On the Rim Tori Refinement of Relative Gromov-Witten Invariants

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Abstract

We construct Ionel-Parker's proposed refinement of the standard relative Gromov-Witten invariants in terms of abelian covers of the symplectic divisor and discuss in what sense it gives rise to invariants. We use it to obtain some vanishing results for the standard relative Gromov-Witten invariants. In a separate paper, we describe to what extent this refinement sharpens the usual symplectic sum formula and give further qualitative applications.

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1   Introduction

Gromov-Witten invariants of symplectic manifolds, which include nonsingular projective varieties, are certain counts of pseudo-holomorphic curves that play prominent roles in symplectic topology, algebraic geometry, and string theory. The decomposition formulas, known as symplectic sum formulas in symplectic topology and degeneration formulas in algebraic geometry, are one of the main tools used to compute Gromov-Witten invariants; they relate Gromov-Witten invariants of one symplectic manifold to Gromov-Witten invariants of two simpler symplectic manifolds. Unfortunately, the formulas of [10, 11] do not completely determine the former in terms of the latter in many cases because of the so-called vanishing cycles: second homology classes in the first manifold which vanish when projected to the union of the other two manifolds; see (2.2). A refinement to the usual relative Gromov-Witten invariants of [9, 11] is sketched in [7]; the aim of this refinement is to resolve the unfortunate deficiency of the formulas of [10, 11] in [8]. In this paper, we formally construct the refinement of [7], discuss the invariance and computability aspects of the resulting curve counts, and obtain some vanishing results for the usual relative GW-invariants. In the sequel [4], we describe the applicability of this refinement to computing the Gromov-Witten invariants of symplectic sums and obtain further qualitative applications.

1.1   Relative GW-invariants

Let (X, ω) be a compact symplectic manifold and J be an ω-tame almost complex structure on X. For $g, k \in \mathbb{Z}^{\geq 0}$ and $A \in H_2(X; \mathbb{Z})$, we denote by $\overline{\mathfrak{M}}_{g,k}(X, A)$ the moduli space of stable J-holomorphic k-marked degree A maps from connected nodal curves of genus g. By [12, 5, 2], this moduli space carries a virtual class, which is independent of J and of representative ω in a deformation equivalence class of symplectic forms on X. If $V \subset X$ is a compact symplectic divisor (symplectic submanifold of real codimension 2), $\ell \in \mathbb{Z}^{\geq 0}$, $\mathbf{s} \equiv (s_1, \ldots, s_\ell)$ is an ℓ-tuple of positive integers such that

$$s_1 + \ldots + s_\ell = A \cdot V, \tag{1.1}$$

and J restricts to an almost complex structure on V, let $\overline{\mathfrak{M}}_{g,k;\mathbf{s}}^{V}(X, A)$ denote the moduli space of stable J-holomorphic $(k+\ell)$-marked maps from connected nodal curves of genus g that have contact with V at the last ℓ marked points of orders $s_1, \ldots, s_\ell$. According to [11, 9], this moduli space carries a virtual class, which is independent of J and of representative ω in a deformation equivalence class of symplectic forms on (X, V).

There are natural evaluation morphisms

$$\mathrm{ev}_X \equiv \mathrm{ev}_1 \times \ldots \times \mathrm{ev}_k \colon \overline{\mathfrak{M}}_{g,k}(X, A), \overline{\mathfrak{M}}_{g,k;\mathbf{s}}^{V}(X, A) \longrightarrow X^k, \tag{1.2}$$

$$\mathrm{ev}_X^V \equiv \mathrm{ev}_{k+1} \times \ldots \times \mathrm{ev}_{k+\ell} \colon \overline{\mathfrak{M}}_{g,k;\mathbf{s}}^{V}(X, A) \longrightarrow V_{\mathbf{s}} \equiv V^\ell, \tag{1.3}$$

sending each stable map to its values at the marked points. The (absolute) GW-invariants of (X, ω) are obtained by pulling back elements of $H^*(X^k; \mathbb{Q})$ by the morphism (1.2) and integrating them

and other natural classes on $\overline{\mathfrak{M}}_{g,k}(X,A)$ against the virtual class of $\overline{\mathfrak{M}}_{g,k}(X,A)$. The (relative) GW-invariants of (X,V,ω) are obtained by pulling back elements of $H^*(X^k;\mathbb{Q})$ and $H^*(V_{\mathbf{s}};\mathbb{Q})$ by the morphisms (1.2) and (1.3), and integrating them and other natural classes on $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$ against the virtual class of $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$.

As emphasized in [7, Section 5], two preimages of the same point in $V_{\mathbf{s}}$ under (1.3) determine an element of
$$\mathcal{R}^V_X \equiv \ker\left\{\iota^X_{X-V*}\colon H_2(X-V;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})\right\}, \tag{1.4}$$
where $\iota^X_{X-V}\colon X-V \longrightarrow X$ is the inclusion; see Section 2.1. The elements of $\mathcal{R}^V_X$, called rim tori in [7], can be represented by circle bundles over loops γ in V; see Section 3.1. By standard topological considerations,
$$\mathcal{R}^V_X \approx H_1(V;\mathbb{Z})_X \equiv \frac{H_1(V;\mathbb{Z})}{H^V_X}, \qquad \text{where} \quad H^V_X \equiv \left\{A \cap V\colon A \in H_3(X;\mathbb{Z})\right\}; \tag{1.5}$$
see Corollary 3.2.

The main claim of [7, Section 5] is that the above observations can be used to lift (1.3) over some regular (Galois), possibly disconnected (unramified) covering
$$\pi^V_{X;\mathbf{s}}\colon \widehat{V}_{X;\mathbf{s}} \longrightarrow V_{\mathbf{s}}, \tag{1.6}$$
though the topology of this cover is not specified and the group of its deck transformation is described incorrectly as $\mathcal{R}^V_X$ in [7]. Since
$$\mathrm{ev}^V_X = \pi^V_{X;\mathbf{s}} \circ \widetilde{\mathrm{ev}}^V_X\colon \overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A) \longrightarrow V_{\mathbf{s}} \tag{1.7}$$
for some morphism
$$\widetilde{\mathrm{ev}}^V_X\colon \overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A) \longrightarrow \widehat{V}_{X;\mathbf{s}}, \tag{1.8}$$
the numbers obtained by pulling back elements of $H^*(\widehat{V}_{X;\mathbf{s}};\mathbb{Q})$ by (1.8), instead of elements of $H^*(V_{\mathbf{s}};\mathbb{Q})$ by (1.3), and integrating them and other natural classes on $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$ against the virtual class of $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$ refine the usual GW-invariants of (X,V,ω). We will call these numbers the IP-counts for (X,V,ω).

The covering (1.6), which is completely determined by (X,V) and $\mathbf{s}$, is defined in Section 6.1 based on the sketch in [7, Section 5] and after some preparation in Section 5.1. For example,
$$\widehat{V}_{X;()} = \mathcal{R}^V_X \times V_{()} = \mathcal{R}^V_X,$$
where $() \in \mathbb{Z}^0_+$ is the empty vector. If V is connected, then
$$\widehat{V}_{X;(1)} = \widehat{V}_X$$
is the abelian covering corresponding to the quotient $H_1(V;\mathbb{Z})_X$ of $H_1(V;\mathbb{Z})$, i.e. to the preimage of H^V_X under the Hurewicz homomorphism $\pi_1(X) \longrightarrow H_1(X;\mathbb{Z})$. The group of deck transformations of the covering (1.6) is given by
$$\mathrm{Deck}\left(\pi^V_{X;\mathbf{s}}\right) = \frac{\mathcal{R}^V_X}{\mathcal{R}'^V_{X;\mathbf{s}}} \times \mathcal{R}'^V_{X;\mathbf{s}} \tag{1.9}$$

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for a certain submodule $\mathcal{R}'^V_{X;\mathbf{s}}$ of $\mathcal{R}^V_X$. For example,

$$\mathcal{R}'^V_{X;\mathbf{s}} = \begin{cases} \{0\}, & \text{if } \ell = 0; \\ \gcd(\mathbf{s})\mathcal{R}^V_X, & \text{if } |\pi_0(V)| = 1. \end{cases}$$

In general, the deck group (1.9) is different from $\mathcal{R}^V_X$ (contrary to an explicit statement in [7, Section 5]). By [14, Assertion 6] in the case $H_1(V;\mathbb{Z})_X$ is of rank 1 and its extension [15], $H_*(\widehat{V}_{X;\mathbf{s}};\mathbb{Q})$ is not finitely generated if V is connected, $\chi(V) \neq 0$, and $H_1(V;\mathbb{Z})_X$ is not a torsion group (so that the covering (1.6) is infinite).

By standard covering spaces considerations, the total relative evaluation map (1.3) lifts over the covering (1.6); see Lemma 6.3. The lift (1.8) of (1.3) is not unique and involves choices of base points in various spaces. However, these choices can be made in a systematic manner and the lift (1.8) extends over the space of stable smooth maps (and L_1^p-maps with $p > 2$); see Theorem 6.5 and Remark 6.7. This ensures that the IP-counts for (X, V, ω) are independent of J and of representative ω in a deformation equivalence class of symplectic forms on (X, V).

If $V = V_1 \sqcup V_2$ for some symplectic divisors $V_1, V_2 \subset X$, it is natural to consider the moduli spaces

$$\overline{\mathfrak{M}}^{V_1,V_2}_{g,k;\mathbf{s}_1\mathbf{s}_2}(X, A) \subset \bigcup_{\mathbf{s}} \overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X, A)$$

that keep track of contacts with V_1 and V_2 separately. There is then a forgetful morphism

$$f : \overline{\mathfrak{M}}^{V_1,V_2}_{g,k;\mathbf{s}_1\mathbf{s}_2}(X, A) \longrightarrow \overline{\mathfrak{M}}^{V_1}_{g,k;\mathbf{s}_1}(X, A)$$

which drops the relative contacts with V_2. The total relative evaluation morphisms (1.3) corresponding to the two moduli spaces above are compatible with f and the projection

$$\pi_1 : (V_1)_{\mathbf{s}_1} \times (V_2)_{\mathbf{s}_2} \longrightarrow (V_1)_{\mathbf{s}_1} ,$$

i.e. the part of the diagram in Figure 1 involving only the corners commutes. The projection π_1 lifts to a smooth map $\widetilde{\pi}_1$ between the total spaces of the coverings

$$\pi^{V_1,V_2}_{X;\mathbf{s}_1\mathbf{s}_2} : \widehat{V}_{X;\mathbf{s}_1\mathbf{s}_2} \longrightarrow (V_1)_{\mathbf{s}_1} \times (V_2)_{\mathbf{s}_2} \quad \text{and} \quad \pi^{V_1}_{X;\mathbf{s}_1} : (\widehat{V}_1)_{X;\mathbf{s}_1} \longrightarrow (V_1)_{\mathbf{s}_1},$$

so that the right square in Figure 1 commutes. The relevant base points determining the lifts (1.8) can be chosen so that the resulting lifted evaluation morphisms are compatible with f and $\widetilde{\pi}_1$, i.e. the left square in Figure 1 commutes.

1.2 Qualitative applications

The set of all IP-counts for (X, V) for elements in an orbit for the $\mathrm{Deck}(\pi^V_{X;\mathbf{s}})$-action on $H^*(\widehat{V}_{X;\mathbf{s}};\mathbb{Q})$ depends only on (X, V, ω), the cohomology class on X^k, and intrinsic classes on $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X, A)$, such as descendants. However, the individual IP-counts also depend on the precise choice of the lift (1.8). If $\ell = 0$, the cover (1.6) is trivial and these numbers can be indexed by the elements of $\mathcal{R}^V_X$. This is generally not the case if $\ell \neq 0$, including in the last claim of [8, Lemma 14.5] and in [8, Lemma 14.8]; see [4, Remarks 6.5, 6.8]. Because the IP-counts generally depend on the choice of the lift (1.8) and the homology of $\widehat{V}_{X;\mathbf{s}}$ is usually very complicated, they appear to be of little quantitative use outside of very rare cases. On the other hand, they can sometimes provide qualitative information, as indicated by Theorem 1.1 below.
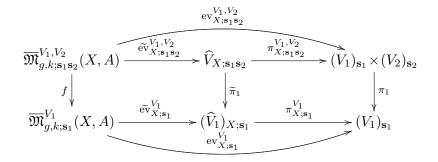
Figure 1: The potential compatibility of the lifted relative evaluation morphisms (1.8).

Theorem 1.1. *Let (X,ω) be a compact symplectic manifold and $V \subset X$ be a compact symplectic divisor which admits a fibration $q \colon V \longrightarrow (S^1)^m$ with a connected fiber F such that $H_1(F;\mathbb{Q}) = \{0\}$. If $\ell \in \mathbb{Z}^+$ and $\mathbf{s} \in \mathbb{Z}_+^\ell$, then*

$$\mathrm{ev}_X^{V*}\alpha \cap \left[\overline{\mathfrak{M}}_{g,k;\mathbf{s}}^V(X,A)\right]^{\mathrm{vir}} = 0 \qquad \forall\ \alpha \in H^r(V_{\mathbf{s}};\mathbb{Q}),\ r > (\dim_{\mathbb{R}} V)\ell - \mathrm{rk}_{\mathbb{Z}} H_1(V;\mathbb{Z})_X\,,$$

i.e. all relative GW-invariants of (X,V,ω) with non-trivial contacts with V and relative insertions α as above vanish.

The proof of this theorem readily extends to disconnected divisors V, after replacing the rank of $H_1(V;\mathbb{Z})_X$ with the rank of the appropriate submodule of $H_1(V;\mathbb{Z})_X$, depending on $\mathbf{s}$; see Remark 6.10.

If $V = (S^1)^{2n-2}$ and $H^3(X;\mathbb{Z}) = 0$, then

$$H_1(V;\mathbb{Z})_X = H_1(V;\mathbb{Z}) \approx \mathbb{Z}^{2n-2}\,.$$

In this case, the relative GW-invariants of (X,V,ω) with non-trivial contacts with V vanish whenever the degree of the relative insertion exceeds $(2n-2)(\ell-1)$. In particular, the only relative GW-invariants of (X,V,ω) with a single (but arbitrary order) contact that may be nonzero are those that involve no relative constraint (insertion $1 \in H^*(V;\mathbb{Q})$). This particular observation is immediate from (1.7), because $\widehat{V}_{X;(s)} \approx \mathbb{C}^{n-1}$ and thus $\widehat{V}_{X;(s)}$ has no positive-degree cohomology for any $s \in \mathbb{Z}^+$. We use this fact in [4, Section 6.3] to streamline the proof of [8, (15.4)], after correcting its statement; this formula computes some GW-invariants of the blowup $\widehat{\mathbb{P}}_9^2$ of $\mathbb{P}^2$ at 9 points.

If V is any topological space, a loop of homeomorphisms

$$\Psi_t \colon V \longrightarrow V, \qquad t \in [0,1], \quad \Psi_0 = \Psi_1,$$

and a point $x \in V$ determines a loop $t \longrightarrow \Psi_t(x)$ in V and thus an element of $H_1(V;\mathbb{Z})$. The latter is independent of the choice of $x \in V$. We denote the set of all elements of $H_1(V;\mathbb{Z})$ obtained in this way by $\mathrm{Flux}(V)$. It is a subgroup of $H_1(V;\mathbb{Z})$, usually called the flux subgroup (or group). If in addition $V \subset X$ is a compact oriented submanifold of a compact oriented manifold and $H_1(V;\mathbb{Z})_X$ is as in (1.5), let

$$\mathrm{Flux}(V)_X \subset H_1(V;\mathbb{Z})_X$$

denote the image of $\mathrm{Flux}(V)$ under the quotient projection.

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Theorem 1.2. *Let (X, ω) be a compact symplectic manifold and $V \subset X$ be a compact connected symplectic divisor such that*

$$\mathrm{Flux}(V)_X = H_1(V; \mathbb{Z})_X. \tag{1.10}$$

Suppose $A \in H_2(X; \mathbb{Z})$ and $\mathbf{s} \in \mathbb{Z}_+^\ell$ with $\ell > 0$. If $\gcd(\mathbf{s})$ and $|\mathcal{R}_X^V|$ are relatively prime, then the IP-counts for (X, V, ω) in degree A with relative contacts $\mathbf{s}$ are independent of the choice of the lift (1.8) and are thus determined by (X, V, ω). If in addition

$$\mathrm{rk}_{\mathbb{Z}} H_1(V; \mathbb{Z})_X \in \{0, 1\}, \tag{1.11}$$

then these IP-counts are the same as the corresponding GW-invariants of (X, V, ω).

If the rim tori module $\mathcal{R}_X^V \approx H_1(V; \mathbb{Z})_X$ is infinite, we call $\gcd(\mathbf{s})$ and $|\mathcal{R}_X^V|$ relatively prime if $\gcd(\mathbf{s}) = 1$. Let

$$H^*\big(\widehat{V}_{X;\mathbf{s}}; \mathbb{Q}\big)^{\pi_{X;\mathbf{s}}^V} = \big\{\eta \in H^*\big(\widehat{V}_{X;\mathbf{s}}; \mathbb{Q}\big) \colon g^*\eta = \eta \ \forall \ g \in \mathrm{Deck}(\pi_{X;\mathbf{s}}^V)\big\}.$$

In general, the set of GW-invariants of (X, V, ω) in degree A with relative contacts $\mathbf{s}$ with all possible cohomology insertions can be identified with the subset of IP-counts with the cohomology insertions in

$$\pi_{X;\mathbf{s}}^{V*} H^*\big(V_{\mathbf{s}}; \mathbb{Q}\big) \subset H^*\big(\widehat{V}_{X;\mathbf{s}}; \mathbb{Q}\big)^{\pi_{X;\mathbf{s}}^V} \subset H^*\big(\widehat{V}_{X;\mathbf{s}}; \mathbb{Q}\big); \tag{1.12}$$

the GW-invariants and IP-counts with such insertions are the same by (1.7). The substance of the first conclusion of Theorem 1.2 is that the second inclusion in (1.12) is an equality, as any two lifts (1.8) are related by an element of $\mathrm{Deck}(\pi_{X;\mathbf{s}}^V)$. The substance of the second conclusion of Theorem 1.2 is that both inclusions in (1.12) are equalities. The cohomology homomorphism $\pi_{X;\mathbf{s}}^{V*}$ may still not be injective; the GW-invariants of (X, V, ω) with insertions in its kernel vanish and thus can be disregarded.

Theorem 1.2 is established at the end of Section 6.1. We show that (1.10) implies that the second inclusion in (1.12) is in fact an equality. The first inclusion in (1.12) is an equality if (1.11) holds; see Corollary 5.8. Both inclusions in (1.12) are equalities if $V = \mathbb{T}^{2n-2}$ and the cover $\widehat{V}_{X;\mathbf{s}}$ is connected, as can be seen by considering all connected covers of tori. From this observation, we obtain the following conclusion concerning IP-counts relative to tori.

Proposition 1.3. *Suppose (X, ω) is a compact symplectic manifold and $V \subset X$ is a symplectic divisor such that $V \approx \mathbb{T}^{2n-2}$. Let $A \in H_2(X; \mathbb{Z})$ and $\mathbf{s} \in \mathbb{Z}_+^\ell$ with $\ell > 0$. If $\gcd(\mathbf{s})$ and $|\mathcal{R}_X^V|$ are relatively prime, then the IP-counts for (X, V, ω) in degree A with relative contacts $\mathbf{s}$ are the same as the corresponding GW-invariants of (X, V, ω).*

Remark 1.4. As pointed out by B. Wieland on *MathOverflow*, the first inclusion in (1.12) can fail to be an equality as soon as $H_1(V; \mathbb{Z})_X$ is at least $\mathbb{Z}^2$. It can fail to be an equality even if $\pi_1(V)$ is free abelian.

Theorem 1.2 and Proposition 1.3 do not provide any new information about the GW-invariants of (X, V, ω). However, they can be useful in refining the usual symplectic sum formula in a narrow set of cases. This formula expresses certain sums of GW-invariants of one symplectic manifold in terms of relative GW-invariants of simpler manifolds; Theorem 1.2 and Proposition 1.3 imply that all the summands in each given sum are the same in these cases.

Generalizations of Theorem 1.2 and of Proposition 1.3 to a disconnected divisor V are described in Remark 6.4.

1.3 Outline of the paper

The relevant setting for relative GW-invariants and the symplectic sum formula is the codimension $\mathfrak{c}=2$ case of the topological setup of Section 2. As restricting to the $\mathfrak{c}=2$ case carries no benefit, we consider the general case to the extent possible. Sections 3.1, 4.1, and 4.4 are concerned with changes in the topology of manifolds under surgery that are directly relevant in the symplectic sum context. The rim tori module $\mathcal{R}_X^V$ and the vanishing cycles module $\mathcal{R}_{X,Y}^V$ are described explicitly in Sections 3.2 and 4.2, respectively, with the aim of easily computing them in many situations. Section 4.3 compares the rim tori modules before and after surgery. The notation for the abelian covers relevant for our purposes is introduced in Section 5.1; some of their properties, focusing on finite generation of the (co)homology, are discussed in Section 5.2. In Section 6.1, we define the intended rim tori coverings (1.6) of [7, Section 5] as special cases of the abelian covers of Section 5.1, show that the evaluation morphisms (1.3) lift to these covers, and establish Theorem 1.2. In Section 6.2, we show that these lifts can be chosen systematically, in respect to the intended applications in [8] and the diagram in Figure 1; see Theorem 6.5. Theorem 1.1 is established in Section 6.3.

The main purpose of this paper is to investigate the topological aspects of the rim tori refinement to the standard relative GW-invariants in preparation for considering its applicability in the symplectic sum context in [4]. We pre-suppose a suitable analytic framework for the construction of relative GW-invariants and describe the necessary steps to implement the idea of [7, Section 5] as an enhancement on existing constructions. We deduce some qualitative applications arising from this refinement and discuss its usability for quantitative purposes. A significant number of examples are included for illustrative purposes.

2 General topological context

The symplectic sum construction is a surgery operation that cuts out tubular neighborhoods of a common submanifold from two manifolds and glues the remainders along the boundaries of the two tubular neighborhoods. Below we discuss central topological aspects of this construction from a more general perspective. Throughout this paper, by a manifold we will mean a smooth manifold.

2.1 Complement of submanifold

Let X be an oriented manifold, $V \subset X$ be a compact oriented submanifold of codimension $\mathfrak{c}$, $S_X V \subset \mathcal{N}_X V$ be the sphere subbundle of the normal bundle of V in X, and

$$\mathcal{R}_X^V \equiv \ker \left\{ \iota_{X-V*}^X \colon H_{\mathfrak{c}}(X-V; \mathbb{Z}) \longrightarrow H_{\mathfrak{c}}(X; \mathbb{Z}) \right\}. \tag{2.1}$$

By Lemma 3.1, each element of $\mathcal{R}_X^V$ can be represented by a cycle of the form $\iota_{S_X V}^{X-V}(S_X V|_\gamma)$ for some loop $\gamma \subset V$; see the end of Section 3.1. In the $\mathfrak{c}=2$ case, i.e. as in (1.4), these cycles are called rim tori in [7, 8].

Suppose in addition that $f\colon Z\longrightarrow X$ is an $\mathfrak{c}$-pseudocycle, as in [21, Section 1.1], and $x\in f^{-1}(V)$ is an isolated point. We define the order of contact of f with V at x, $\mathrm{ord}_x^V f\in\mathbb{Z}$, as follows. On a small neighborhood of x, f can be homotoped without changing its intersection with V so that it takes a small sphere $S_Z x$ in $T_x Z$ to a small sphere $S_X V|_{f(x)}\subset\mathcal{N}_V X$; the number $\mathrm{ord}_x^V f$ is the degree of this map. This definition agrees with the definition used in the construction of $\overline{\mathfrak{M}}_{g,k;\mathbf{s}}^V(X,A)$.

We now combine two pseudocycles with the same contacts with V into a pseudocycle to $X-V$. Suppose

$$f\colon(Z,x_1,\ldots,x_\ell)\longrightarrow(X,V)\qquad\text{and}\qquad f'\colon(Z',x_1',\ldots,x_\ell')\longrightarrow(X,V)$$

are two $\mathfrak{c}$-pseudocycles such that

$$f^{-1}(V)=\{x_1,\ldots,x_\ell\},\qquad f'^{-1}(V)=\{x_1',\ldots,x_\ell'\},$$
$$f(x_i)=f'(x_i'),\quad\mathrm{ord}_{x_i}^V f=\mathrm{ord}_{x_i'}^V f'\qquad\forall\, i=1,2,\ldots,\ell.$$

We can then obtain a smooth map $f\#(-f')\colon Z\#Z'\longrightarrow X-V$ by

- removing small balls B_{x_i} and $B_{x_i'}$ around each of the points x_i and x_i' to form manifolds with boundary $\hat Z$ and $\hat Z'$,

- forming a smooth oriented manifold $Z\#(-Z')$ by identifying the i-th boundary components of $\hat Z$ and $\hat Z'$ by an orientation-preserving diffeomorphism $\varphi_i\colon(\partial\hat Z)_i\longrightarrow(\partial\hat Z')_i$ for each i,

- homotoping f and f' on small neighborhoods of ∂B_{x_i} and $\partial B_{x_i'}$ within a small ball around $f(x_i)=f'(x_i')$ in X so that $f=f'\circ\varphi_i$ for all i.

The last condition is achievable because the degrees of $f, f'\circ\varphi_i\colon S^{\mathfrak{c}-1}\longrightarrow S^{\mathfrak{c}-1}$ are the same and the degree homomorphism $\pi_{\mathfrak{c}-1}(S^{\mathfrak{c}-1})\longrightarrow\mathbb{Z}$ is an isomorphism if $\mathfrak{c}\geq 2$.

The above construction of $f\#(-f')$ depends only on f, f', and choices of degree 1 maps from ℓ disjoint copies of $[0,1]\times S^{\mathfrak{c}-1}$ to $[0,1]\times S^{\mathfrak{c}-1}$. Thus, f and f' completely determine the homology class of $f\#(-f')$. If in addition $[f]=[f']$ in $H_{\mathfrak{c}}(X;\mathbb{Z})$, then $[f\#(-f')]\in\mathcal{R}_X^V$.

2.2 Splice of two manifolds

If X and Y are manifolds, $V\subset X,Y$ is a closed submanifold, and $\varphi\colon S_X V\longrightarrow S_Y V$ is a diffeomorphism commuting with the projections to V, let $X\#_\varphi Y$ be the manifold obtained by gluing the complements of tubular neighborhoods of V in X and Y by φ along their common boundary. If X and Y are oriented and φ is orientation-reversing, then $X\#_\varphi Y$ is oriented as well.

We denote by

$$q_\varphi\colon X\#_\varphi Y\longrightarrow X\cup_V Y$$

a continuous map which restricts to the identity outside of a tubular neighborhood of $S_X V=_\varphi S_Y V$, is a diffeomorphism on the complement of $q_\varphi^{-1}(V)$, and restricts to the bundle projection $S_X V\longrightarrow V$. We will call such a map q_φ a collapsing map. If $\mathfrak{c}$ is the codimension of V in X and Y, let

$$\mathcal{R}_{X,Y}^V\equiv\ker\bigl\{q_{\varphi*}\colon H_{\mathfrak{c}}(X\#_\varphi Y;\mathbb{Z})\longrightarrow H_{\mathfrak{c}}(X\cup_V Y;\mathbb{Z})\bigr\}. \tag{2.2}$$

By the $m = \mathfrak{c}$ case of Lemma 4.1, this collection of vanishing cycles is the span of the images of $\mathcal{R}_X^V$ and $\mathcal{R}_Y^V$ under the homology homomorphisms induced by the inclusions

$$\iota_{X-V}^{X\#_\varphi Y} : X - V \longrightarrow X\#_\varphi Y \qquad \text{and} \qquad \iota_{Y-V}^{X\#_\varphi Y} : Y - V \longrightarrow X\#_\varphi Y,$$

respectively.

Suppose in addition X, Y, and V are compact and oriented and $V_1, \ldots, V_N$ are the topological components of V. Let

$$H_\mathfrak{c}(X;\mathbb{Z}) \times_V H_\mathfrak{c}(Y;\mathbb{Z}) = \big\{ (A_X, A_Y) \in H_\mathfrak{c}(X;\mathbb{Z}) \times H_\mathfrak{c}(Y;\mathbb{Z}) : \\ A_X \cdot_X V_r = A_Y \cdot_Y V_r \quad \forall\, r = 1, \ldots, N \big\}, \tag{2.3}$$

where $\cdot_X$ and $\cdot_Y$ denote the homology intersection pairings in X and Y, respectively. Given an orientation-reversing diffeomorphism φ, we describe below an operation gluing $\mathfrak{c}$-cycles in X and Y into $\mathfrak{c}$-cycles in $X\#_\varphi Y$. It induces a homomorphism

$$H_\mathfrak{c}(X;\mathbb{Z}) \times_V H_\mathfrak{c}(Y;\mathbb{Z}) \longrightarrow H_\mathfrak{c}(X\#_\varphi Y;\mathbb{Z})/\mathcal{R}_{X,Y}^V, \qquad (A_X, A_Y) \longrightarrow A_X \#_\varphi A_Y, \tag{2.4}$$

which is central to the symplectic sum formula for GW-invariants.

Suppose

$$f_X : (Z_X, x_1, \ldots, x_\ell) \longrightarrow (X, V) \qquad \text{and} \qquad f_Y : (Z_Y, y_1, \ldots, y_\ell) \longrightarrow (Y, V)$$

are $\mathfrak{c}$-pseudocycles with boundary disjoint from V such that

$$f_X^{-1}(V) = \{x_1, \ldots, x_\ell\}, \qquad f_Y^{-1}(V) = \{y_1, \ldots, y_\ell\}, \\ f_X(x_i) = f_Y(y_i), \quad \mathrm{ord}_{x_i}^V f_X = \mathrm{ord}_{y_i}^V f_Y \qquad \forall\, i = 1, 2, \ldots, \ell.$$

We can then obtain a smooth map $f_X \#_\varphi f_Y : Z_X \# Z_Y \longrightarrow X\#_\varphi Y$ by

- removing small balls B_{x_i} and B_{y_i} around each of the points x_i and y_i to form manifolds with boundary $\hat{Z}_X$ and $\hat{Z}_Y$,

- forming a smooth oriented manifold $Z_X \# Z_Y$ by identifying the i-th boundary components of $\hat{Z}_X$ and $\hat{Z}_Y$ by an orientation-reversing diffeomorphism $\varphi_i : (\partial \hat{Z}_X)_i \longrightarrow (\partial \hat{Z}_Y)_i$ for each i,

- homotoping f_X and f_Y on small neighborhoods of ∂B_{x_i} and ∂B_{y_i} within small balls around $f_X(x_i)$ in X and $f_Y(y_i)$ in Y so that $\varphi \circ f_X = f_Y \circ \varphi_i$ for all i.

The last condition is achievable because the degrees of

$$\varphi \circ f_X \circ \varphi_i^{-1}, f_Y : S^{\mathfrak{c}-1} \longrightarrow S^{\mathfrak{c}-1}$$

are the same.

The above construction of $f_X \#_\varphi f_Y$ depends only on f_X, f_Y, and choices of degree -1 maps from ℓ disjoint copies of $[0,1] \times S^{\mathfrak{c}-1}$ to $[0,1] \times S^{\mathfrak{c}-1}$. Thus, f_X and f_Y completely determine the homology class of $f_X \#_\varphi f_Y$. Furthermore,

$$q_{\varphi*}([f_X \#_\varphi f_Y]) = \iota_{X*}^{X \cup_V Y}([f_X]) + \iota_{Y*}^{X \cup_V Y}([f_Y]) \in H_\mathfrak{c}(X \cup_\varphi Y; \mathbb{Z}).$$

9

If $[f_X] = [f'_X]$ in $H_{\mathfrak{c}}(X; \mathbb{Z})$, $[f_X \# (-f'_X)] \in \mathcal{R}^V_X$ by Section 2.1. Thus, the homology class of $f_X \#_\varphi f_Y$ in $X \#_\varphi Y$ as above is determined by the homology classes of f_X in X and f_Y in Y only up to an element of $\mathcal{R}^V_{X,Y}$.

Suppose (X, ω_X) and (Y, ω_Y) are symplectic manifolds with a common compact symplectic divisor $V \subset X, Y$ such that
$$e(\mathcal{N}_X V) = -e(\mathcal{N}_Y V) \in H^2(V; \mathbb{Z}).$$

The symplectic sum construction of [6, 18] then produces a symplectic manifold $(X \#_V Y, \omega_\#)$ of the form $X \#_\varphi Y$. Let η be a coset of $H_2(X \#_V Y; \mathbb{Z})$ modulo $\mathcal{R}^V_{X,Y}$. According to the symplectic sum formulas of [11, 10], the sum of the genus g GW-invariants of $X \#_V Y$ in degrees $A \in \eta$ is the same as the sum of the genus g GW-invariants of (X, V, ω_X) and (Y, V, ω_Y) of degrees A_X and A_Y such that $A_X \#_\varphi A_Y = \eta$. It would of course be preferable to express individual GW-invariants of $X \#_V Y$ in terms of relative GW-invariants of (X, V, ω_X) and (Y, V, ω_Y). The rim tori refinement of standard relative invariants is suggested in [7] with the aim of resolving this deficiency in [8]. In [4], we discuss to what extent this is achieved.

3 Cutting out a submanifold

We discuss changes in the homology after cutting out a submanifold in Section 3.1. Lemma 3.1 contains [7, Lemma 5.2] and the corresponding part of the proof of the former is essentially the same as the proof of the latter. We use it in Section 3.2 to give an explicit description of the rim tori module $\mathcal{R}^V_X$ and to compare it with the rim tori module $\mathcal{R}^{U \cup V}_X$ for a submanifold with additional connected components.

3.1 Changes in homology

Given a manifold X and a closed submanifold $V \subset X$, we will view the sphere subbundle $S_X V$ of the normal bundle $\mathcal{N}_X V$ of V in X as a hypersurface in X. If in addition V is compact oriented and the codimension of V in X is $\mathfrak{c}$, we define

$$\Delta^V_X : H_m(V; \mathbb{Z}) \longrightarrow H_{m+\mathfrak{c}-1}(S_X V; \mathbb{Z}), \qquad \Delta^V_X(\gamma) = \mathrm{PD}_{S_X V}\big(q_X^{V^*}(\mathrm{PD}_V \gamma)\big), \qquad (3.1)$$

where $q_X^V : S_X V \longrightarrow V$ is the projection map. If X is also compact, let

$$\cap V : H_*(X; \mathbb{Z}) \longrightarrow H_{*-\mathfrak{c}}(V; \mathbb{Z}), \qquad A \cap V = \mathrm{PD}_V\big((\mathrm{PD}_X A)|_V\big).$$

If $U \subset X - V$ is another subset (possibly empty), we take

$$\cap V : H_*(X - U; \mathbb{Z}) \xrightarrow{\iota^X_{X-U*}} H_*(X; \mathbb{Z}) \xrightarrow{\cap V} H_{*-\mathfrak{c}}(V; \mathbb{Z})$$

to be the composition. Let

$$H^V_{X-U} = \big\{A \cap V : A \in H_{\mathfrak{c}+1}(X - U; \mathbb{Z})\big\} \subset H_1(V; \mathbb{Z}), \quad H_1(V; \mathbb{Z})_{X-U} = \frac{H_1(V; \mathbb{Z})}{H^V_{X-U}}. \qquad (3.2)$$

Lemma 3.1. *Suppose X is a compact oriented manifold, $V \subset X$ is a compact oriented submanifold of codimension $\mathfrak{c}$, and $U \subset X - V$ is a compact subset. Then, the sequence*

$$\ldots \xrightarrow{} H_m(X - U \cup V) \xrightarrow{\iota^{X-U}_{X-U\cup V*}} H_m(X - U) \xrightarrow{\cap V} H_{m-\mathfrak{c}}(V)$$
$$\xrightarrow{\iota^{X-U\cup V}_{S_X V*} \circ \Delta^V_X} H_{m-1}(X - U \cup V) \xrightarrow{} \ldots \tag{3.3}$$

is exact for any coefficient ring.

Proof. Taking the Poincare dual of the Gysin sequence for $S_X V \longrightarrow V$, we obtain an exact sequence

$$\ldots \xrightarrow{\Delta^V_X} H_m(S_X V) \xrightarrow{q^V_{X*}} H_m(V) \xrightarrow{e(\mathcal{N}_X V) \cap} H_{m-\mathfrak{c}}(V) \xrightarrow{\Delta^V_X} H_{m-1}(S_X V) \xrightarrow{q^V_{X*}} \ldots. \tag{3.4}$$

By the proof of Mayer-Vietoris for $X - U = (X - U \cup V) \cup_{S_X V} V$,

$$\ldots \xrightarrow{\delta_X} H_m(S_X V) \xrightarrow{(\iota^{X-U\cup V}_{S_X V*}, -q^V_{X*})} H_m(X - U \cup V) \oplus H_m(V) \xrightarrow{\iota^{X-U}_{X-U\cup V*} + \iota^{X-U}_{V*}} H_m(X - U)$$
$$\xrightarrow{\delta_X} H_{m-1}(S_X V) \xrightarrow{(\iota^{X-U\cup V}_{S_X V*}, -q^V_{X*})} \ldots, \tag{3.5}$$

the connecting homomorphism δ_X is the composition $\Delta^V_X \circ (\cdot \cap V)$, at least up to sign. Since

$$\iota^{X-U}_{X-U\cup V*}(A) \cap V = 0 \qquad \forall\, A \in H_m(X - U \cup V), \tag{3.6}$$

the claim now follows from the observation that

$$\iota^{X-U}_{V*}(A) \cap V = e(\mathcal{N}_X V) \cap A \qquad \forall\, A \in H_m(V), \tag{3.7}$$

at least up to sign (dependent on one's definitions of cup and cap products and Poincare dual); see below for details.

The composition of the first two labeled homomorphisms in (3.3) being 0 is equivalent to (3.6). If $A_{X-U} \cap V = 0$ for some $A_{X-U} \in H_m(X - U)$, then $\delta_X(A_{X-U}) = 0$ and so

$$A_{X-U} = \iota^{X-U}_{X-U\cup V*}(A_{X-U\cup V}) + \iota^{X-U}_{V*}(A_V) \quad \text{for some } A_{X-U\cup V} \in H_m(X - U \cup V),\ A_V \in H_m(V),$$

by the exactness of (3.5). By $A_{X-U} \cap V = 0$, (3.6), and (3.7), $e(\mathcal{N}_X V) \cap A_V = 0$ and so

$$A_V = q^V_{X*}(A_{S_X V}) \quad \text{for some } A_{S_X V} \in H_m(S_X V)$$

by the exactness of (3.4). By the last two displayed expressions,

$$A_{X-U} = \iota^{X-U}_{X-U\cup V*}(A_{X-U\cup V} + \iota^{X-U\cup V}_{S_X V*}(A_{S_X V})),$$

which establishes the exactness of (3.3) at $H_m(X - U)$.

The composition of the last two labeled homomorphisms in (3.3) being 0 follows from the exactness of (3.5) and $\delta_X = \Delta^V_X \circ (\cdot \cap V)$. Suppose

$$\{\iota^{X-U\cup V}_{S_X V*} \circ \Delta^V_X\}(A_V) = 0 \quad \text{for some } A_V \in H_{m-\mathfrak{c}}(V).$$

Since $q_{X*}^V \circ \Delta_X^V = 0$ by the exactness of (3.4),

$$\Delta_X^V(A_V) = \delta_X(A_{X-U}) = \Delta_X^V(A_{X-U} \cap V) \quad \text{for some} \ \ A_{X-U} \in H_m(X{-}U)$$

by the exactness of (3.5). Thus,

$$A_V = A_{X-U} \cap V + e(\mathcal{N}_X V) \cap A_V' = \big(A_{X-U} + \iota_{V*}^{X-U}(A_V')\big) \cap V \quad \text{for some} \ \ A_V' \in H_m(V)$$

by the exactness of (3.4) and (3.7). This establishes the exactness of (3.3) at $H_{m-\mathfrak{c}}(V)$.

The vanishing of the composition

$$\iota_{X-U\cup V*}^{X-U} \circ \big\{ \iota_{S_X V*}^{X-U\cup V} \circ \Delta_X^V \big\} \colon H_{m-\mathfrak{c}}(V) \longrightarrow H_{m-1}(X{-}U)$$

follows from the exactness of (3.5). If $\iota_{X-U\cup V*}^{X-U}(A_{X-U\cup V}) = 0$ for some $A_{X-U\cup V} \in H_{m-1}(X{-}U\cup V)$, then

$$A_{X-U\cup V} = \iota_{S_X V*}^{X-U\cup V}(A_{S_X V}), \quad q_{X*}^V(A_{S_X V}) = 0 \quad \text{for some} \ \ A_{S_X V} \in H_{m-1}(S_X V)$$

by the exactness of (3.5). By the exactness of (3.4) and $q_{X*}^V(A_{S_X V}) = 0$,

$$A_{S_X V} = \Delta_X^V(A_V) \quad \text{for some} \ \ A_V \in H_{m-\mathfrak{c}}(V).$$

This establishes the exactness of (3.3) at $H_{m-1}(X{-}U\cup V)$. $\hfill\square$

By [21, Theorem 1.1], every integral homology class in a manifold can be represented by a pseudocycle. If V is as in (3.1) and $f \colon Z \longrightarrow V$ is a pseudocycle, then the pseudocycle

$$\pi_2 \colon f^* S_X V \equiv \big\{ (z,v) \in Z \times S_X V \colon f(z) = q_X^V(v) \big\} \longrightarrow S_X V,$$

where $\pi_2 \colon Z \times S_X V \longrightarrow S_X V$ is the projection on the second coordinate, represents $\Delta_X^V([f])$. If $Z = S^1$, $f^* S_X V \longrightarrow S^1$ is a trivial $S^{\mathfrak{c}-1}$-bundle and thus f lifts to a map

$$\tilde{f} \colon S^1 \times S^{\mathfrak{c}-1} \longrightarrow \{q_X^V\}^{-1}(f(S^1)) \subset S_X V \qquad \text{s.t.} \quad q_X^V \circ \tilde{f} = f \circ \pi_1,$$

where $\pi_1 \colon S^1 \times S^{\mathfrak{c}-1} \longrightarrow S^1$ is the projection on the first component. Thus, the elements of the module $\mathcal{R}_{X-U}^V$ as in (2.1) can be represented by cycles of the form $\iota_{S_X V}^{X-U\cup V}(S_X V|_\gamma)$ for loops $\gamma \subset V$, according to Lemma 3.1.

3.2   The rim tori

We now use Lemma 3.1 to describe the rim tori module $\mathcal{R}_{X-U}^V$ explicitly and to compare it with other such modules. The first corollary below is an immediate consequence of the $m = \mathfrak{c}$ case of this lemma.

Corollary 3.2. *Suppose X is a compact oriented n-manifold, $V \subset X$ is a compact oriented submanifold of codimension $\mathfrak{c}$, and $U \subset X{-}V$ is a compact subset. Then, the induced homomorphism*

$$\iota_{S_X V*}^{X-U\cup V} \circ \Delta_X^V \colon H_1(V;\mathbb{Z})_{X-U} \longrightarrow \mathcal{R}_{X-U}^V \subset H_{\mathfrak{c}}(X{-}U\cup V;\mathbb{Z})$$

is well-defined and is an isomorphism. In particular, if the restriction homomorphism

$$H^{n-\mathfrak{c}-1}(X;\mathbb{Z}) \longrightarrow H^{n-\mathfrak{c}-1}(V;\mathbb{Z}), \tag{3.8}$$

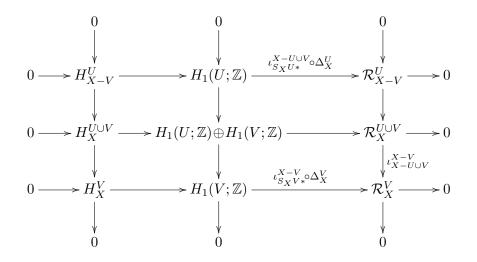*is zero, then $\mathcal{R}_{X-U}^V \approx H_1(V;\mathbb{Z})$.*

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^U_{X-V} & \longrightarrow & H_1(U;\mathbb{Z}) & \xrightarrow{\iota^{X-U\cup V}_{S_X U*}\circ\Delta^U_X} & \mathcal{R}^U_{X-V} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{U\cup V}_X & \longrightarrow & H_1(U;\mathbb{Z})\oplus H_1(V;\mathbb{Z}) & \longrightarrow & \mathcal{R}^{U\cup V}_X & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow{\scriptstyle \iota^{X-V}_{X-U\cup V}} & & \\
0 & \longrightarrow & H^V_X & \longrightarrow & H_1(V;\mathbb{Z}) & \xrightarrow{\iota^{X-V}_{S_X V*}\circ\Delta^V_X} & \mathcal{R}^V_X & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 2: Comparison of rim tori.

Corollary 3.3. *Suppose X is a compact oriented manifold and $U,V\subset X$ are compact oriented disjoint submanifolds of codimension $\mathfrak{c}$. Then, the homology homomorphisms induced by inclusions give rise to the commutative square of short exact sequences of Figure 2.*

Proof. The rows in this diagram are exact by Corollary 3.2. The exactness of the middle column is clear, as is the exactness of the last column at $\mathcal{R}^U_{X-V}$. The last column is exact at $\mathcal{R}^{U\cup V}_X$ by (2.1), with (X,V) replaced by $(X-V,U)$. By the exactness of (3.5),

$$
\mathcal{R}^V_X = \big\{\iota^{X-V}_{S_X V*}(A_{S_X V})\colon A_{S_X V}\in H_{\mathfrak{c}}(S_X V;\mathbb{Z}),\ q^V_{X*}(S_X V)=0\big\},
$$
$$
\mathcal{R}^{U\cup V}_X \supset \big\{\iota^{X-U\cup V}_{S_X V*}(A_{S_X V})\colon A_{S_X V}\in H_{\mathfrak{c}}(S_X V;\mathbb{Z}),\ q^V_{X*}(S_X V)=0\big\}.
$$

This implies that the last column is exact at $\mathcal{R}^V_X$.

By (3.2) and (3.6) with $U=\emptyset$,

$$
H^U_{X-V}\oplus 0 \subset H^{U\cup V}_X \cap H_1(U;\mathbb{Z})\oplus 0 = \big\{(A\cap U,0)\colon A\in H_{\mathfrak{c}+1}(X;\mathbb{Z})\ \text{s.t.}\ A\cap V=0\big\}.
$$

In particular, the left column in the diagram is exact at H^U_{X-V}. By the exactness of (3.3) with $m=\mathfrak{c}+1$ and $U=\emptyset$ at the second position, the above inclusion is in fact an equality and the left column is exact at $H^{U\cup V}_X$. The exactness of the left column at $H^{U\cup V}_X$ is immediate from (3.2).

The commutativity of the bottom right square is equivalent to the homomorphism

$$
\iota^{X-V}_{X-U\cup V*}\circ\iota^{X-U\cup V}_{S_X U*}\circ\Delta^U_X\colon H_1(U;\mathbb{Z}) \longrightarrow H_{\mathfrak{c}}(X-V;\mathbb{Z})
$$

being zero. This is immediate from the exactness of (3.5) with U and V interchanged, since $q^U_{X*}\circ\Delta^U_X=0$ by the exactness of (3.4). The commutativity of the other three squares is clear. $\square$

Example 3.4. Let Z be a compact oriented manifold and $E_1, E_2 \longrightarrow Z$ be complex vector bundles of ranks r_1 and r_2, respectively. By [19, Theorem 5.7.9], there is a commutative diagram

$$
\begin{array}{ccc}
H_*(\mathbb{P}E_2; \mathbb{Z}) & \longrightarrow & H_*(\mathbb{P}(E_1 \oplus E_2); \mathbb{Z}) \\
\approx \downarrow & & \downarrow \approx \\
H_*(Z; \mathbb{Z}) \otimes H_*(\mathbb{P}^{r_2-1}; \mathbb{Z}) & \longrightarrow & H_*(Z; \mathbb{Z}) \otimes H_*(\mathbb{P}^{r_1+r_2-1}; \mathbb{Z})
\end{array}
$$

of homomorphisms of modules. In particular, the homomorphism

$$
\iota^{\mathbb{P}(E_1 \oplus E_2)}_{\mathbb{P}(E_1 \oplus E_2) - \mathbb{P}E_1 *} : H_{2r_2}\big(\mathbb{P}(E_1 \oplus E_2) - \mathbb{P}E_1; \mathbb{Z}\big) \approx H_{2r_2}\big(\mathbb{P}E_2; \mathbb{Z}\big) \longrightarrow H_{2r_2}\big(\mathbb{P}(E_1 \oplus E_2); \mathbb{Z}\big)
$$

is injective. Thus,

$$
\mathcal{R}^{\mathbb{P}E_1}_{\mathbb{P}(E_1 \oplus E_2)} = \{0\}.
$$

In the case $r_2 = 1$, which is the most relevant for our purposes, this statement follows from $\mathbb{P}E_2$ being a section of the fiber bundle $\mathbb{P}(E_1 \oplus E_2) \longrightarrow Z$.

Example 3.5. Let $\widehat{\mathbb{P}}^2_9$ denote the blowup of $\mathbb{P}^2$ at the 9 intersection points of a general pair of smooth cubic curves C_1 and C_2, i.e. the base locus of a general pencil of cubics in $\mathbb{P}^2$. The proper transforms of these cubics in $\widehat{\mathbb{P}}^2_9$ are pairwise disjoint and form a fibration $\pi\colon \widehat{\mathbb{P}}^2_9 \longrightarrow \mathbb{P}^1$, obtained by sending each point in $\widehat{\mathbb{P}}^2_9$ to the cubic in the pencil passing through it. A smooth fiber F of π is a torus $\mathbb{T}^2$; there are also 12 singular fibers, each of which is a sphere with a transverse self-intersection. By Corollary 3.2,

$$
\mathcal{R}^F_{\widehat{\mathbb{P}}^2_9} \approx H_1(F; \mathbb{Z}) \approx \mathbb{Z}^2,
$$

because $H^1(\widehat{\mathbb{P}}^2_9; \mathbb{Z}) = 0$.

Example 3.6. Let F be a compact oriented manifold, $X = \mathbb{P}^1 \times F$, $F_0 = \{0\} \times F$, and $F_\infty = \{\infty\} \times F$. Then,

$$
H^{F_0}_X = H_1(F_0; \mathbb{Z}) = H_1(F; \mathbb{Z}), \qquad H^{F_\infty}_{X-F_0} = \{0\} \subset H_1(F_\infty; \mathbb{Z}) = H_1(F; \mathbb{Z}),
$$
$$
H^{F_0 \cup F_\infty}_X = H_\Delta \subset H_1(F_0 \cup F_\infty; \mathbb{Z}) = H_1(F; \mathbb{Z}) \oplus H_1(F; \mathbb{Z}), \tag{3.9}
$$

where H_Δ is the diagonal subgroup. By Lemma 3.1 or Corollary 3.2, the homomorphism

$$
\iota^{X-F_0 \cup F_\infty}_{S_X F_0 *} \circ \Delta^{F_0}_X : H_1(F_0; \mathbb{Z}) = H_1(F; \mathbb{Z}) \longrightarrow \mathcal{R}^{F_0 \cup F_\infty}_X \subset H_{\mathfrak{c}}(X - F_0 \cup F_\infty; \mathbb{Z})
$$

is an isomorphism. Under this identification, the last labeled homomorphism in (3.3) with $m = 3$ corresponds to

$$
H_1(F_0 \cup F_\infty; \mathbb{Z}) = H_1(F; \mathbb{Z}) \oplus H_1(F; \mathbb{Z}) \longrightarrow H_1(F; \mathbb{Z}), \quad (\gamma_0, \gamma_\infty) \longrightarrow \gamma_0 - \gamma_\infty. \tag{3.10}
$$

4 Gluing along a common submanifold

We discuss changes in the homology and cohomology after gluing two manifolds along a common submanifold in Sections 4.1 and 4.4, respectively. We use Lemma 4.1 in Section 4.2 to express the vanishing cycles module $\mathcal{R}^V_{X,Y}$ defined in (2.2) in terms of the rim tori modules $\mathcal{R}^V_X$ and $\mathcal{R}^V_Y$ defined in (2.1). This lemma is used in Section 4.3 to compare the rim tori modules before and after gluing. Lemma 4.11 contains the precise statements of (a) and (b) at the bottom of [8, p996]; it is useful for determining the cohomology insertions compatible with the symplectic sum formula for GW-invariants.

4.1 Changes in homology

Continuing with the notation introduced in Section 2.2, we relate the analogue of the module (2.2) for homology of any dimension to such analogues of the modules (2.1) for the two pieces.

Lemma 4.1. *If X and Y are manifolds, $V \subset X, Y$ is a closed submanifold, $\varphi\colon S_X V \longrightarrow S_Y V$ is a diffeomorphism commuting with the projections to V, and $q_\varphi\colon X\#_\varphi Y \longrightarrow X\cup_V Y$ is a collapsing map, then*

$$\ker\big\{q_{\varphi*}\colon H_m(X\#_\varphi Y)\longrightarrow H_m(X\cup_\varphi Y)\big\}$$
$$= \big\{\iota^{X\#_\varphi Y}_{X-V*}(A_{X-V})\colon\ A_{X-V}\in H_m(X-V),\ \iota^X_{X-V*}(A_{X-V})=0\big\} \tag{4.1}$$
$$= \big\{\iota^{X\#_\varphi Y}_{Y-V*}(A_{Y-V})\colon\ A_{Y-V}\in H_m(Y-V),\ \iota^Y_{Y-V*}(A_{Y-V})=0\big\}$$

for all $m\in\mathbb{Z}$ and for any coefficient ring.

Proof. Denote the codimension of V in X and Y by $\mathfrak{c}$, $S_X V \approx S_Y V$ by SV, and the bundle projection map $SV \longrightarrow V$ by q_V. Mayer-Vietoris for $X\#_\varphi Y = (X-V)\cup_{SV}(Y-V)$ and $X\cup_V Y$ give a commutative pair of long exact sequences

$$
\begin{array}{ccccccc}
H_m(SV) & \longrightarrow & H_m(X-V)\oplus H_m(Y-V) & \xrightarrow{\iota^{X\#_\varphi Y}_{X-V*}+\iota^{X\#_\varphi Y}_{Y-V*}} & H_m(X\#_\varphi Y) & \xrightarrow{\delta_\varphi} & H_{m-1}(SV) \\
\downarrow{\scriptstyle q_{V*}} & & \downarrow{\scriptstyle \iota^X_{X-V*}\oplus\iota^Y_{Y-V*}} & & \downarrow{\scriptstyle q_{\varphi*}} & & \downarrow{\scriptstyle q_{V*}} \\
H_m(V) & \xrightarrow{(\iota^X_{V*},-\iota^Y_{V*})} & H_m(X)\oplus H_m(Y) & \xrightarrow{\iota^{X\cup_V Y}_{X*}+\iota^{X\cup_V Y}_{Y*}} & H_m(X\cup_V Y) & \xrightarrow{\delta_\cup} & H_{m-1}(V)
\end{array}
$$

The commutativity of the middle square above implies that the second and third expressions in (4.1) are contained in the first.

Suppose $A_\#\in H_m(X\#_V Y)$ and $q_{\varphi*}(A_\#)=0$. By the proof of Mayer-Vietoris for $X\#_\varphi Y$, there exist bordered pseudocycles

$$f_X\colon (Z_X,\partial Z_X) \longrightarrow (X-V,SV) \qquad\text{and}\qquad f_Y\colon (Z_Y,\partial Z_Y) \longrightarrow (Y-V,SV)$$

such that $\partial Z_X = -\partial Z_Y$ and

$$f_X \underset{\partial Z_X=-\partial Z_Y}{\cup} f_Y\colon Z_X \underset{\partial Z_X=-\partial Z_Y}{\cup} Z_Y \longrightarrow X\#_\varphi Y$$

represents the homology class $A_\#$. Since

$$q_{V*}\big[f_X|_{\partial Z_X}\big] = q_{V*}\delta_\varphi(A_\#) = \delta_\cup q_{\varphi*}(A_\#) = 0,$$

by (3.4) we can choose f_X and f_Y so that $f_X(\partial Z_X) = f_Y(\partial Z_Y)$ equals $SV|_{B_V}$ for some class $B_V\in H_{m-\mathfrak{c}}(V)$. The smooth maps

$$\iota^X_{X-V}\circ f_X\colon Z_X \longrightarrow X \qquad\text{and}\qquad \iota^Y_{Y-V}\circ f_Y\colon Z_Y \longrightarrow Y$$

then determine homology classes A_X on X and A_Y on Y; in the exceptional $\mathfrak{c}=1$ case, the two boundary components of these maps come with opposite signs and thus cancel. By the commutativity of the diagram on the chain level inducing the above diagram in homology,

$$\iota^{X\cup_V Y}_{X*}(A_X) + \iota^{X\cup_V Y}_{Y*}(A_Y) = q_{\varphi*}(A_\#) = 0.$$

Thus, there exists $A_V \in H_m(V)$ such that

$$A_X = \iota^X_{V*}(A_V), \qquad A_Y = -\iota^Y_{V*}(A_V).$$

The Mayer-Vietoris sequence (3.5) with $U = \emptyset$ then gives

$$[f_X|_{\partial Z_X}] = \delta_X(A_X) = 0 \qquad \Longrightarrow \qquad \delta_\varphi(A_\#) = [f_X|_{\partial Z_X}] = 0.$$

The first Mayer-Vietoris sequence now implies that

$$A_\# = \iota^{X\#_\varphi Y}_{X-V*}(A_{X-V}) + \iota^{X\#_\varphi Y}_{Y-V*}(A_{Y-V}) \tag{4.2}$$

for some $A_{X-V} \in H_m(X-V)$ and $A_{Y-V} \in H_m(Y-V)$. Since

$$\iota^{X\cup_V Y}_{X*}\big(\iota^X_{X-V*}(A_{X-V})\big) + \iota^{X\cup_V Y}_{Y*}\big(\iota^Y_{Y-V*}(A_{Y-V})\big)$$
$$= q_{\varphi*}\big(\iota^{X\#_\varphi Y}_{X-V*}(A_{X-V})\big) + q_{\varphi*}\big(\iota^{X\#_\varphi Y}_{Y-V*}(A_{Y-V})\big) = q_{\varphi*}\big(A_\#\big) = 0,$$

the second Mayer-Vietoris sequence above implies that

$$\iota^X_{X-V*}(A_{X-V}) = \iota^X_{V*}(A_V), \qquad \iota^Y_{Y-V*}(A_{Y-V}) = -\iota^Y_{V*}(A_V) \tag{4.3}$$

for some $A_V \in H_m(V)$. By the first equality above and the exactness of (3.5), there exists

$$A_{SV} \in H_m(SV) \qquad \text{s.t.} \qquad \iota^{X-V}_{SV*}(A_{SV}) = A_{X-V}, \quad q_{V*}(A_{SV}) = A_V. \tag{4.4}$$

By (4.2) and the first equality in (4.4),

$$A_\# = \iota^{X\#_\varphi Y}_{X-V*}\big(A_{X-V} - \iota^{X-V}_{SV*}(A_{SV})\big) + \iota^{X\#_\varphi Y}_{Y-V*}\big(A_{Y-V} + \iota^{Y-V}_{SV*}(A_{SV})\big)$$
$$= \iota^{X\#_\varphi Y}_{Y-V*}\big(A_{Y-V} + \iota^{Y-V}_{SV*}(A_{SV})\big).$$

By the second equalities in (4.3) and (4.4),

$$\iota^Y_{Y-V*}\big(A_{Y-V} + \iota^{Y-V}_{SV*}(A_{SV})\big) = -\iota^Y_{V*}(A_V) + \iota^Y_{V*}\big(q_{V*}(A_{SV})\big) = 0.$$

Thus, the first expression in (4.1) is contained in the third expression and by symmetry in the second. $\square$

Taking $m = \mathfrak{c}$ in the statement of Lemma 4.1, we find that

$$\mathcal{R}^V_{X,Y} = \iota^{X\#_\varphi Y}_{X-V*}(\mathcal{R}^V_X) = \iota^{X\#_\varphi Y}_{Y-V*}(\mathcal{R}^V_Y); \tag{4.5}$$

see (2.1) and (2.2) for the notation.

4.2   The vanishing cycles

We now focus on the $m = \mathfrak{c}$ case of Lemma 4.1; it relates $\mathcal{R}^V_{X,Y}$ to $\mathcal{R}^V_X$ and $\mathcal{R}^V_Y$. Let

$$H_\mathfrak{c}(SV;\mathbb{Z})_{X,Y} = \big\{A_{SV} \in H_\mathfrak{c}(SV;\mathbb{Z}): q_{V*}(A_{SV}) \in \ker \iota^X_{V*} \cap \ker \iota^Y_{V*}\big\}. \tag{4.6}$$

By the exactness of (3.5), the homomorphisms

$$\iota^{X-V}_{SV*}: H_\mathfrak{c}(SV;\mathbb{Z})_{X,Y} \longrightarrow \mathcal{R}^V_X \subset H_\mathfrak{c}(X-V;\mathbb{Z}) \qquad \text{and}$$
$$\iota^{Y-V}_{SV*}: H_\mathfrak{c}(SV;\mathbb{Z})_{X,Y} \longrightarrow \mathcal{R}^V_Y \subset H_\mathfrak{c}(Y-V;\mathbb{Z}) \tag{4.7}$$

are surjective.

Corollary 4.2. *Let X, Y, V and φ be as in Lemma 4.1 and $\mathfrak{c}$ be the codimension of V in X and Y.*

(1) The subgroup $\mathcal{R}^V_{X,Y} \subset H_{\mathfrak{c}}(X\#_{\varphi}Y; \mathbb{Z})$ is isomorphic to the cokernel of the homomorphism

$$H_{\mathfrak{c}}(SV; \mathbb{Z})_{X,Y} \longrightarrow \mathcal{R}^V_X \oplus \mathcal{R}^V_Y, \quad A_{SV} \longrightarrow \left(\iota^{X-V}_{SV*}(A_{SV}), -\iota^{Y-V}_{SV*}(A_{SV}) \right). \tag{4.8}$$

(2) If either $\mathcal{R}^V_X = \{0\}$ or $\mathcal{R}^V_Y = \{0\}$, then $\mathcal{R}^V_{X,Y} = \{0\}$ and the homomorphism

$$\#\colon H_{\mathfrak{c}}(X; \mathbb{Z}) \times_V H_{\mathfrak{c}}(Y; \mathbb{Z}) \longrightarrow H_{\mathfrak{c}}(X\#_{\varphi}Y; \mathbb{Z}), \quad (A_X, A_Y) \longrightarrow A_X\#_{\varphi}A_Y, \tag{4.9}$$

induced by gluing representatives of homology classes along V, is well-defined.

Proof. (1) By the $m=\mathfrak{c}$ case of Lemma 4.1, this claim is equivalent to

$$\begin{aligned}
&\left\{ (A_{X-V}, A_{Y-V}) \in \mathcal{R}^V_X \oplus \mathcal{R}^V_Y \colon \iota^{X\#_{\varphi}Y}_{X-V*}(A_{X-V}) + \iota^{X\#_{\varphi}Y}_{Y-V*}(A_{Y-V}) = 0 \right\} \\
&= \left\{ \left(\iota^{X-V}_{SV*}(A_{SV}), -\iota^{Y-V}_{SV*}(A_{SV}) \right) \colon A_{SV} \in H_{\mathfrak{c}}(SV; \mathbb{Z}), \ q_{V*}(A_{SV}) \in \ker \iota^X_{V*} \cap \ker \iota^Y_{V*} \right\}.
\end{aligned} \tag{4.10}$$

Let $A_{X-V} \in H_{\mathfrak{c}}(X-V; \mathbb{Z})$ and $A_{Y-V} \in H_{\mathfrak{c}}(Y-V; \mathbb{Z})$. By the Mayer-Vietoris sequence for $X\#_{\varphi}Y$ in the proof of Lemma 4.1,

$$\begin{aligned}
\iota^{X\#_{\varphi}Y}_{X-V*}(A_{X-V}) + \iota^{X\#_{\varphi}Y}_{Y-V*}(A_{Y-V}) = 0 \quad &\Longleftrightarrow \\
(A_{X-V}, A_{Y-V}) = \left(\iota^{X-V}_{SV*}(A_{SV}), -\iota^{Y-V}_{SV*}(A_{SV}) \right) \quad &\text{for some } A_{SV} \in H_{\mathfrak{c}}(SV; \mathbb{Z}).
\end{aligned}$$

For any $A_{SV} \in H_{\mathfrak{c}}(SV; \mathbb{Z})$, the commutativity of the first square in the diagram of short exact sequences in the proof of Lemma 4.1 implies that

$$\left(\iota^{X-V}_{SV*}(A_{SV}), -\iota^{Y-V}_{SV*}(A_{SV}) \right) \in \mathcal{R}^V_X \oplus \mathcal{R}^V_Y \quad \Longleftrightarrow \quad q_{V*}(A_{SV}) \in \ker \iota^X_{V*} \cap \ker \iota^Y_{V*}.$$

The last two statements give (4.10).

(2) The second claim of this corollary follows from the first and the surjectivity of the homomorphisms (4.7). $\qquad\square$

Remark 4.3. The patching map φ covering the identity on V does not effect the homomorphism (4.8), as the former corresponds to a trivialization of an S^{c-1}-bundle over S^1 for each fixed element of $\mathcal{R}^V_X \oplus \mathcal{R}^V_Y$. Thus, $\mathcal{R}^V_{X,Y}$ does not depend on the choice of φ. However, it may depend on the identification of the copies of V in X and Y, as illustrated in Example 4.7.

We next restrict to the setting of Corollary 3.2 with $U = \emptyset$; the last restriction is not necessary, but the case $U = \emptyset$ suffices for our purposes. Thus, suppose that X, Y, and V are compact and oriented. Define

$$\Delta^V_{X,Y}\colon H_1(V; \mathbb{Z})_X \oplus H_1(V; \mathbb{Z})_Y \longrightarrow \frac{H_1(V; \mathbb{Z})}{H^V_X + H^V_Y}, \quad \left([\gamma_X]_{H^V_X}, [\gamma_Y]_{H^V_Y} \right) \longrightarrow [\gamma_X - \gamma_Y]_{H^V_X + H^V_Y}.$$

Denote by $\overline{H}^V_{X,Y}$ the image of the composition

$$H_{\mathfrak{c}}(SV; \mathbb{Z})_{X,Y} \xrightarrow{(\iota^{X-V}_{SV*}, -\iota^{Y-V}_{SV*})} \mathcal{R}^V_X \oplus \mathcal{R}^V_Y \xrightarrow{\approx} H_1(V; \mathbb{Z})_X \oplus H_1(V; \mathbb{Z})_Y \xrightarrow{\Delta^V_{X,Y}} \frac{H_1(V; \mathbb{Z})}{H^V_X + H^V_Y},$$

with the second arrow above given by the isomorphisms of Corollary 3.2. Let $H^V_{X,Y} \subset H_1(V;\mathbb{Z})$ be the preimage of $\overline{H}^V_{X,Y}$ under the quotient projection

$$H_1(V;\mathbb{Z}) \longrightarrow \frac{H_1(V;\mathbb{Z})}{H^V_X + H^V_Y}\,.$$
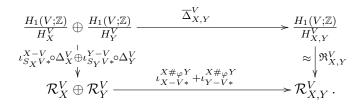
In particular,

$$H^V_X + H^V_Y \subset H^V_{X,Y} \subset H_1(V;\mathbb{Z}) \tag{4.11}$$

and the first inclusion is an equality if either ι^X_{V*} or ι^Y_{V*} is injective on $H_{\mathfrak{c}}$; see the proof of Corollary 4.4 below.

Corollary 4.4. *Let X and Y be compact oriented manifolds, $V \subset X, Y$ be a compact oriented submanifold, and $\varphi\colon S_X V \longrightarrow S_Y V$ be an orientation-reversing diffeomorphism commuting with the projections to V.*

(1) The isomorphisms of Corollary 3.2 for (X,V) and (Y,V) induce a commutative diagram

$$
\begin{array}{ccc}
\dfrac{H_1(V;\mathbb{Z})}{H^V_X} \oplus \dfrac{H_1(V;\mathbb{Z})}{H^V_Y} & \xrightarrow{\ \overline{\Delta}^V_{X,Y}\ } & \dfrac{H_1(V;\mathbb{Z})}{H^V_{X,Y}} \\[2ex]
{\scriptstyle \iota^{X-V}_{S_X V*}\circ\Delta^V_X \oplus \iota^{Y-V}_{S_Y V*}\circ\Delta^V_Y}\Big\downarrow & & \approx\Big\downarrow{\scriptstyle \mathfrak{R}^V_{X,Y}} \\[2ex]
\mathcal{R}^V_X \oplus \mathcal{R}^V_Y & \xrightarrow[\ \iota^{X\#_\varphi Y}_{X-V*} + \iota^{X\#_\varphi Y}_{Y-V*}\ ]{} & \mathcal{R}^V_{X,Y}\,.
\end{array}
$$

(2) If either of ι^X_{V*} or ι^Y_{V*} is injective on $H_{\mathfrak{c}}$, then $\mathcal{R}^V_{X,Y}$ is isomorphic to the cokernel of the homomorphism

$$H_1(V;\mathbb{Z}) \longrightarrow H_1(V;\mathbb{Z})_X \oplus H_1(V;\mathbb{Z})_Y, \quad \gamma \longrightarrow ([\gamma]_X, [\gamma]_Y), \tag{4.12}$$

with $[\gamma]_X$ and $[\gamma]_Y$ denoting the corresponding cosets of γ.

Proof. (1) The image of $\ker \Delta^V_{X,Y}$ under the isomorphisms of Corollary 3.2 is given by

$$
\begin{aligned}
\big\{ \big(\iota^{X-V}_{S_X V*}(\Delta^V_X(\gamma)), \iota^{Y-V}_{S_Y V*}(\Delta^V_Y(\gamma))\big)\colon\ \gamma \in H_1(V;\mathbb{Z})\big\} \\
= \big\{ \big(\iota^{X-V}_{S_X V*}(\Delta^V_X(\gamma)), -\iota^{Y-V}_{S_Y V*}(\Delta^V_X(\gamma))\big)\colon\ \gamma \in H_1(V;\mathbb{Z})\big\},
\end{aligned}
\tag{4.13}
$$

since φ is orientation-reversing. Thus, this image is contained in the image of the homomorphism (4.8). The first claim now follows from Corollary 4.2(1) and the definition of $H^V_{X,Y}$ above.

(2) If either ι^X_{V*} or ι^Y_{V*} is injective on $H_{\mathfrak{c}}$,

$$H_{\mathfrak{c}}(SV;\mathbb{Z})_{X,Y} = \ker q_{V*} = \big\{\Delta^V_X(\gamma)\colon\ \gamma \in H_1(V;\mathbb{Z})\big\}; \tag{4.14}$$

the last equality holds by the exactness of (3.4). Thus, the image of the homomorphism (4.8) is given by (4.13) and $H^V_{X,Y} = H^V_X + H^V_Y$. The second claim of this corollary now follows from the first. $\square$

Corollary 4.5. *If X is a compact oriented manifold, $V \subset X$ is a compact oriented submanifold, and $\varphi\colon S_X V \longrightarrow S_X V$ is an orientation-reversing diffeomorphism commuting with the projection to V, then $\mathcal{R}^V_{X,X} \approx H_1(V;\mathbb{Z})_X$.*

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Proof. Since $\iota_{V*}^X \circ q_{V*} = \iota_{X-V*}^X \circ \iota_{SV*}^{X-V}$,

$$H_{\mathfrak{c}}(SV;\mathbb{Z})_{X,X} = \{A_{SV} \in H_{\mathfrak{c}}(SV;\mathbb{Z}): \iota_{SV*}^{X-V}(A_{SV}) \in \mathcal{R}_X^V\}.$$

Thus, $\overline{H}_{X,Y}^V$ is the image of the diagonal subgroup under the homomorphism

$$H_1(V;Z)_X \oplus H_1(V;Z)_X \longrightarrow H_1(V;Z)_X, \qquad ([\gamma_1],[\gamma_2]) = [\gamma_1 - \gamma_2].$$

Therefore, $H_{X,X}^V = H_X^V$ and the claim follows from Corollary 4.4(1). $\qquad\square$

Example 4.6. Let $\widehat{\mathbb{P}}_9^2$ be a rational elliptic surface and $F \subset \widehat{\mathbb{P}}_9^2$ be a smooth fiber as in Example 3.5. For the standard identification of F in two copies of $\widehat{\mathbb{P}}_9^2$, the homomorphism (4.12) can be written as

$$\mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \oplus \mathbb{Z}^2, \qquad (a,b) \longrightarrow ((a,b),(a,b)).$$

Thus, $\mathcal{R}_{\widehat{\mathbb{P}}_9^2,\widehat{\mathbb{P}}_9^2}^F \approx \mathbb{Z}^2$; this also follows from Corollary 4.5.

Example 4.7. Let F, X, and $F_0, F_\infty \subset X$ be as in Example 3.6 and $Y = X$. For the standard identification of the copies of $F_0 \cup F_\infty$ in X and in Y, the homomorphism (4.12) can be written as

$$H_1(F;\mathbb{Z}) \oplus H_1(F;\mathbb{Z}) \longrightarrow H_1(F;\mathbb{Z}) \oplus H_1(F;\mathbb{Z}), \quad (\gamma_0,\gamma_\infty) \longrightarrow (\gamma_0-\gamma_\infty, \gamma_0-\gamma_\infty).$$

Thus, $\mathcal{R}_{X,X}^{F_0\cup F_\infty} \approx H_1(F;\mathbb{Z})$ in this case, with the isomorphism induced by the homomorphism

$$H_1(F;\mathbb{Z}) \oplus H_1(F;\mathbb{Z}) \longrightarrow H_1(F;\mathbb{Z}), \qquad (\alpha,\beta) \longrightarrow \alpha-\beta.$$

For an arbitrary identification of the two copies of $F_0 \cup F_\infty$, the above homomorphism becomes

$$H_1(F;\mathbb{Z}) \oplus H_1(F;\mathbb{Z}) \longrightarrow H_1(F;\mathbb{Z}) \oplus H_1(F;\mathbb{Z}), \quad (\gamma_0,\gamma_\infty) \longrightarrow (\gamma_0-\gamma_\infty, \gamma_0-\phi_*\gamma_\infty),$$

for some diffeomorphism $\phi\colon F \longrightarrow F$. For example, if $F = \mathbb{T}^2$ is the two-torus, $\mathcal{R}_{X,X}^V \approx \mathbb{Z}^2$ for the standard identification, but $\mathcal{R}_{X,X}^{F_0\cup F_\infty}$ can be $\mathbb{Z}$ or $\{0\}$ for other identifications.

Example 4.8. Suppose X is an oriented manifold and $Z \subset X$ is a compact submanifold so that the normal bundle $\mathcal{N}_X Z$ admits a complex structure. Fix a complex structure in $\mathcal{N}_X Z$ and an identification of the unit disk bundle $D(\mathcal{N}_X Z)$ of $\mathcal{N}_X Z$ with a neighborhood of Z in X. Let

$$\mathbb{P}_X Z = \mathbb{P}(\mathcal{N}_X Z \times \mathbb{C}) = \mathbb{P}(\mathcal{N}_X Z \oplus Z \times \mathbb{C}), \qquad V = \mathbb{P}(\mathcal{N}_X Z) \subset \mathbb{P}(\mathcal{N}_X Z \times \mathbb{C}),$$

and $\mathrm{Bl}_Z X$ be the manifold obtained from X by replacing $D(\mathcal{N}_X Z) \subset X$ with the disk bundle of the complex tautological line bundle $\gamma \longrightarrow V$ (which has the same boundary consisting of the unit vectors in $\mathcal{N}_X V$). Thus,

$$\mathcal{N}_{\mathrm{Bl}_Z X} V = \gamma, \qquad \mathcal{N}_{\mathbb{P}_X Z} V = \gamma^*, \quad \text{and} \quad X = \mathrm{Bl}_Z X \#_\varphi \mathbb{P}_X Z,$$

for an orientation-reversing diffeomorphism $\varphi\colon S_{\mathrm{Bl}_Z X} V \longrightarrow S_{\mathbb{P}_X Z} V$ induced by the canonical isomorphism $\gamma \otimes \gamma^* = V \times \mathbb{C}$ (e.g. $\{\varphi(v)\}v = 1$ for all $v \in S_{\mathrm{Bl}_Z X} V$). By Corollary 4.2(2) and Example 3.4,

$$\mathcal{R}_{\mathrm{Bl}_Z X,\mathbb{P}_X Z}^V = \{0\},$$

i.e. there are no rim tori in this case. A geometric reasoning for this conclusion is given in the proof of [11, Lemma 2.11]. If (X,ω) is a symplectic manifold and $Z \subset X$ is a symplectic submanifold, the construction of [13, Section 7.1] endows $\mathrm{Bl}_Z X$ with a symplectic form $\omega_{Z,\epsilon}$; $(\mathrm{Bl}_Z X, \omega_{Z,\epsilon})$ is then called a symplectic blowup of (X,ω) along Z.

4.3 Changes in rim tori

We continue with the setup of Lemma 4.1. If $U \subset X$ and $W \subset Y$ are closed submanifolds of codimension $\mathfrak{c}$ disjoint from V, then $U, W \subset X\#_\varphi Y$ are also disjoint submanifolds of codimension $\mathfrak{c}$. Below we relate the rim tori for $(X\#_\varphi Y, U \cup W)$ to rim tori in X and Y and to the vanishing cycles in $X\#_\varphi Y$. This relation is described by the squares of exact sequences in Figures 3 and 4. Such a relation is needed to make sense of the rim tori refinement to the symplectic sum formula for relative GW-invariants suggested by [8, (12.7)] and of the convolution product on rim tori covers appearing above [8, (11.5)]; see [4, Section 5.2] for details.

With notation as in (4.6), let

$$\overset{\bullet}{H}{}^{SV}_{X,Y} = \left\{ \left(\iota^{X-V}_{SV*}(A_{SV}), -\iota^{Y-V}_{SV*}(A_{SV}) \right) \colon A_{SV} \in H_{\mathfrak{c}}(SV; \mathbb{Z})_{X,Y} \right\} \subset \mathcal{R}^V_X \oplus \mathcal{R}^V_Y.$$

With U and W as in the previous paragraph fixed, define

$$\begin{aligned}
\widetilde{H}^{SV}_{X,Y} &= \left\{ \left(\iota^{X-U\cup V}_{SV*}(A_{SV}), -\iota^{Y-V\cup W}_{SV*}(A_{SV}) \right) \colon A_{SV} \in H_{\mathfrak{c}}(SV; \mathbb{Z})_{X,Y} \right\}, \\
\overset{\circ}{H}{}^{SV}_{X,Y} &= \widetilde{H}^{SV}_{X,Y} \cap \mathcal{R}^U_{X-V} \oplus \mathcal{R}^W_{Y-V} \subset \mathcal{R}^{U\cup V}_X \oplus \mathcal{R}^{V\cup W}_Y, \\
\widetilde{\mathcal{R}}^{U\cup W}_{X\#_\varphi Y} &= \left\{ \iota^{X\#_\varphi Y - U\cup W}_{X-U\cup V*}(A_X) + \iota^{X\#_\varphi Y - U\cup W}_{Y-V\cup W*}(A_Y) \colon (A_X, A_Y) \in \mathcal{R}^{U\cup V}_X \oplus \mathcal{R}^{V\cup W}_Y \right\}.
\end{aligned} \tag{4.15}$$

By Proposition 4.9 below and (2.2),

$$\widetilde{\mathcal{R}}^{U\cup W}_{X\#_\varphi Y} = \ker \left\{ q_{\varphi*} \circ \iota^{X\#_\varphi Y}_{X\#_\varphi Y - U\cup W*} \colon H_{\mathfrak{c}}(X\#_\varphi Y - U\cup W; \mathbb{Z}) \longrightarrow H_{\mathfrak{c}}(X\cup_V Y; \mathbb{Z}) \right\}.$$

Proposition 4.9. *Suppose X and Y are manifolds, $U, V \subset X$ and $V, W \subset Y$ are closed disjoint submanifolds of the same codimension $\mathfrak{c}$, and $\varphi \colon S_X V \longrightarrow S_Y V$ is a diffeomorphism commuting with the projections to V. Then, the homology homomorphisms induced by inclusions give rise to the commutative square of short exact sequences of Figure 3.*

Proof. The commutativity of all four squares is immediate. The left column is exact by (4.15) and (2.1), while the middle column is exact because the right column in Figure 2 is (which does not require the additional assumptions of Corollary 3.3). The bottom row is exact by Corollary 4.2(1). The middle row is exact at $\widetilde{H}^{SV}_{X,Y}$ and $\widetilde{\mathcal{R}}^{U\cup W}_{X\#_\varphi Y}$ by the definitions of the two modules. It is exact at $\mathcal{R}^{U\cup V}_X \oplus \mathcal{R}^{V\cup W}_Y$ by the exactness of the Mayer-Vietoris sequence for $X\#_\varphi Y$ in the proof of Lemma 4.1 with (X,Y) replaced by $(X-U, Y-W)$; see the proof of Corollary 4.2.

The top row is exact at $\overset{\circ}{H}{}^{SV}_{X,Y}$ by the definition of this module. It is exact at $\mathcal{R}^U_{X-V} \oplus \mathcal{R}^W_{Y-V}$ by the same reasoning as in the proof of Corollary 4.2. In order to establish the exactness of the top row at the last position, we use the commutative diagram

$$\begin{array}{ccccccc}
H_{\mathfrak{c}}(SV) & \longrightarrow & H_{\mathfrak{c}}(X-U\cup V) \oplus H_{\mathfrak{c}}(Y-V\cup W) & \longrightarrow & H_{\mathfrak{c}}(X\#_\varphi Y - U\cup W) & \xrightarrow{\delta_\varphi} & H_{\mathfrak{c}-1}(SV) \\
\| & & \Big\downarrow{\scriptstyle \iota^{X-V}_{X-U\cup V*} \oplus \iota^{Y-V}_{Y-V\cup W*}} & & \Big\downarrow{\scriptstyle \iota^{X\#_\varphi Y}_{X\#_\varphi Y - U\cup W*}} & & \| \\
H_{\mathfrak{c}}(SV) & \longrightarrow & H_{\mathfrak{c}}(X-V) \oplus H_m(Y-V) & \xrightarrow{\iota^{X\#_\varphi Y}_{X-V*} + \iota^{X\#_\varphi Y}_{Y-V*}} & H_{\mathfrak{c}}(X\#_\varphi Y) & \xrightarrow{\delta_\varphi} & H_{\mathfrak{c}-1}(SV)
\end{array}$$

of the Mayer-Vietoris sequences for $X\#_\varphi Y$ and $(X-U)\#_\varphi(Y-W)$.

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & 0 \\
& & \downarrow & & \downarrow & & & & \downarrow \\
0 \longrightarrow & & \mathring{H}^{SV}_{X,Y} \longrightarrow & & \mathcal{R}^U_{X-V}\oplus\mathcal{R}^W_{Y-V} & \xrightarrow{\ \iota^{X\#_\varphi Y-U\cup W}_{X-U\cup V*}+\iota^{X\#_\varphi Y-U\cup W}_{Y-V\cup W*}\ } & & & \mathcal{R}^{U\cup W}_{X\#_\varphi Y} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & & & \downarrow \\
0 \longrightarrow & & \widetilde{H}^{SV}_{X,Y} \longrightarrow & & \mathcal{R}^{U\cup V}_X\oplus\mathcal{R}^{V\cup W}_Y & \xrightarrow{\ \iota^{X\#_\varphi Y-U\cup W}_{X-U\cup V*}+\iota^{X\#_\varphi Y-U\cup W}_{Y-V\cup W*}\ } & & & \widetilde{\mathcal{R}}^{U\cup W}_{X\#_\varphi Y} \longrightarrow 0 \\
& & \downarrow & & {\scriptstyle \iota^{X-V}_{X-U\cup V*}\oplus\iota^{Y-V}_{Y-V\cup W*}}\ \downarrow & & & & \downarrow\ {\scriptstyle \iota^{X\#_\varphi Y}_{X\#_\varphi Y-U\cup W*}} \\
0 \longrightarrow & & \dot{H}^{SV}_{X,Y} \longrightarrow & & \mathcal{R}^V_X\oplus\mathcal{R}^V_Y & \xrightarrow{\ \iota^{X\#_\varphi Y}_{X-V*}+\iota^{X\#_\varphi Y}_{Y-V*}\ } & & & \mathcal{R}^V_{X,Y} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & & & \downarrow \\
& & 0 & & 0 & & & & 0
\end{array}
$$

Figure 3: Rim tori under gluing: general case.

If $A\in\mathcal{R}^{U\cup W}_{X\#_\varphi Y}$, then
$$\delta_\varphi(A) = \delta_\varphi\big(\iota^{X\#_\varphi Y}_{X\#_\varphi Y-U\cup W*}(A)\big) = \delta_\varphi(0) = 0.$$
By the exactness of the first sequence above, this implies that
$$A = \iota^{X\#_\varphi Y-U\cup W}_{X-U\cup V*}(A_X) + \iota^{X\#_\varphi Y-U\cup W}_{Y-V\cup W*}(A_Y) \tag{4.16}$$
for some
$$A_X\in H_{\mathfrak{c}}(X-U\cup V;\mathbb{Z}), \quad A_Y\in H_{\mathfrak{c}}(Y-V\cup W;\mathbb{Z}).$$
By the commutativity of the middle square in the Mayer-Vietoris diagram,
$$
\begin{aligned}
\iota^{X\#_\varphi Y}_{X-V*}\big(\iota^{X-V}_{X-U\cup V*}(A_X)\big) &+ \iota^{V\#_\varphi Y}_{Y-V*}\big(\iota^{Y-V}_{Y-V\cup W*}(A_Y)\big) \\
&= \iota^{X\#_\varphi Y}_{X\#_\varphi Y-U\cup W*}\big(\iota^{X\#_\varphi Y-U\cup W}_{X-U\cup V*}(A_X)\big) + \iota^{X\#_\varphi Y}_{X\#_\varphi Y-U\cup W*}\big(\iota^{X\#_\varphi Y-U\cup W}_{Y-V\cup W*}(A_Y)\big) \\
&= \iota^{X\#_\varphi Y}_{X\#_\varphi Y-U\cup W*}(A) = 0.
\end{aligned}
$$
Thus, the exactness of the bottom row implies that
$$\iota^{X-V}_{X-U\cup V*}(A_X) = \iota^{X-V}_{SV*}(A_{SV}), \quad \iota^{Y-V}_{Y-V\cup W*}(A_Y) = -\iota^{Y-V}_{SV*}(A_{SV}) \tag{4.17}$$
for some $A_{SV}\in H_{\mathfrak{c}}(SV;\mathbb{Z})$. By the commutativity of the left square and (4.17),
$$A_X = A'_X + \iota^{X-U\cup V}_{SV*}(A_{SV}), \qquad A_Y = A'_Y - \iota^{Y-V\cup W}_{SV*}(A_{SV}) \tag{4.18}$$
for some $A'_X\in\mathcal{R}^U_{X-V}$ and $A'_Y\in\mathcal{R}^W_{Y-V}$. By (4.16) and (4.18),
$$A = \iota^{X\#_\varphi Y-U\cup W}_{X-U\cup V*}(A'_X) + \iota^{X\#_\varphi Y-U\cup W}_{Y-V\cup W*}(A'_Y),$$
and so the top row is exact at the last position.

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$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \mathring{H}^V_{X,Y} & \longrightarrow & H_1(U;\mathbb{Z})\oplus H_1(W;\mathbb{Z}) & \longrightarrow & \mathcal{R}^{U\cup W}_{X\#_\varphi Y} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & \widetilde{H}^V_{X,Y} & \longrightarrow & H_1(U;\mathbb{Z})\oplus H_1(V;\mathbb{Z})\oplus H_1(V;\mathbb{Z})\oplus H_1(W;\mathbb{Z}) & \longrightarrow & \widetilde{\mathcal{R}}^{U\cup W}_{X\#_\varphi Y} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow\ \iota^{X\#_\varphi Y}_{X\#_\varphi Y-U\cup W*} & & \\
0 \longrightarrow & \mathring{H}_{X,Y} V & \longrightarrow & H_1(V;\mathbb{Z})\oplus H_1(V;\mathbb{Z}) & \longrightarrow & \mathcal{R}^V_{X,Y} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
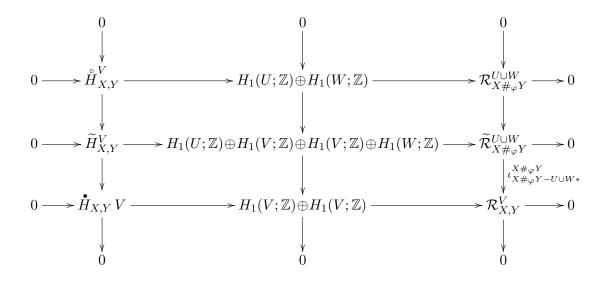& & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 4: Rim tori under gluing: compact oriented case.

Finally, the first homomorphism in the right column is the inclusion of a submodule. This column is exact at the middle and last positions by the commutativity of the diagram and the exactness of the remaining sequences. $\qquad\square$

We now restrict to the settings of Corollaries 3.2 and 4.4. Thus, suppose that X, Y, U, V, and W are compact and oriented. Let

$$
\mathring{H}^V_{X,Y} \subset H_1(U;\mathbb{Z})\oplus H_1(W;\mathbb{Z}), \qquad \overset{\bullet}{H}{}^V_{X,Y} \subset H_1(V;\mathbb{Z})\oplus H_1(V;\mathbb{Z}),
$$
$$
\text{and} \qquad \widetilde{H}^V_{X,Y} \subset H_1(U;\mathbb{Z})\oplus H_1(V;\mathbb{Z})\oplus H_1(V;\mathbb{Z})\oplus H_1(W;\mathbb{Z})
$$

be the preimages of

$$
\mathring{H}^{SV}_{X,Y} \subset \mathcal{R}^U_{X-V}\oplus\mathcal{R}^W_{Y-V}, \qquad H^{SV}_{X,Y} \subset \mathcal{R}^V_X\oplus\mathcal{R}^V_Y, \quad \text{and} \quad \widetilde{H}^V_{X,Y} \subset \mathcal{R}^{U\cup V}_X\oplus\mathcal{R}^{V\cup W}_Y,
$$

respectively, under the homomorphisms as in Corollary 3.2. The commutative square of short exact sequences of Figure 3 then induces the commutative square of short exact sequences of Figure 4.

Example 4.10. Let $X=\widehat{\mathbb{P}}^2_9$ be the rational elliptic surface of Examples 3.5 and 4.6 with smooth fiber $F\subset\widehat{\mathbb{P}}^2_9$, $Y=\mathbb{P}^1\times\mathbb{T}^2$, and $F_0,F_\infty\subset\mathbb{P}^1\times\mathbb{T}^2$ be as in Examples 3.6 and 4.7. We take $U=\emptyset$, $W=F_\infty$, $V=F\subset X$, and $V=F_0\subset Y$. In this case, $\mathfrak{c}=2$, the homomorphisms

$$
\iota^X_{V*}\colon H_2(V;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z}) \qquad \text{and} \qquad \iota^Y_{V*}\colon H_2(V;\mathbb{Z}) \longrightarrow H_2(Y;\mathbb{Z})
$$

are injective, and the homomorphisms

$$
\Delta^V_X\colon H_1(V;\mathbb{Z}) \longrightarrow H_2(SV;\mathbb{Z})_{X,Y}, \qquad \iota^{Y-V\cup W}_{SV*}\colon H_2(SV;\mathbb{Z})_{X,Y} \longrightarrow H_2(Y-V\cup W;\mathbb{Z}),
$$
$$
\text{and} \qquad \iota^{X-U\cup V}_{SV*}=\iota^{X-V}_{SV*}\colon H_2(SV;\mathbb{Z})_{X,Y} \longrightarrow H_2(X-U\cup V;\mathbb{Z})=H_2(X-V;\mathbb{Z}),
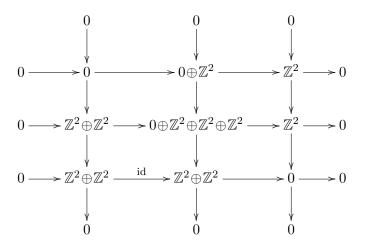$$

Figure 5: Rim tori under gluing for $(\widehat{\mathbb{P}}_9^2, F) = (\widehat{\mathbb{P}}_9^2, \emptyset) \#_\varphi (\mathbb{P}^1 \times \mathbb{T}^2, F_\infty)$.

are isomorphisms. The exact square in Figure 4 then specializes to the exact square of Figure 5. The two homomorphisms in the middle column are given by

$$(0, \gamma) \longrightarrow (0, 0, 0, \gamma) \qquad \text{and} \qquad (0, \gamma_1, \gamma_2, \gamma_3) \longrightarrow (\gamma_1, \gamma_2).$$

The two homomorphisms in the middle row are given by

$$(\gamma_1, \gamma_2) \longrightarrow (0, \gamma_1, \gamma_1 + \gamma_2, \gamma_2) \qquad \text{and} \qquad (0, \gamma_1, \gamma_2, \gamma_3) \longrightarrow \gamma_1 - \gamma_2 + \gamma_3;$$

the restrictions of the first homomorphism to the last component and of the second homomorphism to the last two components correspond to (3.9) and (3.10), respectively. For the standard identification φ, $\widehat{\mathbb{P}}_9^2 = \widehat{\mathbb{P}}_9^2 \#_\varphi (\mathbb{P}^1 \times \mathbb{T}^2)$.

4.4 Changes in cohomology

By Lemma 4.1 and the exactness of (3.5),

$$
\begin{aligned}
\ker\left\{ q_{\varphi*} \colon H_m(X \#_\varphi Y; \mathbb{Z}) \longrightarrow H_m(X \cup_V Y; \mathbb{Z}) \right\} \\
= \left\{ \iota_{SV*}^{X \#_\varphi Y}(A_{SV}) \colon A_{SV} \in \ker\{ q_V \colon H_m(SV; \mathbb{Z}) \longrightarrow H_m(V; \mathbb{Z}) \} \right\}.
\end{aligned}
\tag{4.19}
$$

Lemma 4.11 below, which describes cohomology classes used as primary inputs for GW-invariants in the symplectic sum formula, can be seen as the dual of (4.19). The analogue of this lemma with field coefficients, which would be sufficient for the purposes of the symplectic sum formula, follows immediately by dualizing (4.1) with coefficients in the same field. Similarly, the proof of Lemma 4.11 can be viewed as the dual version of the proof of Lemma 4.1, but we include it for the sake of completeness; like the proof of Lemma 4.1, it contains a delicate step.

Lemma 4.11. *If X and Y are manifolds, $V \subset X, Y$ is a closed submanifold, $\varphi \colon S_X V \longrightarrow S_Y V$ is a diffeomorphism commuting with the projections to V, and $q_\varphi \colon X \#_\varphi Y \longrightarrow X \cup_V Y$ is a collapsing map, then*

$$\left\{ q_\varphi^* \alpha_\cup \colon \alpha_\cup \in H^*(X \cup_V Y; \mathbb{Z}) \right\} = \left\{ \alpha_\# \in H^*(X \#_\varphi Y; \mathbb{Z}) \colon \alpha_\#|_{SV} \in q_V^*(H^*(V; \mathbb{Z})) \right\}, \tag{4.20}$$

where $SV \subset X \#_\varphi Y$ is the sphere bundle $S_X V \approx S_Y V$.

Proof. The commutativity of the diagram

$$
\begin{array}{ccc}
SV & \xrightarrow{\iota_{SV}^{X\#_\varphi Y}} & X\#_\varphi Y \\
\Big\downarrow{\scriptstyle q_V} & & \Big\downarrow{\scriptstyle q_\varphi} \\
V & \xrightarrow[\iota_V^{X\cup_V Y}]{} & X\cup_V Y
\end{array}
$$

implies that the left-hand side of (4.20) is contained in the right-hand side. Below we confirm the opposite inclusion.

We will use the commutative diagram of the Mayer-Vietoris cohomology sequences for $X\cup_V Y$ and $X\#_\varphi Y = (X-V)\cup_{SV}(Y-V)$,

$$
\begin{array}{ccccccc}
H^{m-1}(V) & \xrightarrow{\delta_\cup^*} & H^m(X\cup_V Y) & \xrightarrow{(\iota_X^{X\cup_V Y*},\,\iota_Y^{X\cup_V Y*})} & H^m(X)\oplus H^m(Y) & \xrightarrow{\iota_V^{X*}-\iota_V^{Y*}} & H^m(V) \\
\Big\downarrow{\scriptstyle q_V^*} & & \Big\downarrow{\scriptstyle q_\varphi^*} & & \Big\downarrow{\scriptstyle \iota_{X-V}^{X*}\oplus\iota_{Y-V}^{Y*}} & & \Big\downarrow{\scriptstyle q_V^*} \\
H^{m-1}(SV) & \xrightarrow{\delta_\varphi^*} & H^m(X\#_\varphi Y) & \xrightarrow{(\iota_{X-V}^{X\#_\varphi Y*},\,\iota_{Y-V}^{X\#_\varphi Y*})} & H^m(X-V)\oplus H^m(Y-V) & \xrightarrow{\iota_{SV}^{X-V*}-\iota_{SV}^{Y-V*}} & H^m(SV)
\end{array}
$$

where H^* denotes integral cohomology groups. Suppose

$$
\alpha_\# \in H^*(X\#_\varphi Y;\mathbb{Z}), \quad \alpha_V \in H^*(V;\mathbb{Z}), \quad \alpha_\#|_{SV} = q_V^*\alpha_V.
$$

By Mayer-Vietoris for $M = (M-V)\cup_{SV} V$, where $M = X, Y$,

$$
H^m(M;\mathbb{Z}) \xrightarrow{(\iota_{M-V}^{M*},\,\iota_V^{M*})} H^m(M-V;\mathbb{Z})\oplus H^m(V;\mathbb{Z}) \xrightarrow{\iota_{SV}^{M-V*}-q_V^*} H^m(SV;\mathbb{Z}),
$$

there exist $\alpha_X \in H^m(X;\mathbb{Z})$ and $\alpha_Y \in H^m(Y;\mathbb{Z})$ such that

$$
\alpha_X|_{X-V} = \alpha_\#|_{X-V}, \quad \alpha_Y|_{Y-V} = \alpha_\#|_{Y-V}, \quad \alpha_X|_V, \alpha_Y|_V = \alpha_V. \tag{4.21}
$$

By the last equality in (4.21) and the Mayer-Vietoris sequence for $X\cup_V Y$ above, there exists

$$
\alpha_\cup \in H^m(X\cup_V Y;\mathbb{Z}) \qquad \text{s.t.} \quad \alpha_\cup|_X = \alpha_X, \qquad \alpha_\cup|_Y = \alpha_Y. \tag{4.22}
$$

By the commutativity of the middle square in the above diagram, the two equalities in (4.22), and the first two equalities in (4.21),

$$
\big(\alpha_\# - q_\varphi^*\alpha_\cup\big)\big|_{X-V} = 0 \quad \text{and} \quad \big(\alpha_\# - q_\varphi^*\alpha_\cup\big)\big|_{Y-V} = 0.
$$

Along with the exactness of the bottom row, this implies that

$$
\alpha_\# - q_\varphi^*\alpha_\cup \in \big\{\delta_\varphi^*(\beta_{SV}) \colon \beta_{SV} \in H^{m-1}(SV;\mathbb{Z})\big\}.
$$

The claim then follows from the observation that

$$
\big\{\delta_\varphi^*(\beta_{SV}) \colon \beta_{SV} \in H^{m-1}(SV;\mathbb{Z})\big\} \subset \big\{q_\varphi^*(\alpha_\cup) \colon \alpha_\cup \in H^m(X\cup_V Y;\mathbb{Z})\big\}, \tag{4.23}
$$

which is established below.

Choose an open subset $\widetilde{Y}$ of $X\cup_V Y$ consisting of Y and a tubular neighborhood of V in X. Let $\mathcal{S}_\cup^*$ denote the cochain complex of $\mathbb{Z}$-valued homomorphisms on the sub-complex of singular chains generated by simplices in $X\cup_V Y$ with images in either $X-V$ or $\widetilde{Y}$. Similarly, let $\mathcal{S}_\#^*$ denote the cochain complex of $\mathbb{Z}$-valued homomorphisms on the sub-complex of singular chains generated by simplices in $X\#_\varphi Y$ with images in either $q_\varphi^{-1}(X-V)$ or $q_\varphi^{-1}(\widetilde{Y})$. By [20, Section 5.32], the restriction homomorphisms from the usual singular cochain complexes,

$$\mathcal{S}^*(X\cup_V Y) \longrightarrow \mathcal{S}_\cup^* \qquad \text{and} \qquad \mathcal{S}^*(X\#_\varphi Y) \longrightarrow \mathcal{S}_\#^*,$$

induce isomorphisms in cohomology. Thus, we can replace the domains of these homomorphisms by their targets in order to verify (4.23). Let $V_\cup = (X-V)\cap\widetilde{Y}$ and $SV_\# = q_\varphi^{-1}(V_\cup)$.

For any $\eta \in \mathcal{S}^*(SV_\#)$, define

$$\eta_{q_\varphi^{-1}(X-V)} \in \mathcal{S}^*\big(q_\varphi^{-1}(X-V)\big)\,, \qquad \eta_{q_\varphi^{-1}(X-V)}(\sigma) = \begin{cases} \eta(\sigma), & \text{if } \operatorname{Im}\sigma \subset SV_\#; \\ 0, & \text{otherwise}; \end{cases}$$

$$\eta_\# \in \mathcal{S}_\#^*\,, \qquad \eta_\#(\sigma) = \begin{cases} \eta_{q_\varphi^{-1}(X-V)}(\partial\sigma), & \text{if } \operatorname{Im}\sigma \subset q_\varphi^{-1}(X-V); \\ 0, & \text{if } \operatorname{Im}\sigma \subset q_\varphi^{-1}(\widetilde{Y}); \end{cases}$$

$$\eta_\cup \in \mathcal{S}_\cup^*\,, \qquad \eta_\cup(\sigma) = \begin{cases} \eta_{q_\varphi^{-1}(X-V)}(\partial(q_\varphi^{-1}\circ\sigma)), & \text{if } \operatorname{Im}\sigma \subset X-V; \\ 0, & \text{if } \operatorname{Im}\sigma \subset \widetilde{Y}; \end{cases}$$

where σ denotes an appropriate singular simplex. The homomorphisms $\eta_\#$ and $\eta_\cup$ are well-defined on the overlaps if $\delta\eta = 0$, i.e. η determines an element $[\eta]$ in $H^*(SV_\#)$. In such a case,

$$\delta_\varphi^*[\eta] = [\eta_\#], \qquad q_\varphi^*[\eta_\cup] = [\eta_\#],$$

by the construction of the connecting homomorphism in the Snake Lemma and the definition of pull-back homomorphisms. This establishes (4.23). $\qquad\square$

In [8, Section 13], the cohomology classes on $X\#_\varphi Y$ not contained in the left-hand side of (4.20) are described as *cutting through neck*, i.e. $SV \subset X\#_\varphi Y$. The next statement makes this terminology precise.

Corollary 4.12. *Suppose X, Y, V, φ, and q_φ are as in Lemma 4.11, X and Y are compact, $\mathfrak{c}$ is the codimension of V in X and Y, and $\alpha_\# \in H^*(X\#_\varphi Y;\mathbb{Z})$. Then, $\alpha_\# = q_\varphi^*\alpha_\cup$ for some $\alpha_\cup \in H^*(X\cup_V Y;\mathbb{Z})$ if and only if $\mathrm{PD}_{X\#_\varphi Y}(\alpha_\#)$ can be represented by a pseudocycle $f_\# : Z_\# \longrightarrow X\#_\varphi Y$ transverse to SV such that $f_\#^{-1}(SV) = f_V^* SV$ for some pseudocycle $f_V : Z_V \longrightarrow V$ of dimension $\mathfrak{c}$ less.*

Proof. Let $f_\# : Z_\# \longrightarrow X\#_\varphi Y$ be a pseudocycle representative for the Poincare dual of $\alpha_\#$ transverse to SV. The restriction of $f_\#$ to $f_\#^{-1}(SV)$ then represents the Poincare dual of $\alpha_\#|_{SV}$.

(1) If $f_\#^{-1}(SV) = f_V^* SV$ for some pseudocycle $f_V : Z_V \longrightarrow V$ of dimension $\mathfrak{c}$ less, $\alpha_\#|_{SV} = q_V^*\alpha_V$, where $\alpha_V \in H^*(V;\mathbb{Z})$ is the Poincare dual of the class represented by f_V. Lemma 4.11 then implies

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that $\alpha_\# = q^*_\varphi \alpha_\cup$ for some $\alpha_\cup \in H^*(X \cup_V Y; \mathbb{Z})$.

(2) If $\alpha_\# = q^*_\varphi \alpha_\cup$ for some $\alpha_\cup \in H^*(X \cup_V Y; \mathbb{Z})$, then $\alpha_\#|_{SV} = q^*_V \alpha_V$ for some $\alpha_V \in H^*(V; \mathbb{Z})$; see Lemma 4.11. Let $f_V : Z_V \longrightarrow V$ be a pseudocycle representing the Poincare dual of α_V and $\widetilde{f}_V : f^*_V SV \longrightarrow SV$ be the induced pseudocycle from the total space of the bundle $SV \longrightarrow V$ pulled back by f_V; see the end of Section 3.1. Thus, there exists a pseudocycle equivalence $\widetilde{f} : \widetilde{Z} \longrightarrow SV$ so that

$$\partial \widetilde{f} = f_\#\big|_{f^{-1}_\#(S_V)} - \widetilde{f}_V .$$

Cutting $Z_\#$ along the hypersurface $f^{-1}_\#(SV)$, gluing in $\widetilde{f}$ and $-\widetilde{f}$ along the resulting cuts, identifying $\widetilde{f}$ and $-\widetilde{f}$ along $\widetilde{f}_V$, and moving $\pm\widetilde{f}$ on the complement of $\widetilde{f}_V$ outside SV, we obtain a pseudocycle representative $\widehat{f}_\# : \widehat{Z}_\# \longrightarrow X \#_\varphi Y$ for the Poincare dual of $\alpha_\#$ transverse to SV such that $\widehat{f}^{-1}_\#(SV) = f^*_V SV$. $\square$

Remark 4.13. There is a slight misstatement in part (a) at the bottom of page 996 in [8] related to the $\mathfrak{c} = 2$ case of Lemma 4.11, since the first map in [8, (10.13)] is never injective for dimensional reasons. The statement in (a) should instead be that $\alpha \in H^m(Z_\lambda; \mathbb{Z})$ separates if

$$\cup c_1 : H^{m-1}(V; \mathbb{Z}) \longrightarrow H^{m+1}(V; \mathbb{Z})$$

is injective. In (b), $j^* : H^*(Z_\lambda; \mathbb{Z}) \longrightarrow H^*(SV; \mathbb{Z})$ is the restriction map.

5 Abelian covers of topological spaces

The notation for the abelian covers of topological spaces relevant in our context is introduced in Section 5.1. Section 5.2 is concerned with their topological properties, focusing on whether their (co)homology is finitely generated or not.

By a topological space V, we will mean a locally path-connected and a semilocally simply connected topological space V, as in [16, §25,82]; all manifolds and more generally CW-complexes fall in this category. The first assumption implies that the connected and path-connected components of V are the same; see [16, Theorem 25.5]. The two assumptions together imply the connected covers of the connected components V_r of V are classified by their fundamental subgroups $\pi_1(V_r)$; see [16, Theorems 79.4, 82.1].

5.1 Notation and examples

Let $\mathbb{Z}_\pm \subset \mathbb{Z}$ denote the nonzero integers. For a tuple $\mathbf{s} = (s_1, \ldots, s_\ell) \in \mathbb{Z}_\pm^\ell$ with $\ell \in \mathbb{Z}^{\geq 0}$, we denote by $\gcd(\mathbf{s})$ the greatest common divisor of $s_1, \ldots, s_\ell$; if $\ell = 0$, we set $\gcd(\mathbf{s}) = 0$.

Let V be a topological space. For any submodule $H \subset H_1(V; \mathbb{Z})$, let

$$q_H : H_1(V; \mathbb{Z}) \longrightarrow \mathcal{R}_H \equiv \frac{H_1(V; \mathbb{Z})}{H} \tag{5.1}$$

be the projection to the corresponding quotient module. If $V_1, \ldots, V_N$ are the topological components of V, $\ell_1, \ldots, \ell_N \in \mathbb{Z}^{\geq 0}$, and $\mathbf{s}_1 \in \mathbb{Z}_\pm^{\ell_1}, \ldots, \mathbf{s}_N \in \mathbb{Z}_\pm^{\ell_N}$, then the topological space

$$V_{\mathbf{s}_1 \ldots \mathbf{s}_N} \equiv V_1^{\ell_1} \times \ldots \times V_N^{\ell_N}$$

is connected.

With V and $\mathbf{s}_1, \ldots, \mathbf{s}_N$ as above, define

$$\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}\colon H_1\big(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z}\big) = \bigoplus_{r=1}^{N} H_1(V_r;\mathbb{Z})^{\oplus\ell_r} \longrightarrow H_1(V;\mathbb{Z}), \tag{5.2}$$

$$\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}\big((\gamma_{r;i})_{i\leq\ell_r, r\leq N}\big) = \sum_{r=1}^{N}\sum_{i=1}^{\ell_r} s_{r;i}\gamma_{r;i}\,.$$

For any submodule $H \subset H_1(V;\mathbb{Z})$, let

$$H_{\mathbf{s}_1\ldots\mathbf{s}_N} = \Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}^{-1}(H) \subset H_1\big(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z}\big), \tag{5.3}$$

$$\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N} = \mathrm{Im}\big\{q_H\circ\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}\big\} \subset \mathcal{R}_H, \qquad \mathcal{R}_{H;\mathbf{s}_1\ldots\mathbf{s}_N} = \frac{\mathcal{R}_H}{\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}} \times \mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\,. \tag{5.4}$$

If $\gcd(\mathbf{s}_r)=1$ for every $r=1,\ldots,N$, then

$$\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N} = \mathcal{R}_{H;\mathbf{s}_1\ldots\mathbf{s}_N} = \mathcal{R}_H\,.$$

If V is connected, then $\mathcal{R}'_{H;\mathbf{s}} = \gcd(\mathbf{s})\mathcal{R}_H$ for any $\mathbf{s}\in\mathbb{Z}_{\pm}^{\ell}$ and $H_{(1)}=H$.

For each $r=1,\ldots,N$, let $\widehat{V}_r \longrightarrow V_r$ be the maximal abelian cover of V_r, i.e. the covering projection corresponding to the commutator subgroup of $\pi_1(V)$. The group of deck transformations of this regular covering is $H_1(V_r;\mathbb{Z})$. The maximal abelian cover of $V_{\mathbf{s}_1\ldots\mathbf{s}_N}$ is given by

$$\widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \prod_{r=1}^{N} \widehat{V}_r^{\ell_r} \longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N}\,; \tag{5.5}$$

there is a natural action of $H_1(V_{\mathbf{s}};\mathbb{Z})$ on this space. For any submodule $H\subset H_1(V;\mathbb{Z})$, let

$$\pi'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\colon \widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}\big/ H_{\mathbf{s}_1\ldots\mathbf{s}_N} \longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N}\,,$$

$$\pi_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\colon \widehat{V}_{H;\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \frac{\mathcal{R}_H}{\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}} \times \widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N} \longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N}\,. \tag{5.6}$$

We will write elements of the second covering as

$$\big([\gamma]_{H;\mathbf{s}_1\ldots\mathbf{s}_N}, [\widehat{x}]_H\big) \in \frac{\mathcal{R}_H}{\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}} \times \widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}, \tag{5.7}$$

with the first component denoting the image of $\gamma\in H_1(V;\mathbb{Z})$ under the homomorphism

$$H_1(V;\mathbb{Z}) \longrightarrow \mathcal{R}_H \longrightarrow \frac{\mathcal{R}_H}{\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}}$$

and the second component denoting the image of $\widehat{x}\in\widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}$.

The groups of deck transformations of these regular coverings are

$$\mathrm{Deck}\big(\pi'_{H;\mathbf{s}_1...\mathbf{s}_N}\big) = \mathcal{R}'_{H;\mathbf{s}_1...\mathbf{s}_N} \qquad \text{and} \qquad \mathrm{Deck}\big(\pi_{H;\mathbf{s}_1...\mathbf{s}_N}\big) = \mathcal{R}_{H;\mathbf{s}_1...\mathbf{s}_N}, \tag{5.8}$$

respectively. The action of $\mathcal{R}'_{H;\mathbf{s}_1...\mathbf{s}_N}$ is induced from the default action of $H_1(V_\mathbf{s};\mathbb{Z})$ on $\widehat{V}_\mathbf{s}$ via the surjective homomorphism

$$q_H \circ \Phi_{V;\mathbf{s}_1...\mathbf{s}_N} \colon H_1(V_\mathbf{s};\mathbb{Z}) \longrightarrow \mathcal{R}'_{H;\mathbf{s}_1...\mathbf{s}_N};$$

the kernel of this homomorphism, i.e. $H_{\mathbf{s}_1...\mathbf{s}_N}$, acts trivially on $\widehat{V}'_{H;\mathbf{s}_1...\mathbf{s}_N}$.

If V is connected, then

$$\pi_H \equiv \pi_{H;(1)} \colon \widehat{V}_H \equiv \widehat{V}_{H;(1)} = \widehat{V}/H \longrightarrow V$$

is the abelian covering corresponding to the subgroup $H \subset H_1(V;\mathbb{Z})$, i.e. the covering corresponding to the normal subgroup $\mathrm{Hur}^{-1}(H) \subset \pi_1(V)$, where

$$\mathrm{Hur} \colon \pi_1(V_{\mathbf{s}_1...\mathbf{s}_N}) \longrightarrow H_1(V_{\mathbf{s}_1...\mathbf{s}_N};\mathbb{Z})$$

is Hurewicz homomorphism; see [19, Section 7.4].

A collection $\{\gamma_j\} \subset H_1(V;\mathbb{Z})$ of representatives for the elements of $\mathcal{R}_H / \mathcal{R}'_{H;\mathbf{s}_1...\mathbf{s}_N}$ induces a homomorphism

$$H_1(V;\mathbb{Z}) \longrightarrow \mathrm{Deck}(\pi_{H;\mathbf{s}_1...\mathbf{s}_N}), \qquad \eta \longrightarrow \Theta_\eta,$$

as follows. For every $\eta \in H_1(V;\mathbb{Z})$ and a coset representative γ_j, let $\gamma_j(\eta)$ be the unique coset representative from the chosen collection such that

$$\gamma_j + \eta - \gamma_j(\eta) - \Phi_{V;\mathbf{s}_1...\mathbf{s}_N}(\eta_j) \in H \tag{5.9}$$

for some $\eta_j \in H_1(V_{\mathbf{s}_1...\mathbf{s}_N};\mathbb{Z})$. Define

$$\Theta_\eta \colon \widehat{V}_{H;\mathbf{s}_1...\mathbf{s}_N} \longrightarrow \widehat{V}_{H;\mathbf{s}_1...\mathbf{s}_N}, \quad \Theta_\eta\big([\gamma_j]_{H;\mathbf{s}_1...\mathbf{s}_N}, [\widehat{x}]_H\big) = \big([\gamma_j(\eta)]_{H;\mathbf{s}_1...\mathbf{s}_N}, [\eta_j \cdot \widehat{x}]_H\big).$$

Since (5.9) determines η_j up to an element of $H_{\mathbf{s}_1...\mathbf{s}_N}$, the last component of Θ_η is well-defined.

Suppose in addition that $V' \subset V$ is the union of $V_1, \ldots, V_{N'}$ for some $N' \leq N$ and $H' \subset H_1(V';\mathbb{Z})$ is a submodule. Let

$$q \colon V_{\mathbf{s}_1...\mathbf{s}_N} \longrightarrow V'_{\mathbf{s}_1...\mathbf{s}_{N'}}, \qquad (x_{r;i})_{i \leq \ell_r, r \leq N} \longrightarrow (x_{r;i})_{i \leq \ell_r, r \leq N'},$$
$$\widehat{q} \colon \widehat{V}_{\mathbf{s}_1...\mathbf{s}_N} \longrightarrow \widehat{V}'_{\mathbf{s}_1...\mathbf{s}_{N'}}, \qquad (\widehat{x}_{r;i})_{i \leq \ell_r, r \leq N} \longrightarrow (\widehat{x}_{r;i})_{i \leq \ell_r, r \leq N'}$$

denote the projections to the V'- and $\widehat{V}'$-components. If H' contains the image of H under the projection

$$q_* \colon H_1(V;\mathbb{Z}) = \bigoplus_{r=1}^{N} H_1(V_r;\mathbb{Z}) \longrightarrow H_1(V';\mathbb{Z}) = \bigoplus_{r=1}^{N'} H_1(V_r;\mathbb{Z}), \tag{5.10}$$
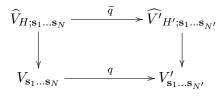
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then q_* induces a commutative diagram of homomorphisms (not exact sequences)

$$
\begin{array}{ccccccc}
H_{\mathbf{s}_1\ldots\mathbf{s}_N} & \hookrightarrow & H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z}) & \xrightarrow{\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}} & H_1(V;\mathbb{Z}) & \longrightarrow \mathcal{R}_H \longrightarrow & \dfrac{\mathcal{R}_H}{\mathcal{R}_{H;\mathbf{s}_1\ldots\mathbf{s}_N}} \\
\downarrow q_* & & \downarrow q_* & & \downarrow q & \downarrow & \downarrow \\
H'_{\mathbf{s}_1\ldots\mathbf{s}_{N'}} & \hookrightarrow & H_1(V'_{\mathbf{s}_1\ldots\mathbf{s}_{N'}};\mathbb{Z}) & \xrightarrow{\Phi_{V';\mathbf{s}_1\ldots\mathbf{s}_{N'}}} & H_1(V';\mathbb{Z}) & \longrightarrow \mathcal{R}_{H'} \longrightarrow & \dfrac{\mathcal{R}_{H'}}{\mathcal{R}'_{H';\mathbf{s}_1\ldots\mathbf{s}_{N'}}}
\end{array}
$$

The continuous map

$$
\big([\gamma]_{H;\mathbf{s}_1\ldots\mathbf{s}_N}, [\widehat{x}]_H\big) \longrightarrow \big([q_*(\gamma)]_{H';\mathbf{s}_1\ldots\mathbf{s}_{N'}}, [\widehat{q}(\widehat{x})]_{H'}\big) \tag{5.11}
$$

then induces a commutative diagram

$$
\begin{array}{ccc}
\widehat{V}_{H;\mathbf{s}_1\ldots\mathbf{s}_N} & \xrightarrow{\ \widetilde{q}\ } & \widehat{V'}_{H';\mathbf{s}_1\ldots\mathbf{s}_{N'}} \\
\downarrow & & \downarrow \\
V_{\mathbf{s}_1\ldots\mathbf{s}_N} & \xrightarrow{\ q\ } & V'_{\mathbf{s}_1\ldots\mathbf{s}_{N'}}
\end{array}
$$

of fiber bundles.

Example 5.1. If $V = \mathbb{T}^2$, $\ell \in \mathbb{Z}^+$, and $H = \{0\}$, then

$$
\widehat{V}_{H;\mathbf{s}} = \mathbb{C} \times \mathbb{T}_{\mathbf{s}}^{2(\ell-1)}, \quad \text{where} \quad \mathbb{T}_{\mathbf{s}}^{2(\ell-1)} = \Big\{ (z_i)_{i\leq\ell} \in \mathbb{C}^\ell : \sum_{i=1}^{\ell} s_i z_i \in \mathbb{Z} \oplus \mathrm{i}\mathbb{Z} \Big\} / \mathbb{Z}^{2\ell} \subset \mathbb{T}^{2\ell} = V_{\mathbf{s}}.
$$

The second covering in (5.6) can be written as

$$
\mathbb{C} \times \mathbb{T}_{\mathbf{s}}^{2(\ell-1)} \longrightarrow \mathbb{T}^{2\ell}, \qquad (z, [z_i]_{i\leq\ell}) \longrightarrow \Big[z_i - \frac{z}{s_i} \Big]_{i\leq\ell}. \tag{5.12}
$$

A path $t \longrightarrow [\gamma_{i'}(t)]_{i'\leq\ell}$ in $\mathbb{T}^{2\ell}$ lifts to the path

$$
t \longrightarrow \Big(\frac{1}{\ell} \sum_{i'=1}^{\ell} s_{i'}\gamma_{i'}(t), \Big[\gamma_i(t) + \frac{1}{\ell s_i} \sum_{i'=1}^{\ell} s_{i'}\gamma_{i'}(t) \Big]_{i\leq\ell} \Big)
$$

in $\mathbb{C} \times \mathbb{T}_{\mathbf{s}}^{2(\ell-1)}$. Under the standard identification of $H_1(\mathbb{T}^2;\mathbb{Z})$ with $\mathbb{Z} \oplus \mathrm{i}\mathbb{Z}$, the action of $H_1(\mathbb{T}^2;\mathbb{Z})^{\oplus\ell}$ on this cover is thus given by

$$
(\gamma_{i'})_{i'\leq\ell} \cdot (z, [z_i]_{i\leq\ell}) = \Big(z + \frac{1}{\ell} \sum_{i'=1}^{\ell} s_{i'}\gamma_{i'}, \Big[z_i + \frac{1}{\ell s_i} \sum_{i'=1}^{\ell} s_{i'}\gamma_{i'} \Big]_{i\leq\ell} \Big).
$$

The group of deck transformations of this cover is $\mathbb{Z}_{\gcd(\mathbf{s})}^2 \oplus \gcd(\mathbf{s})\mathbb{Z}^2$. The action of the second component is induced by the action of $H_1(\mathbb{T}^2;\mathbb{Z})^{\oplus\ell}$ via the surjective homomorphism

$$
H_1(\mathbb{T}^2;\mathbb{Z})^{\oplus\ell} \longrightarrow \gcd(\mathbf{s})H_1(\mathbb{T}^2;\mathbb{Z}), \qquad (\gamma_{i'})_{i'\leq\ell} \longrightarrow \sum_{i'=1}^{\ell} s_{i'}\gamma_{i'}.
$$

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5.2 Some properties

We now describe some cases when the (co)homology of the abelian covers $\widehat{V}_{H;\mathbf{s}}$ is finitely generated. As indicated in [4, Sections 1.2,1.3], the refinement to the usual GW-invariants suggested in [7] is more likely to lead to qualitative applications in the symplectic sum context in such cases. We continue with the notation of Section 5.1.

Lemma 5.2. *Let V be a finite connected CW-complex, $H \subset H_1(V;\mathbb{Z})$ be a submodule, and $\mathbf{s} \in \mathbb{Z}_{\pm}^{\ell}$ with $\ell \in \mathbb{Z}^+$. If $H_*(\widehat{V}_H;\mathbb{Q})$ is finitely generated, then so is $H_*(\widehat{V}_{H;\mathbf{s}};\mathbb{Q})$.*

Proof. Since $\mathcal{R}_H/\mathcal{R}'_{H;\mathbf{s}}$ is finite, it is sufficient to show that $H_*(\widehat{V}'_{H;\mathbf{s}};\mathbb{Q})$ is finitely generated. By the Universal Coefficient Theorem [17, Theorem 53.5], H_* is finitely generated if and only if H^* is. Since $H_{\mathbf{s}/\gcd(\mathbf{s})} \subset H_{\mathbf{s}}$ and $H \subset H_1(V;\mathbb{Z})$ is finitely generated,

$$q \colon \widehat{V}'_{H;\mathbf{s}/\gcd(\mathbf{s})} = \widehat{V}^{\ell}/H_{\mathbf{s}/\gcd(\mathbf{s})} \longrightarrow \widehat{V}'_{H;\mathbf{s}} = \widehat{V}^{\ell}/H_{\mathbf{s}}$$

is a finite covering and so the homomorphism

$$q^* \colon H^*\big(\widehat{V}'_{H;\mathbf{s}};\mathbb{Q}\big) \longrightarrow H^*\big(\widehat{V}'_{H;\mathbf{s}/\gcd(\mathbf{s})};\mathbb{Q}\big)$$

is injective. In particular, the claim of the lemma holds if $\ell = 1$.

Suppose $\ell > 1$. Let $\mathbf{s}'$ denote the tuple consisting of the first $(\ell-1)$ components of $\mathbf{s}$ and

$$H' = H + s_{\ell} H_1(V;\mathbb{Z}) \subset H_1(V;\mathbb{Z}). \tag{5.13}$$

The projection $\widehat{V}^{\ell} \longrightarrow V$ onto the last component induces a fiber bundle

$$q_{\ell} \colon \widehat{V}'_{H;\mathbf{s}} = \widehat{V}^{\ell}/H_{\mathbf{s}} \longrightarrow V = \widehat{V}/H_1(V;\mathbb{Z})$$

with fiber $\widehat{V}_{H';\mathbf{s}'}$. By Serre's Spectral Sequence (e.g. Theorem 9.2.1, 9.2.17, or 9.3.1 in [19] applied with $\mathbb{Z}_2$-coefficients in the last two cases), $H^*(\widehat{V}'_{H;\mathbf{s}};\mathbb{Q})$ is thus finitely generated if $H^*(V;\mathbb{Q})$ and $H^*(\widehat{V}'_{H';\mathbf{s}'};\mathbb{Q})$ are finitely generated. This is the case for $H^*(V;\mathbb{Q})$ because V is a finite CW-complex. By induction on ℓ, we can assume that this is also the case for $H^*(\widehat{V}'_{H';\mathbf{s}'};\mathbb{Q})$. $\qquad\square$

Remark 5.3. The statement and proof of Lemma 5.2 can be adapted to a disconnected V. For each $r = 1, \ldots, N$, let

$$\mathcal{R}_{H;r} = q_H\big(H_1(V_r;\mathbb{Z})\big) \subset \mathcal{R}_H \,;$$

these modules span $\mathcal{R}_H$. The first factor in the definition of $\widehat{V}_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$ in (5.6) is finite if and only if the submodule

$$\widetilde{\mathcal{R}}_{H;\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \sum_{\substack{1 \leq r \leq N \\ \ell_r \neq 0}} \mathcal{R}_{H;r} \subset \mathcal{R}_H \tag{5.14}$$

has finite index. This index is finite if $\ell_r \neq 0$ whenever $H_1(V_r;\mathbb{Q}) \neq \{0\}$ or if $V = \{0,\infty\} \times F$ for some connected F and

$$H = H_{\Delta} \subset H_1(V;\mathbb{Z}) = H_1(F;\mathbb{Z}) \oplus H_1(F;\mathbb{Z})$$

is the diagonal. If $\widetilde{\mathcal{R}}_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$ does not have finite index in $\mathcal{R}_H$, then $H_*(\widehat{V}_{H;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q})$ is clearly not finitely generated. If the index is finite, $H_*(\widehat{V}_{H;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q})$ is finitely generated. For the purposes of establishing this statement, V can be replaced by the union of V_r with $\ell_r \neq 0$ and H by its intersection with the H_1 of this subspace. Thus, we can assume that $\ell_r \neq 0$ for all $r = 1,\ldots,N$. The proof of Lemma 5.2 then applies by projecting to V_N and replacing $s_\ell H_1(V;\mathbb{Z})$ in (5.13) by $s_{N;\ell_N} H_1(V_N;\mathbb{Z})$ if $\ell_N \geq 2$ and by $H_1(V_N;\mathbb{Z})$ if $\ell_N = 1$ (in this case, V_N no longer appears in the fiber).

We next relate the action of $\mathrm{Deck}(\pi'_{H;\mathbf{s}_1\ldots\mathbf{s}_N})$ on the cohomology of $\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$ to the flux subgroup $\mathrm{Flux}(V) \subset H_1(V;\mathbb{Z})$ defined in Section 1.2. Let $V_1,\ldots,V_N$ be the connected components of V, $\mathbf{s}_1 \in \mathbb{Z}_\pm^{\ell_1},\ldots,\mathbf{s}_N \in \mathbb{Z}_\pm^{\ell_N}$, and $H \subset H_1(V;\mathbb{Z})$ be a submodule. Define

$$\mathrm{Flux}(V)_{H;\mathbf{s}_1\ldots\mathbf{s}_N} = \left\{ q_H\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}\big((\gamma_{r;i})_{i\leq\ell_r,r\leq N}\big)\big) : \gamma_{r;i} \in \mathrm{Flux}(V_r) \; \forall\, i \leq \ell_r,\, r \leq N \right\}$$
$$\subset \mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N} \subset \mathcal{R}_H .$$

Lemma 5.4. *Let V, $V_1,\ldots,V_N$, $\mathbf{s}_1,\ldots,\mathbf{s}_N$, and H be as above. If*

$$\gamma \in H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z}) \qquad and \qquad q_H\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma)\big) \in \mathrm{Flux}(V)_{H;\mathbf{s}_1\ldots\mathbf{s}_N},$$

then the isomorphism

$$\{\gamma \cdot \}^* \colon H^*\big(\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z}\big) \longrightarrow H^*\big(\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z}\big)$$

is the identity.

Proof. Let

$$\gamma' \equiv (\gamma'_{r;i})_{i\leq\ell_r,r\leq N} \in \bigoplus_{r=1}^N \mathrm{Flux}(V_r)^{\oplus\ell_r}$$

be such that $q_H(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma)) = q_H(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma'))$. Since $\gamma' - \gamma \in H_{\mathbf{s}_1\ldots\mathbf{s}_N}$, the actions of γ and γ' on $\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$ are the same and we can assume that $\gamma' = \gamma$.

For each $r = 1,\ldots,N$ and $i = 1,\ldots,\ell_r$, let $\Psi_{r;i;t} \colon V_r \longrightarrow V_r$ be a loop of homeomorphisms generating $\gamma_{r;i}$ such that $\Psi_{r;i;0} = \mathrm{id}$. These loops lift to paths of homeomorphisms

$$\widehat{\Psi}_{r;i;t} \colon \widehat{V}_r \longrightarrow \widehat{V}_r, \qquad t \in [0,1], \quad \widehat{\Psi}_{r;i;0} = \mathrm{id}_{\widehat{V}_r}, \quad \widehat{\Psi}_{r;i;1}(\widehat{x}_{r;i}) = \gamma_{r;i} \cdot \widehat{x}_{r;i},$$
$$\widetilde{\Psi}_t \colon \widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N} \longrightarrow \widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}, \quad t \in [0,1], \quad \widetilde{\Psi}_t\big([(\widehat{x}_{r;i})_{i\leq\ell_r,r\leq N}]_H\big) = \big[\big(\widehat{\Psi}_{r;i;t}(\widehat{x}_{r;i})\big)_{i\leq\ell_r,r\leq N}\big]_H.$$

Since $\widetilde{\Psi}_1 = \gamma\cdot$, the homeomorphism $\gamma\cdot$ of $\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$ is homotopic to the identity. This implies the claim. $\qquad\square$

Corollary 5.5. *Let V be a finite CW-complex with connected components $V_1,\ldots,V_N$, $H \subset H_1(V;\mathbb{Z})$ be a submodule, $\ell_1,\ldots\ell_N \in \mathbb{Z}^+$, and $\mathbf{s}_1 \in \mathbb{Z}_\pm^{\ell_1},\ldots,\mathbf{s}_N \in \mathbb{Z}_\pm^{\ell_N}$.*

(1) If the index of $\mathrm{Flux}(V)_H$ in $\mathcal{R}_H$ is finite, then $H^*(\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q})$ is finitely generated.

(2) If $\mathrm{Flux}(V)_H = \mathcal{R}_H$, then $H^*(\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q})^{\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}} = H^*(\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q})$.

(3) If $\mathrm{Flux}(V)_H = \mathcal{R}_H$ and $\mathrm{rk}_\mathbb{Z}\mathcal{R}_H \leq 1$, then $\pi'^*_{H;\mathbf{s}_1\ldots\mathbf{s}_N} H^*(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q}) = H^*(\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q})$.

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Proof. (1) The vector space $H^*(\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q})$ is finitely generated over the group ring of

$$\mathrm{Deck}\big(\pi'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}\big) = \mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}. \tag{5.15}$$

By Lemma 5.4, the elements of $\mathrm{Flux}(V)_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$ act trivially on the cohomology of $\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$. Thus, $H^*(\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q})$ is finitely generated over the group ring of the quotient $\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}/\mathrm{Flux}(V)_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$. If the index of $\mathrm{Flux}(V)_H$ in $\mathcal{R}_H$ is finite, then the index of $\mathrm{Flux}(V)_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$ in $\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$ is also finite and so $H^*(\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q})$ is finitely generated over $\mathbb{Q}$.

(2) If $\mathrm{Flux}(V)_H = \mathcal{R}_H$, then $\mathrm{Flux}(V)_{H;\mathbf{s}_1\ldots\mathbf{s}_N} = \mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$. By Lemma 5.4, $\mathcal{R}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N}$ thus acts trivially on $H^*(\widehat{V}'_{H;\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Q})$.

(3) The last claim of this corollary follows from the second claim and Corollary 5.8 below. □

In the remainder of this section, we establish Lemmas 5.6 and 5.7 below. They imply Corollary 5.8, which is used in the proof of Corollary 5.5 above. The statement and proof of Lemma 5.6 are well-known. As pointed out by M. Wendt and D. Ruberman on *MathOverflow*, the statement of Lemma 5.7 can be obtained either from a spectral sequence applied to the Borel construction associated to the $\mathbb{Z}$-covering $\widetilde{V}$ or from the short exact sequence in the proof of [14, Assertion 5]. As we are not aware of any published reference for these statements, we include their proofs for the sake of completeness. In the proof of Lemma 5.7, we represent each $\mathbb{Z}$-invariant cohomology class on a regular $\mathbb{Z}$-covering by an explicit cohomology class on the associated Borel construction (the argument suggested by D. Ruberman is more efficient, but does not produce such a cocycle).

For a cochain complex (C^*,δ) with an action of a group G, let

$$(C^*,\delta)^G \equiv \big((C^*)^G, \delta|_{(C^*)^G}\big), \qquad \text{where} \quad (C^*)^G = \big\{\eta\in G^*\colon g\cdot\eta=\eta \ \forall\, g\in G\big\},$$

be the G-invariant subcomplex of (C^*,δ) and

$$H^*(C^*,\delta)^G \equiv \big\{[\eta]\in H^*(C^*,\delta)\colon g\cdot[\eta]=[\eta] \ \forall\, g\in G\big\}$$

be the G-invariant part of the cohomology of (C^*,δ). The inclusion $(C^*,\delta)^G$ into (C^*,δ) induces a homomorphism

$$H^*\big((C^*,\delta)^G\big) \longrightarrow H^*(C^*,\delta)^G. \tag{5.16}$$

Lemma 5.6. *Let (C^*,δ) be a cochain complex over $\mathbb{Q}$ with an action of a group G. If G is finite, then the homomorphism (5.16) is an isomorphism.*

Proof. Let $\eta\in C^*$ be a cocycle such that $[\eta]\in H^*(C^*,\delta)^G$. Then, the cocycle

$$\eta_G \equiv \frac{1}{|G|}\sum_{g\in G} g\cdot\eta \in (C^*)^G$$

also represents $[\eta]$ and so the homomorphism (5.16) is surjective. If $\eta\in(C^*)^G$ is a cocycle such that $\eta=\delta\mu$ for some cochain $\mu\in C^*$, then $\eta=\delta\mu_G$ and so the homomorphism (5.16) is injective. □

Let $\widetilde{V}$ be a topological space with an action of a group G and

$$\pi \colon \widetilde{V} \longrightarrow V \equiv \widetilde{V}/G$$

be the projection to the quotient. Since π commutes with the group action,

$$\pi^* H^*(V;\mathbb{Q}) \subset H^*(\widetilde{V};\mathbb{Q})^G \equiv H^*\big(C^*(\widetilde{V};\mathbb{Q}),\delta\big)^G. \qquad (5.17)$$

In some important cases, the above inclusion is an equality. If $\pi \colon \widetilde{V} \longrightarrow V$ is a regular covering, i.e. the group G of its deck transformations acts transitively on the fibers, then $V = \widetilde{V}/G$. In such a case, every G-invariant cochain (resp. cycle) on $\widetilde{V}$ descends to a cochain (resp. cycle) on V, i.e.

$$\pi^*\big(C^*(V;\mathbb{Q}),\delta\big) = \big(C^*(\widetilde{V};\mathbb{Q}),\delta\big)^G, \qquad (5.18)$$

provided the topological space V is sufficiently nice and a suitable cohomology theory is used (so that all elements of $C_*(V;\mathbb{Q})$ lie inside evenly covered open subsets of V). The inclusion in (5.17) is an equality if the homomorphism (5.16) with $C^* = C^*(\widetilde{V};\mathbb{Q})$ is surjective. The two statements below describe regular coverings for which this is the case. For the CW-complexes appearing in these statements, we use the (co)chain complexes generated by the cells of a CW-structure subordinate to the evenly covered open subsets of V or of its pullback to $\widetilde{V}$. A regular covering $\pi \colon \widetilde{V} \longrightarrow V$ is called abelian if its group of deck transformations is abelian.

Lemma 5.7. *Let $\pi \colon \widetilde{V} \longrightarrow V$ be an abelian covering of a connected CW-complex with the group of deck transformations $\mathbb{Z}$. Suppose G is a finite group that acts on $\widetilde{V}$ so that its action commutes with the action of $\mathbb{Z}$ and thus descends to V. Then,*

$$\pi^*\big(H^*(V;\mathbb{Q})^G\big) = H^*\big(\widetilde{V};\mathbb{Q}\big)^{\mathbb{Z}\times G}.$$

Proof. Let $u \colon \widetilde{V} \longrightarrow \widetilde{V}$ be a generator of the $\mathbb{Z}$-action and

$$B_{\mathbb{Z}}\widetilde{V} = \big(\mathbb{R}\times\widetilde{V}\big)/\mathbb{Z}, \qquad (t,x) \sim (t-1,u\cdot x),$$

be the corresponding Borel construction for $\widetilde{V}$. Since the actions of $\mathbb{Z}$ and G on $\widetilde{V}$ commute, the latter induces a G-action on $B_{\mathbb{Z}}\widetilde{V}$. Since $\mathbb{Z}$ acts freely on $\widetilde{V}$, the projection

$$q \colon B_{\mathbb{Z}}\widetilde{V} \longrightarrow V, \qquad q\big([t,x]\big) = \pi(x) \equiv [x],$$

is a G-equivariant homotopy equivalence. The composition of the inclusion $\iota \colon \widetilde{V} \longrightarrow B_{\mathbb{Z}}\widetilde{V}$ of a fiber for the fibration $B_{\mathbb{Z}}\widetilde{V} \longrightarrow S^1$ with q is the covering $\pi \colon \widetilde{V} \longrightarrow V$. Thus, it is sufficient to show that the inclusion

$$\iota^*\big(H^*(B_{\mathbb{Z}}\widetilde{V};\mathbb{Q})^G\big) \subset H^*(\widetilde{V};\mathbb{Q})^{\mathbb{Z}\times G}$$

is in fact an equality.

Let $\eta \in C^k(\widetilde{V};\mathbb{Q})$ be a cocycle such that $[\eta] \in H^k(\widetilde{V};\mathbb{Q})^{\mathbb{Z}\times G}$. Since

$$\eta_G \equiv \frac{1}{|G|}\sum_{g\in G} g\cdot\eta \in C^k(\widetilde{V};\mathbb{Q})^G$$

determines the same element of $H^k(B_{\mathbb{Z}}\widetilde{V};\mathbb{Q})$, we can assume that $\eta\in C^k(\widetilde{V};\mathbb{Q})^G$. Thus,

$$u^*\eta - \eta = \delta\mu \qquad \text{for some} \quad \mu \in C^{k-1}(\widetilde{V};\mathbb{Q})^G.$$

The chain groups $C_k(\widetilde{V},\mathbb{Q})$ and $C_{k-1}(\widetilde{V},\mathbb{Q})$ are freely generated by the simplices

$$\{u^s{\circ}\sigma_i\colon s\in\mathbb{Z},\ i\} \qquad \text{and} \qquad \{u^s{\circ}\tau_j\colon s\in\mathbb{Z},\ j\}$$

for some k-cells σ_i and $(k{-}1)$-cells τ_j on $\widetilde{V}$. Define

$$\widetilde{\eta} \in C^k(\mathbb{R}{\times}\widetilde{V};\mathbb{Q})^{\mathbb{Z}{\times}G} \quad \text{by} \quad \widetilde{\eta}(\{r\}{\times}u^s{\circ}\sigma_i) = \eta(u^{r+s}{\circ}\sigma_i), \ \ \widetilde{\eta}([r,r{+}1]{\times}u^s{\circ}\tau_j) = \mu(u^{r+s}{\circ}\tau_j).$$

If σ_i is a k-cell and ϖ_ℓ is a $(k{+}1)$-cell on $\widetilde{V}$, then

$$\begin{aligned}
\widetilde{\eta}\big(\partial([r,r{+}1]{\times}u^s{\circ}\sigma_i)\big) &= \widetilde{\eta}\big(\{r{+}1\}{\times}u^s{\circ}\sigma_i\big) - \widetilde{\eta}\big(\{r\}{\times}u^s{\circ}\sigma_i\big) - \widetilde{\eta}\big([r,r{+}1]{\times}u^s{\circ}\partial\sigma_i\big) \\
&= \{u^*\eta\}\big(u^{r+s}{\circ}\sigma_i\big) - \eta\big(u^{r+s}{\circ}\sigma_i\big) - \partial\mu\big(u^{r+s}{\circ}\sigma_i\big) = 0, \\
\widetilde{\eta}\big(\partial(\{r\}{\times}u^s{\circ}\varpi_\ell)\big) &= \widetilde{\eta}\big(\{r\}{\times}u^s{\circ}\partial\varpi_\ell\big) = \{\delta\eta\}\big(u^{r+s}{\circ}\varpi_\ell\big) = 0.
\end{aligned}$$

Thus, $\widetilde{\eta}$ is a $\mathbb{Z}{\times}G$-invariant cocycle and so descends to a G-invariant cocycle on $B_{\mathbb{Z}}\widetilde{V}$. The latter restricts to η along the fiber of $B_{\mathbb{Z}}\widetilde{V}\longrightarrow S^1$ over $0\in S^1$. $\qquad\square$

Corollary 5.8. *Let $\pi\colon \widetilde{V}\longrightarrow V$ be an abelian covering of a connected CW-complex and G be its group of deck transformations. If G is finitely generated and $\mathrm{rk}_{\mathbb{Z}}G\le 1$, then*

$$\pi^*H^*(V;\mathbb{Q}) = H^*\big(\widetilde{V};\mathbb{Q}\big)^G.$$

Proof. By [1, Theorem 12.6.4],

$$G \approx \mathbb{Z}^r \times G_f$$

for some $r\in\mathbb{Z}^{\ge 0}$ and some finite (abelian) group G_f. If $r=0$, the claim follows from (5.18) and Lemma 5.6.

Suppose $r=1$. Let $\widetilde{V}_f=\widetilde{V}/\mathbb{Z}$ and

$$\pi_{\mathbb{Z}}\colon \widetilde{V}\longrightarrow \widetilde{V}_f \qquad \text{and} \qquad \pi_f\colon \widetilde{V}_f\longrightarrow V$$

be the quotient projection maps. Thus, $\pi=\pi_f{\circ}\pi_{\mathbb{Z}}$. By the $r=0$ case above and Lemma 5.7,

$$\pi_f^*H^*(V;\mathbb{Q}) = H^*\big(\widetilde{V}_f;\mathbb{Q}\big)^{G_f} \qquad \text{and} \qquad \pi_{\mathbb{Z}}^*\big(H^*\big(\widetilde{V}_f;\mathbb{Q}\big)^{G_f}\big) = H^*\big(\widetilde{V};\mathbb{Q}\big)^G,$$

respectively. Combining the two equations, we obtain the $r=1$ case of the claim. $\qquad\square$

6   The refined relative GW-counts

We now provide the details needed to refine the standard relative GW-invariants of (X,V,ω), as suggested in [7, Section 5] and outlined in Section 1.1. The coverings (1.6) are special cases of the abelian covers described in Section 5.1 and are specified at the beginning of Section 6.1. In the remainder of Section 6.1, we show that the total relative evaluation morphisms (1.3) lift to the these coverings and establish Theorem 1.2. The lifts (1.8) are not unique, but can be chosen consistently; see Theorem 6.5 in Section 6.2. Theorem 1.1 is proved in Section 6.3.

6.1 The rim tori covers

Let X be a compact oriented manifold and $V \subset X$ be a compact oriented submanifold of codimension $\mathfrak{c}$ with topological components $V_1, \ldots, V_N$. With $H_X^V \subset H_1(V; \mathbb{Z})$ as in (3.2),

$$\mathcal{R}_{H_X^V} \equiv H_1(V; \mathbb{Z})_X \approx \mathcal{R}_X^V \, ;$$

see Corollary 3.2. For $\ell_1, \ldots, \ell_N \in \mathbb{Z}^{\geq 0}$ and $\mathbf{s}_1 \in \mathbb{Z}_{\pm}^{\ell_1}, \ldots, \mathbf{s}_N \in \mathbb{Z}_{\pm}^{\ell_N}$, define

$$H_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V = \left(H_X^V\right)_{\mathbf{s}_1\ldots\mathbf{s}_N} \subset H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N}; \mathbb{Z}),$$

$$\mathcal{R'}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V = \mathcal{R'}_{H_X^V;\mathbf{s}_1\ldots\mathbf{s}_N} \subset H_1(V; \mathbb{Z})_X, \qquad \mathcal{R}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V = \mathcal{R}_{H_X^V;\mathbf{s}_1\ldots\mathbf{s}_N} \, .$$

The rim tori covers (1.6) are the abelian covers

$$
\begin{aligned}
\pi'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N} &\equiv \pi'^V_{H_X^V;\mathbf{s}_1\ldots\mathbf{s}_N} : \widehat{V}'_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \widehat{V}'_{H_X^V;\mathbf{s}_1\ldots\mathbf{s}_N} \longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N}, \\
\pi^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N} &\equiv \pi^V_{H_X^V;\mathbf{s}_1\ldots\mathbf{s}_N} : \widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \widehat{V}_{H_X^V;\mathbf{s}_1\ldots\mathbf{s}_N} \longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N}.
\end{aligned}
\tag{6.1}
$$

We will write elements of the second covering as

$$\left([\gamma]_{X;\mathbf{s}_1\ldots\mathbf{s}_N}, [\widehat{x}]_X\right) \in \frac{\mathcal{R}_X^V}{\mathcal{R'}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V} \times \widehat{V}'_{X;\mathbf{s}_1\ldots\mathbf{s}_N}, \tag{6.2}$$

with notation as in (5.7) for $H = H_X^V$.

By (5.8), the groups of deck transformations of these regular coverings are

$$\mathrm{Deck}\left(\pi'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\right) = \mathcal{R'}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V \qquad \text{and} \qquad \mathrm{Deck}\left(\pi^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\right) = \mathcal{R}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V, \tag{6.3}$$

respectively. If V is connected,

$$\mathrm{Deck}\left(\pi^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}\right) \approx \frac{H_1(V; \mathbb{Z})_X}{\gcd(\mathbf{s})H_1(V; \mathbb{Z})_X} \times \gcd(\mathbf{s})H_1(V; \mathbb{Z})_X \, .$$

In general, the second group in (6.3) is different from $\mathcal{R}_X^V$ (contrary to an explicit statement in [7, Section 5] and the spirit of the description). In the $\ell_1, \ldots, \ell_N = 0$ case, $\widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$ is a discrete set of points identified with $\mathcal{R}_X^V$. In most other cases, the coverings (6.1) are non-trivial (i.e. the first one is not $V_{\mathbf{s}_1\ldots\mathbf{s}_N} \times \mathcal{R'}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V$).

Suppose in addition that $V' \subset V$ is the union of $V_1, \ldots, V_{N'}$ for some $N' \leq N$. By Corollary 3.3, $H_X^{V'} \subset H_1(V'; \mathbb{Z})$ is the image of H_X^V under the projection (5.10). Thus, the continuous map (5.11) with $H = H_X^V$ and $H' = H_X^{V'}$ induces a commutative diagram

$$
\begin{array}{ccc}
\widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N} & \xrightarrow{\;\;\widetilde{q}\;\;} & \widehat{V}'_{X;\mathbf{s}_1\ldots\mathbf{s}_{N'}} \\
\downarrow & & \downarrow \\
V_{\mathbf{s}_1\ldots\mathbf{s}_N} & \xrightarrow{\;\;q\;\;} & V'_{\mathbf{s}_1\ldots\mathbf{s}_{N'}}
\end{array}
\tag{6.4}
$$

of fiber bundles. It corresponds to the right square in the diagram of Figure 1.

Example 6.1. Suppose $\widehat{\mathbb{P}}_9^2$ is a rational elliptic surface as in Example 3.5, $F \subset \widehat{\mathbb{P}}_9^2$ is a smooth fiber, $\ell \in \mathbb{Z}^+$, and $\mathbf{s} \in \mathbb{Z}^\ell$. In this case, $N = 1$, $H_X^V = \{0\}$, and the first covering in (6.1) is isomorphic to the restriction of (5.12) to any of the connected components of $\mathbb{C} \times \mathbb{T}_{\mathbf{s}}^{2(n-1)}$. Its group of deck transformations is $\mathcal{R}'^F_{\widehat{\mathbb{P}}_9^2;\mathbf{s}} \approx \mathbb{Z}^2$ and can be identified with

$$\gcd(\mathbf{s})\mathcal{R}^F_{\widehat{\mathbb{P}}_9^2} \subset \mathcal{R}^F_{\widehat{\mathbb{P}}_9^2} \approx \mathbb{Z}^2 \,.$$

The second covering in (6.1) is (5.12) itself; its group of deck transformations is isomorphic to $(\mathbb{Z}_{\gcd(\mathbf{s})})^2 \oplus \mathbb{Z}^2$.

Example 6.2. Let F, X, and $F_0, F_\infty \subset X$ be as in Example 3.6 with F connected and $V = F_0 \cup F_\infty$. In this case, $N = 2$,
$$H_1(V; \mathbb{Z}) = H_1(F; \mathbb{Z}) \oplus H_1(F; \mathbb{Z}),$$
and $H_X^V \subset H_1(V; \mathbb{Z})$ is the diagonal. With the identifications of Example 3.6, the composition of the homomorphism (5.2) with the projection to $\mathcal{R}_X^V$ can be written as

$$H_1(F;\mathbb{Z})^{\ell_1} \oplus H_1(F;\mathbb{Z})^{\ell_2} \longrightarrow H_1(F;\mathbb{Z}), \quad \big((\gamma_{0;i})_{i \leq \ell_1}, (\gamma_{\infty;i})_{i \leq \ell_2}\big) \longrightarrow \sum_{i=1}^{\ell_1} s_{1;i}\gamma_{0;i} - \sum_{i=1}^{\ell_2} s_{2;i}\gamma_{\infty;i} \,.$$

The first covering in (6.1) is thus the first covering in (5.6) with V replaced by F, $H = \{0\}$, and $\mathbf{s}$ being the merged tuple of $\mathbf{s}_1$ and $-\mathbf{s}_2$. If $F = \mathbb{T}^2$ and $(\ell_1, \ell_2) \neq \mathbf{0}$, the second covering in (6.1) is described by (5.12) with $\mathbf{s}$ replaced by the merged tuple of $\mathbf{s}_1$ and $-\mathbf{s}_2$. With $V' = F_0$ in (6.4), $\widehat{V'}_{X;\mathbf{s}_1} = V^{\ell_1}$ and $\widetilde{q} = q$.

If Σ is a compact oriented m-dimensional manifold, $A \in H_m(X; \mathbb{Z})$, $k \in \mathbb{Z}^{\geq 0}$, and $p > m$, let $\mathfrak{X}_{\Sigma,k}(X, A)$ be the space of tuples $(z_1, \ldots, z_k, f)$ such that $f \in L_1^p(\Sigma; X)$, $f_*[\Sigma] = A$, and $z_1, \ldots, z_k \in \Sigma$ are distinct points. If in addition $m = \mathfrak{c}$, $V_1, \ldots, V_N$ and $\mathbf{s}_1, \ldots, \mathbf{s}_N$ are as before, and (1.1) holds for each $(V, \mathbf{s}) = (V_r, \mathbf{s}_r)$, let

$$\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}(X, A) \subset \mathfrak{X}_{\Sigma,k+\ell_1+\ldots+\ell_N}(X, A)$$

be the subspace of tuples $(z_1, \ldots, z_{k+\ell_1+\ldots+\ell_N}, f)$ such that

$$f^{-1}(V_r) = \big\{z_{k+\ell_1+\ldots+\ell_{r-1}+1}, \ldots, z_{k+\ell_1+\ldots+\ell_r}\big\} \qquad \forall\, r = 1, \ldots, N,$$
$$\mathrm{ord}^{V_r}_{z_{k+\ell_1+\ldots+\ell_{r-1}+i}} f = s_{r;i} \qquad \forall\, i = 1, 2, \ldots, \ell_r,\ r = 1, \ldots, N.$$

We denote by

$$\mathrm{ev}_X^V = \mathrm{ev}_{k+1} \times \ldots \times \mathrm{ev}_{k+\ell_1+\ldots+\ell_N} : \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}(X, A) \longrightarrow V_{\mathbf{s}_1\ldots\mathbf{s}_N} \tag{6.5}$$

the total relative evaluation morphism. The $\mathfrak{c} = 2$ case of the next lemma is the implied claim of [7, Section 5].

Lemma 6.3. *Suppose X is a compact oriented manifold, $V \subset X$ is a compact oriented submanifold of codimension $\mathfrak{c}$ with connected components $V_1, \ldots, V_N$, $A \in H_{\mathfrak{c}}(X; \mathbb{Z})$, and $\mathbf{s}_r \in \mathbb{Z}^{\ell_r}_\pm$ for $r = 1, \ldots, N$. If Σ is a compact oriented $\mathfrak{c}$-dimensional manifold, $k \in \mathbb{Z}^{\geq 0}$, and $r = 1, \ldots, N$, then the morphism (6.5) lifts over the first covering in (6.1) to a continuous map*

$$\widetilde{\mathrm{ev}}'^V_X : \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}(X, A) \longrightarrow \widehat{V}'_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \,.$$

Proof. Let
$$\gamma\colon [0,1] \longrightarrow \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A), \qquad t \longrightarrow (z_{t;1},\ldots,z_{t;k+\ell_1+\ldots+\ell_N}, f_t),$$

be a loop. For each $r=1,\ldots,N$ and $i=1,\ldots,\ell_r$,

$$\gamma_{r;i} \equiv \mathrm{ev}_{k+\ell_1+\ldots+\ell_{r-1}+i} \circ \gamma\colon [0,1] \longrightarrow V_r$$

is also a loop. For each $i=1,\ldots,\ell_1+\ldots+\ell_N$, let $B_{z_{k+i}} \subset \Sigma$ be a small ball around z_{k+i}. As in the construction of $f\#(-f')$ in Section 2.1, it can be assumed that

$$f_t\colon \partial B_{z_{k+\ell_1+\ldots+\ell_{r-1}+i}} \longrightarrow S_X V|_{f_t(z_{k+\ell_1+\ldots+\ell_{r-1}+i})}\,.$$

Since γ is a loop,

$$
\begin{aligned}
0 = [f_1\#(-f_0)] &= \sum_{r=1}^{N}\sum_{i=1}^{\ell_r} \iota^{X-V}_{S_X V_r *}\big(\Delta^{V_r}_X(s_{r;i}\gamma_{r;i})\big) \\
&= \iota^{X-V}_{S_X V *}\big(\Delta^V_X\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\mathrm{ev}^V_X\circ\gamma)\big) \in \mathcal{R}^V_X\,.
\end{aligned}
$$

By Corollary 3.2, this implies that $\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\mathrm{ev}^V_X\circ\gamma) \in H^V_X$. Thus, the image of the fundamental group of $\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A)$ under (6.5) in $\pi_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N})$ lies in the image of the fundamental group of $\widehat{V}'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$ under $\pi'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$. By [16, Lemma 79.1], this implies the claim. $\square$

Proof of Theorem 1.2. Since $\gcd(\mathbf{s})$ and $|\mathcal{R}^V_X|$ are relatively prime, $\widehat{V}_{H;\mathbf{s}} = \widehat{V}'_{H;\mathbf{s}}$. The second inclusion in (1.12) is then an equality by Corollary 5.5(2) with $H = H^V_X$. Under the additional assumption (1.11), both inclusions in (1.12) are equalities by Corollary 5.5(3). $\square$

Remark 6.4. Let V be possibly disconnected with topological components $V_1,\ldots,V_N$. The conclusions of Theorem 1.2 then hold if (1.10) and the relatively prime condition are replaced by

$$\Big\{\sum_{\ell_r\neq 0}[\gamma_r]_{H^V_X}\colon \gamma_r \in \mathrm{Flux}(V_r)_X \;\forall r\Big\} = \mathrm{Flux}(V)_X \qquad \text{and} \qquad \mathcal{R}'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N} = \mathcal{R}^V_X, \tag{6.6}$$

respectively. The latter is the case if $\gcd(\mathbf{s}_r)$ and $|H_1(V_r;\mathbb{Z})|$ are relatively prime for every r. It is also the case if F is connected, $X=\mathbb{P}^1\times F$ and $V=\{0,\infty\}\times F$ as in Example 6.2, and $\gcd(\mathbf{s}_1,\mathbf{s}_2)$ and $|H_1(F;\mathbb{Z})|$ are relatively prime. The conclusion of Proposition 1.3 holds if $V_r \approx \mathbb{T}^{2n-2}$ for every $r=1,\ldots,N$ and (6.6) holds.

6.2 Consistent choices of lifts

We next show that the lifts in Lemma 6.3 can be chosen in a systematic way, consistent with their use in [8] for refining the symplectic sum formula for GW-invariants and with the diagram in Figure 1. The significance of Theorem 6.5 for the former is demonstrated by [4, Proposition 4.2]. We continue with the notation of Section 6.1.

Theorem 6.5. *Suppose X is a compact oriented manifold, $V \subset X$ is a compact oriented submanifold of codimension $\mathfrak{c}$ with connected components $V_1,\ldots,V_N$, $A \in H_{\mathfrak{c}}(X;\mathbb{Z})$, and $\mathbf{s}_r \in \mathbb{Z}^{\ell_r}_{\pm}$ for $r=1,\ldots,N$. Let $\{\gamma_j\} \subset H_1(V;\mathbb{Z})$ be a collection of representatives for the elements of $\mathcal{R}^V_X/\mathcal{R}'^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$. If Σ is a compact oriented $\mathfrak{c}$-dimensional manifold and $k\in\mathbb{Z}^{\geq 0}$, there exists a lift*

$$\widetilde{\mathrm{ev}}^V_X\colon \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A) \longrightarrow \widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \tag{6.7}$$

of the morphism ev_X^V *in (6.5) over the covering* $\pi_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V$ *in (6.1) with the following property. For any* $\mathbf{f}, \mathbf{f}' \in \mathfrak{X}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}^{V_1,\ldots,V_N}(X,A)$ *with*

$$\widetilde{\mathrm{ev}}_X^V(\mathbf{f}) = \big([\gamma_j]_{X;\mathbf{s}_1\ldots\mathbf{s}_N}, [\gamma\cdot\widehat{x}]_X\big) \quad and \quad \widetilde{\mathrm{ev}}_X^V(\mathbf{f}') = \big([\gamma_{j'}]_{X;\mathbf{s}_1\ldots\mathbf{s}_N}, [\widehat{x}]_X\big) \tag{6.8}$$

for some $\widehat{x} \in \widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}$, $\gamma \in H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z})$, *and* j, j' *indexing the coset representatives, the map components of* $\mathbf{f}$ *and* $\mathbf{f}'$ *satisfy*

$$\big[f\#(-f')\big] = \iota_{S_X V*}^{X-V}\big(\Delta_X^V\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma) + \gamma_j - \gamma_{j'}\big)\big) \in H_{\mathfrak{c}}(X-V;\mathbb{Z}). \tag{6.9}$$

Furthermore, $\widetilde{\mathrm{ev}}_X^V(\mathbf{f}')$ *is the unique point in* $\pi_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^{V\;-1}(\mathrm{ev}_X^V(\mathbf{f}'))$ *so that (6.9) holds for a given value of* $\widetilde{\mathrm{ev}}_X^V(\mathbf{f})$.

Proof. We can assume that $\mathfrak{X}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}^{V_1,\ldots,V_N}(X,A) \neq \emptyset$; otherwise, there is nothing to prove. Choose a base point $\widehat{x}_{\mathbf{s}_1\ldots\mathbf{s}_N} \in \widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}$ over some $x_{\mathbf{s}_1\ldots\mathbf{s}_N} \in V_{\mathbf{s}_1\ldots\mathbf{s}_N}$, an overall base map

$$\mathbf{f}_0 \equiv (z_{0;1}, \ldots, z_{0;k+\ell_1+\ldots+\ell_N}, f_0) \in \mathfrak{X}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}^{V_1,\ldots,V_N}(X,A) \quad \text{s.t.} \quad \mathrm{ev}_X^V(\mathbf{f}_0) = x_{\mathbf{s}_1\ldots\mathbf{s}_N}, \tag{6.10}$$

and a base point

$$\mathbf{f}_m \equiv (z_{m;1}, \ldots, z_{m;k+\ell_1+\ldots+\ell_N}, f_m) \in \mathfrak{X}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}^{V_1,\ldots,V_N}(X,A)$$

for each topological component so that

$$\mathrm{ev}_X^V(\mathbf{f}_m) = x_{\mathbf{s}_1\ldots\mathbf{s}_N}, \qquad \big[f_m\#(-f_0)\big] = \iota_{S_X V*}^{X-V}\big(\Delta_X^V(\gamma_j)\big) \in H_{\mathfrak{c}}(X-V;\mathbb{Z}) \tag{6.11}$$

for some $j = j(m)$. We define

$$\widetilde{\mathrm{ev}}_X^V : \mathfrak{X}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}^{V_1,\ldots,V_N}(X,A) \longrightarrow \widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N} \equiv \frac{\mathcal{R}_X^V}{\mathcal{R}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^{\prime V}} \times \widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}' \tag{6.12}$$

to be the lift of ev_X^V over $\pi_{X;\mathbf{s}_1\ldots\mathbf{s}_N}^V$ so that

$$\widetilde{\mathrm{ev}}_X^V(\mathbf{f}_m) = \big([\gamma_{j(m)}]_{X;\mathbf{s}_1\ldots\mathbf{s}_N}, [\widehat{x}_{\mathbf{s}_1\ldots\mathbf{s}_N}]_X\big) \qquad \forall\, m;$$

see [16, Lemma 79.1].

It remains to verify that this lift has the claimed properties. Suppose $\mathbf{f}, \mathbf{f}' \in \mathfrak{X}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}^{V_1,\ldots,V_N}(X,A)$ satisfy (6.8) and thus $\mathrm{ev}_X^V(\mathbf{f}) = \mathrm{ev}_X^V(\mathbf{f}')$. Let $\mathbf{f}_m$ and $\mathbf{f}_{m'}$ be the base points of the topological components of $\mathfrak{X}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}^{V_1,\ldots,V_N}(X,A)$ containing $\mathbf{f}$ and $\mathbf{f}'$, respectively, and α and α' be paths from $\mathbf{f}_m$ to $\mathbf{f}$ and from $\mathbf{f}_{m'}$ to $\mathbf{f}'$, respectively. By (6.8), $j(m) = j$ and $j(m') = j'$. Along with (6.11), this gives

$$\big[f_m\#(-f_{m'})\big] = \big[f_m\#(-f_0)\big] - \big[f_{m'}\#(-f_0)\big] = \iota_{S_X V*}^{X-V}\big(\Delta_X^V(\gamma_j - \gamma_{j'})\big) \in H_{\mathfrak{c}}(X-V;\mathbb{Z}). \tag{6.13}$$

Since $\mathrm{ev}_X^V(\mathbf{f}_m) = \mathrm{ev}_X^V(\mathbf{f}_{m'})$ and $\mathrm{ev}_X^V(\mathbf{f}) = \mathrm{ev}_X^V(\mathbf{f}')$,

$$\gamma' \equiv \big(-\mathrm{ev}_X^V \circ \alpha'\big) * \big(\mathrm{ev}_X^V \circ \alpha\big) : [0,1] \longrightarrow V_r$$

is a well-defined loop. Furthermore,

$$[f\#(-f')] - [f_m\#(-f_{m'})] = \iota_{S_X V*}^{X-V}\big(\Delta_X^V(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma'))\big) \in H_{\mathfrak{c}}(X-V;\mathbb{Z}).$$

Combining this statement with (6.13), we find that

$$[f\#(-f')] = \iota_{S_X V*}^{X-V}\big(\Delta_X^V\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma') + \gamma_j - \gamma_{j'}\big)\big) \in H_{\mathfrak{c}}(X-V;\mathbb{Z}). \tag{6.14}$$

Let $\widehat{\alpha}, \widehat{\alpha}', \widehat{\gamma}' \colon [0,1] \longrightarrow \widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}$ be the lifts of $\mathrm{ev}_X^V \circ \alpha, \mathrm{ev}_X^V \circ \alpha', \gamma'$ such that

$$\widehat{\alpha}(0), \widehat{\alpha}'(0) = \widehat{x}_{\mathbf{s}_1,\ldots,\mathbf{s}_N}, \qquad \widehat{\gamma}'(0) = \widehat{\alpha}'(1).$$

Thus,

$$\big[\widehat{\alpha}(1)\big]_X = \widetilde{\mathrm{ev}}_X'^V(\mathbf{f}) = [\gamma\cdot\widehat{x}]_X, \qquad \big[\widehat{\alpha}'(1)\big]_X = \widetilde{\mathrm{ev}}_X'^V(\mathbf{f}') = [\widehat{x}]_X,$$

where $\widetilde{\mathrm{ev}}_X'^V$ is the composition of (6.12) with the projection to the second component. Since $\widehat{\alpha} = \widehat{\alpha}' * \widehat{\gamma}' \colon [0,1] \longrightarrow \widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}$,

$$[\gamma\cdot\widehat{x}]_X = \big[\widehat{\alpha}(1)\big]_X = \big[\gamma'\cdot\widehat{\alpha}'(1)\big]_X = [\gamma'\cdot\widehat{x}]_X\,.$$

Since $\widehat{V}'_{X;\mathbf{s}_1\ldots\mathbf{s}_N} = \widehat{V}_{\mathbf{s}_1\ldots\mathbf{s}_N}/H^V_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$, it follows that

$$\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma) - \Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma') \in H_X^V \subset H_1(V;\mathbb{Z})\,.$$

The claim (6.9) now follows from (6.14) and Corollary 3.2 with $U = \emptyset$. The latter also implies the uniqueness claim. $\qquad\square$

We will call a lift (6.7) satisfying the properties of Theorem 6.5 $\{\gamma_j\}$-compatible. The next proposition describes all such lifts.

Proposition 6.6. *Suppose* X, V, $\mathfrak{c}$, A, $\mathbf{s}_r$, $\{\gamma_j\}$, Σ, *and* k *are as in the statement of Theorem 6.5.*

(1) *Let* $x_{\mathbf{s}_1\ldots\mathbf{s}_N} \in V_{\mathbf{s}_1\ldots\mathbf{s}_N}$. *If a lift (6.7) of (6.5) satisfies (6.9) for all* $\mathbf{f}, \mathbf{f}' \in \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A)$ *with* $\mathrm{ev}_X^V(\mathbf{f}) = x_{\mathbf{s}_1\ldots\mathbf{s}_N}$, *then it is* $\{\gamma_j\}$-*compatible.*

(2) *Let* $\eta \in H_1(X;\mathbb{Z})$. *If* $\widetilde{\mathrm{ev}}_X^V$ *is a lift of (6.5) compatible with* $\{\gamma_j\}$, *then so is the lift* $\Theta_\eta \circ \widetilde{\mathrm{ev}}_X^V$.

(3) *If the lifts* $\widetilde{\mathrm{ev}}_X^V$ *and* $\widetilde{\mathrm{ev}}_X'^V$ *as in (6.5) are compatible with* $\{\gamma_j\}$, *then* $\widetilde{\mathrm{ev}}_X'^V = \Theta_\eta \circ \widetilde{\mathrm{ev}}_X^V$ *for some* $\eta \in H_1(X;\mathbb{Z})$.

Proof. (1) This is immediate because the two sides of (6.9) take discrete values, but depend continuously on $(\mathbf{f}, \mathbf{f}')$.

(2) If $\mathbf{f}, \mathbf{f}' \in \mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A)$ satisfy (6.8), then

$$\Theta_\eta\big(\widetilde{\mathrm{ev}}_X^V(\mathbf{f})\big) = \big([\gamma_j(\eta)]_{X;\mathbf{s}_1\ldots\mathbf{s}_N}, [\eta_j\gamma\cdot\widehat{x}]_X\big), \quad \Theta_\eta\big(\widetilde{\mathrm{ev}}_X^V(\mathbf{f}')\big) = \big([\gamma_{j'}(\eta)]_{X;\mathbf{s}_1\ldots\mathbf{s}_N}, [\eta_{j'}\cdot\widehat{x}]_X\big),$$

with notation as in (5.9). By (5.9) and (6.9),

$$\iota_{S_X V*}^{X-V}\big(\Delta_X^V\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\eta_j + \gamma - \eta_{j'}) + \gamma_j(\eta) - \gamma_{j'}(\eta)\big)\big) = \iota_{S_X V*}^{X-V}\big(\Delta_X^V\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma) + \gamma_j - \gamma_{j'}\big)\big)$$
$$= \big[f\#(-f')\big] \in H_{\mathfrak{c}}(X-V;\mathbb{Z}).$$

This establishes the second claim.

(3) Let $x_{\mathbf{s}_1\ldots\mathbf{s}_N}$ and $\mathbf{f}_0$ be as in (6.10). Suppose

$$\widetilde{\mathrm{ev}}_X^V(\mathbf{f}_0) = \big([\gamma_j]_{X;\mathbf{s}_1\ldots\mathbf{s}_N},[\widehat{x}]_X\big), \qquad \widetilde{\mathrm{ev}}_X^{\prime V}(\mathbf{f}_0) = \big([\gamma_j']_{X;\mathbf{s}_1\ldots\mathbf{s}_N},[\widehat{x}']_X\big),$$

where γ_j,γ_j' are among the chosen coset representatives. Since $\widetilde{\mathrm{ev}}_X^V$ and $\widetilde{\mathrm{ev}}_X^{\prime V}$ are lifts of (6.5),

$$[\widehat{x}']_X = [\eta_j\cdot\widehat{x}]_X$$

for some $\eta_j\in H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z})$. Let

$$\eta = \gamma_j' - \gamma_j + \Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\eta_j).$$

In particular, $\gamma_j(\eta)=\gamma_j'$. Suppose $\mathbf{f}'\in\mathfrak{X}^{V_1,\ldots,V_N}_{\Sigma,k;\mathbf{s}_1\ldots\mathbf{s}_N}(X,A)$,

$$\widetilde{\mathrm{ev}}_X^V(\mathbf{f}') = \big([\gamma_{j'}]_{X;\mathbf{s}_1\ldots\mathbf{s}_N},[\gamma^{-1}\cdot\widehat{x}]_X\big), \quad \widetilde{\mathrm{ev}}_X^{\prime V}(\mathbf{f}') = \big([\gamma_{j'}']_{X;\mathbf{s}_1\ldots\mathbf{s}_N},[\eta_{j'}\gamma^{-1}\cdot\widehat{x}]_X\big)$$

for some $\gamma,\eta_{j'}\in H_1(V_{\mathbf{s}_1\ldots\mathbf{s}_N};\mathbb{Z})$. Since $\widetilde{\mathrm{ev}}_X^V$ and $\widetilde{\mathrm{ev}}_X^{\prime V}$ are lifts of (6.5) compatible with $\{\gamma_j\}$,

$$\iota^{X-V}_{S_XV*}\big(\Delta_X^V\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\gamma)+\gamma_j-\gamma_{j'}\big)\big) = \big[f_0\#(-f')\big]$$
$$= \iota^{X-V}_{S_XV*}\big(\Delta_X^V\big(\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\eta_j+\gamma-\eta_{j'})+\gamma_j'-\gamma_{j'}'\big)\big).$$

It follows that

$$\gamma_{j'} + \eta - \gamma_{j'}' - \Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}(\eta_{j'}) \in H_X^V, \qquad \gamma_{j'}(\eta)=\gamma_{j'}',$$
$$\Theta_\eta\big([\gamma_{j'}]_{X;\mathbf{s}_1\ldots\mathbf{s}_N},[\gamma^{-1}\cdot\widehat{x}]_X\big) = \big([\gamma_{j'}']_{X;\mathbf{s}_1\ldots\mathbf{s}_N},[\eta_{j'}\gamma^{-1}\cdot\widehat{x}]_X\big).$$

Thus, $\widetilde{\mathrm{ev}}_X^{\prime V}=\Theta_\eta\circ\widetilde{\mathrm{ev}}_X^V$. $\qquad\square$

Remark 6.7. The above choices of $\{\gamma_j\}$, $\mathbf{f}_0$, and $\widehat{x}_{\mathbf{s}_1\ldots\mathbf{s}_N}$ roughly correspond to the two set-theoretic descriptions of $\widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$ in [7] as equivalence classes of cycles in $X-V$ that are standard near V. These descriptions in [7, Section 5] do not specify a topology, especially when the contact points come together. A hands-on description of the topology of $\widehat{V}_{X;(1)}$ is given at the end of [7, Section 5]; our definition of $\widehat{V}_{X;\mathbf{s}_1\ldots\mathbf{s}_N}$ via the homomorphisms $\Phi_{V;\mathbf{s}_1\ldots\mathbf{s}_N}$ is based on this description. Lemma 6.3 makes it apparent that the lifts of the evaluation maps can be extended over the compactified moduli spaces of relative stable maps once these spaces are shown to be compatible with gluing; the latter is necessary for any virtual construction of the moduli cycle and ensures that the boundary strata of the configuration spaces are of real codimension at least 2 (and so the loop γ as in the proof of Lemma 6.3 can be assumed to lie completely in the main stratum).

Example 6.8. If $X=\widehat{\mathbb{P}}^2_9$ is a rational elliptic surface, $V=F$ is a smooth fiber as in Examples 3.5 and 6.1, and $\ell\in\mathbb{Z}^+$, then $H_X^V=0$. We can identify $H_1(V;\mathbb{Z})$ with $\mathbb{Z}\oplus i\mathbb{Z}$ and take

$$x_{\mathbf{s}} = [\mathbf{0}_{2\ell}] \in V^\ell=\mathbb{T}^{2\ell}, \qquad \widehat{x}_{\mathbf{s}} = \mathbf{0}_{2\ell} \in \mathbb{C}^\ell.$$

Let $\{\gamma_j\}\subset\mathbb{Z}\oplus i\mathbb{Z}$ be a collection of representatives for the elements of $\mathbb{Z}_{\gcd(\mathbf{s})}\oplus i\mathbb{Z}_{\gcd(\mathbf{s})}$. As indicated in [4, Example 3.1], it is convenient to identify the corresponding base points for the components of $\widehat{V}_{X;\mathbf{s}}$ as

$$\big([\gamma_j]_{X;\mathbf{s}},[\widehat{x}_{\mathbf{s}}]_X\big) = \big(\ell^{-1}\gamma_j,[\ell^{-1}s_i^{-1}\gamma_j]_{i\leq\ell}\big) \in \widehat{V}_{X;\mathbf{s}} = \mathbb{C}\times\mathbb{T}^{2(\ell-1)}_{\mathbf{s}}.$$

Example 6.9. In the setting of Examples 3.6 and 6.2, only the tuples $\mathbf{s}_1 \in \mathbb{Z}_{\pm}^{\ell_1}$ and $\mathbf{s}_2 \in \mathbb{Z}_{\pm}^{\ell_2}$ with

$$\sum_{i=1}^{\ell_1} s_{1;i} = \sum_{i=1}^{\ell_2} s_{2;i}$$

are relevant in the context of Theorem 6.5. We first choose a base point $x_1 \in F$. For all $\mathbf{s}_1$ and $\mathbf{s}_2$ as above, we then choose a base point $\widehat{x}_{\mathbf{s}_1 \mathbf{s}_2}$ in $\widehat{F}_{\mathbf{s}}$ over $x_1^{\ell_1 + \ell_2}$ in $F^{\ell_1 + \ell_2}$, where $\mathbf{s}$ is the merged tuple of $\mathbf{s}_1$ and $\mathbf{s}_2$ as before, and a collection $\{\gamma_j\} \subset H_1(F; \mathbb{Z})$ of representatives for the elements of

$$\frac{H_1(V; \mathbb{Z})_X}{\mathcal{R}'^V_{X;\mathbf{s}_1 \mathbf{s}_2}} = \frac{H_1(F; \mathbb{Z})}{\gcd(\mathbf{s}_1, \mathbf{s}_2) H_1(F; \mathbb{Z})} .$$

If $F = \mathbb{T}^2$ and $(\ell_1, \ell_2) \neq \mathbf{0}$, then we can identify $H_1(F; \mathbb{Z})$ with $\mathbb{Z} \oplus \mathrm{i}\mathbb{Z}$. As indicated in [4, Example 3.3], it is then convenient to identify the corresponding base points for the components of $\widehat{V}_{X;\mathbf{s}}$ as

$$\left([\gamma_j]_{X;\mathbf{s}_1 \mathbf{s}_2}, [\widehat{x}_{\mathbf{s}_1 \mathbf{s}_2}]_X\right) = \left(\ell^{-1} \gamma_j, \left([\ell^{-1} s_{1;i}^{-1} \gamma_j]_{i \leq \ell_1}, [-\ell^{-1} s_{2;i}^{-1} \gamma_j]_{i \leq \ell_2}\right)\right) \in \mathbb{C} \times \mathbb{T}^{2(\ell_1 + \ell_2 - 1)}_{(\mathbf{s}_1, -\mathbf{s}_2)}.$$

6.3   Proof of Theorem 1.1

Let $n_V = \dim_{\mathbb{R}} V$ and $r_0 = \mathrm{rk}_{\mathbb{Z}} H_1(V; \mathbb{Z})_X$. In light of (1.7) and (1.8), it is sufficient to show that

$$\widehat{V}'_{X;\mathbf{s}} \approx \mathbb{R}^{r_0} \times Y \tag{6.15}$$

for some manifold Y of dimension $n_V \ell - r_0$.

With $B = (S^1)^m$ and H_X^V as in (3.2), let

$$H_X^B = q_*(H_X^V) \subset H_1(B; \mathbb{Z}), \qquad \widetilde{H}_X^B = q_*^{-1}(H_X^B) \subset H_1(V; \mathbb{Z}).$$

Denote by $\mathcal{K}_{X;\mathbf{s}}^B \subset \pi_1(B^\ell)$ and $\mathcal{K}_{X;\mathbf{s}}^V, \widetilde{\mathcal{K}}_{X;\mathbf{s}}^B \subset \pi_1(V^\ell)$ the preimages of H_X^B and $H_X^V, \widetilde{H}_X^B$ under the homomorphisms

$$\pi_1(B^\ell) \xrightarrow{\mathrm{Hur}} H_1(B^\ell; \mathbb{Z}) \xrightarrow{\Phi_{B;\mathbf{s}}} H_1(B; \mathbb{Z}) \quad \text{and} \quad \pi_1(V^\ell) \xrightarrow{\mathrm{Hur}} H_1(V^\ell; \mathbb{Z}) \xrightarrow{\Phi_{V;\mathbf{s}}} H_1(V; \mathbb{Z}),$$

with $\Phi_{B;\mathbf{s}}$ and $\Phi_{V;\mathbf{s}}$ as in (5.2). From the commutativity of the diagram

$$\begin{array}{ccc}
\pi_1(V^\ell) & \xrightarrow{\Phi_{V;\mathbf{s}} \circ \mathrm{Hur}} & H_1(V; \mathbb{Z}) \\
{\scriptstyle q_*} \downarrow & & \downarrow {\scriptstyle q_*} \\
\pi_1(B^\ell) & \xrightarrow{\Phi_{B;\mathbf{s}} \circ \mathrm{Hur}} & H_1(B; \mathbb{Z}) ,
\end{array}$$

we find that $\widetilde{\mathcal{K}}_{X;\mathbf{s}}^B = q_*^{-1}(\mathcal{K}_{X;\mathbf{s}}^B)$.

Let

$$\pi'^B_{X;\mathbf{s}} \colon \widehat{B}'_{X;\mathbf{s}} \equiv \widehat{B}'_{H^B_{X;\mathbf{s}}} \longrightarrow B^\ell \qquad \text{and} \qquad \tilde{q} \colon \widetilde{V}^B_{X;\mathbf{s}} \equiv \pi'^{B*}_{X;\mathbf{s}} V^\ell \longrightarrow \widehat{B}'_{X;\mathbf{s}}$$

be the covering corresponding to the normal subgroup $\mathcal{K}_{X;\mathbf{s}}^B$ of $\pi_1(B^\ell)$ and the pullback of the fibration $q^\ell \colon V^\ell \longrightarrow B^\ell$, respectively. Since $\tilde{q}_*$ is surjective on π_1, the natural projection

$$\widetilde{\pi}'^B_{X;\mathbf{s}} \colon \widetilde{V}^B_{X;\mathbf{s}} \longrightarrow V^\ell$$

is the covering corresponding to the normal subgroup $q_*^{-1}(\mathcal{K}_{X;\mathbf{s}}^B) = \widetilde{\mathcal{K}}_{X;\mathbf{s}}^B$ of $\pi_1(V^\ell)$. Since $\mathcal{K}_{X;\mathbf{s}}^V \subset \widetilde{\mathcal{K}}_{X;\mathbf{s}}^B$, the total space of the covering

$$\pi_{X;\mathbf{s}}'^V \colon \widehat{V}'_{X;\mathbf{s}} \longrightarrow V_{\mathbf{s}} = V^\ell$$

corresponding to $\mathcal{K}_{X;\mathbf{s}}^V$ is also a covering of $\widetilde{V}_{X;\mathbf{s}}^B$.

By the homotopy exact sequence [19, Theorem 7.2.10] for the fibration $q \colon V \longrightarrow B$ and the Hurewicz isomorphism [19, Theorem 7.5.5], the sequence

$$H_1(F;\mathbb{Z}) \longrightarrow H_1(V;\mathbb{Z}) \xrightarrow{q_*} H_1(B;\mathbb{Z}) \longrightarrow 0$$

is exact. Since $H_1(F;\mathbb{Q}) = \{0\}$,

$$\mathrm{rk}_{\mathbb{Z}}\big(H_1(B;\mathbb{Z})/H_X^B\big) = \mathrm{rk}_{\mathbb{Z}} H_1(V;\mathbb{Z})_X = r_0.$$

Similarly to Example 5.1, this implies that

$$\widehat{B}'_{X;\mathbf{s}} \approx \mathbb{R}^{r_0} \times (S^1)^{m\ell - r_0}.$$

Thus, $\widetilde{V}_{X;\mathbf{s}}^B \approx \mathbb{R}^{r_0} \times Y'$ for some manifold Y' of dimension $n_V\ell - r_0$ and so (6.15) holds for some covering Y of Y'.

Remark 6.10. Theorem 1.1 extends to disconnected divisors V with topological components $V_1, \ldots, V_N$ by replacing the rank of $H_1(V;\mathbb{Z})_X$ with the rank of the submodule in (5.14) with $H = H_X^V$.

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