times\widehat{W}'_{X;\mathbf{s}_1\ldots\mathbf{s}_{N'}}\times\widehat{W}'_{Y;\mathbf{s}_1\ldots\mathbf{s}_{N'}};\mathbb{Q}\big)\subset H^*\big(\widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{Y;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q}\big)$$

be the cohomology class obtained by restricting $\mathrm{PD}^{V,A}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta$ to the topological component

$$\big\{[\gamma]\big\}\times\widehat{W}'_{X;\mathbf{s}_1\ldots\mathbf{s}_{N'}}\times\widehat{W}'_{Y;\mathbf{s}_1\ldots\mathbf{s}_{N'}}\subset\widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\times\widehat{V}_{Y;\mathbf{s}_1\ldots\mathbf{s}_N}$$

and then extending it by zero over the remaining components. By (4.27),

$$\widetilde{\mathrm{GW}}^{X\cup_V Y}_{g,(A_X,A_Y);\mathbf{s}_1\ldots\mathbf{s}_N}\big(\kappa_\#;\mathrm{PD}^{V,A;\gamma^*}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta\big)\neq 0 \tag{4.29}$$

for some γ^*.

Suppose the homology of $\widehat{V}_X$ is finitely generated. The natural projections

$$\prod_{r=1}^{N}\widehat{V}_r \longrightarrow \prod_{r=1}^{N'}\widehat{V}_r \qquad \text{and} \qquad H_X^V \longrightarrow H_X^W$$

induce covering maps

$$\widehat{V}_X \longrightarrow \widehat{W}_X \times \prod_{r=N'+1}^{N} V_r \longrightarrow \prod_{r=1}^{N} V_r\,.$$

By [4, Theorem 1], the $\mathbb{Q}$-homology of the middle space above is finitely generated; thus, so is $H_*(\widehat{W}_X;\mathbb{Q})$. From [5, Remark 5.3], we then conclude that $H_*(\widehat{W}'_{X;\mathbf{s}_1\ldots\mathbf{s}_{N'}};\mathbb{Q})$ is finitely generated as well. By the Kunneth formula for cohomology [23, Corollary 60.7], this implies that

$$\mathrm{PD}^{V,A;\gamma^*}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta = \sum_{i=1}^{m}\widetilde{\kappa}_{X;i}\otimes\widetilde{\kappa}_{Y;i} \in H^*(\widehat{W}'_{X;\mathbf{s}_1\ldots\mathbf{s}_{N'}}\times\widehat{W}'_{Y;\mathbf{s}_1\ldots\mathbf{s}_{N'}};\mathbb{Q}) \tag{4.30}$$

for some $\widetilde{\kappa}_{X;i}\in H^*(\widehat{W}'_{X;\mathbf{s}_1\ldots\mathbf{s}_{N'}};\mathbb{Q})$ and $\widetilde{\kappa}_{Y;i}\in H^*(\widehat{W}'_{Y;\mathbf{s}_1\ldots\mathbf{s}_{N'}};\mathbb{Q})$. By (4.29) and (4.30),

$$\mathrm{Contr}^{A;\gamma^*}_{\Gamma}\big(\kappa_{\#;X},\kappa_{\#;Y}\big) \equiv \sum_{i=1}^{m}\widetilde{\mathrm{GW}}^{X,V}_{\Gamma}\big(\kappa_{\#;X};\widetilde{\kappa}_{X;i}\big)\,\widetilde{\mathrm{GW}}^{Y,V}_{\Gamma}\big(\kappa_{\#;Y};\widetilde{\kappa}_{Y;i}\big) \neq 0 \tag{4.31}$$

for some absolute insertions $\kappa_{\#;X}$ for X and $\kappa_{\#;Y}$ for Y, with $\widetilde{\mathrm{GW}}^{X,V}_{\Gamma}$ and $\widetilde{\mathrm{GW}}^{Y,V}_{\Gamma}$ denoting the disconnected IP-counts associated with the moduli spaces in (1.14).

Since $\mathcal{R}^V_{X,Y}$ is infinite, $H_1(V;\mathbb{Z})$ contains an infinite cyclic subgroup $G\approx\mathbb{Z}$ on which the quotient projection

$$H_1(V;\mathbb{Z}) \longrightarrow \frac{H_1(V;\mathbb{Z})}{H^V_{X,Y}} \overset{\mathfrak{R}^V_{X,Y}}{\approx} \mathcal{R}^V_{X,Y}$$

is injective. In particular, all classes

$$A-\eta \equiv A-\mathfrak{R}^V_{X,Y}\big([\eta]_{H^V_{X,Y}}\big) \in H_2(X\#_V Y;\mathbb{Z}), \qquad \eta\in G,$$

are distinct. Since the first quotient on the right-hand side in (4.28) is finite, G contains an infinite cyclic subgroup G_0 so that the deck transformation

$$\Theta_\eta\colon \widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \longrightarrow \widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$$

maps each topological component of $\widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$ to itself whenever $\eta\in G_0$. In light of (4.10) and (3.5), this implies that

$$\mathrm{PD}^{V,A-\eta;\gamma^*}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta = \big\{\Theta_\eta\times\mathrm{id}\big\}^*\big(\mathrm{PD}^{V,A;\gamma^*}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}\Delta\big) = \sum_{i=1}^{m}\big(\Theta_\eta^*\widetilde{\kappa}_{X;i}\big)\otimes\widetilde{\kappa}_{Y;i}$$

and that the $(\Gamma,\gamma^*,\kappa_{\#;X},\kappa_{\#;Y})$ contribution to $\mathrm{GW}^{X\#_V Y}_{g,A-\eta}(\kappa_\#)$ is given by

$$\mathrm{Contr}^{A-\eta;\gamma^*}_{\Gamma}\big(\kappa_{\#;X},\kappa_{\#;Y}\big) \equiv \sum_{i=1}^{m}\widetilde{\mathrm{GW}}^{X,V}_{\Gamma}\big(\kappa_{\#;X};\Theta_\eta^*\widetilde{\kappa}_{X;i}\big)\,\widetilde{\mathrm{GW}}^{Y,V}_{\Gamma}\big(\kappa_{\#;Y};\widetilde{\kappa}_{Y;i}\big). \tag{4.32}$$

Since $H^*(\widehat{W}_{X;\mathbf{s}_1\ldots\mathbf{s}_{N'}};\mathbb{Q})$ is finitely generated and $\widetilde{\mathrm{GW}}_\Gamma^{X,V}$ is linear in its (relative) inputs, (4.31) implies that infinitely many of the numbers (4.32) with $\eta\in G_0$ are nonzero (if the dimension of $H^*(\widehat{W}_{X;\mathbf{s}_1\ldots\mathbf{s}_{N'}};\mathbb{Q})$ is d, every set of d consecutive numbers (4.32) contains at least one nonzero number). However, this contradicts Gromov's Compactness, since all classes $A-\eta$ have the same symplectic energy. Therefore, (4.26) cannot hold. $\square$

Remark 4.8. Let W be the union of the first $N'\le N$ connected components of V. The conclusion of Theorem 4.8 remains valid if

$$A\in H_2(X\#_V Y;\mathbb{Z})-\iota_{X-V*}^{X\#_V Y}\big(\mathrm{Eff}_{\omega_X}(X,W)\big)-\iota_{Y-V*}^{X\#_V Y}\big(\mathrm{Eff}_{\omega_Y}(Y,V)\big)$$

and the cokernel of the homomorphism (1.23) is surjective for every $r=1,\ldots,N'$.

Proof of Corollary 1.2. Let

$$A=[\mathbb{T}^2\times\mathrm{pt}]\in H_2(\mathbb{T}^2\times F).$$

The moduli space of genus 1 degree A stable morphisms for a product almost complex structure on $\mathbb{T}^2\times F$ and its obstruction bundle are described by

$$\overline{\mathfrak{M}}_1(\mathbb{T}^2\times F;A)\approx F \qquad\text{and}\qquad \mathrm{Obs}\approx\mathcal{H}_{\mathbb{T}^2}^{0,1}\otimes TF,$$

where $\mathcal{H}_{\mathbb{T}^2}^{0,1}$ is the space of harmonic $(0,1)$-forms on $\mathbb{T}^2$. Thus, the genus 1 degree A GW-invariant of $\mathbb{T}^2\times F$ is

$$\mathrm{GW}_{1,A}^{\mathbb{T}^2\times F}()=\big\langle TF,F\big\rangle=\chi(F). \tag{4.33}$$

On the other hand, $\mathbb{T}^2\times F$ is the symplectic sum of $\mathbb{P}^1\times F$ with itself along $V=\{0,\infty\}\times F$ with respect to the canonical isomorphism (1.11) and

$$A\notin\iota_{X-V*}^{X\#_V Y}\big(H_2(X-V;\mathbb{Z})\big)+\iota_{Y-V*}^{X\#_V Y}\big(H_2(Y-V;\mathbb{Z})\big),$$

where X,Y are the two copies of $\mathbb{P}^1\times F$. By [5, Example 3.6], the homomorphism (1.23) is surjective for $r=1,2$ and so $V\subset X$ is virtually connected. By [5, Example 6.2],

$$\widehat{V}_X=\widehat{F}\times\widehat{F}/H_1(V;\mathbb{Z}),\qquad \big(\widehat{x}_1,\widehat{x}_2\big)\sim\big(\gamma\cdot\widehat{x}_1,\gamma\cdot\widehat{x}_2\big),$$

where $\widehat{F}\longrightarrow F$ is the maximal abelian cover. By Serre's Spectral Sequence (e.g. Theorem 9.2.1, 9.2.17, or 9.3.1 in [26] applied with $\mathbb{Z}_2$-coefficients in the last two cases) for the fiber bundle

$$\widehat{F}\longrightarrow\widehat{V}_X\longrightarrow F,$$

the $\mathbb{Q}$-cohomology of $\widehat{V}_X$ is finitely generated if this is the case for $\mathbb{Q}$-cohomology of $\widehat{F}$. By [5, Example 4.7],

$$\mathcal{R}_{X,Y}^V\approx H_1(F;\mathbb{Z});$$

thus, $\mathcal{R}_{X,Y}^V$ is infinite if $H_1(F;\mathbb{Q})\neq0$. Combining these observations with Theorem 1.1, we conclude that

$$\mathrm{GW}_{1,A}^{\mathbb{T}^2\times F}()=0$$

if $H_1(F;\mathbb{Q})\neq0$ and $H_*(\widehat{F};\mathbb{Q})$ is finitely generated. Comparing with (4.33), we conclude that the latter is not the case if $\chi(F)\neq0$. $\square$

We next turn to the assertions made in Section 1.4. With X, Y, V, and Φ as in Propositions 1.3 and 1.4, let

$$\mathrm{Flux}(V_r)_{X,Y} \subset \mathcal{R}_{X,Y}^V \subset H_2(X\#_V Y;\mathbb{Z})$$

be as in the $\mathfrak{c}=2$ case of (4.24).

Theorem 4.9. *Let (X,ω_X) and (Y,ω_Y) be compact symplectic manifolds, $V\subset X,Y$ be a common compact symplectic divisor with topological components $V_1,\ldots,V_N$, and Φ be an isomorphism of complex line bundles as in (1.11). Suppose $\mathcal{N}_X V_r\approx V_r\times\mathbb{C}$ for all $r=1,\ldots,N'$ and some $N'\le N$. If*

$$A_1,A_2\in H_2(X\#_V Y;\mathbb{Z}) \qquad \textit{and} \qquad A_1-A_2\in\bigoplus_{r=1}^{N'}|A_1|_{V_r}\mathrm{Flux}(V_r)_{X,Y}, \tag{4.34}$$

then the GW-invariants of $X\#_V Y$ of degrees A_1 and A_2 with Φ-admissible inputs are the same.

Proof. By (1.21), $\mathrm{GW}_{g,A_1}^{X\#_\varphi Y}(\kappa_\#)$ and $\mathrm{GW}_{g,A_2}^{X\#_\varphi Y}(\kappa_\#)$ are sums of the IP-counts for $X\cup_V Y$ of the form

$$\widetilde{\mathrm{GW}}_{g,(A_X,A_Y);\mathbf{s}_1\ldots\mathbf{s}_N}^{X\cup_V Y}\big(\kappa_\#;\mathrm{PD}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}^{V,A_1}\Delta\big) \;\text{ and }\; \widetilde{\mathrm{GW}}_{g,(A_X,A_Y);\mathbf{s}_1\ldots\mathbf{s}_N}^{X\cup_V Y}\big(\kappa_\#;\mathrm{PD}_{X,Y;\mathbf{s}_1\ldots\mathbf{s}_N}^{V,A_2}\Delta\big), \tag{4.35}$$

respectively. These sums are taken over all (A_X,A_Y) in (4.2) and $\mathbf{s}_1\in\mathbb{Z}^{\ell_1},\ldots,\mathbf{s}_N\in\mathbb{Z}^{\ell_N}$ such that

$$A_1,A_2\in A_X\#_V A_Y, \quad |\mathbf{s}_r|=A_X\cdot_X V=A_Y\cdot_Y V=|A_1|_{V_r}=|A_2|_{V_r} \;\;\forall\, r=1,\ldots,N'. \tag{4.36}$$

By the second assumptions in (4.34) and (4.36), the second assumption in (4.25) is satisfied. By Corollary 4.7, the two numbers in (4.35) are thus the same for all relevant (A_X,A_Y) and $\mathbf{s}_1\in\mathbb{Z}^{\ell_1},\ldots,\mathbf{s}_N\in\mathbb{Z}^{\ell_N}$. □

Under the assumptions of Proposition 1.3, $N,N'=1$ and $\mathrm{Flux}(V_1)_{X,Y}=\mathcal{R}_{X,Y}^V$. Thus, the second condition in (4.34) reduces to the second condition in (1.26) in this case. Under the assumptions of Proposition 1.4,

$$|A_1|_{V_r}\mathrm{Flux}(V_r)_{X,Y}=\mathrm{Flux}(V_r)_{X,Y} \;\;\forall\, r=1,\ldots,N, \qquad \bigoplus_{r=1}^{N}\mathrm{Flux}(V_r)_{X,Y}=\mathcal{R}_{X,Y}^V.$$

Thus, the second condition in (4.34) reduces to the second condition in (1.27) in this case. Combining Theorem 4.9 with Gromov's Compactness, we obtain the following statement.

Corollary 4.10. *Let (X,ω_X) and (Y,ω_Y) be compact symplectic manifolds, $V\subset X,Y$ be a common compact symplectic divisor with topological components $V_1,\ldots,V_N$, and Φ be an isomorphism of complex line bundles as in (1.11). Suppose $\mathcal{N}_X V_{r^*}\approx V_{r^*}\times\mathbb{C}$ and $\mathrm{Flux}(V_{r^*})_{X,Y}$ is infinite for some $r^*=1,\ldots,N$. If $A\in H_2(X\#_V Y;\mathbb{Z})$ is such that $|A|_{V_{r^*}}\neq 0$, then all degree A GW-invariants of $X\#_V Y$ with Φ-admissible inputs vanish.*

5 The convolution product on covers

In Section 5.1, we describe a convolution-like operation on abelian covers that takes a product of covers for $V'\cup V$ and for $V\cup V''$ to a cover for $V'\cup V''$. In the symplectic sum context, this operation takes a product of rim tori covers for $(X,V'\cup V)$ and $(Y,V\cup V'')$ to a rim tori cover for $(X\#_V Y,V'\cup V'')$; see Section 5.2. This operation is needed to make sense of [12, (10.8)], which expresses GW-invariants of $(X\#_V Y,V'\cup V'')$ in terms of GW-invariants of $(X,V'\cup V)$ and $(Y,V\cup V'')$. Throughout Sections 5.1 and 5.2, we let $H_1(V)=H_1(V;\mathbb{Z})$ for any topological space V. We continue with the notation of Section 2.

5.1 Abelian covers

Let V, V', V'' be topological spaces,

$$W = V' \sqcup V'', \qquad W' = V' \sqcup V, \qquad W'' = V \sqcup V'',$$

and H_Δ be the diagonal submodule of $H_1(V) \oplus H_1(V)$. Denote by

$$\iota\colon H_1(V') \oplus H_1(V'') \longrightarrow H_1(V') \oplus H_1(V) \oplus H_1(V) \oplus H_1(V''),$$
$$\pi_V\colon H_1(V') \oplus H_1(V) \longrightarrow H_1(V), \quad \pi_V\colon H_1(V) \oplus H_1(V'') \longrightarrow H_1(V),$$
$$\pi_V\colon H_1(V') \oplus H_1(V) \oplus H_1(V) \oplus H_1(V'') \longrightarrow H_1(V) \oplus H_1(V)$$

the canonical inclusion and projections.

Fix submodules

$$H_1, H_2 \subset H_1(V), \quad \overset{\bullet}{H}_{12} \subset H_1(V) \oplus H_1(V), \quad \overset{\circ}{H}_{12} \subset H_1(V') \oplus H_1(V''),$$
$$\widetilde{H}_1 \subset H_1(V') \oplus H_1(V), \quad \widetilde{H}_2 \subset H_1(V) \oplus H_1(V''), \quad \widetilde{H}_{12} \subset H_1(V') \oplus H_1(V) \oplus H_1(V) \oplus H_1(V''),$$

such that

$$H_1 \oplus H_2, H_\Delta \subset \overset{\bullet}{H}_{12}, \quad \widetilde{H}_1 \oplus \widetilde{H}_2, 0 \oplus H_\Delta \oplus 0 \subset \widetilde{H}_{12}, \tag{5.1}$$
$$\iota(\overset{\circ}{H}_{12}) \supset \widetilde{H}_{12} \cap \ker \pi_V, \quad \pi_V(\widetilde{H}_1) \subset H_1, \quad \pi_V(\widetilde{H}_2) \subset H_2, \quad \pi_V(\widetilde{H}_{12}) \supset \overset{\bullet}{H}_{12}. \tag{5.2}$$

Choose a collection of representatives

$$\{\overset{\bullet}{\gamma}_j\} \subset H_1(V) \oplus H_1(V) \tag{5.3}$$

for the cosets of $\overset{\bullet}{H}_{12}$.

Let $V_1, \ldots, V_N$, $V'_1, \ldots, V'_{N'}$, and $V''_1, \ldots, V''_{N''}$ be the topological components of V, V', and V'', respectively, and

$$\mathbf{s} \equiv (s_{r;i})_{i \le \ell_r, r \le N} \in \prod_{r=1}^{N} \mathbb{Z}_\pm^{\ell_r}, \quad \mathbf{s}' \equiv (s'_{r;i})_{i \le \ell'_r, r \le N'} \in \prod_{r=1}^{N'} \mathbb{Z}_\pm^{\ell'_r}, \quad \mathbf{s}'' \equiv (s''_{r;i})_{i \le \ell''_r, r \le N''} \in \prod_{r=1}^{N''} \mathbb{Z}_\pm^{\ell''_r}. \tag{5.4}$$

Denote by

$$\pi_{\mathbf{s}}\colon W'_{\mathbf{s}'\mathbf{s}} \equiv V'_{\mathbf{s}'} \times V_{\mathbf{s}} \longrightarrow V_{\mathbf{s}} \qquad \text{and} \qquad \pi_{\mathbf{s}}\colon W''_{\mathbf{s}\mathbf{s}''} \equiv V_{\mathbf{s}} \times V''_{\mathbf{s}''} \longrightarrow V_{\mathbf{s}}$$

the projection maps. Let

$$\widehat{W}_{\widetilde{H}_1, \widetilde{H}_2; \mathbf{s}'\mathbf{s}\mathbf{s}''} = \widehat{W'}_{\widetilde{H}_1; \mathbf{s}'\mathbf{s}} \times_{V_{\mathbf{s}}} \widehat{W''}_{\widetilde{H}_2; \mathbf{s}\mathbf{s}''} \equiv \big\{\pi_{\mathbf{s}} \circ \pi_{\widetilde{H}_1; \mathbf{s}'\mathbf{s}} \times \pi_{\mathbf{s}} \circ \pi_{\widetilde{H}_2; \mathbf{s}\mathbf{s}''}\big\}^{-1}\big(\Delta_{\mathbf{s}}^V\big).$$

We will describe a continuous map

$$\Xi_{\widetilde{H}_1, \widetilde{H}_2}^{H_{12}, \widetilde{H}_{12}}\colon \widehat{W}_{\widetilde{H}_1, \widetilde{H}_2; \mathbf{s}'\mathbf{s}\mathbf{s}''} \longrightarrow \widehat{W}_{\overset{\circ}{H}_{12}; \mathbf{s}'\mathbf{s}''} \tag{5.5}$$

so that the diagram in Figure 3 commutes.

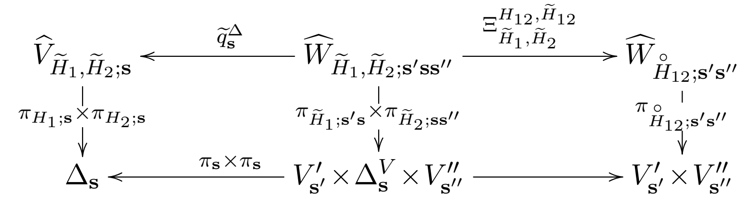

Figure 3: The convolution product and related morphisms.

Choose collections

$$\{(\gamma'_{j_1},\gamma_{1;j_1})\}\subset H_1(V')\oplus H_1(V),\quad \{(\gamma_{2;j_2},\gamma''_{j_2})\}\subset H_1(V)\oplus H_1(V''), \tag{5.6}$$
$$\big\{\mathring{\gamma}_j\big\}\subset H_1(V')\oplus H_1(V'') \tag{5.7}$$

of representatives for the elements of

$$\frac{\mathcal{R}_{\widetilde{H}_1}}{\mathcal{R}'_{\widetilde{H}_1;\mathbf{s}'\mathbf{s}}},\qquad \frac{\mathcal{R}_{\widetilde{H}_2}}{\mathcal{R}'_{\widetilde{H}_2;\mathbf{s}\mathbf{s}''}},\qquad \frac{\mathcal{R}_{\mathring{H}_{12}}}{\mathcal{R}'_{\mathring{H}_{12};\mathbf{s}'\mathbf{s}''}},$$

respectively. Suppose

$$\mathbf{x}\equiv\big(\big([\gamma'_{j_1},\gamma_{1;j_1}]_{\widetilde{H}_1;\mathbf{s}'\mathbf{s}},[\widehat{x}',\gamma\cdot\widehat{x}]_{\widetilde{H}_1}\big),\big([\gamma_{2;j_2},\gamma''_{j_2}]_{\widetilde{H}_2;\mathbf{s}\mathbf{s}''},[\widehat{x},\widehat{x}'']_{\widetilde{H}_2}\big)\big)$$

for some $\gamma\in H_1(V_{\mathbf{s}})$. Let $\overset{\bullet}{h}\in\overset{\bullet}{H}_{12}$ and $\overset{\bullet}{\gamma}$ be a coset representative from the collection in (5.3) such that

$$\big(\gamma_{1;j_1},\gamma_{2;j_2}\big)+\big(\Phi_{V;\mathbf{s}}(\gamma),0\big)=\overset{\bullet}{\gamma}+\overset{\bullet}{h}\,. \tag{5.8}$$

By the last assumption in (5.2),

$$\big(h',\overset{\bullet}{h},h''\big)\in\widetilde{H}_{12} \tag{5.9}$$

for some (h',h''). Let $(\gamma',\gamma'')\in H_1(V'_{\mathbf{s}'})\oplus H_1(V''_{\mathbf{s}''})$, $\mathring{h}\in\mathring{H}_{12}$, and $\mathring{\gamma}$ be a coset representative from the collection in (5.7) such that

$$\big(\gamma'_{j_1}-h',\gamma''_{j_2}-h''\big)=\big(\Phi_{V';\mathbf{s}'}(\gamma'),\Phi_{V'';\mathbf{s}''}(\gamma'')\big)+\mathring{\gamma}+\mathring{h}\,. \tag{5.10}$$

We set

$$\Xi^{\overset{\bullet}{H}_{12},\widetilde{H}_{12}}_{\widetilde{H}_1,\widetilde{H}_2}(\mathbf{x})=\big([\mathring{\gamma}]_{\mathring{H}_{12};\mathbf{s}'\mathbf{s}''},[\gamma'\cdot\widehat{x}',\gamma''\cdot\widehat{x}'']_{\mathring{H}_{12}}\big). \tag{5.11}$$

Below we show that the right-hand side of the above expression depends only on $\mathbf{x}$.

With the tuple (5.9) fixed, (γ',γ'') is defined by (5.10) up to an element of $(\mathring{H}_{12})_{\mathbf{s}'\mathbf{s}''}$; such an element has no effect on the right-hand side of (5.11). By the first assumption in (5.2), (5.9) is determined by the left-hand side in (5.8) up to an element of $\mathring{H}_{12}$; such an element has no effect on (γ',γ'') in (5.10).

Suppose $(\alpha,\beta)\in(\widetilde{H}_1)_{\mathbf{s}'\mathbf{s}}$ and so

$$\mathbf{x}=\big(\big([\gamma'_{1;j_1},\gamma_{1;j_1}]_{\widetilde{H}_1;\mathbf{s}'\mathbf{s}},[\alpha\cdot\widehat{x}',\beta\gamma\cdot\widehat{x}]_{\widetilde{H}_1}\big),\big([\gamma_{2;j_2},\gamma''_{2;j_2}]_{\widetilde{H}_2;\mathbf{s}\mathbf{s}''},[\widehat{x},\widehat{x}'']_{\widetilde{H}_2}\big)\big).$$

By the second assumption in (5.2), $\Phi_{V;\mathbf{s}}(\beta)\in H_1$. By the first inclusion in the first assumption in (5.1), this change thus adds $(\Phi_{V;\mathbf{s}}(\beta),0)$ to $\overset{\bullet}{h}$ in (5.8). By the first inclusion in the second assumption in (5.1), it adds $\Phi_{V';\mathbf{s}'}(\alpha)$ to h' in (5.9) and subtracts α from γ' in (5.10). Thus, (5.11) becomes

$$\Xi^{\overset{\bullet}{H}_{12},\widetilde{H}_{12}}_{\widetilde{H}_1,\widetilde{H}_2}(\mathbf{x}) = \big([\overset{\circ}{\gamma}]_{\overset{\circ}{H}_{12};\mathbf{s}'\mathbf{s}''},[\gamma'\alpha^{-1}\cdot(\alpha\cdot\widehat{x}'),\gamma''\cdot\widehat{x}'']_{\overset{\circ}{H}_{12}}\big),$$

which agrees with (5.11).

Suppose $(\alpha,\beta)\in(\widetilde{H}_2)_{\mathbf{s}\mathbf{s}''}$ and so

$$\mathbf{x} = \big(([\gamma'_{1;j_1},\gamma_{1;j_1}]_{\widetilde{H}_1;\mathbf{s}'\mathbf{s}},[\widehat{x}',\gamma\cdot\widehat{x}]_{\widetilde{H}_1}\big),\big([\gamma_{2;j_2},\gamma''_{2;j_2}]_{\widetilde{H}_2;\mathbf{s}\mathbf{s}''},[\alpha\cdot\widehat{x},\beta\cdot\widehat{x}'']_{\widetilde{H}_2})\big).$$

By the third assumption in (5.2), $\Phi_{V;\mathbf{s}}(\alpha)\in H_2$. By the first assumption in (5.1), this change thus subtracts $(\Phi_{V;\mathbf{s}}(\alpha),0)$ from $\overset{\bullet}{h}$ in (5.8). By the second assumption in (5.1), it adds $\Phi_{V'';\mathbf{s}''}(\beta)$ to h'' in (5.9) and subtracts β from γ'' in (5.10). Thus, (5.11) becomes

$$\Xi^{\overset{\bullet}{H}_{12},\widetilde{H}_{12}}_{\widetilde{H}_1,\widetilde{H}_2}(\mathbf{x}) = \big([\overset{\circ}{\gamma}]_{\overset{\circ}{H}_{12};\mathbf{s}'\mathbf{s}''},[\gamma'\cdot\widehat{x}',\gamma''\beta^{-1}\cdot(\beta\cdot\widehat{x}'')]_{\overset{\circ}{H}_{12}}\big),$$

which agrees with (5.11). It follows that the right-hand side of (5.11) depends only on $\mathbf{x}$.

Let $H_{12}\subset H_1(V)$ denote the image of $\overset{\bullet}{H}_{12}$ under the homomorphism

$$H_1(V)\oplus H_1(V)\longrightarrow H_1(V), \qquad (\gamma_1,\gamma_2)\longrightarrow\gamma_1-\gamma_2.$$

Thus, the homomorphism

$$\Delta_{H_{12},\overset{\bullet}{H}_{12}}:\frac{H_1(V)\oplus H_1(V)}{\overset{\bullet}{H}_{12}}\longrightarrow\mathcal{R}_{H_{12}}\equiv\frac{H_1(V)}{H_{12}}, \quad \Delta_{H_{12},\overset{\bullet}{H}_{12}}\big([\gamma_1,\gamma_2]_{\overset{\bullet}{H}_{12}}\big) = \big[\gamma_1-\gamma_2\big]_{H_{12}}, \tag{5.12}$$

is well-defined and is an isomorphism. By the middle two assumptions in (5.2), there is a natural projection map

$$\widetilde{\pi}_{\mathbf{s}}\times\widetilde{\pi}_{\mathbf{s}}:\widehat{W'}_{\widetilde{H}_1;\mathbf{s}'\mathbf{s}}\times\widehat{W''}_{\widetilde{H}_2;\mathbf{s}\mathbf{s}''}\longrightarrow\widehat{V}_{H_1;\mathbf{s}}\times\widehat{V}_{H_2;\mathbf{s}},$$

which restricts to a map

$$\widetilde{q}^{\Delta}_{\mathbf{s}}:\widehat{W}_{\widetilde{H}_1,\widetilde{H}_2;\mathbf{s}'\mathbf{s}\mathbf{s}''}\longrightarrow\widehat{V}_{H_1,H_2;\mathbf{s}};$$

see [5, (5.11)]. If the (unparametrized) collections $\{\gamma_{1;j_1}\},\{\gamma_{2;j_2}\}\subset H_1(V)$ obtained from (5.6) are the same as the collections (3.3) used in the construction of (3.4), then

$$\widetilde{\Psi}^{H_{12}}_{H_1,H_2}\big(\widetilde{q}^{\Delta}_{\mathbf{s}}(\mathbf{x})\big) = \Delta_{H_{12},\overset{\bullet}{H}_{12}}\big([\overset{\bullet}{\gamma}]_{\overset{\bullet}{H}_{12}}\big)\in\mathcal{R}_{H_{12}},$$

with $\overset{\bullet}{\gamma}\equiv\overset{\bullet}{\gamma}(\mathbf{x})$ given by (5.8).

The collections (5.6) can be chosen so that the induced collections $\{\gamma_{1;j_1}\}$ and $\{\gamma_{2;j_2}\}$ contain precisely one representative for each coset in $\mathcal{R}_{H_1}/\mathcal{R}'_{H_1;\mathbf{s}}$ and $\mathcal{R}_{H_2}/\mathcal{R}'_{H_2;\mathbf{s}}$, respectively, if and only if the inclusions in the two middle conditions in (5.2) are equalities. In such a case,

$$\widehat{W}^{\widetilde{\eta}}_{\widetilde{H}_1,\widetilde{H}_2;\mathbf{s}'\mathbf{s}\mathbf{s}''}\equiv\big\{\widetilde{q}^{\Delta}_{\mathbf{s}}\big\}^{-1}\big(\{\widetilde{\Psi}^{H_{12}}_{H_1,H_2}\}^{-1}(\widetilde{\eta})\big)\subset\widehat{W'}_{\widetilde{H}_1;\mathbf{s}'\mathbf{s}}\times\widehat{W''}_{\widetilde{H}_2;\mathbf{s}\mathbf{s}''} \tag{5.13}$$

is a closed subset for every $\widetilde{\eta}\in\mathcal{R}_{H_{12}}$. If in addition V,V',V'' are oriented manifolds, then so is the subset (5.13). The cohomology class determined by this submanifold as in Section 3.1 satisfies

$$\mathrm{PD}^{\widetilde{\eta}}_{\widetilde{H}_1,\widetilde{H}_2;\mathbf{s}'\mathbf{s}\mathbf{s}''}\Delta=\big\{\widetilde{\pi}_{\mathbf{s}}\times\widetilde{\pi}_{\mathbf{s}}\big\}^*\big(\mathrm{PD}^{\widetilde{\eta}}_{H_1,H_2;\mathbf{s}}\Delta\big)\in H^*\big(\widehat{W'}_{\widetilde{H}_1;\mathbf{s}'\mathbf{s}}\times\widehat{W''}_{\widetilde{H}_2;\mathbf{s}\mathbf{s}''};\mathbb{Q}\big),\tag{5.14}$$

with $\mathrm{PD}^{\widetilde{\eta}}_{H_1,H_2;\mathbf{s}}\Delta$ as in (3.9).

Example 5.1. Let F be a connected topological space, $V,V''=F$, $V'=\emptyset$,

$$H_1,\mathring{H}_{12},\widetilde{H}_1=0,\quad H_2=H_1(V),\quad \overset{\bullet}{H}_{12}=H_1(V)\oplus H_1(V),\quad \widetilde{H}_2=H_\Delta\subset H_1(V)\oplus H_1(V''),$$
$$\widetilde{H}_{12}=\big\{(0,\alpha,\alpha+\beta,\beta)\colon\ \alpha,\beta\in H_1(F)\big\}\subset H_1(V')\oplus H_1(V)\oplus H_1(V)\oplus H_1(V'').$$

In this case, the collection (5.3) consists of a single element, which we can take $\overset{\bullet}{\gamma}=0$. Let $\mathbf{s}_1\in\mathbb{Z}^{\ell_1}_{\pm}$, $\mathbf{s}_2\in\mathbb{Z}^{\ell_2}_{\pm}$, and

$$\{\gamma_{1;j_1}\},\{\gamma_{2;j_2}\},\{\gamma_{12;j_{12}}\}\subset H_1(F)\tag{5.15}$$

be collections of representatives for the cosets of $\gcd(\mathbf{s}_1)H_1(F)$, $\gcd(\mathbf{s}_2)H_1(F)$, and $\gcd(\mathbf{s}_1,\mathbf{s}_2)H_1(F)$, respectively. With identifications as in Example 3.3, the map (5.11) becomes

$$\frac{H_1(F)}{\gcd(\mathbf{s}_1)H_1(F)}\times\frac{H_1(F)}{\gcd(\mathbf{s}_1,\mathbf{s}_2)H_1(F)}\times\widehat{F}'_{0;\mathbf{s}_1}\times_{F_{\mathbf{s}_1}}\widehat{F}'_{0;\mathbf{s}_1(-\mathbf{s}_2)}\longrightarrow\frac{H_1(F)}{\gcd(\mathbf{s}_2)H_1(F)}\times\widehat{F}'_{0;\mathbf{s}_2},$$
$$\big([\gamma_{1;j_1}]_{0;\gcd(\mathbf{s}_1)},[\gamma_{12;j_{12}}]_{0;\gcd(\mathbf{s}_1\mathbf{s}_2)},[\gamma\cdot\widehat{x}]_0,[\widehat{x},\widehat{x}'']_0\big)\longrightarrow\big([\gamma_{2;j_2}]_{0;\gcd(\mathbf{s}_2)},[\gamma''\cdot\widehat{x}'']_0\big),$$

where $\gamma''\in H_1(F_{\mathbf{s}_2})$ and $\gamma_{2;j_2}$ is an element of the second collection in (5.15) such that

$$\Phi_{F;\mathbf{s}_1}(\gamma)+\gamma_{1;j_1}-\gamma_{12;j_{12}}=\Phi_{F;\mathbf{s}_2}(\gamma'')+\gamma_{2;j_2}\in H_1(F).$$

Example 5.2. Suppose in addition that $F=\mathbb{T}^2$ and $\ell_1,\ell_2\ge1$. With the identifications as in Examples 2.1 and 3.1 and in the second half of Example 3.3,

$$\begin{split}&\widehat{F}_{0;\mathbf{s}_1}\times_{F_{\mathbf{s}_1}}\widehat{F}_{0;\mathbf{s}_1(-\mathbf{s}_2)}\\&\quad=\big\{\big(x,[z_i+s_{1;i}^{-1}x]_{i\le\ell_1},y,\big([z_i+s_{1;i}^{-1}y]_{i\le\ell_1},[z_{2;i}-s_{2;i}^{-1}y]_{i\le\ell_2}\big)\big)\in\mathbb{C}\times\mathbb{T}^{2(\ell_1-1)}_{\mathbf{s}_1}\times\mathbb{C}\times\mathbb{T}^{2(\ell-1)}_{\mathbf{s}_1(-\mathbf{s}_2)}\big\},\end{split}$$

where $\ell=\ell_1+\ell_2$. The map (5.11) becomes

$$\begin{split}&\big(x,[z_i+s_{1;i}^{-1}x]_{i\le\ell_1},y,\big([z_i+s_{1;i}^{-1}y]_{i\le\ell_1},[z_{2;i}-s_{2;i}^{-1}y]_{i\le\ell_2}\big)\big)\\&\qquad\longrightarrow\Big(\ell_2^{-1}\big(\ell_1x-\ell y\big),\big[z_{2;i}+\ell_2^{-1}s_{2;i}^{-1}\big(\ell_1x-\ell y\big)\big]_{i\le\ell_2}\Big)\in\mathbb{C}\times\mathbb{T}^{2(\ell_2-1)}_{\mathbf{s}_2}.\end{split}\tag{5.16}$$

By (5.14),

$$\mathrm{PD}^0_{0,H_\Delta;()\mathbf{s}_1\mathbf{s}_2}\Delta=\big\{\mathrm{id}\times\widetilde{\pi}_{\mathbf{s}_1}\big\}^*\big(\mathrm{PD}^0_{0,H;\mathbf{s}_1}\Delta\big)\in H^*\big(\mathbb{C}\times\mathbb{T}^{2(\ell_1-1)}_{\mathbf{s}_1}\times\mathbb{C}\times\mathbb{T}^{2(\ell-1)}_{\mathbf{s}_1(-\mathbf{s}_2)};\mathbb{Q}\big),\tag{5.17}$$

where $H=H_1(F)$,

$$\widetilde{\pi}_{\mathbf{s}_1}\colon\mathbb{C}\times\mathbb{T}^{2(\ell-1)}_{\mathbf{s}_1(-\mathbf{s}_2)}\longrightarrow\mathbb{T}^{2\ell_1},\qquad\widetilde{\pi}_{\mathbf{s}_1}\big(y,[z_{1;i}]_{i\le\ell_1},[z_{2;i}]_{i\le\ell_2}\big)=\big[z_{1;i}-s_{1;i}^{-1}y\big]_{i\le\ell_1},$$

and

$$\mathrm{PD}^0_{0,H;\mathbf{s}_1}\Delta\in H^*\big(\mathbb{C}\times\mathbb{T}^{2(\ell_1-1)}_{\mathbf{s}_1}\times\mathbb{T}^{2\ell_1};\mathbb{Q}\big)$$

is described by (3.12).

5.2 Rim tori covers

We begin this section by defining the rim tori convolution [12, (10.8)] as a special case of the continuous map (5.5); see (5.24). We then show that its composition with the lifted evaluation morphisms (2.21) for the input divisors is compatible with lifted evaluation morphisms for the output divisor, provided the relevant coset representatives are chosen consistently; see Proposition 5.3 and Figure 4. We continue with the notation introduced at the beginning of Section 4.1.

Throughout this section, X and Y denote compact oriented manifolds, $V,V'\subset X$ and $V,V''\subset Y$ are compact oriented disjoint submanifolds of codimension $\mathfrak{c}$, and

$$W=V'\sqcup V'', \qquad W'=V'\sqcup V, \qquad W''=V\sqcup V''.$$

Let $\varphi\colon S_XV\longrightarrow S_YV$ be an orientation-reversing diffeomorphism commuting with the projections to V and $X\#_\varphi Y$ be the manifold obtained by gluing the complements of tubular neighborhoods of V in X and Y by φ along their common boundary. Define

$$\widetilde{\mathcal{R}}^W_{X\#_\varphi Y}=\ker\big\{q_{\varphi*}\circ\iota^{X\#_\varphi Y}_{X\#_\varphi Y-W*}\colon H_{\mathfrak{c}}(X\#_\varphi Y-W;\mathbb{Z})\longrightarrow H_{\mathfrak{c}}(X\cup_V Y;\mathbb{Z})\big\},$$

where $q_\varphi\colon X\#_\varphi Y\longrightarrow X\cup_V Y$ is a smooth map obtained by collapsing circles in the boundaries of the two complements; see [5, Section 2.2].

With $H_{\mathfrak{c}}(SV;\mathbb{Z})_{X,Y}$ given by (4.1), denote by $\dot{H}^{SV}_{X,Y}$ and $\widetilde{H}^{SV}_{X,Y}$ the images of the homomorphisms

$$H_{\mathfrak{c}}(SV;\mathbb{Z})_{X,Y}\xrightarrow{(\iota^{X-V}_{SV*},-\iota^{Y-V}_{SV*})}\mathcal{R}^V_X\oplus\mathcal{R}^V_Y\xrightarrow{\approx}\frac{H_1(V)}{H^V_X}\oplus\frac{H_1(V)}{H^V_Y}\qquad\text{and}$$

$$H_{\mathfrak{c}}(SV;\mathbb{Z})_{X,Y}\xrightarrow{(\iota^{X-W'}_{SV*},-\iota^{Y-W''}_{SV*})}\mathcal{R}^{W'}_X\oplus\mathcal{R}^{W''}_Y\xrightarrow{\approx}\frac{H_1(V')\oplus H_1(V)}{H^{W'}_X}\oplus\frac{H_1(V)\oplus H_1(V'')}{H^{W''}_Y},$$

respectively; the second homomorphisms above are the isomorphisms of [5, Corollary 3.2]. Let $\dot{H}^V_{X,Y}$ and $\widetilde{H}^V_{X,Y}$ be the preimages of $\dot{H}^{SV}_{X,Y}$ and $\widetilde{H}^{SV}_{X,Y}$, respectively, under the quotient projections

$$H_1(V)\oplus H_1(V)\longrightarrow\frac{H_1(V)}{H^V_X}\oplus\frac{H_1(V)}{H^V_Y}\qquad\text{and}$$

$$H_1(V')\oplus H_1(V)\oplus H_1(V)\oplus H_1(V'')\longrightarrow\frac{H_1(V')\oplus H_1(V)}{H^{W'}_X}\oplus\frac{H_1(V)\oplus H_1(V'')}{H^{W''}_Y}.$$

In particular,

$$H^V_X\oplus H^V_Y,H_\Delta\subset\dot{H}^V_{X,Y},\quad H^{W'}_X\oplus H^{W''}_Y,0\oplus H_\Delta\oplus 0\subset\widetilde{H}^V_{X,Y},\quad \pi_V\big(\widetilde{H}^V_{X,Y}\big)=\dot{H}^V_{X,Y}\,. \tag{5.18}$$

By (2.15),

$$\pi_V\big(H^{W'}_X\big)=H^V_X\subset H_1(V),\qquad \pi_V\big(H^{W''}_Y\big)=H^V_Y\subset H_1(V). \tag{5.19}$$

Define

$$\mathring{H}^V_{X,Y}=\widetilde{H}^V_{X,Y}\cap H_1(V')\oplus 0\oplus 0\oplus H_1(V'')\,.$$

Thus, the modules

$$H_1=H^V_X,\quad H_2=H^V_Y,\quad \dot{H}_{12}=\dot{H}^V_{X,Y},\quad \mathring{H}_{12}=\mathring{H}^V_{X,Y},\quad \widetilde{H}_1=H^{W'}_X,\quad \widetilde{H}_2=H^{W''}_Y,\quad \widetilde{H}_{12}=\widetilde{H}^V_{X,Y}$$

satisfy the conditions (5.1) and (5.2); all four inclusions in (5.2) are in fact equalities in this case.

By [5, Proposition 4.9], $\mathring{H}^V_{X,Y}=H^W_{X\#_\varphi Y}$ and there is a natural isomorphism

$$\widetilde{\mathfrak{R}}^V_{X,Y}\colon \frac{H_1(W')\oplus H_1(W'')}{\widetilde{H}^V_{X,Y}} \longrightarrow \widetilde{\mathcal{R}}^W_{X\#_\varphi Y}\subset H_{\mathfrak{c}}(X\#_\varphi Y-W;\mathbb{Z})\,;$$

the $V',V''=\emptyset$ case of this isomorphism is the composition of the right vertical arrow in (4.4) with (5.12). Let

$$\big\{\gamma_{\#;j}\big\}\subset H_1(V)\oplus H_1(V) \tag{5.20}$$

be a collection of coset representatives for

$$\frac{H_1(V)\oplus H_1(V)}{\dot{H}^V_{X,Y}}\approx \mathcal{R}^V_{X,Y}\subset H_{\mathfrak{c}}(X\#_\varphi Y;\mathbb{Z})\,.$$

With $\mathbf{s},\mathbf{s}',\mathbf{s}''$ as in (5.4), let

$$\widehat{W}_{X,Y;\mathbf{s}'\mathbf{s}\mathbf{s}''}=\widehat{W}_{H^{W'}_X,H^{W''}_Y;\mathbf{s}'\mathbf{s}\mathbf{s}''}\,.$$

Choose collections

$$\big\{\gamma_{X;j_1}\big\},\big\{\gamma_{Y;j_2}\big\}\subset H_1(V)\qquad\text{and}\qquad \big\{\mathring{\gamma}_j\big\}\subset H_1(V')\oplus H_1(V'') \tag{5.21}$$

of representatives for the elements of

$$\frac{\mathcal{R}^V_X}{\mathcal{R}'^V_{X;\mathbf{s}}}\approx\frac{\mathcal{R}_{H^V_X}}{\mathcal{R}'_{H^V_X;\mathbf{s}}},\qquad \frac{\mathcal{R}^V_Y}{\mathcal{R}'^V_{Y;\mathbf{s}}}\approx\frac{\mathcal{R}_{H^V_Y}}{\mathcal{R}'_{H^V_Y;\mathbf{s}}},\quad\text{and}\quad \frac{\mathcal{R}^W_{X\#_\varphi Y}}{\mathcal{R}'^W_{X\#_\varphi Y;\mathbf{s}'\mathbf{s}''}}\approx\frac{\mathcal{R}_{\mathring{H}^V_{X,Y}}}{\mathcal{R}'_{\mathring{H}^V_{X,Y};\mathbf{s}'\mathbf{s}''}}.$$

By (5.19),

$$\begin{aligned}\pi_V\big(H^{W'}_X+\operatorname{Im}\Phi_{W';\mathbf{s}'\mathbf{s}}\big)&=H^V_X+\operatorname{Im}\Phi_{V;\mathbf{s}}\subset H_1(V),\\ \pi_V\big(H^{W''}_Y+\operatorname{Im}\Phi_{W'';\mathbf{s}\mathbf{s}''}\big)&=H^V_Y+\operatorname{Im}\Phi_{V;\mathbf{s}}\subset H_1(V).\end{aligned}$$

Thus, there exist collections

$$\big\{(\gamma'_{j_1},\gamma_{X;j_1})\big\}\subset H_1(V')\oplus H_1(V),\quad \big\{(\gamma_{Y;j_2},\gamma''_{j_2})\big\}\subset H_1(V)\oplus H_1(V'') \tag{5.22}$$

of representatives for the elements of

$$\frac{\mathcal{R}^{W'}_X}{\mathcal{R}'^{W'}_{X;\mathbf{s}'\mathbf{s}}}\approx\frac{\mathcal{R}_{H^{W'}_X}}{\mathcal{R}'_{H^{W'}_X;\mathbf{s}'\mathbf{s}}}\qquad\text{and}\qquad \frac{\mathcal{R}^{W''}_Y}{\mathcal{R}'^{W''}_{Y;\mathbf{s}\mathbf{s}''}}\approx\frac{\mathcal{R}_{H^{W''}_Y}}{\mathcal{R}'_{H^{W''}_Y;\mathbf{s}\mathbf{s}''}}$$

so that the corresponding (unparametrized) collections $\{\gamma_{X;j_1}\},\{\gamma_{Y;j_2}\}\subset H_1(V)$ are the same as the first two collections in (5.21). As described in Sections 4.1 and 5.1, the collections (5.20), (5.21), and (5.22) determine smooth maps

$$\widetilde{\Psi}^V_{X,Y}\equiv\mathfrak{R}^V_{X,Y}\circ\widetilde{\Psi}^{H^V_{X,Y}}_{H^V_X,H^V_Y}\colon \widehat{V}_{X,Y;\mathbf{s}}\longrightarrow\mathcal{R}^V_{X,Y}\subset H_{\mathfrak{c}}(X\#_\varphi Y;\mathbb{Z}), \tag{5.23}$$

$$\Xi^{V',V,V''}_{X,Y}\equiv\Xi^{\dot{H}^V_{X,Y},\widetilde{H}^V_{X,Y}}_{H^{W'}_X,H^{W''}_Y}\colon\widehat{W}_{X,Y;\mathbf{s}'\mathbf{s}\mathbf{s}''}\longrightarrow\widehat{W}_{X\#_\varphi Y;\mathbf{s}'\mathbf{s}''}\,. \tag{5.24}$$

We show below that the second map can be compatible with the lifted evaluation morphisms of Proposition 2.2.

Let Σ_X,Σ_Y be compact oriented $\mathfrak{c}$-dimensional manifolds, $k_X,k_Y\in\mathbb{Z}^{\ge0}$, and (A_X,A_Y) be an element of $H_{\mathfrak{c}}(X;\mathbb{Z})\times_V H_{\mathfrak{c}}(Y;\mathbb{Z})$. Denote by

$$\begin{split}\widetilde{\mathrm{ev}}_X^{V'\cup V}\colon\mathfrak{X}^{V',V}_{\Sigma_X,k_X;\mathbf{s}'\mathbf{s}}(X,A_X)\longrightarrow\widehat{W'}_{X;\mathbf{s}'\mathbf{s}},\qquad \widetilde{\mathrm{ev}}_Y^{V\cup V''}\colon\mathfrak{X}^{V,V''}_{\Sigma_Y,k_Y;\mathbf{s}\mathbf{s}''}(Y,A_Y)\longrightarrow\widehat{W''}_{Y;\mathbf{s}\mathbf{s}''},\\ \text{and}\qquad \widetilde{\mathrm{ev}}^{V'\cup V''}_{X\#_\varphi Y}\colon\mathfrak{X}^{V',V''}_{\Sigma_X\#\Sigma_Y;\mathbf{s}'\mathbf{s}''}(X\#_\varphi Y,A_\#)\longrightarrow\widehat{W}_{X\#_\varphi Y;\mathbf{s}'\mathbf{s}''}\end{split}\tag{5.25}$$

the lifted relative evaluation morphisms of Proposition 2.2 for the relative pairs

$$(X,V'\cup V),\qquad (Y,V\cup V''),\quad\text{and}\quad (X\#_\varphi Y,V'\cup V'')$$

compatible with the two collections in (5.22) and the last collection in (5.21). Let

$$\begin{split}\mathfrak{X}^{V',V,V''}_{(\Sigma_X,\Sigma_Y),(k_X,k_Y);\mathbf{s}'\mathbf{s}\mathbf{s}''}\big((X,Y),(A_X,A_Y)\big)&=\big\{\widetilde{\mathrm{ev}}_X^{V'\cup V}\times\widetilde{\mathrm{ev}}_Y^{V\cup V''}\big\}^{-1}\big(\widehat{W}_{X,Y;\mathbf{s}'\mathbf{s}\mathbf{s}''}\big)\\ &=\big\{\pi_{\mathbf{s}}\circ\mathrm{ev}_X^{V'\cup V}\times\pi_{\mathbf{s}}\circ\mathrm{ev}_Y^{V\cup V''}\big\}^{-1}\big(\Delta^V_{\mathbf{s}}\big).\end{split}\tag{5.26}$$

Each element $(\mathbf{f}_X,\mathbf{f}_Y)$ of this space gives rise to a marked map

$$\mathbf{f}_X\#_\varphi\mathbf{f}_Y\in\bigsqcup_{A_\#\in A_X\#_\varphi A_Y}\mathfrak{X}^{V',V''}_{\Sigma_X\#\Sigma_Y,k_X+k_Y;\mathbf{s}'\mathbf{s}''}(X\#_\varphi Y,A_\#);\tag{5.27}$$

see [5, Section 2.2].

Proposition 5.3. *Suppose X, Y, $V\subset X,Y$, φ, and (A_X,A_Y) are as in Proposition 4.2, $V'\subset X-V$ and $V''\subset Y-V$ are compact oriented submanifolds of codimension $\mathfrak{c}$, and $\mathbf{s},\mathbf{s}',\mathbf{s}''$ are as in (5.4). Let $\widetilde{\mathrm{ev}}_X^{V'\cup V}$, $\widetilde{\mathrm{ev}}_Y^{V\cup V''}$, and $\widetilde{\mathrm{ev}}^{V'\cup V''}_{X\#_\varphi Y}$ be the lifted evaluation morphisms as in (5.25) compatible with the two collections in (5.22) and the last collection in (5.21). Then there exists a collection*

$$\{\eta_{\#;j}\}\subset H_1(V')\oplus H_1(V'')$$

indexed as the collection in (5.20) with the following property. If $(\mathbf{f}_X,\mathbf{f}_Y)$ is an element of the fiber product (5.26) such that

$$\widetilde{\Psi}^V_{X,Y}\big(\widetilde{\mathrm{ev}}_X^{V'\cup V}(\mathbf{f}_X),\widetilde{\mathrm{ev}}_Y^{V\cup V''}(\mathbf{f}_Y)\big)=[\gamma_{\#;j}]_{\overset{\bullet}{H}{}^V_{X,Y}}\in\mathcal{R}^V_{X,Y}\,,\tag{5.28}$$

then

$$\Xi^{V',V,V''}_{X,Y}\big(\widetilde{\mathrm{ev}}_X^{V'\cup V}(\mathbf{f}_X),\widetilde{\mathrm{ev}}_Y^{V\cup V''}(\mathbf{f}_Y)\big)=\Theta_{\eta_{\#;j}}\big(\widetilde{\mathrm{ev}}^{V'\cup V''}_{X\#_\varphi Y}(\mathbf{f}_X\#_\varphi\mathbf{f}_Y)\big).\tag{5.29}$$

Proof. Fix base points $x_{\mathbf{s}}\in V_{\mathbf{s}}, x'_{\mathbf{s}'}\in V'_{\mathbf{s}'}, x''_{\mathbf{s}''}\in V''_{\mathbf{s}''}$. Let

$$\begin{split}\mathbf{f}_X^\bullet&\equiv\big(z^\bullet_{X;1},\ldots,z^\bullet_{X;k_X+\ell_1+\ldots+\ell'_N+\ell_N},f_X^\bullet\big)\in\mathfrak{X}^{V',V}_{\Sigma_X,k_X;\mathbf{s}'\mathbf{s}}(X,A_X),\\ \mathbf{f}_Y^\bullet&\equiv\big(z^\bullet_{Y;1},\ldots,z^\bullet_{Y;k_Y+\ell_1+\ldots+\ell_N+\ell''_N},f_Y^\bullet\big)\in\mathfrak{X}^{V,V''}_{\Sigma_Y,k_Y;\mathbf{s}\mathbf{s}''}(Y,A_Y)\end{split}$$

be such that

$$\mathrm{ev}_X^{V'\cup V}(\mathbf{f}_X^\bullet)=\big(x'_{\mathbf{s}'},x_{\mathbf{s}}\big),\qquad \mathrm{ev}_Y^{V\cup V''}(\mathbf{f}_Y^\bullet)=\big(x_{\mathbf{s}},x''_{\mathbf{s}''}\big),\tag{5.30}$$

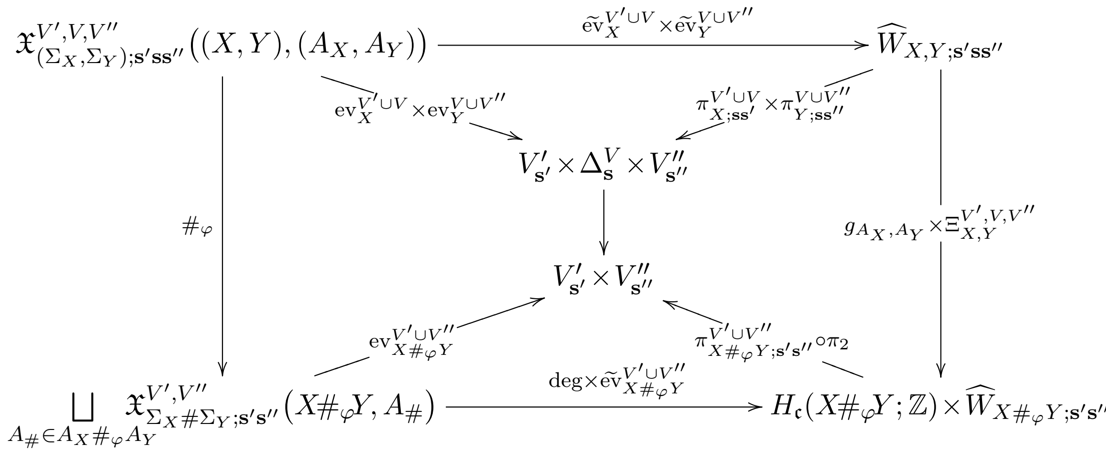

Figure 4: The evaluation and gluing maps of Proposition 5.3; see Remark 5.4 regarding the commutativity of this diagram.

and (5.28) holds for $(\mathbf{f}_X,\mathbf{f}_Y)=(\mathbf{f}_X^\bullet,\mathbf{f}_Y^\bullet)$. We choose $\eta_{\#;j}$ so that (5.29) holds for $(\mathbf{f}_X,\mathbf{f}_Y)=(\mathbf{f}_X^\bullet,\mathbf{f}_Y^\bullet)$. We then show that (5.29) holds for all $(\mathbf{f}_X,\mathbf{f}_Y)$ satisfying (5.30), with $(\mathbf{f}_X^\bullet,\mathbf{f}_Y^\bullet)$ replaced by $(\mathbf{f}_X,\mathbf{f}_Y)$, and (5.28). Since both sides of (5.29) are continuous lifts of the evaluation morphism

$$\mathfrak{X}^{V',V,V''}_{(\Sigma_X,\Sigma_Y);\mathbf{s}'\mathbf{s}\mathbf{s}''}\big((X,Y),(A_X,A_Y)\big)\longrightarrow V'_{\mathbf{s}'}\times V''_{\mathbf{s}''},\quad (\mathbf{f}_X,\mathbf{f}_Y)\longrightarrow\big(\mathrm{ev}_X^{V'}(\mathbf{f}_X),\mathrm{ev}_Y^{V''}(\mathbf{f}_Y)\big),\tag{5.31}$$

over the covering projection

$$\widehat{W}_{X\#_\varphi Y;\mathbf{s}'\mathbf{s}''}\longrightarrow V'_{\mathbf{s}'}\times V''_{\mathbf{s}''}$$

and this morphism is surjective on every topological component of the domain, it follows that (5.29) holds for all $(\mathbf{f}_X,\mathbf{f}_Y)$ satisfying (5.28). If a pair $(\mathbf{f}_X^\bullet,\mathbf{f}_Y^\bullet)$ satisfying (5.30) and (5.28) does not exist, then the value of $\eta_{\#;j}$ does not matter.

By (5.30)

$$\widetilde{\mathrm{ev}}_X^{V'\cup V}(\mathbf{f}_X^\bullet)=\big([\gamma_j'^\bullet,\gamma_{X;j}^\bullet]_{X;\mathbf{s}'\mathbf{s}},[\widehat{x}',\gamma\cdot\widehat{x}]_X\big),\quad \widetilde{\mathrm{ev}}_Y^{V\cup V''}(\mathbf{f}_Y^\bullet)=\big([\gamma_{Y;j}^\bullet,\gamma_j''^\bullet]_{Y;\mathbf{s}\mathbf{s}''},[\widehat{x},\widehat{x}'']_Y\big)\tag{5.32}$$

for some $\gamma\in H_1(V_{\mathbf{s}})$ and elements $(\gamma_j'^\bullet,\gamma_{X;j}^\bullet)$ and $(\gamma_{Y;j}^\bullet,\gamma_j''^\bullet)$ of the two collections in (5.22). Since (5.28) holds for $(\mathbf{f}_X,\mathbf{f}_Y)=(\mathbf{f}_X^\bullet,\mathbf{f}_Y^\bullet)$, there exists $(h'^\bullet,h^\bullet,h''^\bullet)\in\widetilde{H}^V_{X,Y}$ such that

$$h^\bullet\in\mathring{H}^V_{X,Y},\qquad \big(\gamma_{X;j}^\bullet,\gamma_{Y;j}^\bullet\big)+\big(\Phi_{V;\mathbf{s}}(\gamma),0\big)=\gamma_{\#;j}+h^\bullet\,.\tag{5.33}$$

Let $(\gamma'^\bullet,\gamma''^\bullet)\in H_1(V'_{\mathbf{s}'})\oplus H_1(V''_{\mathbf{s}''})$, $\mathring{h}_j\in\mathring{H}^V_{X,Y}$, and $\mathring{\gamma}_j$ be a coset representative from the last collection in (5.21) such that

$$\big(\gamma_j'^\bullet-h'^\bullet,\gamma_j''^\bullet-h''^\bullet\big)=\big(\Phi_{V';\mathbf{s}'}(\gamma'^\bullet),\Phi_{V'';\mathbf{s}''}(\gamma''^\bullet)\big)+\mathring{\gamma}_j+\mathring{h}_j\,.\tag{5.34}$$

By (5.11) and (5.32)-(5.34),

$$\Xi^{V',V,V''}_{X,Y}\big(\widetilde{\mathrm{ev}}_X^{V'\cup V}(\mathbf{f}_X^\bullet),\widetilde{\mathrm{ev}}_Y^{V\cup V''}(\mathbf{f}_Y^\bullet)\big)=\big([\mathring{\gamma}_j]_{X\#_\varphi Y;\mathbf{s}'\mathbf{s}''},[\gamma'^\bullet\cdot\widehat{x}',\gamma''^\bullet\cdot\widehat{x}'']_{X\#_\varphi Y}\big).\tag{5.35}$$

Since both sides of (5.29) are lifts of (5.31), there exist $\alpha\in H(V'_{\mathbf{s}'})$ and $\beta\in H(V''_{\mathbf{s}''})$ such that

$$\widetilde{\mathrm{ev}}^{V'\cup V''}_{X\#_\varphi Y}(\mathbf{f}^\bullet_X\#_\varphi\mathbf{f}^\bullet_Y) = \big([\mathring{\gamma}'_j]_{X\#_\varphi Y;\mathbf{s}'\mathbf{s}''},[\alpha\gamma'^\bullet\cdot\widehat{x}',\beta\gamma''^\bullet\cdot\widehat{x}'']_{X\#_\varphi Y}\big) \tag{5.36}$$

for some $\alpha\in H_1(V'_{\mathbf{s}'})$, $\beta\in H_1(V''_{\mathbf{s}''})$, and a coset representative $\mathring{\gamma}'_j$ from the last collection in (5.21). By (2.10), (5.35), and (5.36), (5.29) with

$$\eta_{\#;j} = \mathring{\gamma}_j - \mathring{\gamma}'_j - \big(\Phi_{V';\mathbf{s}'}(\alpha),\Phi_{V'';\mathbf{s}''}(\beta)\big) \in H_1(V')\oplus H_1(V'') \tag{5.37}$$

holds for $(\mathbf{f}_X,\mathbf{f}_Y)=(\mathbf{f}^\bullet_X,\mathbf{f}^\bullet_Y)$.

Let $(\mathbf{f}_X,\mathbf{f}_Y)$ be any pair satisfying (5.30) and (5.28). By the first assumption and (5.32),

$$\begin{aligned}\widetilde{\mathrm{ev}}^{V'\cup V}_{X}(\mathbf{f}_X) &= \big([\gamma'_j,\gamma_{X;j}]_{X;\mathbf{s}'\mathbf{s}},[\alpha'\cdot\widehat{x}',\alpha\gamma\cdot\widehat{x}]_X\big),\\ \widetilde{\mathrm{ev}}^{V\cup V''}_{Y}(\mathbf{f}_Y) &= \big([\gamma_{Y;j},\gamma''_j]_{Y;\mathbf{s}\mathbf{s}''},[\beta\cdot\widehat{x},\beta''\cdot\widehat{x}'']_Y\big),\end{aligned} \tag{5.38}$$

for some $\alpha,\beta\in H_1(V_{\mathbf{s}})$, $\alpha'\in H_1(V'_{\mathbf{s}'})$, $\beta''\in H_1(V''_{\mathbf{s}''})$, and elements $(\gamma'_j,\gamma_{X;j})$ and $(\gamma_{Y;j},\gamma''_j)$ of the two collections in (5.22). By (5.28), there exists $(h',h,h'')\in\widetilde{H}^V_{X,Y}$ such that

$$h\in\overset{\bullet}{H}{}^V_{X,Y}, \qquad \big(\gamma_{X;j},\gamma_{Y;j}\big)+\big(\Phi_{V;\mathbf{s}}(\gamma+\alpha-\beta),0\big) = \gamma_{\#;j}+h\,. \tag{5.39}$$

Let $(\gamma',\gamma'')\in H_1(V'_{\mathbf{s}'})\oplus H_1(V''_{\mathbf{s}''})$, $\mathring{h}\in\mathring{H}^V_{X,Y}$, and $\mathring{\gamma}$ be a coset representative from the last collection in (5.21) such that

$$\big(\gamma'_j-h',\gamma''_j-h''\big) = \big(\Phi_{V';\mathbf{s}'}(\gamma'),\Phi_{V'';\mathbf{s}''}(\gamma'')\big)+\mathring{\gamma}+\mathring{h}\,. \tag{5.40}$$

By (5.11) and (5.38)-(5.40),

$$\Xi^{V',V,V''}_{X,Y}\big(\widetilde{\mathrm{ev}}^{V'\cup V}_{X}(\mathbf{f}_X),\widetilde{\mathrm{ev}}^{V\cup V''}_{Y}(\mathbf{f}_Y)\big) = \big([\mathring{\gamma}]_{X\#_\varphi Y;\mathbf{s}'\mathbf{s}''},[\gamma'\alpha'\cdot\widehat{x}',\gamma''\beta''\cdot\widehat{x}'']_{X\#_\varphi Y}\big). \tag{5.41}$$

By (2.23), (5.32), and (5.38),

$$\begin{aligned}\big[f_X\#(-f^\bullet_X)\big] &= \iota^{X-W'}_{S_XW'*}\big(\Delta^{W'}_X\big(\Phi_{V';\mathbf{s}'}(\alpha')+\gamma'_j-\gamma'^\bullet_j,\Phi_{V;\mathbf{s}}(\alpha)+\gamma_{X;j}-\gamma^\bullet_{X;j}\big)\big),\\ \big[f_Y\#(-f^\bullet_Y)\big] &= \iota^{Y-W''}_{S_YW''*}\big(\Delta^{W''}_Y\big(\Phi_{V;\mathbf{s}}(\beta)+\gamma_{Y;j}-\gamma^\bullet_{Y;j},\Phi_{V'';\mathbf{s}''}(\beta'')+\gamma''_j-\gamma''^\bullet_j\big)\big).\end{aligned} \tag{5.42}$$

By (5.42), (5.33), and (5.39),

$$\begin{aligned}\big[(f_X\#_\varphi f_Y)\#(-(f^\bullet_X\#_\varphi f^\bullet_Y))\big] &= \iota^{X\#_\varphi Y-W}_{X-W'*}\big(\big[f_X\#(-f^\bullet_X)\big]\big)+\iota^{X\#_\varphi Y-W}_{Y-W''*}\big(\big[f_Y\#(-f^\bullet_Y)\big]\big)\\ &= \widetilde{\mathfrak{R}}^V_{X,Y}\big(\big(\Phi_{V';\mathbf{s}'}(\alpha')+\gamma'_j-\gamma'^\bullet_j,\Phi_{V;\mathbf{s}}(\beta),\Phi_{V;\mathbf{s}}(\beta),\Phi_{V'';\mathbf{s}''}(\beta'')+\gamma''_j-\gamma''^\bullet_j\big)+(0,h-h^\bullet,0)\big)\\ &= \widetilde{\mathfrak{R}}^V_{X,Y}\big(\Phi_{V';\mathbf{s}'}(\alpha')+(\gamma'_j-h')-(\gamma'^\bullet_j-h'^\bullet),0,0,\Phi_{V'';\mathbf{s}''}(\beta'')+(\gamma''_j-h'')-(\gamma''^\bullet_j-h''^\bullet)\big);\end{aligned}$$

the last equality makes use of the second inclusion in the middle statement in (5.18). Combining the above with (5.34) and (5.40), we find that

$$\begin{aligned}&\big[(f_X\#_\varphi f_Y)\#(-(f^\bullet_X\#_\varphi f^\bullet_Y))\big]\\ &\qquad = \iota^{X\#_\varphi Y-W}_{S_WX\#_\varphi Y*}\big(\Delta^W_{X\#_\varphi Y}\big(\Phi_{W;\mathbf{s}'\mathbf{s}''}(\alpha'+\gamma'-\gamma'^\bullet,\beta''+\gamma''-\gamma''^\bullet)+\mathring{\gamma}-\mathring{\gamma}_j\big)\big).\end{aligned} \tag{5.43}$$

By [5, Proposition 6.6], Proposition 2.2 applied to $(X\#_\varphi Y,W)$, (5.35), and (5.43),

$$\Theta_{\eta_{\#;j}}\big(\widetilde{\mathrm{ev}}^{V'\cup V''}_{X\#_\varphi Y}(\mathbf{f}_X\#_\varphi\mathbf{f}_Y)\big) = \big([\mathring{\gamma}]_{X\#_\varphi Y;\mathbf{s}'\mathbf{s}''},[\gamma'\alpha'\cdot\widehat{x}',\gamma''\beta''\cdot\widehat{x}'']_{X\#_\varphi Y}\big).$$

Comparing with (5.41), we obtain (5.29). $\square$

Remark 5.4. In order to make the diagram in Figure 4 fully commutative, the second component in the bottom morphism should be twisted by suitable deck transformations $\Theta_{\eta_{A_\#}}$ of $\widehat{W}_{X\#_\varphi Y;\mathbf{s}'\mathbf{s}''}$. By [5, Proposition 6.6], these deck transformations correspond to the differences between the possible lifts for $(X\#_\varphi Y, W)$. An intention of [11, Section 5] is to take $\Theta_{\eta_{A_\#}}=\mathrm{id}$ by choosing the lifts (1.9) consistently across different relative pairs. This property, which is used in [12, (10.8)], is implicit in the two set-theoretic descriptions of the rim tori covers; see [5, Remark 6.7] for more details.

For each $A\in A_X\#_\varphi A_Y$, let

$$\widehat{W}^A_{X,Y;\mathbf{s}'\mathbf{s}\mathbf{s}''}=\{\widetilde{\pi}_{\mathbf{s}}\times\widetilde{\pi}_{\mathbf{s}}\}^{-1}\big(\widehat{V}^A_{X,Y;\mathbf{s}}\big)\subset\widehat{W'}_{X;\mathbf{s}'\mathbf{s}}\times\widehat{W''}_{Y;\mathbf{s}\mathbf{s}''}\,, \tag{5.44}$$

with $\widehat{V}^A_{X,Y;\mathbf{s}}$ as in (4.17) and

$$\widetilde{\pi}_{\mathbf{s}}\colon\widehat{W'}_{X;\mathbf{s}'\mathbf{s}}\longrightarrow\widehat{V}_{X;\mathbf{s}}\qquad\text{and}\qquad\widetilde{\pi}_{\mathbf{s}}\colon\widehat{W''}_{Y;\mathbf{s}\mathbf{s}''}\longrightarrow\widehat{V}_{X;\mathbf{s}}$$

being the natural projection maps; see [5, (6.4)]. The cohomology class determined by the submanifold (5.44) satisfies

$$\mathrm{PD}^{V,A}_{X,Y,W;\mathbf{s}'\mathbf{s}\mathbf{s}''}\Delta=\{\widetilde{\pi}_{\mathbf{s}}\times\widetilde{\pi}_{\mathbf{s}}\}^*\big(\mathrm{PD}^{V,A}_{X,Y;\mathbf{s}}\Delta\big)\in H^*\big(\widehat{W'}_{X;\mathbf{s}'\mathbf{s}}\times\widehat{W''}_{Y;\mathbf{s}\mathbf{s}''};\mathbb{Q}\big), \tag{5.45}$$

with $\mathrm{PD}^{V,A}_{X,Y;\mathbf{s}}\Delta$ as in (4.18).

Example 5.5. With $X=\widehat{\mathbb{P}}^2_9$, $Y=\mathbb{P}^1\times\mathbb{T}^2$, and $V=F\subset X,Y$ as in Example 4.6, we take $V'=\emptyset$ and $V''=F''\subset Y$ to be a fiber different from F. In this case,

$$\begin{gathered}H^V_X,\mathring{H}^V_{X,Y},H^{W'}_X=0\quad H_2=H_1(V),\quad \dot{H}^V_{X,Y}=H_1(V)\oplus H_1(V),\quad H^{W''}_Y=H_\Delta\subset H_1(V)\oplus H_1(V''),\\ \widetilde{H}^V_{X,Y}=\big\{(0,\alpha,\alpha+\beta,\beta)\colon\ \alpha,\beta\in H_1(F)\big\}\subset H_1(V')\oplus H_1(V)\oplus H_1(V)\oplus H_1(V'').\end{gathered}$$

The smooth map (5.24) can be described as in Example 5.2. By (5.17) and (4.21),

$$\mathrm{PD}^{V,A}_{X,Y,V'';()(1)(1)}\Delta=1\times1\times\mathrm{PD}_{\mathbb{T}^2}(\mathrm{pt})\in H^*\big(\mathbb{C}\times\mathbb{C}\times\mathbb{T}^2;\mathbb{Q}\big), \tag{5.46}$$

if the $(Y,V\cup V'')$ covering is written as

$$\mathbb{C}\times\mathbb{T}^2\longrightarrow\mathbb{T}^2\times\mathbb{T}^2,\qquad (z_1,[z_2])\longrightarrow\big([z_2-z_1],[z_2+z_1]\big).$$

Applying (5.46) to the decomposition

$$(\widehat{\mathbb{P}}^2_9,F)=(\widehat{\mathbb{P}}^2_9,F)\#_F\big(\mathbb{P}^1\times\mathbb{T}^2,\{0,\infty\}\times\mathbb{T}^2\big)$$

as in Example 4.6, we obtain

$$\widetilde{\mathrm{GW}}^{\widehat{\mathbb{P}}^2_9,F}_{g,\mathfrak{s}_i+d\mathfrak{f};(1)}\big(\mathrm{pt}^g;1\big)=\sum_{\substack{d_1,d_2\in\mathbb{Z}^{\ge0}\\ d_1+d_2=d}}\widetilde{\mathrm{GW}}^{\widehat{\mathbb{P}}^2_9,F}_{g,\mathfrak{s}_i+d_1\mathfrak{f};(1)}\big(\mathrm{pt}^g;1\big)\widetilde{\mathrm{GW}}^{\mathbb{P}^1\times\mathbb{T}^2,\{0,\infty\}\times F}_{0,\mathfrak{s}+d_2\mathfrak{f};(1),(1)}\big(;\mathrm{PD}_{\mathbb{T}^2}(\mathrm{pt}),1\big).$$

By the same considerations as in Example 4.6, this identity reduces to

$$\mathrm{GW}^{\widehat{\mathbb{P}}^2_9,F}_{g,\mathfrak{s}_i+d\mathfrak{f};(1)}\big(\mathrm{pt}^g;1\big)=\sum_{\substack{d_1,d_2\in\mathbb{Z}^{\ge0}\\ d_1+d_2=d}}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9,F}_{g,\mathfrak{s}_i+d_1\mathfrak{f};(1)}\big(\mathrm{pt}^g;1\big)\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2,\{0,\infty\}\times F}_{0,\mathfrak{s}+d_2\mathfrak{f};(1),(1)}\big(;\mathrm{pt},1\big).$$

This equation is consistent with

$$\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2,\{0,\infty\}\times F}_{0,\mathfrak{s}+d_2\mathfrak{f};(1),(1)}\big(;\mathrm{pt},1\big)=\begin{cases}1, & \text{if } d_2\!=\!0;\\ 0, & \text{if } d_2\!\neq\!0;\end{cases}$$

the last statement can be seen directly from the moduli space consisting of the sections in the first case and being empty in the second case. Putting one point on the Y side, we similarly obtain

$$\mathrm{GW}^{\widehat{\mathbb{P}}^2_9,F}_{g,\mathfrak{s}_i+d\mathfrak{f};(1)}\big(\mathrm{pt}^g;1\big)=\sum_{\substack{d_1,d_2\in\mathbb{Z}^{\ge0}\\ d_1+d_2=d}}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9,F}_{g-1,\mathfrak{s}_i+d_1\mathfrak{f};(1)}\big(\mathrm{pt}^{g-1};1\big)\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2,\{0,\infty\}\times F}_{1,\mathfrak{s}+d_2\mathfrak{f};(1),(1)}\big(\mathrm{pt};\mathrm{pt},1\big).$$

In light of Theorem 6.1 and Lemma 6.7, this identity is consistent with

$$\sum_{d=0}^{\infty}\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2,\{0,\infty\}\times F}_{1,\mathfrak{s}+d\mathfrak{f};(1),(1)}\big(\mathrm{pt};\mathrm{pt},1\big)q^d=qG'(q),$$

with $G(q)$ given by (6.3). A direct reason for the last equation, which corrects the statement in the middle case of the last claim in [11, Lemma 14.5], is indicated in the proof of this lemma; see Remarks 6.5 and 6.8 for related comments.

6 The Bryan-Leung formula

One of the three applications of the symplectic sum formula appearing in [12] is an alternative proof of [3, Theorem 1.2], a closed formula for the GW-invariants of the blowup $\widehat{\mathbb{P}}^2_9$ of $\mathbb{P}^2$ at 9 points. The approach of [12] is significantly more efficient than the original proof in [3], though less direct, as it relies heavily on the symplectic sum formula. Unfortunately, the argument in [12] contains some unnecessary statements and several incorrect statements, including one which results in the incorrect main conclusion, [12, (15.4)]. It is also missing a crucial intermediate observation; see Lemma 6.4 and Remark 6.13. Some of the incorrect claims in [11] concern basic points regarding IP-counts. The approach in [12] in fact indicates an effective proof of [3, Theorem 1.2] via the standard symplectic sum formula. In this section, we correct and slightly streamline the argument in [12] by making use of the vanishing result of [5, Theorem 1.1] in a case when it is an obvious consequence of the existence of the lift (1.9); see (6.16).

Let $\widehat{\mathbb{P}}^2_9$ be a blowup of $\mathbb{P}^2$ at the 9 intersection points of a general pair of smooth cubic curves C_1 and C_2 and $\pi\colon\widehat{\mathbb{P}}^2_9\longrightarrow\mathbb{P}^1$ be the projection to the pencil parametrizing the cubics spanned by C_1 and C_2. This fibration has 9 sections $S_1,\ldots,S_9$ corresponding to the 9 exceptional divisors. The homology classes $\mathfrak{s}_1,\ldots,\mathfrak{s}_9$ of $S_1,\ldots,S_9$ and the homology class $\mathfrak{f}$ of a smooth fiber F form a basis for an index 3 sublattice of $H_2(\widehat{\mathbb{P}}^2_9;\mathbb{Z})\approx\mathbb{Z}^{10}$. For $g\in\mathbb{Z}^{\ge0}$, define

$$\mathcal{F}_g(q)=\sum_{d=0}^{\infty}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{g,\mathfrak{s}_i+d\mathfrak{f}}(\mathrm{pt}^g)q^d, \tag{6.1}$$

where the summand denotes the genus g degree $\mathfrak{s}_i+d\mathfrak{f}$ GW-invariant of $\widehat{\mathbb{P}}^2_9$ with g point constraints. Since $\mathfrak{s}_i^2=-1$, there is only one holomorphic curve in the homology class $\mathfrak{s}_i$ and thus

$$\mathcal{F}_0(q)\in 1+q\mathbb{Q}[[q]],\qquad \mathcal{F}_g(q)\in q\mathbb{Q}[[q]]\ \ \forall\, g\!\in\!\mathbb{Z}^+. \tag{6.2}$$

Let

$$G(q) = \sum_{d=1}^{\infty} \sigma(d) q^d, \qquad \text{where} \quad \sigma(d) = \sum_{r|d} r. \tag{6.3}$$

Theorem 6.1 ([3, Theorem 1.2])**.** *For every* $g \in \mathbb{Z}^{\geq 0}$,

$$\mathcal{F}_g(q) = \left(\prod_{d=1}^{\infty} (1-q^d) \right)^{-12} \big(qG'(q)\big)^g. \tag{6.4}$$

6.1 Genus 1 GW-invariants of $\mathbb{P}^1 \times \mathbb{T}^2$

We begin with some observations concerning the genus 1 GW-invariants of $\mathbb{P}^1 \times \mathbb{T}^2$ and $(\mathbb{P}^1 \times \mathbb{T}^2, F)$, where $F = p \times \mathbb{T}^2$ is a fiber of the projection to the first component. We denote by $\mathfrak{s}$ and $\mathfrak{f}$ the homology classes of $\mathbb{P}^1 \times q$ and F, respectively.

Lemma 6.2 ([12, Lemma 14.4])**.** *The genus 1 GW-invariants of* $\mathbb{P}^1 \times \mathbb{T}^2$ *satisfy*

$$\sum_{d=1}^{\infty} d \, \mathrm{GW}_{1,d\mathfrak{f}}^{\mathbb{P}^1 \times \mathbb{T}^2}() q^d = 2G(q).$$

Proof. Let $L = \mathcal{O}_{\mathbb{T}^2}(p-q) \longrightarrow \mathbb{T}^2$ be a non-torsion line bundle ($L^{\otimes k} \not\approx \mathcal{O}_{\mathbb{T}^2}$ for all $k \in \mathbb{Z}^+$). The only holomorphic maps in $\mathbb{P}(L \oplus \mathcal{O}_{\mathbb{T}^2}) \approx \mathbb{P}^1 \times \mathbb{T}^2$ in the homology class $d\mathfrak{f}$ are then covers of

$$F_0 \equiv \mathbb{P}\big(0 \oplus \mathcal{O}_{\mathbb{T}^2}\big) \qquad \text{and} \qquad F_\infty \equiv \mathbb{P}\big(L \oplus 0\big),$$

and these maps are regular. Since the number of degree d covers $\mathbb{T}^2 \longrightarrow \mathbb{T}^2$ (or equivalently of subgroups of $\mathbb{Z}^2$ of index d) is $\sigma(d)$, $\overline{\mathfrak{M}}_{1,0}(\mathbb{P}^1 \times \mathbb{T}^2, d\mathfrak{f})$ consists of $2\sigma(d)$ elements. Since the order of the automorphism group of each of these elements is d, we conclude that

$$\mathrm{GW}_{1,d\mathfrak{f}}^{\mathbb{P}^1 \times \mathbb{T}^2}() = 2\sigma(d)/d,$$

as claimed. ☐

Lemma 6.3 ([12, Lemma 14.5])**.** *The genus 1 GW-invariants of* $(\mathbb{P}^1 \times \mathbb{T}^2, F)$ *with two point constraints satisfy*

$$\sum_{d=0}^{\infty} \mathrm{GW}_{1,\mathfrak{s}+d\mathfrak{f};(1)}^{\mathbb{P}^1 \times \mathbb{T}^2, F}\big(\mathrm{pt}; \mathrm{pt}\big) q^d = qG'(q).$$

Proof. Suppose Σ is a connected nodal genus 1 curve and $u\colon \Sigma \longrightarrow \mathbb{P}^1 \times \mathbb{T}^2$ is a degree $\mathfrak{s}+d\mathfrak{f}$ stable map. Since $\pi_1 \circ u : \Sigma \longrightarrow \mathbb{P}^1$ has degree 1 and every holomorphic map $\mathbb{P}^1 \longrightarrow \mathbb{T}^2$ is constant, Σ contains a unique irreducible component $\Sigma_0 \approx \mathbb{P}^1$ such that $u\colon \Sigma_0 \longrightarrow \mathbb{P}^1 \times q_2$ is an isomorphism for some $q_2 \in \mathbb{T}^2$. If Σ_i is another irreducible rational component of Σ, then $u|_{\Sigma_i}$ is constant. Since Σ is of genus 1, Σ contains at most one (precisely one if $d > 0$) irreducible genus 1 component Σ_1; furthermore, $u|_{\Sigma_1}$ is a degree d (unbranched) cover of $q_1 \times \mathbb{T}^2$ for some $q_1 \in \mathbb{P}^1$. Every such stable map is regular.

Thus, the subspace

$$\big\{[u, x_1, y_1] \in \overline{\mathfrak{M}}_{1,1;(1)}^F(\mathbb{P}^1 \times \mathbb{T}^2, \mathfrak{s}+d\mathfrak{f}) \colon u(x_1) = \mathrm{pt}_1, \; u(y_1) = \mathrm{pt}_2\big\}$$

consists of maps $u: \Sigma_0 \cup \Sigma_1 \longrightarrow \mathbb{P}^1 \times \mathbb{T}^2$ such that $u: \Sigma_0 \longrightarrow \mathbb{P}^1 \times \pi_2(\mathrm{pt}_2)$ is an isomorphism and $u: \Sigma_1 \longrightarrow \pi_1(\mathrm{pt}_1) \times \mathbb{T}^2$ is a degree d cover. There are $\sigma(d)$ such maps u, each of which has an automorphism of order d. For each choice of the map u, there are d choices for the preimage of pt_1 and d choices for the nodes on Σ_1. Thus,

$$\mathrm{GW}_{1,d\mathfrak{f};(1)}^{\mathbb{P}^1 \times \mathbb{T}^2, F}\big(\mathrm{pt}; \mathrm{pt}\big) = \sigma(d)/d \cdot d \cdot d = d\sigma(d);$$

this establishes the claim. □

For any compact symplectic manifold X, we denote by

$$\psi_1 \in H^2\big(\overline{\mathfrak{M}}_{g,k}(X, A); \mathbb{Q}\big)$$

the first chern class of the universal cotangent line bundle for the first marked point. For each $d \in \mathbb{Z}^{\geq 0}$, let

$$\begin{aligned}
\mathrm{GW}_{1,\mathfrak{s}+d\mathfrak{f}}^{\mathbb{P}^1 \times \mathbb{T}^2}\big(\tau_1[\mathfrak{f}], \mathrm{pt}\big) &= \int_{[\overline{\mathfrak{M}}_{1,2}(\mathbb{P}^1 \times \mathbb{T}^2, \mathfrak{s}+d\mathfrak{f})]^{\mathrm{vir}}} \psi_1\big(\mathrm{ev}_1^* \mathrm{PD}_{\mathbb{P}^1 \times \mathbb{T}^2} \mathfrak{f}\big)\big(\mathrm{ev}_2^* \mathrm{PD}_{\mathbb{P}^1 \times \mathbb{T}^2} \mathrm{pt}\big), \\
\mathrm{GW}_{1,\mathfrak{s}+d\mathfrak{f};(1)}^{\mathbb{P}^1 \times \mathbb{T}^2, F}\big(\tau_1[\mathfrak{f}]; \mathrm{pt}\big) &= \int_{[\overline{\mathfrak{M}}_{1,1;(1)}^F(\mathbb{P}^1 \times \mathbb{T}^2, \mathfrak{s}+d\mathfrak{f})]^{\mathrm{vir}}} \psi_1\big(\mathrm{ev}_1^* \mathrm{PD}_{\mathbb{P}^1 \times \mathbb{T}^2} \mathfrak{f}\big)\big(\mathrm{ev}_2^* \mathrm{PD}_F \mathrm{pt}\big).
\end{aligned}$$

Lemma 6.4. *For every* $d \in \mathbb{Z}^{\geq 0}$,

$$\mathrm{GW}_{1,\mathfrak{s}+d\mathfrak{f};(1)}^{\mathbb{P}^1 \times \mathbb{T}^2, F}\big(\tau_1[\mathfrak{f}]; \mathrm{pt}\big) = \mathrm{GW}_{1,\mathfrak{s}+d\mathfrak{f}}^{\mathbb{P}^1 \times \mathbb{T}^2}\big(\tau_1[\mathfrak{f}], \mathrm{pt}\big). \tag{6.5}$$

Proof. As we explained below,

$$\begin{aligned}
&\big\{[u, x, y] \in \overline{\mathfrak{M}}_{1,1;(1)}^F(\mathbb{P}^1 \times \mathbb{T}^2, \mathfrak{s}+d\mathfrak{f}):\ u(x) \in \mathfrak{f},\ u(y) = \mathrm{pt}\big\} \\
&\qquad\qquad \approx \big\{[u, x_1, x_2] \in \overline{\mathfrak{M}}_{1,2}(\mathbb{P}^1 \times \mathbb{T}^2, \mathfrak{s}+d\mathfrak{f}):\ u(x_1) \in \mathfrak{f},\ u(x_2) = \mathrm{pt}\big\},
\end{aligned}$$

where $\mathrm{pt} \in F$ is a fixed point. Both spaces contain three irreducible components, which we describe below and which have essentially the same deformation/obstruction theory (after capping with ψ_1 in the third case); see Figure 5. This implies the claim.

One of the components common to both spaces is isomorphic to

$$\mathbb{P}^1 \times \big\{[u, x_1'] \in \overline{\mathfrak{M}}_{1,1}(\mathbb{T}^2, d):\ u(x_1') = \mathrm{pt}_2\big\},$$

if $\mathrm{pt} \equiv (\mathrm{pt}_1, \mathrm{pt}_2) \in \mathbb{P}^1 \times \mathbb{T}^2$. A generic element of this component is a morphism from a smooth genus 1 curve and a rational tail carrying the two marked points which restricts to a degree d morphism to a fiber of π_1 (specified by $\mathbb{P}^1$) on the genus 1 curve and an isomorphism from the tail to the section s_{pt} through pt.

Another component is isomorphic to

$$\big\{[u, x_1, x_2'] \in \overline{\mathfrak{M}}_{1,2}(\mathfrak{f}, d):\ u(x_2') = \mathrm{pt}_2\big\}.$$

A generic element of this component is a morphism from a smooth genus 1 curve carrying the first marked point and a rational tail carrying the second marked point which restricts to a degree d

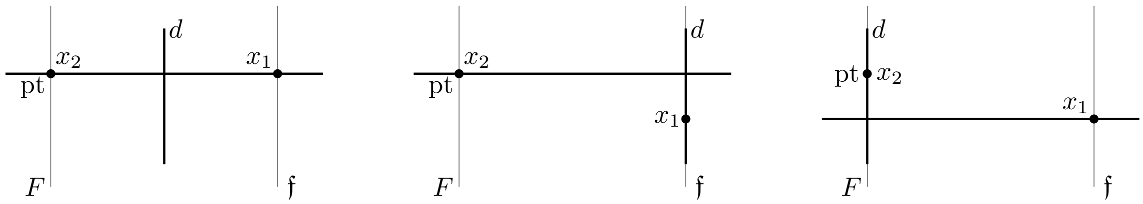

Figure 5: The three components of $\overline{\mathfrak{M}}_{1,2}(\mathbb{P}^1\times\mathbb{T}^2,\mathfrak{s}+d\mathfrak{f})$

morphism to the fiber $\mathfrak{f}$ of π_1 on the genus 1 curve and an isomorphism from the tail to $s_{\rm pt}$.

The last component of the absolute moduli space is isomorphic to

$$\big\{[u,x_1',x_2]\in\overline{\mathfrak{M}}_{1,2}(F,d):\ u(x_2)=\mathrm{pt}\big\}.$$

A generic element of this component is a morphism from a smooth genus 1 curve carrying the second marked point and a rational tail carrying the first marked point which restricts to a degree d morphism to the fiber F of π_1 on the genus 1 curve and an isomorphism from the tail to a section of π_1 (through the image of the first marked point of an element of $\overline{\mathfrak{M}}_{1,2}(F,d)$). The last component of the relative moduli space is described in the same way, except the morphism on the genus 1 component above is replaced by the $\mathbb{C}^*$-equivalence class of a morphism into the rubber $\mathbb{P}^1\times\mathbb{T}^2$ from a smooth genus 1 component with a rational tail carrying two marked points which restricts to a degree d morphism into a fiber of π_1, but not over $0,\infty\in\mathbb{P}^1$, and an isomorphism from the tail to a section of π_1. The restriction of ψ_1 to this component of the moduli space is the pullback of the first chern class of the conormal bundle to $\mathfrak{f}$ by the first evaluation map. Thus, this restriction vanishes in the absolute and relative cases. □

Remark 6.5. Lemma 6.2 corrects the first statement of [12, Lemma 14.4]; the other two, one of which is similarly off, are never used. Lemma 6.3 is the second statement of [12, Lemma 14.5]; the other two are never used. The proof of [12, Lemma 14.5] has two mutually canceling errors, ignoring the automorphisms of the cover and the choices of the node on the genus 1 component. The statement at the end of the first paragraph of the proof in [12] is true only generically or after imposing the constraints; otherwise, there could be maps with a component mapped into the rubber. The third statement of [12, Lemma 14.5] and its proof incorrectly describe the IP-counts of $(\mathbb{P}^1\times\mathbb{T}^2, V\equiv F_0\cup F_\infty)$ as being indexed by the rim tori, suggesting that the rim tori cover $\widehat{V}_{\mathbb{P}^1\times\mathbb{T}^2;(1),(1)}$ is $\mathbb{Z}^2\times F_0\times F_\infty$. As explained in [5, Example 6.9], $\widehat{V}_{\mathbb{P}^1\times\mathbb{T}^2;(1),(1)}\approx\mathbb{C}\times\mathbb{T}^2$ and there is only one IP-count of each type appearing in the third statement of [12, Lemma 14.5]; there is no indexing by the rim tori.

6.2 GW-invariants of $\widehat{\mathbb{P}}^2_9$

We next make some observations concerning absolute GW-invariants of $\widehat{\mathbb{P}}^2_9$ and relative GW-invariants of $(\widehat{\mathbb{P}}^2_9,F)$, where $F\approx\mathbb{T}^2$ is a fiber of the projection $\widehat{\mathbb{P}}^2_9\longrightarrow\mathbb{P}^1$. From

$$(\mathfrak{s}_i+d\mathfrak{f})\cdot\mathfrak{f}=1,\qquad \big\langle c_1(T\widehat{\mathbb{P}}^2_9),\mathfrak{f}\big\rangle=0,\qquad\text{and}\quad \big\langle c_1(T\widehat{\mathbb{P}}^2_9),\mathfrak{s}_i\big\rangle=1,$$

we find that

$$\dim_{\mathbb{C}}\overline{\mathfrak{M}}_{1,0}(\widehat{\mathbb{P}}^2_9,d\mathfrak{f})=0+(2-3)(1-1)=0,$$
$$\dim_{\mathbb{C}}\overline{\mathfrak{M}}_{g,0}(\widehat{\mathbb{P}}^2_9,\mathfrak{s}_i+d\mathfrak{f})=\dim_{\mathbb{C}}\overline{\mathfrak{M}}^F_{g,0;(1)}(\widehat{\mathbb{P}}^2_9,\mathfrak{s}_i+d\mathfrak{f})=1+(2-3)(1-g)=g.$$

Thus, $\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{1,d\mathfrak{f}}()$, $\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{g,\mathfrak{s}_i+d\mathfrak{f}}(\mathrm{pt}^g)$, and $\mathrm{GW}^{\widehat{\mathbb{P}}^2_9,F}_{g,\mathfrak{s}_i+d\mathfrak{f};(1)}(\mathrm{pt}^g)$, where pt^g denotes g absolute point constraints, are rational numbers. These invariants are independent of the choice of the complex structure on $\widehat{\mathbb{P}}^2_9$ and $(\widehat{\mathbb{P}}^2_9, F)$.

Lemma 6.6 ([12, p1019])**.** *The genus 1 GW-invariants of $\widehat{\mathbb{P}}^2_9$ in the fiber classes are described by*

$$\sum_{d=1}^{\infty} d\,\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{1,d\mathfrak{f}}()q^d = G(q).$$

Proof. If $\widehat{\mathbb{P}}^2_9$ is obtained by blowing up $\mathbb{P}^2$ at 9 general points, there is only one degree $\mathfrak{f}$ holomorphic curve; this is the proper transform of the unique cubic passing through the 9 points. In this case, $\overline{\mathfrak{M}}_{1,0}(\widehat{\mathbb{P}}^2_9, d\mathfrak{f})$ consists of the $\sigma(d)$ unbranched covers of this cubic, all of which are regular and have an automorphism of order d. Thus,

$$d\,\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{1,d\mathfrak{f}} = \sigma(d),$$

as claimed. □

Lemma 6.7 ([12, Lemma 14.8])**.** *Let $d, g \in \mathbb{Z}^{\geq 0}$. The absolute and relative degree $\mathfrak{s}_i+d\mathfrak{f}$ genus g GW-invariants of $\widehat{\mathbb{P}}^2_9$ and $(\widehat{\mathbb{P}}^2_9, F)$ with g point insertions satisfy*

$$\mathrm{GW}^{\widehat{\mathbb{P}}^2_9,F}_{g,\mathfrak{s}_i+d\mathfrak{f};(1)}(\mathrm{pt}^g;\mathfrak{f}) = \mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{g,\mathfrak{s}_i+d\mathfrak{f}}(\mathrm{pt}^g).$$

Proof. Let J be a generic almost complex structure on $(\widehat{\mathbb{P}}^2_9, F)$. Suppose Σ is a connected nodal genus g curve and $u\colon \Sigma \longrightarrow \widehat{\mathbb{P}}^2_9$ is a degree $\mathfrak{s}_i+d\mathfrak{f}$ J-holomorphic stable map. If Σ_i is an irreducible component of Σ such that $u\colon \Sigma_i \longrightarrow F$ is not constant, then the genus of Σ_i is at least one and the sum of the genera of the remaining components of Σ is at most $g-1$. Therefore, if the g points are in general position, $u(\Sigma)$ does not contain all of them. It follows that all of the maps contributing to the absolute invariant with g point insertions are F-regular and thus contribute in the same way to the relative invariant. □

Remark 6.8. Lemma 6.7 corrects the statement of [12, Lemma 14.8]. The latter and its proof incorrectly describe the IP-counts of $(\widehat{\mathbb{P}}^2_9, F)$ as being indexed by the rim tori, suggesting that the rim tori cover $\widehat{F}_{\widehat{\mathbb{P}}^2_9;(1)}$ is $\mathbb{Z}^2 \times F$. As explained in [5, Example 6.8], $\widehat{F}_{\widehat{\mathbb{P}}^2_9;(1)} \approx \mathbb{C}$ and there is only one IP-count appearing in the statement of [12, Lemma 14.8]; it is the count appearing in (4.22). Its relative constraint is the pullback of $1 \in H^0(\widehat{F}_{\widehat{\mathbb{P}}^2_9;(1)};\mathbb{Q})$ by a lifted evaluation map (1.9); there is no indexing by the rim tori.

6.3 Proof of Theorem 6.1

For each $d \in \mathbb{Z}^{\geq 0}$, let

$$\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{1,\mathfrak{s}_i+d\mathfrak{f}}\big(\tau_1[\mathfrak{f}]\big) = \deg\big([\overline{\mathfrak{M}}_{1,1}(\widehat{\mathbb{P}}^2_9, \mathfrak{s}_i+d\mathfrak{f})]^{\mathrm{vir}} \cap \psi_1 \cap \mathrm{ev}_1^*1\big) \equiv \int_{[\overline{\mathfrak{M}}_{1,1}(\widehat{\mathbb{P}}^2_9,\mathfrak{s}_i+d\mathfrak{f})]^{\mathrm{vir}}} \psi_1 \mathrm{ev}_1^* 1\,.$$

The $g=0$ case of (6.4) is proved in [12] by obtaining two different expressions for

$$H(q) \equiv \sum_{d=0}^{\infty} \mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{1,\mathfrak{s}_i+d\mathfrak{f}}\big(\tau_1[\mathfrak{f}]\big)q^d \tag{6.6}$$

and setting them equal.

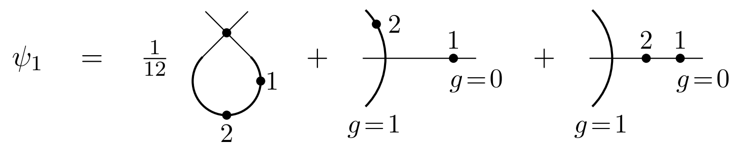

Figure 6: The genus 1 TRR on $\overline{\mathfrak{M}}_{1,2}(X,A)$

Lemma 6.9 ([12, Lemma 15.1]). *Let X be a symplectic 4-manifold with canonical class K_X.*

(a) For every $f\in H^2(X;\mathbb{Q})$,

$$\mathrm{GW}^X_{1,0}(f)=\frac{1}{24}K_X\cdot f\,.$$

(b) For every $A\in H_2(X;\mathbb{Z})$ with $A\cdot K_X<0$ and $f\in H^2(X;\mathbb{Q})$,

$$\begin{aligned}\mathrm{GW}^X_{1,A}\big(\tau_1(f),\mathrm{pt}^{-K_X\cdot A-1}\big)&=\frac{f\cdot A}{24}\big(A\cdot A+K_X\cdot A\big)\mathrm{GW}^X_{0,A}\big(\mathrm{pt}^{-K_X\cdot A-1}\big)\\&+\sum_{\substack{A_0,A_1\in H_2(X;\mathbb{Z})-0\\A_0+A_1=A}}\binom{-K_X\cdot A-1}{-K_X\cdot A_0-1}(f\cdot A_0)(A_0\cdot A_1)\mathrm{GW}^X_{0,A_0}\big(p^{-K_X\cdot A_0-1}\big)\mathrm{GW}^X_{1,A_1}\big(p^{-K_X\cdot A_1}\big).\end{aligned}$$

Proof. (a) If $h\colon Y\longrightarrow X$ represents the Poincare dual of f (after passing to a multiple if necessary),

$$\big\{(y,[u,x_1])\in Y\times\overline{\mathfrak{M}}_{1,1}(X,0):\ h(y)=u(x_1)\big\}\approx Y\times\overline{\mathcal{M}}_{1,1}$$

and the obstruction bundle is isomorphic to $\pi_1^*h^*TX\otimes\pi_2^*\mathbb{E}^*$, where $\mathbb{E}\longrightarrow\overline{\mathcal{M}}_{1,1}$ is the Hodge line bundle. Thus,

$$\begin{aligned}\mathrm{GW}^X_{1,0}(f)&=\big\langle e\big(\pi_1^*h^*TX\otimes\pi_2^*\mathbb{E}^*\big),Y\times\overline{\mathcal{M}}_{1,1}\big\rangle\\&=-\langle h^*c_1(TX),Y\rangle\langle c_1(\mathbb{E}),\overline{\mathcal{M}}_{1,1}\rangle=\frac{1}{24}K_X\cdot f.\end{aligned}$$

(b) Let $k=-K_X\cdot A$ and $\{H_i\},\{\check{H}_i\}\subset H^2(X;\mathbb{Q})$ be dual bases. Choose a representative $F\subset X$ for f and $k-1$ general points $\mathrm{pt}_2,\ldots,\mathrm{pt}_k\in X$. By the genus 1 topological recursion relation, illustrated in Figure 6 and explained in [17],

$$\psi_1=\frac{1}{12}\Delta_0+\Delta_{;1}\,,$$

where $\Delta_0,\Delta_{;1}\subset\overline{\mathfrak{M}}_{1,k}(X,A)$ are the virtual divisors whose virtually generic elements are morphisms from the genus 1 irreducible nodal curve and from a smooth genus 1 curve with a rational tail which carries the first marked point.

By the Kunneth decomposition of the diagonal in X^2 and the divisor relation, the degree of the intersection of

$$\overline{\mathfrak{M}}'_{1,k}(X,A)\equiv\big\{[u,x_1,\ldots,x_k]\in\overline{\mathfrak{M}}_{1,k}(X,A):\ u(x_1)\in f,\ u(x_2)=\mathrm{pt}_2,\ldots,u(x_k)=\mathrm{pt}_k\big\}$$

with Δ_0 is

$$\frac{1}{2}\sum_i \mathrm{GW}^X_{0,A}(H_i,\check{H}^i,f,\mathrm{pt}^{k-1}) = \frac{1}{2}\sum_i (H_i\cdot A)(\check{H}_i\cdot A)(f\cdot A)\mathrm{GW}^X_{0,A}(\mathrm{pt}^{k-1}) = \frac{1}{2}(A\cdot A)(f\cdot A)\mathrm{GW}^X_{0,A}(\mathrm{pt}^{k-1}).$$

This gives the first term in our formula.

The intersection of $\overline{\mathfrak{M}}'_{1,k}(X,A)$ with the components of $\Delta_{;1}$ whose generic element restricts to a morphism of degree $A_1=0$ on the genus 1 component is the same as with the subset of these components consisting of morphisms from domains with no marked points on the genus 1 component (since the virtual complex dimension of $\overline{\mathfrak{M}}_{1,1}(X,0)$ is 1, it contains no elements passing through any of the points $\mathrm{pt}_2,\ldots,\mathrm{pt}_k$). Thus, similarly to the above, the degree of this intersection is

$$\sum_i \mathrm{GW}^X_{0,A}(H_i,f,\mathrm{pt}^{k-1})\mathrm{GW}^X_{1,0}(\check{H}^i) = \sum_i (H_i\cdot A)(f\cdot A)\mathrm{GW}^X_{0,A}(\mathrm{pt}^{k-1})\frac{1}{24}K_X\cdot\check{H}_i = \frac{1}{24}(f\cdot A)(K_X\cdot A)\mathrm{GW}^X_{0,A}(\mathrm{pt}^{k-1});$$

the first equality follows from part (a). This gives the second term in our formula. The intersection of $\overline{\mathfrak{M}}'_{1,k}(X,A)$ with the components of $\Delta_{;1}$ whose generic element restricts to a morphism of degree $A_1=A$ on the genus 1 component is empty, since the domain of any morphism in the intersection would contain a union of irreducible components on which the morphism is of degree 0 and which carries at least one of the last $k-1$ points (for stability), but F does not contain any of the points $\mathrm{pt}_2,\ldots,\mathrm{pt}_k$.

For dimensional reasons, the intersection of $\overline{\mathfrak{M}}'_{1,k}(X,A)$ with the components of $\Delta_{;1}$ whose generic element restricts to a morphism of degree $A_1\neq 0$ on the genus 1 component and $A_0\neq 0$ on the genus 0 tail consists of morphisms from the domains so that the rational tail carries $-K_X\cdot A_0-1$ of the last $k-1$ marked points. Thus, similarly to the above, the degree of this intersection is

$$\begin{aligned}&\sum_i \binom{-K_X\cdot A-1}{-K_X\cdot A_0-1}\mathrm{GW}^X_{0,A_0}\big(H_i,f,\mathrm{pt}^{-K_X\cdot A_0-1}\big)\mathrm{GW}^X_{1,A_1}\big(\check{H}_i,p^{-K_X\cdot A_1}\big)\\ &=\sum_i \binom{-K_X\cdot A-1}{-K_X\cdot A_0-1}(H_i\cdot A_0)(f\cdot A_0)\mathrm{GW}^X_{0,A_0}\big(\mathrm{pt}^{-K_X\cdot A_0-1}\big)(\check{H}_i\cdot A_1)\mathrm{GW}^X_{1,A_1}\big(\mathrm{pt}^{-K_X\cdot A_1}\big)\\ &=\binom{-K_X\cdot A-1}{-K_X\cdot A_0-1}(f\cdot A_0)(A_0\cdot A_1)\mathrm{GW}^X_{0,A_0}\big(\mathrm{pt}^{-K_X\cdot A_0-1}\big)\mathrm{GW}^X_{1,A_1}\big(\mathrm{pt}^{-K_X\cdot A_1}\big).\end{aligned}$$

This gives the last term in our formula. □

Corollary 6.10 ([12, (15.7)])**.** *The genus 0 and 1 GW-invariants of $\widehat{\mathbb{P}}^2_9$ satisfy*

$$H(q)=\frac{1}{12}\big(q\mathcal{F}_0'(q)-\mathcal{F}_0(q)\big)+\mathcal{F}_0(q)\cdot G(q)\,. \tag{6.7}$$

Proof. We apply Lemma 6.9(b) with $X=\widehat{\mathbb{P}}^2_9$ and $A=\mathfrak{s}_i+d\mathfrak{f}$. In this case,

$$-K_X\cdot(\mathfrak{s}_i+d\mathfrak{f})=-1,\quad (\mathfrak{s}_i+d\mathfrak{f})^2=2d-1,\quad \overline{\mathfrak{M}}_{0,k}(X,d\mathfrak{f})=\emptyset\qquad\forall\ d\in\mathbb{Z}^+\,.$$

Thus,

$$H(q)=\sum_{d=0}^{\infty}\frac{d-1}{12}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{0,\mathfrak{s}+d\mathfrak{f}}()q^d+\sum_{\substack{d_0\in\mathbb{Z}^{\ge0},d_1\in\mathbb{Z}^+\\ d_0+d_1=d}}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{0,\mathfrak{s}+d_0\mathfrak{f}}()q^{d_0}\cdot d_1\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{1,d_1\mathfrak{f}}()q^{d_1}.$$

The claim now follows from the $g=0$ case of (6.1) and Lemma 6.6. □

Corollary 6.11. *The genus 1 relative GW-invariants of* $(\mathbb{P}^1\times\mathbb{T}^2,F)$ *satisfy*

$$\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2,F}_{1,\mathfrak{s}+d\mathfrak{f};(1)}\big(\tau_1[\mathfrak{f}];\mathrm{pt}\big)=\begin{cases}-\frac{1}{12}, & \textit{if } d=0;\\ d\,\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2}_{1,d\mathfrak{f}}(), & \textit{if } d>0.\end{cases}$$

Proof. We apply Lemma 6.9(b) with $X=\mathbb{P}^1\times\mathbb{T}^2$ and $A=\mathfrak{s}+d\mathfrak{f}$ to the right-hand side of (6.5). In this case,

$$-K_X\cdot(\mathfrak{s}+d\mathfrak{f})=-2,\quad (\mathfrak{s}+d\mathfrak{f})^2=2d,\qquad \forall\ d\in\mathbb{Z}.$$

Thus,

$$\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2}_{1,\mathfrak{s}+d\mathfrak{f}}\big(\tau_1[\mathfrak{f}],\mathrm{pt}\big)=\frac{d-1}{12}\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2}_{0,\mathfrak{s}+d\mathfrak{f}}(\mathrm{pt})+\sum_{\substack{d_0\in\mathbb{Z}^{\ge0},d_1\in\mathbb{Z}^+\\ d_0+d_1=d}}\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2}_{0,\mathfrak{s}+d_0\mathfrak{f}}(\mathrm{pt})\cdot d_1\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2}_{1,d_1\mathfrak{f}}(). \tag{6.8}$$

Since the composition of a degree $\mathfrak{s}+d\mathfrak{f}$ morphism to $\mathbb{P}^1\times\mathbb{T}^2$ with the projection to the second factor is a degree d morphism to $\mathbb{T}^2$ and there are no such morphisms from $\mathbb{P}^1$ if $d\in\mathbb{Z}^+$,

$$\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^1\times\mathbb{T}^2,\mathfrak{s}+d\mathfrak{f}),\overline{\mathfrak{M}}^F_{0,1;(1)}(\mathbb{P}^1\times\mathbb{T}^2,\mathfrak{s}+d\mathfrak{f})=\emptyset\qquad\forall\,d\in\mathbb{Z}^+. \tag{6.9}$$

Thus, the first genus 0 term on the right-hand side of (6.8) vanishes unless $d=0$ and the second unless $d_0=0$; in the exceptional cases, they equal 1. The claim now follows from Lemma 6.4. □

We next obtain a second expression for $H(q)$ by applying the symplectic sum formula to the decomposition

$$\widehat{\mathbb{P}}^2_9=\widehat{\mathbb{P}}^2_9\#_F(\mathbb{P}^1\times\mathbb{T}^2) \tag{6.10}$$

and moving the fiber constraint to the $\mathbb{P}^1\times\mathbb{T}^2$ side. Since $\mathcal{R}^F_{\mathbb{P}^1\times\mathbb{T}^2}=0$, the homomorphism

$$\#\colon H_2(\widehat{\mathbb{P}}^2_9;\mathbb{Z})\times_F H_2(\mathbb{P}^1\times\mathbb{T}^2;\mathbb{Z})\longrightarrow H_2(\widehat{\mathbb{P}}^2_9;\mathbb{Z})$$

is well-defined; see [5, Corollary 4.2(2)]. Since

$$(a_1\mathfrak{s}_1+\ldots+a_9\mathfrak{s}_9+d'\mathfrak{f})\cdot F=(a\mathfrak{s}+d''\mathfrak{f})\cdot F$$

if and only if $a_1+\ldots+a_9=a$ and $\mathfrak{s}_i\#\mathfrak{s}=\mathfrak{s}_i$, the symplectic sum formula gives

$$\begin{split}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{1,\mathfrak{s}_i+d\mathfrak{f}}\big(\tau_1[\mathfrak{f}]\big)=&\sum_{\substack{d',d''\in\mathbb{Z}^{\ge0}\\ d'+d''=d}}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9,F}_{0,\mathfrak{s}_i+d'\mathfrak{f};(1)}(;\mathfrak{f})\,\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2,F}_{1,\mathfrak{s}+d''\mathfrak{f};(1)}\big(\tau_1[\mathfrak{f}];\mathrm{pt}\big)\\ &+\sum_{\substack{d',d''\in\mathbb{Z}^{\ge0}\\ d'+d''=d}}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9,F}_{1,\mathfrak{s}_i+d'\mathfrak{f};(1)}(;\mathrm{pt})\,\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2,F}_{0,\mathfrak{s}+d''\mathfrak{f};(1)}\big(\tau_1[\mathfrak{f}];\mathfrak{f}\big),\end{split} \tag{6.11}$$

where the relative constraints are listed after the semi-columns.

By (6.9),

$$\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2,F}_{0,\mathfrak{s}+d''\mathfrak{f};(1)}\big(\tau_1[\mathfrak{f}];\mathfrak{f}\big)=0 \qquad \forall\, d''\in\mathbb{Z}^+\,.$$

On the other hand, the morphism

$$\big\{[u,x,y]\in\overline{\mathfrak{M}}^F_{0,1;(1)}(\mathbb{P}^1\times\mathbb{T}^2,\mathfrak{s})\colon\; u(x)\in\mathfrak{f}\big\}\longrightarrow\mathfrak{f}, \qquad [u,x,y]\longrightarrow u(x),$$

is an isomorphism and the restriction of ψ_1 under this isomorphism is the first chern class of the conormal bundle to a fiber $\mathbb{T}^2$ in $\mathbb{P}^1\times\mathbb{T}^2$, i.e. 0. Thus, the second sum in (6.11) vanishes.

Combining (6.11) with the above conclusion, the $g\!=\!0$ case of Lemma 6.7, and Corollary 6.11, we find that

$$\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{1,\mathfrak{s}_i+d\mathfrak{f}}\big(\tau_1[\mathfrak{f}]\big)=-\frac{1}{12}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{0,\mathfrak{s}_i+d\mathfrak{f}}()+\sum_{\substack{d'\in\mathbb{Z}^{\ge0},d''\in\mathbb{Z}^+\\ d'+d''=d}}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{0,\mathfrak{s}_i+d'\mathfrak{f}}()\,d''\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2}_{1,d''\mathfrak{f}}().$$

Along with (6.6), the $g\!=\!0$ case of (6.1), and Lemma 6.2, this identity gives

$$H(q)=-\frac{1}{12}\mathcal{F}_0(q)+\mathcal{F}_0(q)\cdot 2G(q). \tag{6.12}$$

By (6.2), (6.7), and (6.12),

$$\mathcal{F}_0(0)=1, \qquad q\frac{\mathrm{d}}{\mathrm{d}q}\log\mathcal{F}_0(q)=12G(q). \tag{6.13}$$

Since

$$\frac{1}{12}q\frac{\mathrm{d}}{\mathrm{d}q}\log\bigg(\prod_{d=1}^{\infty}(1-q^d)\bigg)^{-12}=\sum_{d=1}^{\infty}\frac{dq^d}{1-q^d}=\sum_{d=1}^{\infty}\sigma(d)q^d=G(q),$$

(6.13) implies the $g=0$ case of (6.4). The full statement of (6.4) follows from this case and the next lemma.

Lemma 6.12. *For every* $g\!\in\!\mathbb{Z}^+$,

$$\mathcal{F}_g(q)=\mathcal{F}_{g-1}(q)\cdot qG'(q). \tag{6.14}$$

Proof. In light of Lemmas 6.3 and 6.7, this statement is equivalent to (4.23), which was obtained based on the approach to the symplectic sum formula in [12]. We now give a proof by applying the usual symplectic sum theorem to the splitting (6.10) and moving one point to the $\mathbb{P}^1\times\mathbb{T}^2$ side. Since $g\!-\!1$ points stay on the $\widehat{\mathbb{P}}^2_9$ side, the genus on the $\widehat{\mathbb{P}}^2_9$ side in the symplectic sum formula must be at least $g\!-\!1$ for the invariants not to vanish. Thus, similarly to (6.11), we obtain

$$\begin{split}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{g,\mathfrak{s}_i+d\mathfrak{f}}\big(\mathrm{pt}^g\big)=&\sum_{\substack{d',d''\in\mathbb{Z}^{\ge0}\\ d'+d''=d}}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9,F}_{g-1,\mathfrak{s}_i+d'\mathfrak{f};(1)}\big(\mathrm{pt}^{g-1};\mathfrak{f}\big)\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2,F}_{1,\mathfrak{s}+d''\mathfrak{f};(1)}(\mathrm{pt};\mathrm{pt})\\ &+\sum_{\substack{d',d''\in\mathbb{Z}^{\ge0}\\ d'+d''=d}}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9,F}_{g,\mathfrak{s}_i+d'\mathfrak{f};(1)}\big(\mathrm{pt}^{g-1};\mathrm{pt}\big)\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2,F}_{0,\mathfrak{s}+d''\mathfrak{f};(1)}(\mathrm{pt};\mathfrak{f}).\end{split} \tag{6.15}$$

By [5, Theorem 1.1],

$$\mathrm{GW}^{\widehat{\mathbb{P}}^2_9,F}_{g,\mathfrak{s}_i+d'\mathfrak{f};(1)}\big(\mathrm{pt}^{g-1};\mathrm{pt}\big)=0\,, \tag{6.16}$$

This particular statement holds because the relative evaluation morphism (1.3) factors through the lift to $\widehat{F}_{\widehat{\mathbb{P}}^2_9;(1)}\approx\mathbb{C}$; see [5, Example 6.8]. Combining (6.15) with (6.16) and Lemma 6.7, we find that

$$\sum_{d=0}^{\infty}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{g,\mathfrak{s}_i+d\mathfrak{f}}\big(\mathrm{pt}^g\big)q^d=\sum_{d',d''\in\mathbb{Z}^{\ge0}}\mathrm{GW}^{\widehat{\mathbb{P}}^2_9}_{g-1,\mathfrak{s}_i+d'\mathfrak{f}}\big(\mathrm{pt}^{g-1}\big)q^{d'}\cdot\mathrm{GW}^{\mathbb{P}^1\times\mathbb{T}^2,F}_{1,\mathfrak{s}+d''\mathfrak{f};(1)}(\mathrm{pt};\mathrm{pt})q^{d''}.$$

The claim now follows from (6.1) and Lemma 6.3. □

Remark 6.13. Lemma 6.9(b) extends [12, Lemma 15.1(b)] from the $K_X\cdot A=-1$ case, using the same argument; the $K_X\cdot A=-2$ case is needed to obtain the crucial identity [12, (15.8)], i.e. (6.12) above. Our use of (6.16) avoids the need for [12, Lemma 15.2(c)] and directly establishes the last equation in [12, Section 15.3]. The statement of the symplectic sum formula in the middle of [12, p1020] is wrong, as it should involve relative GW-invariants; as stated, the last factor is not even zero-dimensional. The next displayed expression in [12] has the same problem and does not lead to [12, (15.8)]. Because of problems with these formulas, Lemma 6.4 never even arises in [12].

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