Mirror Symmetry: from curve counts to hypergeometric series

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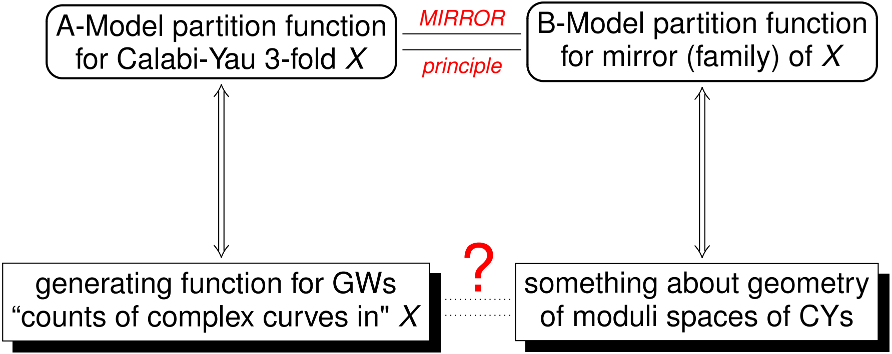
From string theory to enumerative geometry



Basic notions

- Calabi-Yau 3-fold X = (cmpt) complex manifold $\dim_{\mathbb{C}} X = 3$, $c_1(TX) = 0$
- Mirror family $\widehat{X}$ = family of Calabi-Yau 3-folds some singular

What is special about CY 3-folds X?

expected # of genus-g degree-d curves in X is finite, $n_{g,d} \in \mathbb{Z}$

e.g. $n_{0,1} = 2,875$ # of lines on general X_5

$n_{g,1} = n_{g,2} = 0 \ \forall g \geq 1$

genus g degree d GW of X: $N_{g,d} \in \mathbb{Q}$

"linear combination" of $n_{g',d'}$, $g' \leq g$, $d' \leq d$

More generally: $n_{1,d}$ is finite if $c_1(TX) = 0$ (any dim X)

$\Longrightarrow$ genus 1 degree d GW of X: $N_{1,d} \in \mathbb{Q}$

Main example: $X \equiv X_n \subset \mathbb{P}^{n-1}$ hypersurface of degree n

Mirror symmetry for X

$$\mathbb{A}_g^X(Q) \equiv \sum_{d=1}^{\infty} N_{g,d} Q^d \stackrel{?}{=} \mathbb{B}_g^X(q), \qquad Q = Q(q),\ q = q(Q)$$

$\mathbb{B}_g^X(q) =$ explicit function determined by mirror family of X

Mathematical verifications

g = 0: *Givental'96/Lian-Liu-Yau'97/...* (X_n, etc.)
g = 1: *'07* (hypersurfaces $X_n \subset \mathbb{P}^{n-1}$ only)
g ≥ 2: **?**

B-model PFs for $X = X_n$

$$\mathbb{F}_0(x,q) = \sum_{d=0}^{\infty} q^d \frac{\prod_{r=1}^{r=nd}(nx+r)}{\prod_{r=1}^{r=d}(x+r)^n} \in 1 + q\cdot\mathbb{Q}(x)[[q]]$$

$$\mathbb{I}_0(q) = \mathbb{F}_0(0,q), \qquad \mathbb{F}_1(x,q) = \left\{1 + \frac{q}{x}\frac{\partial}{\partial q}\right\}\frac{\mathbb{F}_0(x,q)}{\mathbb{I}_0(q)}$$

$$\mathbb{I}_1(q) = \mathbb{F}_1(0,q), \qquad \mathbb{F}_2(x,q) = \left\{1 + \frac{q}{x}\frac{\partial}{\partial q}\right\}\frac{\mathbb{F}_1(x,q)}{\mathbb{I}_1(q)}$$

$$\mathbb{I}_3(q), \mathbb{I}_4(q), \ldots, \mathbb{I}_{n-1}(q) \in 1 + q\cdot\mathbb{Q}[[q]]$$

$$\mathbb{F}_0(x,q) \equiv \mathbb{I}_0(q)\big(1 + \mathbb{J}(q)x + O(x^2)\big) \implies \mathbb{I}_1(q) = 1 + q\frac{\partial}{\partial q}\mathbb{J}(q)$$

Mirror symmetry in genus 1 for $X = X_n$

$$\mathbb{B}_1^X(q) = \left(\frac{(n-2)(n+1)}{48} + \frac{1-(1-n)^n}{24n^2}\right)\mathbb{J}(q) - \frac{(3n-8)(n-1)}{48}\log(1-n^n q) + \frac{n^2-1+(1-n)^n}{24n}\log\mathbb{I}_0(q) - \frac{1}{2}\sum_{r=0}^{n-1}\binom{r}{2}\log\mathbb{I}_r(q)$$

Mirror Symmetry in genus 1 for $X = X_n \subset \mathbb{P}^{n-1}$

$$\mathbb{A}_1^X(Q) \equiv \sum_{d=1}^{\infty} N_{1,d} Q^d = \mathbb{B}_1^X(q), \qquad Q = q \cdot e^{\mathbb{J}(q)}$$

Some properties of $\mathbb{I}_r(q)$

$$\mathbb{F}_0(x,q)=\sum_{d=0}^{\infty} q^d \frac{\prod_{r=1}^{r=nd}(nx+r)}{\prod_{r=1}^{r=d}(x+r)^n} \in 1+q\cdot\mathbb{Q}(x)[[q]]$$

$$\mathbb{I}_0(q)=\mathbb{F}_0(0,q), \qquad \mathbb{F}_1(x,q)=\left\{1+\frac{q}{x}\frac{\partial}{\partial q}\right\}\frac{\mathbb{F}_0(x,q)}{\mathbb{I}_0(q)}$$

$$\mathbb{I}_1(q)=\mathbb{F}_1(0,q), \qquad \mathbb{F}_2(x,q)=\left\{1+\frac{q}{x}\frac{\partial}{\partial q}\right\}\frac{\mathbb{F}_1(x,q)}{\mathbb{I}_1(q)}$$

$$\mathbb{I}_3(q),\mathbb{I}_4(q),\ldots,\mathbb{I}_{n-1}(q)\in 1+q\cdot\mathbb{Q}[[q]]$$

$$\mathbb{I}_r(q)=\mathbb{I}_{n-1-r}(q), \qquad r=0,1,\ldots,n-1$$

$$\mathbb{I}_0(q)\mathbb{I}_1(q)\ldots\mathbb{I}_{n-1}(q)=(1-n^nq)^{-1}$$

Reality check, I

$$n = 1, 2, 4 : \mathbb{B}_1^X(q) = 0$$

$$n{=}1 : X = \emptyset \subset \mathbb{P}^{1-1} \Longrightarrow N_{g,d}{=}0 \;\forall\, d{\in}\mathbb{Z}^+ \Longrightarrow \mathbb{A}_1^X(Q) = 0 \quad \checkmark$$
$$n{=}2 : X = 2pts \subset \mathbb{P}^1 \Longrightarrow N_{g,d}{=}0 \;\forall\, d{\in}\mathbb{Z}^+ \Longrightarrow \mathbb{A}_1^X(Q) = 0 \quad \checkmark$$

$$n{=}4 : X = K3 \subset \mathbb{P}^3 \Longrightarrow N_{g,d}{=}0 \;\forall\, d{\in}\mathbb{Z}^+ \Longrightarrow \mathbb{A}_1^X(Q) = 0 \quad \checkmark$$

Geometric reason (*Junho Lee'03*): there are no J-holomorphic curves on K3 for some almost complex structure J

Verification of physics predictions: $\mathbb{A}_1^X(Q) \stackrel{?}{=} \mathbb{B}_1^X(q)$

$n{=}5: X = X_5 \subset \mathbb{P}^4$ quintic 3-fold

$$\mathbb{B}_1^X(q) = \frac{25}{12}\mathbb{J}(q) - \frac{1}{12}\log(1-5^5q) - \frac{31}{3}\log\mathbb{I}_0(q) - \frac{1}{2}\log\mathbb{I}_1(q)$$

B*ershadsky*-***C****ecotti*-***O****oguri*-***V****afa'93* ✓

$n{=}6: X = X_6 \subset \mathbb{P}^5$ sextic 4-fold

$$\mathbb{B}_1^X(q) = -\frac{35}{2}\mathbb{J}(q) - \frac{1}{24}\log(1-6^6q) - \frac{423}{4}\log\mathbb{I}_0(q) - \log\mathbb{I}_1(q)$$

Klemm-Pandharipande'07 ✓

A mystery: BPS states in higher dimensions?

Gopakumar-Vafa'98, $\dim X = 3$: $\exists$ “BPS states" $n_{g,d} \in \mathbb{Z}$ s.t.

$$\{N_{g,d}\} = \text{Upper-}\Delta \text{ Transform}(\{n_{g,d}\})$$

Klemm-Pand...'07, $\dim X = 4$: $\exists$ “curve counts" $n_{g,d} \in \mathbb{Z}$ s.t.

$$\{N_{g,d}\} = \text{Upper-}\Delta \text{ Transform}(\{n_{g,d}\})$$

Pandharipande-Z.'08, $\dim X = 5$: same

All conjectures: true for $d \le 100$ in $X_7 \subset \mathbb{P}^6$
Klemm: no physical motivation if $\dim X \ge 5$

Reality check, II: *A*-side

$n = 3 : X \subset \mathbb{P}^2$ cubic curve (2-torus)

$N_{1,d} = \#\left\{(d/3)\!:\!1 \text{ covers } T^2 \longrightarrow X\right\} / \,|\mathrm{Aut}\,|$

$$N_{1,3d} = \frac{\sigma_d}{d}\,, \quad \sigma_d = \sum_{r|d} r \quad \Longleftrightarrow \quad \sum_{d=1}^{\infty} \sigma_d Q^d = \sum_{d=1}^{\infty} d \frac{Q^d}{1-Q^d}$$

$$\mathbb{A}_1^X(Q) = \sum_{d=1}^{\infty} \frac{\sigma_d}{d} Q^{3d} = -\sum_{d=1}^{\infty} \ln(1 - Q^{3d})$$

Reality check, II

$$\mathbb{I}_0(q) \equiv \sum_{d=0}^{\infty} q^d \frac{(3d)!}{(d!)^3}, \quad \mathbb{J}(q) \equiv \frac{1}{\mathbb{I}_0(q)} \sum_{d=1}^{\infty} q^d \left(\frac{(3d)!}{(d!)^3} \sum_{r=d+1}^{3d} \frac{3}{r} \right)$$

$$\mathbb{B}_1^X(q) = \frac{1}{8}\mathbb{J}(q) - \frac{1}{24}\log(1-3^3q) - \frac{1}{2}\log \mathbb{I}_0(q), \quad Q = q \cdot e^{\mathbb{J}(q)}$$

Mirror Symmetry:

$$\mathbb{A}_1^X(Q) = \mathbb{B}_1^X(q) \iff q^3(1-27q)\mathbb{I}_0(q)^{12} = Q^3 \prod_{d=1}^{\infty}(1-Q^{3d})^{24}$$

Scheidegger'09: direct proof (modular forms)

Approach to verifying $\mathbb{A}_g^X = \mathbb{B}_g^X$ for $X \subset \mathbb{P}^{n-1}$

(works for $g=0,1$)

Need to compute each $N_{g,d}$ and all of them (for fixed g):

- Step 1: relate $N_{g,d}$ to GWs of $\mathbb{P}^{n-1} \supset X$
- Step 2: use $(\mathbb{C}^*)^n$-action on $\mathbb{P}^{n-1}$ to compute each $N_{g,d}$ by localization
- Step 3: find some recursive feature(s) to compute $N_{g,d}\ \forall d$ $\iff \mathbb{A}_g^X$

GW-invariants of $X_5 \subset \mathbb{P}^4$

$$\overline{\mathfrak{M}}_g(X_5, d) = \big\{[u\colon \Sigma \longrightarrow X_5] \,\big|\, g(\Sigma)=g, \deg u=d,\ \bar\partial u = 0\big\}$$

$$\begin{aligned} N_{g,d} &\equiv \deg\, \big[\overline{\mathfrak{M}}_g(X_5, d)\big]^{vir} \\ &\equiv \#\big\{[u\colon \Sigma \longrightarrow X_5] \,\big|\, g(\Sigma)=g, \deg u=d,\ \bar\partial u = \nu(u)\big\} \end{aligned}$$

ν = small generic deformation of $\bar\partial$-equation

From $X_5 \subset \mathbb{P}^4$ to $\mathbb{P}^4$

$$\begin{array}{ccc} & & \mathcal{L} \equiv \mathcal{O}(5) \\ & & s \Big\uparrow \Big\downarrow \pi \\ X_5 \equiv s^{-1}(0) & \subset & \mathbb{P}^4 \end{array} \qquad \begin{array}{ccc} & & \mathcal{V}_{g,d} \equiv \overline{\mathfrak{M}}_g(\mathcal{L}, d) \\ & & \tilde{s} \Big\uparrow \Big\downarrow \tilde{\pi} \\ \overline{\mathfrak{M}}_g(X_5, d) \equiv \tilde{s}^{-1}(0) & \subset & \overline{\mathfrak{M}}_g(\mathbb{P}^4, d) \end{array}$$

$$\tilde{\pi}\big([\xi\colon \Sigma \longrightarrow \mathcal{L}]\big) = [\pi \circ \xi\colon \Sigma \longrightarrow \mathbb{P}^4]$$

$$\tilde{s}\big([u\colon \Sigma \longrightarrow \mathbb{P}^4]\big) = [s \circ u\colon \Sigma \longrightarrow \mathcal{L}]$$

From $X_5 \subset \mathbb{P}^4$ to $\mathbb{P}^4$

$$\begin{array}{ccc} & & \mathcal{L} \equiv \mathcal{O}(5) \\ & & s \big\uparrow \big\downarrow \pi \\ X_5 \equiv s^{-1}(0) & \subset & \mathbb{P}^4 \end{array} \qquad \begin{array}{ccc} & & \mathcal{V}_{g,d} \equiv \overline{\mathfrak{M}}_g(\mathcal{L}, d) \\ & & \tilde{s} \big\uparrow \big\downarrow \tilde{\pi} \\ \overline{\mathfrak{M}}_g(X_5, d) \equiv \tilde{s}^{-1}(0) & \subset & \overline{\mathfrak{M}}_g(\mathbb{P}^4, d) \end{array}$$

This suggests: *Hyperplane Property*

$$N_{g,d} \equiv \deg \left[\overline{\mathfrak{M}}_g(X_5, d)\right]^{vir} \equiv \;^{\pm}\big|\tilde{s}^{-1}(0)\big|$$

$$\stackrel{?}{=} \left\langle e(\mathcal{V}_{g,d}), \overline{\mathfrak{M}}_g(\mathbb{P}^4, d)\right\rangle$$

Genus 0 vs. positive genus

$g=0$ everything is as expected:

- $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ is smooth
- $[\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)]^{vir} = [\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)]$
- $\mathcal{V}_{0,d} \longrightarrow \overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ is vector bundle
- hyperplane prop. makes sense and holds

$g \geq 1$ none of these holds

Genus 1 analogue

Thm. A: HP holds for reduced genus 1 GWs

$$\left[\overline{\mathfrak{M}}_1^0(X_5, d)\right]^{vir} = e(\mathcal{V}_{1,d}) \cap \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d).$$

This generalizes to complete intersections $X \subset \mathbb{P}^n$.

- $\overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d) \subset \overline{\mathfrak{M}}_1(\mathbb{P}^4, d)$ main irred. component closure of $\left\{[u\colon \Sigma \longrightarrow \mathbb{P}^4] \in \overline{\mathfrak{M}}_1(\mathbb{P}^4, d) : \Sigma \text{ is smooth}\right\}$
- $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)$ not vector bundle, but $e(\mathcal{V}_{1,d})$ well-defined (0-set of generic section)

Standard vs. reduced GWs

$$\text{Thm. A} \implies N^{0}_{1,d} \equiv \deg\,[\overline{\mathfrak{M}}^{0}_{1}(X,d)]^{vir} = \int_{\overline{\mathfrak{M}}^{0}_{1}(\mathbb{P}^4,d)} e(\mathcal{V}_{1,d})$$

$$\overline{\mathfrak{M}}^{0}_{1}(X,d) \equiv \overline{\mathfrak{M}}^{0}_{1}(\mathbb{P}^4,d) \cap \overline{\mathfrak{M}}_{1}(X,d)$$

Thm. B: $N_{1,d} - N^{0}_{1,d} = \frac{1}{12} N_{0,d}$

This generalizes to all symplectic manifolds:

$$[\text{standard}] - [\text{reduced genus 1 GW}] = f(\text{genus 0 GW})$$

$\therefore$ to check BCOV, enough to compute $\int_{\overline{\mathfrak{M}}^{0}_{1}(\mathbb{P}^4,d)} e(\mathcal{V}_{1,d})$

Torus actions

- $(\mathbb{C}^*)^5$ acts on $\mathbb{P}^4$ (with 5 fixed pts)
- $\Longrightarrow$ on $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ (with simple fixed loci)
  and on $\mathcal{V}_{g,d} \longrightarrow \overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$
- $\int_{\overline{\mathfrak{M}}^0_g(\mathbb{P}^4,d)} e(\mathcal{V}_{g,d})$ localizes to fixed loci

$g = 0$: Atiyah-Bott Localization Thm reduces $\int$ to $\sum_{graphs}$

$g = 1$: $\overline{\mathfrak{M}}^0_g(\mathbb{P}^4, d), \mathcal{V}_{g,d}$ singular $\Longrightarrow$ AB does not apply

Genus 1 bypass

Thm. C: $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)$ admit natural desingularizations:

$$\begin{array}{ccc} \widetilde{\mathcal{V}}_{1,d} & \longrightarrow & \mathcal{V}_{1,d} \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{M}}_1^0(\mathbb{P}^4, d) & \longrightarrow & \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d) \end{array}$$

$$\Longrightarrow \qquad \int_{\overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d)} e(\mathcal{V}_{1,d}) = \int_{\widetilde{\mathfrak{M}}_1^0(\mathbb{P}^4,d)} e(\widetilde{\mathcal{V}}_{1,d})$$

Computation of genus 1 GWs of CIs

Thm. C generalizes to all $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d)$:

$$\begin{array}{ccc} \mathcal{L} \equiv \mathcal{O}(a) & & \mathcal{V}_{1,d} \equiv \overline{\mathfrak{M}}_{1,k}(\mathcal{L}, d) \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ \mathbb{P}^n & & \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n, d) \end{array}$$

$\therefore$ Thms A,B,C provide an algorithm for computing genus 1 GWs of complete intersections $X \subset \mathbb{P}^n$

Computation of $N_{1,d}$ for all d

- split genus 1 graphs into many genus 0 graphs at special vertex
- make use of good properties of genus 0 numbers to eliminate infinite sums
- extract non-equivariant part of elements in $H^*_{\mathbb{T}}(\mathbb{P}^4)$

Key geometric foundation

A sharp Gromov's compactness thm in genus 1

- describes limits of sequences of pseudo-holomorphic maps
- describes limiting behavior for sequences of solutions of a $\bar{\partial}$-equation with limited perturbation
- allows use of topological techniques to study genus 1 GWs

Main tool

Analysis of local obstructions

- study obstructions to smoothing pseudo-holomorphic maps from singular domains
- not just potential existence of obstructions