Mirror Symmetry: from curve counts to hypergeometric series

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joint with Jun Li, R. Vakil, D. Zagier

From string theory to enumerative geometry

$$\boxed{\begin{array}{c}\text{A-Model partition function}\\ \text{for Calabi-Yau 3-fold } X\end{array}} \;\xrightarrow[\text{principle}]{\text{MIRROR}}\; \boxed{\begin{array}{c}\text{B-Model partition function}\\ \text{for mirror (family) of } X\end{array}}$$

generating function for GWs
"counts of complex curves in" X

something about geometry
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$$\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{A-Model partition function}\\ \text{for Calabi-Yau 3-fold } X\end{array} \arrow[r, equal, "\textit{MIRROR}", "\textit{principle}"'] \arrow[d, Leftrightarrow] & \begin{array}{c}\text{B-Model partition function}\\ \text{for mirror (family) of } X\end{array} \arrow[d, Leftrightarrow] \\
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\end{tikzcd}$$

Basic notions

- Calabi-Yau 3-fold X = (cmpt) complex manifold $\dim_{\mathbb{C}} X = 3$, $c_1(TX) = 0$
- Mirror family $\widehat{X}$ = family of Calabi-Yau 3-folds some singular

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What is special about CY 3-folds X?

expected # of genus-g degree-d curves in X is finite, $n_{g,d} \in \mathbb{Z}$

e.g. $n_{0,1} = 2,875$ # of lines on general X_5

$n_{g,1} = n_{g,2} = 0\ \forall g \geq 1$

genus g degree d GW of X: $N_{g,d} \in \mathbb{Q}$

"linear combination" of $n_{g',d'}$, $g' \leq g$, $d' \leq d$

More generally: $n_{1,d}$ is finite if $c_1(TX) = 0$ (any dim X)

$\Longrightarrow$ genus 1 degree d GW of X: $N_{1,d} \in \mathbb{Q}$

Main example: $X \equiv X_n \subset \mathbb{P}^{n-1}$ hypersurface of degree n

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Mirror symmetry for X

$$\mathbb{A}_g^X(Q) \equiv \sum_{d=1}^{\infty} N_{g,d} Q^d \overset{?}{=} \mathbb{B}_g^X(q), \qquad Q = Q(q), \;\; q = q(Q)$$

$\mathbb{B}_g^X(q) =$ explicit function determined by mirror family of X

Mathematical verifications

$g=0$: *Givental'96/Lian-Liu-Yau'97/...* (X_n, etc.)

$g=1$: *'07* (hypersurfaces $X_n \subset \mathbb{P}^{n-1}$ only)

$g \geq 2$: ?

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B-model PFs for $X = X_n$

$$\mathbb{F}_0(x,q) = \sum_{d=0}^{\infty} q^d \frac{\prod_{r=1}^{r=nd}(nx+r)}{\prod_{r=1}^{r=d}(x+r)^n} \in 1 + q\cdot\mathbb{Q}(x)[[q]]$$

$$\mathbb{I}_0(q) = \mathbb{F}_0(0,q), \qquad \mathbb{F}_1(x,q) = \left\{1 + \frac{q}{x}\frac{\partial}{\partial q}\right\}\frac{\mathbb{F}_0(x,q)}{\mathbb{I}_0(q)}$$

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$$\mathbb{B}_1^X(q) = \left(\frac{(n-2)(n+1)}{48} + \frac{1-(1-n)^n}{24n^2}\right)\mathbb{J}(q) - \frac{(3n-8)(n-1)}{48}\log(1-n^n q) + \frac{n^2-1+(1-n)^n}{24n}\log \mathbb{I}_0(q) - \frac{1}{2}\sum_{r=0}^{n-1}\binom{r}{2}\log \mathbb{I}_r(q)$$

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$$n = 1, 2, 4 : \mathbb{B}_1^X(q) = 0$$

$$n=1 : X = \emptyset \subset \mathbb{P}^{1-1} \Longrightarrow N_{g,d}=0 \; \forall\, d\in\mathbb{Z}^+ \Longrightarrow \mathbb{A}_1^X(Q) = 0 \quad \checkmark$$

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Geometric reason (*Junho Lee'03*): there are no J-holomorphic curves on K3 for some almost complex structure J

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A mystery: BPS states in higher dimensions?

Gopakumar-Vafa'98, $\dim X = 3$: $\exists$ "BPS states" $n_{g,d} \in \mathbb{Z}$ s.t.

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All conjectures: true for $d \le 100$ in $X_7 \subset \mathbb{P}^6$

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$n = 3 : X \subset \mathbb{P}^2$ cubic curve (2-torus)

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$$\mathbb{I}_0(q) \equiv \sum_{d=0}^{\infty} q^d \frac{(3d)!}{(d!)^3}, \quad \mathbb{J}(q) \equiv \frac{1}{\mathbb{I}_0(q)} \sum_{d=1}^{\infty} q^d \left(\frac{(3d)!}{(d!)^3} \sum_{r=d+1}^{3d} \frac{3}{r} \right)$$

$$\mathbb{B}_1^X(q) = \frac{1}{8}\mathbb{J}(q) - \frac{1}{24}\log(1-3^3 q) - \frac{1}{2}\log \mathbb{I}_0(q), \qquad Q = q \cdot e^{\mathbb{J}(q)}$$

Mirror Symmetry:

$$\mathbb{A}_1^X(Q) = \mathbb{B}_1^X(q) \iff q^3(1-27q)\mathbb{I}_0(q)^{12} = Q^3 \prod_{d=1}^{\infty}(1-Q^{3d})^{24}$$

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Approach to verifying $\mathbb{A}_g^X = \mathbb{B}_g^X$ for $X \subset \mathbb{P}^{n-1}$

(works for $g=0,1$)

Need to compute each $N_{g,d}$ and all of them (for fixed g):

Step 1: relate $N_{g,d}$ to GWs of $\mathbb{P}^{n-1} \supset X$

Step 2: use $(\mathbb{C}^*)^n$-action on $\mathbb{P}^{n-1}$ to compute each $N_{g,d}$ by localization

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GW-invariants of $X_5 \subset \mathbb{P}^4$

$$\overline{\mathfrak{M}}_g(X_5,d) = \{[u\colon \Sigma \longrightarrow X_5] \mid g(\Sigma)=g, \deg u = d, \bar\partial u = 0\}$$

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From $X_5 \subset \mathbb{P}^4$ to $\mathbb{P}^4$

$\mathcal{L}$ $\qquad$ $\overline{\mathfrak{M}}_g(\mathcal{L}, d)$

$X_5 \equiv s^{-1}(0) \subset \mathbb{P}^4$ $\qquad$ $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$

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Genus 0 vs. positive genus

$g = 0$ everything is as expected:

- $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ is smooth
- $[\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)]^{vir} = [\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)]$
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Thm. A: HP holds for reduced genus 1 GWs

$$\left[\overline{\mathfrak{M}}_1^0(X_5,d)\right]^{vir} = e(\mathcal{V}_{1,d}) \cap \overline{\mathfrak{M}}_1^0(\mathbb{P}^4,d).$$

This generalizes to complete intersections $X \subset \mathbb{P}^n$.

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$$\text{Thm. A} \implies N^0_{1,d} \equiv \deg\,[\overline{\mathfrak{M}}{}^0_1(X,d)]^{vir} = \int_{\overline{\mathfrak{M}}{}^0_1(\mathbb{P}^4,d)} e(\mathcal{V}_{1,d})$$

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This generalizes to all symplectic manifolds:

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Torus actions

- $(\mathbb{C}^*)^5$ acts on $\mathbb{P}^4$ (with 5 fixed pts)
- $\Longrightarrow$ on $\overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$ (with simple fixed loci)
  and on $\mathcal{V}_{g,d} \longrightarrow \overline{\mathfrak{M}}_g(\mathbb{P}^4, d)$
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Genus 1 bypass

Thm. C: $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{P}^4, d)$ admit natural desingularizations:

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Computation of genus 1 GWs of CIs

Thm. C generalizes to all $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d)$:

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Computation of $N_{1,d}$ for all d

- split genus 1 graphs into many genus 0 graphs at special vertex
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- describes limits of sequences of pseudo-holomorphic maps
- describes limiting behavior for sequences of solutions of a $\bar{\partial}$-equation with limited perturbation
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