Some Conjectures on
the Asymptotic Behavior of Gromov-Witten Invariants

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Abstract

The purpose of this note is to share some observations and speculations concerning the asymptotic behavior of Gromov-Witten invariants. They may be indicative of some deep phenomena in symplectic topology that in full generality are outside of the reach of current techniques. On the other hand, many interesting cases can perhaps be treated via combinatorial techniques.

Contents

1 Introduction

Gromov-Witten invariants are certain counts of curves in smooth projective varieties and of pseudoholomorphic maps into symplectic manifolds inspired by [11]. Their applications in symplectic topology have included Gromov's Non-Squeezing Theorem [24, Theorem 9.3.1], distinguishing diffeomorphic symplectic manifolds [28], and the uniruledness of symplectic manifolds with Hamiltonian group actions [23]. On the other hand, the vast literature on Gromov-Witten invariants in

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algebraic geometry has generally concerned striking algebraic properties exhibited by collections of these invariants associated with individual manifolds. These properties have included the associativity of the quantum product on the cohomology, mirror symmetry in genus 0 [8, 18] and in genus 1 [27, 35], and the modularity of some generating functions for Gromov-Witten invariants [3, 33].

The present note concerns three types of asymptotic behavior of Gromov-Witten invariants as the degree of the curves and the energy of the maps being counted increase: upper bounds, asymptotics involving upper and lower bounds, and vanishing statements. While observations in this spirit were first made in the early days of Gromov-Witten theory [6, 31], there has been fairly little progress on these kinds of problems since then. Developments in the theory over the past 25 years have reduced such problems in many important cases to essentially combinatorial questions. Recent works, some of which we review in the present note, have tackled these questions in some cases and appear very promising for treating them in many other cases. While combinatorial techniques should confirm many important special cases of the conjectures stated in Section 2, they are unlikely to explain the geometry behind the conjectured phenomena on their own though. These phenomena may have connections with Strominger-Yau-Zaslow's deep proposal [30] and Gross-Siebert's related program [12] suggesting that the Gromov-Witten invariants of (at least many) symplectic manifolds are "made up" of the Gromov-Witten invariants of some toric pieces.

A conjecture for rather precise asymptotics of the arbitrary-genus Gromov-Witten invariants of the complex projective plane $\mathbb{P}^2$ was formulated in [6]. In Section 2.1, we recall this conjecture, summarize the recent work [32, 37] completing the proof of its genus 0 case initiated in [6] and moving on to the genus 1 case, and discuss extensions to other manifolds. In Section 2.2, we state a conjecture bounding Gromov-Witten invariants by the energy of the underlying maps, summarize an approach for establishing it in some cases, and describe its connection with the proposal of [22] for showing that natural generating functions for Gromov-Witten invariants of many symplectic manifolds have nonzero radii of convergence. In Section 2.3, we state a vanishing conjecture for the (descendant) Gromov-Witten invariants of monotone symplectic manifolds (and smooth Fano varieties) and again summarize an approach for establishing it in some cases. Section 3 contains a detailed proof of the genus 0 and 1 cases of the conjecture of [6] stated in Section 2.1; it combines parts of [6], [37], and [32]. We hope that this proof can be adapted to many other manifolds. Section 4 contains related miscellaneous observations.

The author would like to thank R. Pandharipande for bringing up [6] and for sharing his idea to establish the bound (2.4), G. Tian for raising questions about asymptotics in Gromov-Witten theory, P. Sarnak and D. Grigoriev for enlightening discussions, A. Gathmann for the *growi* program computing Gromov-Witten invariants, and the IAS School of Mathematics for hospitality while [36] and [37] were completed.

2 Conjectures and theorems

The three parts of this section contain conjectures concerning asymptotic behavior of Gromov-Witten invariants of three different flavors. The first conjecture is particularly simple to state, as it concerns only curve counts in the complex projective plane $\mathbb{P}^2$. The last two conjectures require a nominal amount of notation. If true, they should be consequences of fundamental properties of

Gromov-Witten invariants that are yet to be discovered.

2.1 Asymptotic expansions

In the case of the complex projective plane $\mathbb{P}^2$, the classical enumerative counts of complex curves and the modern Gromov-Witten invariants agree. For $g\in\mathbb{Z}^{\geq 0}$ and $d\in\mathbb{Z}^+$, we denote by $N_{g,d}$ the number of degree d genus g curves through $3d-1+g$ general points in $\mathbb{P}^2$.

Conjecture 2.1 ([6, Footnote 2]). *There exist $b\in\mathbb{R}^+$ independent of g and $a_g\in\mathbb{R}^+$ for each $g\in\mathbb{Z}^{\geq 0}$ such that*

$$\frac{N_{g,d}}{(3d-1+g)!} = a_g b^d d^{-1-\frac{5}{2}(1-g)}\big(1+o(1)\big) \qquad as \quad d\longrightarrow\infty$$

for each $g\in\mathbb{Z}^{\geq 0}$.

The $g=0$ case of Conjecture 2.1 is [6, Proposition 3]. The starting point for the reasoning behind [6, Proposition 3] is Kontsevich's recursion for $N_{0,d}$, restated as (3.5) in the present note. It is claimed in [6] that the $g=0$ case of Conjecture 2.1 is a direct consequence of the statement of Proposition 3.3 in the present note. As noted in [37],

(G1) the existence of the expansion (3.13) used to establish the conclusion of Proposition 3.3 was not justified in [6];

(G2) Proposition 3.3 does not directly imply the $g=0$ case of Conjecture 2.1 or even the first statement of Corollary 2.3 below;

see Section 3.2 for details. Following P. Sarnak's suggestion to use the Frobenius method to bypass (G1), [37] established Proposition 3.3 and suggested the use of the Eguchi-Hori-Xiong recursion [26, (8)] to move from the genus 0 case of Conjecture 2.1 to the genus 1 case. Lemmas 3.5 and 3.6 in the present note were first obtained (in slightly different formulations) in [32] with the knowledge of the contents of [37] and completed the proof of the genus 0 and 1 cases of Conjecture 2.1, with even more refined statements for the asymptotics of $N_{0,d}$ and $N_{1,d}$.

Theorem 2.2 ([32, Theorems 1.1,1.2]). *There exist $b\in\mathbb{R}^+$, $a_{0;k},a_{1;k}\in\mathbb{R}$ for each $k\in\mathbb{Z}^{\geq 0}$, and $C_N\in\mathbb{R}$ for each $N\in\mathbb{Z}^+$ such that*

$$\bigg|\frac{N_{0,d}}{(3d-1)!}-b^d\sum_{k=3}^{N-1}a_{0;k}d^{-k-\frac{1}{2}}\bigg| < C_N b^d d^{-N-\frac{1}{2}}\,,$$
$$\bigg|\frac{N_{1,d}}{(3d)!}-b^d\bigg(\frac{1}{48d}+\sum_{k=0}^{N-1}a_{1;k}d^{-k-\frac{1}{2}}\bigg)\bigg| < C_N b^d d^{-N-\frac{1}{2}}$$

for all $d,N\in\mathbb{Z}^+$.

Corollary 2.3 ([32, Corollaries 1.1,1.2]). *There exists $b\in\mathbb{R}^+$ such that*

$$\lim_{d\longrightarrow\infty}\sqrt[d]{\frac{N_{0,d}}{(3d-1)!}} = b = \lim_{d\longrightarrow\infty}\sqrt[d]{\frac{N_{1,d}}{(3d)!}}\,; \tag{2.1}$$

in particular, the above limits exist and are nonzero.

The Eguchi-Hori-Xiong recursion, restated as (3.16) in the present note, determines the numbers $N_{1,d}$ from the numbers $N_{0,d}$. As noted in [37], the second equality in (2.1) is a direct consequence of the first because of this recursion; see the end of Section 3.2. This observation motivated the transition from the $g=0$ to the $g=1$ case in [32]; see Corollary 3.4 and Lemma 3.5 in the present note.

Corollary 2.3 is an immediate consequence of Theorem 2.2. We give a detailed proof of Theorem 2.2 in Section 3 to encourage others to consider similar asymptotics and energy bounds problems. Section 3.1 presents the approach of [6] for bounding recursively defined sequences by essentially geometric sequences. Section 3.2 describes the remainder of the reasoning of [6] behind the claim to establish the genus 0 case of Conjecture 2.1 and discusses the gaps (G1) and (G2). Section 3.3 is a more systematic, though mathematically analogous, version of the treatment of (G2) provided by [32]. Section 3.4 contains a proof of Proposition 3.3, which addresses (G1); it is essentially the same argument as in [37], but is better organized.

By the proof of Theorem 2.2, the asymptotics for $N_{0,d}$ and $N_{1,d}$ are completely determined by b in Conjecture 2.1 and by the coefficients a_0 and a_2 in (3.8). The remaining coefficients in (3.8) are determined by (3.40) and (3.41) and in turn determine the asymptotic coefficients $a_{0;k}$ and $a_{1;k}$ in Theorem 2.2 via Lemma 3.6 and (3.17). It would be interesting to determine these coefficients explicitly and to understand their geometric meaning. The leading coefficient in the genus 1 asymptotic expansion, i.e. a_1 in Conjecture 2.1, is half the first Chern class of the Hodge line bundle $\mathbb{E}_1$ on the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{1,1}$ of elliptic curves. This suggests a potential connection between the asymptotics conjectured in [6] in the early days of Gromov-Witten theory and the behavior of Gromov-Witten invariants under degenerations of the target.

The counts of genus 0 curves in many standard Kähler manifolds agree with the corresponding Gromov-Witten invariants and can be computed recursively. Asymptotics for the growth of such counts for blowups of $\mathbb{P}^2$ are obtained in [14, 15], but these asymptotics are fairly coarse. The $\mathbb{P}^2$ case of these asymptotics is essentially equivalent to the existence of the lower and upper bounds in (3.3), which were originally obtained in [6]. On the other hand, the recursions determining the counts of genus 0 curves in blowups of $\mathbb{P}^2$ have the same general structure as Kontsevich's recursion [29, (10.4)] for $\mathbb{P}^2$. As shown in [26], the Eguchi-Hori-Xiong recursion for counts of genus 1 curves in $\mathbb{P}^2$ is essentially a lift of Getzler's relation [7] from $\overline{\mathcal{M}}_{1,4}$ to the moduli space of genus 1 stable maps to $\mathbb{P}^2$. It thus has analogues for other Kähler and more generally symplectic manifolds. This all suggests that the approach described in Section 3 may extend from $\mathbb{P}^2$ to obtain refined asymptotics for counts of genus 0 and 1 curves in its blowups.

A completely different approach is needed to deal with the $g\geq 2$ cases of Conjecture 2.1 and its potential analogues for other Kähler surfaces. Some kind of geometric degeneration approach remains elusive at this point. On the other hand, the Göttsche-Yau-Zaslow formula [33] essentially enumerates arbitrary-genus curves on smooth algebraic surfaces and appears promising as the starting point for a combinatorial approach to the $g\geq 2$ cases of Conjecture 2.1 and its potential generalizations.

For complex manifolds of dimension 3 and higher, different types of incidence conditions (not just points) should be considered. All counts of genus 0 curves in $\mathbb{P}^n$ are computable from the recursion of [29, Theorem 10.4]. For $\mathbb{P}^3$, *Mathematica* suggests Conjecture 2.4 below; it is based on the numbers up to $d=200$ (the computation of these numbers already takes a long time). As the convergence appears to be very slow (for $N_{0,d}$, it is still going noticeably even for $d=1000$), it is feasible that the limit below is even independent of the slope α/β chosen, but the numbers so far do not suggest this.

Conjecture 2.4. *Let $N_{0,d}(p)$ be the number of degree d rational curves through $2d-p$ points and $2p$ lines in general position in $\mathbb{P}^3$. For all $\alpha,\beta\in\mathbb{Z}^+$ fixed, the limit*

$$\lim_{d\longrightarrow\infty}\sqrt[d]{\frac{N_{0,\alpha d}(\beta d)}{((2\alpha+\beta)d)!}}$$

exists and is nonzero.

An upper bound on the sequences in Conjecture 2.4 can be obtained from a two-variable version of the approach used in the proof of [6, Proposition 3]; see Section 4.1. It also follows immediately from [36, Theorem 1]. A lower bound appears more elusive, since the recursion of [29, Theorem 10.4] for $\mathbb{P}^3$ involves negative coefficients.

A natural extension of Conjecture 2.4 to arbitrary genera g and arbitrary compact symplectic manifolds (X,ω) would be as follows. Fix $H_1,\ldots,H_k\in H^*(X)$. Let $\beta_r\in H_2(X)$ be a sequence such that $\langle\omega,\beta_r\rangle\longrightarrow\infty$ and the lines $\mathbb{R}\beta_r$ converge in the projectivization of $H_2(X;\mathbb{R})$. Suppose $b_{1;r},\ldots,b_{k;r}\in\mathbb{Z}^+$ are sequences such that the lines $[b_{1;r},\ldots,b_{k;r}]$ converge in $\mathbb{RP}^{k-1}$ and

$$\sum_{i=1}^{k}b_{i;r}\big(\deg H_i\big)=2\big\langle c_1(TX),\beta_r\big\rangle+\big(\dim_{\mathbb{R}}X-6\big)(1-g)+2k\qquad\forall\,r\in\mathbb{Z}^+\,.$$

One might then ask whether the sequence of Gromov-Witten invariants

$$\Big(\big\langle\underbrace{H_1,\ldots,H_1}_{b_{1;r}},\ldots,\underbrace{H_k,\ldots,H_k}_{b_{k;r}}\big\rangle_{g,\beta_r}^{X}\Big)^{1/\langle\omega,\beta_r\rangle}$$

converges; see (2.2) for the notation.

2.2 Energy bounds

Enumerative counts of curves in many smooth projective varieties are not well-defined. It is instead natural to consider the asymptotic behavior of Gromov-Witten invariants. For $g,N\in\mathbb{Z}^{\ge0}$, an almost Kähler manifold (X,ω,J), and $\beta\in H_2(X)$, we denote by $\overline{\mathfrak{M}}_{g,N}(X,\beta)$ the moduli space of stable degree β genus g N-marked J-holomorphic maps to X. For each $s=1,\ldots,N$, let

$$\mathrm{ev}_s\colon\overline{\mathfrak{M}}_{g,N}(X,\beta)\longrightarrow X\quad\text{ and }\quad\psi_s\equiv c_1(L_s^*)\in H^2\big(\overline{\mathfrak{M}}_{g,N}(X,\beta);\mathbb{Q}\big)$$

be the evaluation map and the first Chern class of the universal cotangent line bundle at the s-th marked point, respectively. For $b_1,\ldots,b_N\in\mathbb{Z}^{\ge0}$ and $H_1,\ldots,H_N\in H^*(X;\mathbb{Q})$, let

$$\big\langle\tau_{b_1}H_1,\ldots,\tau_{b_N}H_N\big\rangle_{g,\beta}^{X}=\int_{[\overline{\mathfrak{M}}_{g,N}(X,\beta)]^{vir}}\prod_{s=1}^{s=N}\big(\psi_s^{b_s}\mathrm{ev}_s^*H_s\big);\tag{2.2}$$

this rational number is a descendant Gromov-Witten invariant.

Conjecture 2.5 ([36, Conjecture 1]). *Suppose (X,ω) is a compact symplectic manifold and $g\in\mathbb{Z}$. For all $H_1,\ldots,H_k\in H^*(X)$, there exists $C_{X,g}\in\mathbb{R}^+$ such that*

$$\left|\frac{\big\langle b_1!\,\tau_{b_1}H_{c_1},\ldots,b_N!\,\tau_{b_N}H_{c_N}\big\rangle_{g,\beta}^X}{N!}\right|\le C_{X,g}^{\langle\omega,\beta\rangle+N} \tag{2.3}$$

for all $\beta\in H_2(X)$, $N,b_s\in\mathbb{Z}^{\ge 0}$, and $c_s\in\{1,\ldots,k\}$.

The exponent $\langle\omega,\beta\rangle$ in Conjecture 2.5 is the energy of the J-holomorphic maps of class β, while $\langle\omega,\beta\rangle+N$ is essentially the energy of the induced "graph map". The $\beta=0$ case of this conjecture is obtained by induction from the string and dilaton equations [13, p527] for Hodge classes on the Deligne-Mumford moduli spaces $\overline{\mathcal{M}}_{g,k}$ of stable k marked genus g curves. In fact, the limsup of the N-th root of the left-hand side in (2.3) is at most 1. Theorem 1 in [38] combines these relations with the Virtual Equivariant Localization Theorem of [10] to establish Conjecture 2.5 for $X=\mathbb{P}^n$. Theorem 1 in [36] establishes the $g=0$ case of this conjecture for complete intersections $X\subset\mathbb{P}^n$ with each H_s being a power of the hyperplane class $H\in H^2(\mathbb{P}^n)$. For the reasons explained below, Conjecture 2.5 also holds for all Calabi-Yau complete intersections $X\subset\mathbb{P}^n$ of dimension at least 4 with each H_s again being a power of H.

For a Calabi-Yau threefold X, Conjecture 2.5 is equivalent to the existence of $C_{X,g}\in\mathbb{R}^+$ such that

$$\big|\langle\rangle_{g,\beta}^X\big|\le C_{X,g}^{\langle\omega,\beta\rangle}\qquad\forall\ \beta\in H_2(X)\,. \tag{2.4}$$

The existence of such a $C_{X,g}$ in turn corresponds to the string theory presumption that the partition function determined by the genus g Gromov-Witten invariants of X has positive radius of convergence. The bounds (2.4) are implied by mirror symmetry predictions. These predictions have been confirmed mathematically for $g=0$ for many Calabi-Yau threefolds in [8, 18, 19, 20] and for $g=1$ for Calabi-Yau complete intersections $X\subset\mathbb{P}^n$ in [27, 35], but are yet to be confirmed for $g\ge 2$ for any compact Calabi-Yau threefold. The mirror symmetry predictions can also be used to obtain asymptotics for the invariants on the left-hand side of (2.4) in the style of Conjecture 2.1, as is done in [5, 17].

For a Calabi-Yau manifold X of (complex) dimension at least 4, the Gromov-Witten invariants of genus 2 and higher vanish. Conjecture 2.5 then reduces to its cases for the genus 0 Gromov-Witten invariants with arbitrary insertions and for the genus 1 Gromov-Witten invariants with no insertions, i.e. as in (2.4). For complete intersections $X\subset\mathbb{P}^n$, such genus 0 bounds with each H_i being a power of H are provided by Theorem 1 in [36]. The required genus 1 bounds for complete intersections $X\subset\mathbb{P}^n$ are implied by the genus 1 mirror formulas established in [27, 35].

Theorem 1 in [36] and Theorem 1 in [38] referenced above are obtained from the mirror symmetry formulas for the equivariant multi-pointed genus 0 Gromov-Witten invariants of complete intersections $X\subset\mathbb{P}^n$ and for the equivariant multi-pointed genus g Gromov-Witten invariants of $\mathbb{P}^n$, respectively, established in the two papers. The starting inputs for both mirror formulas are the mirror formulas for Givental's J-function for the equivariant one-pointed genus 0 Gromov-Witten invariants of complete intersections $X\subset\mathbb{P}^n$ obtained in [8, 18] and for its two-pointed analogue obtained in [34] from Givental's J-function. Givental's J-functions for the equivariant one-pointed genus 0 Gromov-Witten invariants of many other spaces have been computed in [4, 16, 18, 19, 20] and in other works. The reasoning in [34, 36, 38] can be used to convert these J-functions into mirror formulas for the equivariant multi-pointed genus 0 Gromov-Witten invariants of complete intersections in many spaces with groups actions and the equivariant multi-pointed genus g Gromov-Witten invariants of these spaces themselves. It should then be possible to establish the corresponding cases of Conjecture 2.5 as in [36, 38].

A special case of the bound (2.3) is [31, (7.3)]. This case concerns only primary Gromov-Witten invariants, i.e. $b_s = 0$ for all s, and only in symplectic manifolds (X, ω) with $H_2(X;\mathbb{Z}) = \mathbb{Z}$. It is stated, without a proof, that such a bound follows from the WDVV recursion (analogue of Kontsevich's recursion) of [31, Theorem 7.1]. This does appear to be the case in general, though we do not see a simple argument. We provide a proof of [31, (7.3)] for $X = \mathbb{P}^3$ from the WDVV recursion by adapting the approach of [6] described in Section 3.1.

The paper [6] was in fact first brought to the author's attention by R. Pandharipande in 2008 while describing a scheme [22] to reduce the bounds (2.4) for the renown quintic threefold to the bound of Conjecture 2.5 for the genus 0 Gromov-Witten invariants of $\mathbb{P}^3$ via [9, Theorem 1] and the degeneration approach of [21]. It was an expectation of R. Pandharipande that the bounds for $\mathbb{P}^3$ could be established by adapting the approach of [6]. However, recursive formulas for the descendant invariants of Conjecture 2.5 involve negative coefficients; this makes the approach of [6] unsuitable for these invariants. The proof of [36, Theorem 1], which in particular establishes these bounds for the genus 0 Gromov-Witten invariants of all projective spaces, instead uses the classical equivariant localization theorem of [2]. The proof of [38, Theorem 1], which extends such bounds to arbitrary genus, removes the need to use [9, Theorem 1] in the scheme of [22] to establish the bounds (2.4). It also enables the application of this scheme for establishing Conjecture 2.5 for many other smooth projective varieties (at least with each H_s being a power the hyperplane class H).

Question 2.6. *Are the bounds arising from Conjecture 2.5 sharp? Are they reflexive of the asymptotic behavior of some natural sequences of Gromov-Witten invariants, as in Conjectures 2.1 and 2.4?*

2.3 Vanishing statements

The observations and speculations on the vanishing of certain descendant Gromov-Witten invariants in this section concern monotone symplectic manifolds. We recall that a symplectic manifold (X, ω) is called monotone with minimal Chern number $\nu \in \mathbb{R}^+$ if

$$c_1(X) = \lambda[\omega] \in H^2(X;\mathbb{R}) \tag{2.5}$$

for some $\lambda \in \mathbb{R}^+$ and ν is the minimal value of $c_1(X)$ on the homology classes representable by non-constant J-holomorphic maps $S^2 \longrightarrow X$ for every ω-compatible almost complex structure on X. Perhaps the monotone condition in Conjecture 2.7 below can be weakened to (X, ω) being positive (Fano) with minimal Chern number ν, i.e. dropping the requirement (2.5), or needs to be strengthened with the additional requirement that $H^2(X;\mathbb{R})$ be one-dimensional. Since the Gromov-Witten invariants of a symplectic manifold (X, ω) are invariant under deformations of ω, (2.5) needs to hold only for some symplectic form ω' deformation equivalent to ω and ν below (2.5) can be taken to be the maximum of the corresponding values over all such ω' for which (2.5) holds for some λ.

Conjecture 2.7. *Suppose* (X,ω) *is a compact monotone symplectic manifold with minimal Chern number* ν,

$$g,N\in\mathbb{Z}^{\ge 0}\quad\textit{with } 2g+N\ge 3,\qquad b_s,c_s\in\mathbb{Z}^{\ge 0},\;\; H_s\in H^{2c_s}(X)\quad\textit{for } s=1,\ldots,N.$$

If there exists $S\subset\{1,\ldots,k\}$ *such that*

$$b_s+c_s<\nu\quad\forall\,s\in S\qquad\textit{and}\qquad\sum_{s\in S}b_s>3(g-1)+N,\tag{2.6}$$

then $\langle\tau_{b_1}H_1,\ldots,\tau_{b_N}H_N\rangle_{g,\beta}^X=0$.

For example,

$$\big\langle\underbrace{\tau_bH^{n-b},\ldots,\tau_bH^{n-b}}_{N-2},\cdot,\cdot\big\rangle_{g,d}^{\mathbb{P}^n}=0\quad\forall\,N\ge 3,\;b=1,2,\ldots,n\;\;\text{with }(N-2)b>3(g-1)+N.$$

For $g=0$ and $X=\mathbb{P}^1$, this statement follows from the dilaton relation [13, p527]. For $n\ge 2$ (which is necessary for the assumptions of Conjecture 2.7 to be satisfied if $g\ge 1$), τ_bH^{n-b} is not a (virtual) divisor on $\overline{\mathfrak{M}}_{g,N}(\mathbb{P}^n,d)$ and there appears to be no direct geometric reason for the vanishing above.

The assumption $N\ge 3$ if $g=0$ in Conjecture 2.7 is needed. For example,

$$\langle H^2,H^2\rangle_{0,\ell}^{\mathbb{P}^2},\langle\tau_2,H^2\rangle_{0,\ell}^{\mathbb{P}^2}=1,\quad\langle\tau_1H,H^2\rangle_{0,\ell}^{\mathbb{P}^2}=-1,$$

where $\ell\in H_2(\mathbb{P}^2;\mathbb{Z})$ is the standard generator; the constraints for all invariants above satisfy (2.6). For $g\ge 1$, the condition $2g+N\ge 3$ is forced by the second requirement in (2.6). Both conditions in (2.6) are needed as well. For example,

$$\langle\tau_1H,H^2,H\rangle_{0,\ell}^{\mathbb{P}^2},\langle\tau_2,H^2,H\rangle_{0,\ell}^{\mathbb{P}^2}\langle\tau_2,H^2,H,H\rangle_{0,\ell}^{\mathbb{P}^2}=0,\quad\langle\tau_3,H,H\rangle_{0,\ell}^{\mathbb{P}^2},\langle\tau_1H,H^2,H,H\rangle_{0,\ell}^{\mathbb{P}^2}=1.$$

The constraints for the first three invariants above satisfy both conditions. The constraints for the second-to-last invariant fail the first condition, but satisfy the second. The constraints for the last invariant satisfy the first condition, but fail the second.

Theorem 2 in [38] establishes Conjecture 2.5 for $X=\mathbb{P}^n$. Theorem 2 in [36] establishes the $g=0$ case of this conjecture for complete intersections $X\subset\mathbb{P}^n$ with each H_s being a power of the hyperplane class $H\in H^2(\mathbb{P}^n)$. Because of the conditions on b_s in Conjecture 2.7, the assumptions of this conjecture are never satisfied if $\nu=0,1$ (Calabi-Yau and borderline Fano cases) or if $\nu=2$ and $g\ge 1$. For the same reason, its conclusion is the strongest for $X=\mathbb{P}^n$ (when the Fano index ν is maximal relative to the dimension of X, at least in the category of Kähler manifolds).

Theorem 2 in [36] and Theorem 2 in [38] are obtained from the mirror symmetry formulas for the equivariant multi-pointed genus 0 Gromov-Witten invariants of complete intersections $X\subset\mathbb{P}^n$ and for the equivariant multi-pointed genus g Gromov-Witten invariants of $\mathbb{P}^n$, respectively, established in the two papers. Unlike the situation with Theorem 1 in [36] and Theorem 1 in [38] the derivation of which from the mirror symmetry formulas requires a significant amount of combinatorial analysis, Theorem 2 in [36] and Theorem 2 in [38] are immediate consequences of these formulas. As explained at the end of Section 2.2, it should be possible to extend the mirror symmetry formulas of [36, 38] to many other targets and thus test Conjecture 2.7 for them.

3 Proof of Theorem 2.2

We recall the approach of [6] to bounding recursively defined sequences of the form

$$n_d = \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}} f(d_1,d_2)n_{d_1}n_{d_2} \qquad \forall\, d\ge 2, \tag{3.1}$$

with $n_1, f(d_1,d_2)>0$ in Section 3.1. Section 3.2 describes the attempt in [6] to obtain Proposition 3.3 and to conclude the $g=0$ case of Conjecture 2.1 *directly* from it; we achieve the former in Section 3.4. Section 3.3 presents the observations in [32] that are used to deduce Theorem 2.2 from Propositions 3.3.

3.1 Lower and upper bounds

Let $n_1, n_2, \ldots$ be a sequence of numbers satisfying

$$n_d = a \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}} n_{d_1}n_{d_2} \qquad \forall\, d\ge 2, \tag{3.2}$$

for some $a>0$. The generating function

$$\Phi(q) \equiv \sum_{d=1}^{\infty} n_d q^d$$

then satisfies $\Phi(q) = n_1 q + a\Phi(q)^2$. Thus,

$$\Phi(q) = \frac{1-\sqrt{1-4an_1q}}{2a} = -\frac{1}{2a}\sum_{d=1}^{\infty}\binom{1/2}{d}(-4an_1q)^d = \sum_{d=1}^{\infty}\frac{(2d-2)!}{d!(d-1)!}a^{d-1}n_1^d q^d\,;$$

the middle equality above is the Binomial Theorem.

Lemma 3.1. *If $n_1, n_2, \ldots$ is a sequence of numbers satisfying*

$$n_d = a \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge0}} \frac{f(d_1)f(d_2)}{f(d)} n_{d_1}n_{d_2} \qquad \forall\, d\ge 2,$$

for some $a>0$ and $f:\mathbb{Z}^+\longrightarrow\mathbb{R}$, then

$$n_d = \frac{(2d-2)!}{d!(d-1)!}\frac{a^{d-1}}{f(d)}\big(f(1)n_1\big)^d \qquad \forall\, d\ge 1.$$

Proof. The sequence $\widetilde{n}_d = f(d)n_d$ satisfies the recursion (3.2). □

For each $g\in\mathbb{Z}^{\ge0}$ and $d\in\mathbb{Z}^+$, let

$$n_{g,d} = \frac{N_{g,d}}{(3d-1+g)!}\,.$$

Corollary 3.2. *The numbers $n_{0,d}$ satisfy*

$$\frac{8}{5}\left(\frac{1}{27}\right)^d d^{-7/2} \le n_{0,d} \le \frac{45}{16}\left(\frac{4}{15}\right)^d d^{-7/2}. \tag{3.3}$$

In particular,

$$\frac{1}{27} \le \liminf_{d\longrightarrow\infty} \sqrt[d]{n_{0,d}} \le b_+ \equiv \limsup_{d\longrightarrow\infty} \sqrt[d]{n_{0,d}} \le \frac{4}{15}.$$

Proof. Let

$$\begin{aligned} f(d_1,d_2) &= \frac{d_1d_2((3d_1-2)(3d_2-2)(d+2)+8(d-1))}{6(3d-3)(3d-2)(3d-1)} \\ &= \frac{d_1d_2(3d_1d_2(d+2)-2d^2)}{2(3d-3)(3d-2)(3d-1)} \qquad \text{where} \quad d\equiv d_1+d_2. \end{aligned}$$

We note that

$$\begin{aligned} \frac{1}{54}\frac{d_1d_2(3d_1-2)(3d_2-2)}{d(3d-2)} &= \frac{d_1d_2(3d_1-2)(3d_2-2)(d-1)}{6(3d-3)(3d-2)3d} \\ &\le f(d_1,d_2) \le \frac{d_1d_2\cdot 3d_1d_2(d-\frac{2}{3})}{2\frac{3d}{2}(3d-2)\frac{5d}{2}} = \frac{2}{15}\frac{d_1^2d_2^2}{d^2} \end{aligned} \tag{3.4}$$

for all $d_1,d_2\in\mathbb{Z}^+$ and $d\equiv d_1+d_2$. By [29, (10.4)],

$$n_{0,1}=\frac{1}{2}, \qquad n_{0,d}=\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 1}} f(d_1,d_2)n_{0,d_1}n_{0,d_2} \quad \forall\, d\ge 2. \tag{3.5}$$

By (3.4) and Lemma 3.1,

$$\frac{9}{2}\frac{(2d)!}{(d!)^2}\left(\frac{1}{108}\right)^d d^{-3} \le n_{0,d} \le \frac{15}{4}\frac{(2d)!}{(d!)^2}\left(\frac{1}{15}\right)^d d^{-3}.$$

By Stirling's formula [1, Theorem 15.19],

$$\frac{16}{45}4^d d^{-1/2} \le \frac{4^d}{\sqrt{\pi d}}\left(1+\frac{1}{4d}\right)^{-2} \le \frac{(2d)!}{(d!)^2} \le \frac{4^d}{\sqrt{\pi d}}\left(1+\frac{1}{8d}\right) \le \frac{3}{4}4^d d^{-1/2}. \tag{3.6}$$

Combining the last two statements, we obtain (3.3). □

3.2 The reasoning in [6]

Proposition 3.3 and Corollary 3.4 below describe the behavior the generating series

$$F_0(z)\equiv\frac{1}{3}\sum_{d=1}^{\infty}n_{0,d}e^{dz} \quad \text{and} \quad F_1(z)\equiv\sum_{d=1}^{\infty}n_{1,d}e^{dz}, \qquad z\in\mathbb{C}, \tag{3.7}$$

for the counts of genus 0 and 1 curves in $\mathbb{P}^2$. The statement of Proposition 3.3 appears in [6]. This statement, not established in [6], is behind the claim in [6] to confirm the $g=0$ case of Conjecture 2.1. We prove Proposition 3.3 in Section 3.4.

For $\delta\in\mathbb{R}^+$ and $x_0\in\mathbb{R}$, let

$$B_\delta(0)=\big\{z\in\mathbb{C}\colon |z|<\delta\big\},\quad \mathbb{C}_{x_0}^{<}=\big\{z\in\mathbb{C}\colon \mathrm{Re}\,z<x_0\big\},\quad \mathbb{C}_{x_0}^{\leq}=\big\{z\in\mathbb{C}\colon \mathrm{Re}\,z\leq x_0\big\}.$$

We define $z^{1/2}$ on $\mathbb{C}-\mathbb{R}^+$ by the condition $\mathrm{Im}(z^{1/2})\geq 0$.

Proposition 3.3. *There exists $x_0\in\mathbb{R}$ such that the power series F_0 converges on $\mathbb{C}_{x_0}^{<}$ and diverges outside of $\mathbb{C}_{x_0}^{\leq}$. Furthermore, there exist $\delta\in\mathbb{R}^+$, $a_0,a_2\in\mathbb{R}$, and $a_{2d}\in\mathbb{R}$ and $a_{2d+1}\in\mathrm{i}\mathbb{R}$ with $d\in\mathbb{Z}$, $d\geq 2$, such that $a_5\in\mathrm{i}\mathbb{R}^-$ and*

$$F_0(x_0+z)=a_0+a_2z+a_4z^2+\sum_{d=5}^{\infty}a_dz^{d/2} \tag{3.8}$$

for all $z\in B_\delta(0)$ with $\mathrm{Re}\,(z)\leq 0$.

The first statement of this proposition is immediate from (3.3). Furthermore, $e^{x_0}b_+=1$.

Since $n_{0,d}\in\mathbb{R}^+$ for all d, there is no neighborhood of $z=x_0$ on (all of) which this series converges; otherwise, every point z_0 with $\mathrm{Re}\,z_0=x_0$ would have such a neighborhood. By (3.5),

$$(9+2F_0'-3F_0'')F_0'''=2F_0-11F_0'+18F_0''+(F_0'')^2\,; \tag{3.9}$$

there is a sign typo in [6, (2.55)], which is corrected in [6, (2.57)]. Since $n_{0,d}>0$ for all $d\in\mathbb{Z}^+$,

$$0<F_0(z)<F_0'(z)<F_0''(z)<F_0'''(z)\qquad \forall\, z\in(-\infty,x_0)\,. \tag{3.10}$$

Combined with (3.9), this implies that

$$\begin{gathered}3F_0''(z)-2F_0'(z)<9\qquad \forall\, z\in(-\infty,x_0),\\ 2F_0(x_0)-11F_0'(x_0)+18F_0''(x_0)+F_0''(x_0)^2>0.\end{gathered} \tag{3.11}$$

By (3.10) and the first statement in (3.11), the series for F_0, F_0', and F_0'' converge at $z=x_0$. Along with (3.9), this implies that

$$3F_0''(x_0)-2F_0'(x_0)=9\,; \tag{3.12}$$

otherwise, (3.9) could be used to compute all derivatives of F_0 at $z=x_0$ and F_0 could be extended on a neighborhood of x_0.

Since the power series for F_0'' converges at $z=x_0$ and the power series for F_0''' does not converge,

$$\limsup_{d\longrightarrow\infty}\frac{\ln n_{0,d}-d\ln b_+}{\ln d}\in[-4,-3].$$

According to [6, p170], this also implies that F_0 admits an expansion around $z=x_0$ of the form

$$F_0(x_0+z)=c_0+c_1z+\frac{c_2z^2}{2}+\lambda z^{2+\alpha}+\ldots \tag{3.13}$$

for some $\alpha\in(0,1)$. By (3.13) and (3.9),

$$\big((9+2c_1-3c_2)-3\lambda(1+\alpha)(2+\alpha)z^\alpha\big)\cdot\lambda\alpha(1+\alpha)(2+\alpha)z^{\alpha-1}=2c_0-11c_1+18c_2+c_2^2+o(1).$$

Along with (3.12), this gives

$$9+2c_1-3c_2=0, \quad 2\alpha-1=0, \quad -3\lambda\alpha(1+\alpha)^2(2+\alpha)^2=2c_0-11c_1+18c_2+c_2^2, \tag{3.14}$$

and implies (3.8). However, the existence of the expansion (3.13) does not follow just from the convergence of F_0'' at x_0 and the non-convergence of F_0''' there. In Section 3.4, we justify (3.8) bypassing (3.13).

According to [6, p170], (3.8) *corresponds to* the $g=0$ case of Conjecture 2.1. However, (3.8) by itself can describe *at most* a suitable lim sup. It does not imply even the genus 0 case of Corollary 2.3. For example, replacing the numbers $n_{0;d}$ in (3.7) by the numbers

$$n'_{0,d}=\begin{cases} n_{0,d/2}, & \text{if } d\in 2\mathbb{Z};\\ 0, & \text{if } d\notin 2\mathbb{Z};\end{cases}$$

would break the validity of the first equality in (2.1) without affecting the validity of the conclusion of Proposition 3.3.

Mathematica suggests that the numbers on the left-hand side of (2.1) are increasing (after the first few terms), but it is not clear how this can be proved. In light of Kontsevich's recursion (3.5), this could be a special case of Conjecture 4.1. Combined with the conclusion of Conjecture 4.1 for the numbers $n_{0,d}$ given by (3.5), Proposition 3.3 would at least imply the first statement of Corollary 2.3.

Corollary 3.4. *Let $x_0\in\mathbb{R}^+$ be as in Proposition 3.3. The power series F_1 converges on $\mathbb{C}^{<}_{x_0}$ and diverges outside of $\mathbb{C}^{\le}_{x_0}$. Furthermore, there exist $\delta\in\mathbb{R}^+$ and $b_{2d}\in\mathbb{R}$ and $b_{2d+1}\in\mathfrak{i}\mathbb{R}$ with $d\in\mathbb{Z}$, $d\ge -1$, such that*

$$F_1'(x_0+z)=-\frac{1}{48z}+\sum_{d=-1}^{\infty}b_d z^{d/2} \tag{3.15}$$

for all $z\in B_\delta(0)$ with $\mathrm{Re}\,(z)\le 0$.

Proof. By [26, (8)],

$$n_{1,d}=\frac{(d-1)(d-2)}{216}n_{0,d}+\frac{1}{27d}\sum_{\substack{d_0+d_1=d\\ d_0,d_1\ge 1}}(3d_0^2-2d_0)d_1\,n_{0,d_0}n_{1,d_1}\,. \tag{3.16}$$

This implies that

$$(9+2F_0'-3F_0'')F_1'=\frac{1}{8}\big(F_0'''-3F_0''+2F_0'\big). \tag{3.17}$$

By (3.17) and the first statement of Proposition 3.3, the series F_1 converges on $\mathbb{C}^{<}_{x_0}$.

By (3.8) and (3.12),

$$\begin{aligned} 9+2F_0'(x_0+z)-3F_0''(x_0+z)&=z^{1/2}\sum_{d=0}^{\infty}\frac{d+3}{4}\big(4a_{d+3}-3(d+5)a_{d+5}\big)z^{d/2},\\ F_0'''(x_0+z)&=z^{-1/2}\sum_{d=0}^{\infty}\frac{(d+1)(d+3)(d+5)}{8}a_{d+5}z^{d/2}, \end{aligned} \tag{3.18}$$

with $a_3\equiv 0$. Along with (3.17), this implies (3.15). □

Since the power series F_1 converges for $z\in\mathbb{C}^<_{x_0}$,

$$\limsup_{d\longrightarrow\infty}\sqrt[d]{n_{1,d}}\le e^{-x_0}=\limsup_{d\longrightarrow\infty}\sqrt[d]{n_{0,d}}\,.$$

The opposite inequality follows directly from (3.16); it also holds for lim inf. Thus, the existence of $b\in\mathbb{R}^+$ such that the first equality in (2.1) holds implies that the second equality also holds.

3.3 The observations in [32]

We now describe the statements in [32] that have completed the proof of the $g=0$ case of Conjecture 2.1, initiated in [6] and continued in [37], and have extended it to the $g=1$ case.

Since the functions (3.7) are $2\pi\mathfrak{i}$-periodic, they do not extend analytically over a neighborhood of $x_0+2\pi k\mathfrak{i}$ for any $k\in\mathbb{Z}$. By Lemma 3.5 below, they extend analytically around all other points of the vertical line $\mathrm{Re}\,z=x_0$ in $\mathbb{C}$. The two asymptotic expansions of Theorem 2.2 then follow from Lemma 3.6, Proposition 3.3, and Corollary 3.4.

Lemma 3.5 ([32, Lemma 3.1]). *Let F_0 and F_1 be as in (3.7) and x_0 be as in Proposition 3.3. Then*

$$3F_0''\big(x_0+\mathfrak{i}y_0\big)-2F_0'\big(x_0+\mathfrak{i}y_0\big)\neq 9\qquad\forall\;y_0\in\mathbb{R}-2\pi\mathbb{Z}\,. \tag{3.19}$$

Thus, the functions F_0 and F_1 extend analytically over a neighborhood of every point $x_0+\mathfrak{i}y_0$ with $y_0\in\mathbb{R}-2\pi\mathbb{Z}$.

Proof. The first statement follows from $n_{0,d}>0$ and (3.12), since

$$\begin{aligned}\mathrm{Re}\Big(3F_0''\big(x_0+\mathfrak{i}y_0\big)-2F_0'\big(x_0+\mathfrak{i}y_0\big)\Big)&=\sum_{d=1}^{\infty}\big(3d-2\big)d\,n_{0,d}e^{dx_0}\cos(dy_0)\\&<\sum_{d=1}^{\infty}\big(3d-2\big)d\,n_{0,d}e^{dx_0}=3F_0''(x_0)-2F_0'(x_0)=9\,.\end{aligned}$$

By (3.19), (3.9) and (3.17) can used to compute all derivatives of F_0 and F_1 at $z_0\equiv x_0+\mathfrak{i}y_0$ and to extend F_0 and F_1 around z_0. □

The remaining part of [32, Lemma 3.1] is equivalent to Proposition 3.3, established 5 years earlier in [37]. The proof in [32] provides an alternative argument for Proposition 3.3; this argument is somewhat shorter in length than Section 3.4, but is more ad hoc and does not include recursions for the coefficients a_d in (3.8).

The crucial observation of [32] is that the asymptotic behavior of the coefficients n_d of a generating series F as in (3.20) below is described by expansions around x_0 *if* F has no additional singular points on the vertical line $\mathrm{Re}\,z=x_0$. This observation is reformulated in greater generality as Lemma 3.6 below, which appears similar to some asymptotic analysis statements in combinatorics. Two special cases of this lemma are considered in [32] and treated separately, but the argument in the second case in [32] applies to the general case of Lemma 3.6 without any changes of substance.

For $k\in\mathbb{R}^+$, let

$$\Gamma(k)\equiv\int_0^\infty t^{k-1}e^{-t}\mathrm{d}t$$

denote the value of the Γ function at k.

Lemma 3.6. *Let $x_0\in\mathbb{R}$, $\delta_0\in\mathbb{R}^+$, $n_d\in\mathbb{C}$ for $d\in\mathbb{Z}^+$, and $a_d\in\mathbb{C}$ for $2d\in\mathbb{Z}$. If the power series*

$$F(z) \equiv \sum_{d=1}^{\infty} n_d e^{dz}, \qquad z\in\mathbb{C}, \tag{3.20}$$

converges on $\mathbb{C}^{<}_{x_0}$, extends analytically over a neighborhood of every point $x_0+\mathrm{i}y_0$ with $y_0\in\mathbb{R}-2\pi\mathbb{Z}$, and satisfies

$$F(x_0+z) = \sum_{\substack{2k\in\mathbb{Z}\\ -1\le k}} a_k z^k$$

for all $z\in B_{\delta_0}(0)$ with $\mathrm{Re}\,(z)<0$, then for each $N\in\mathbb{Z}$ there exists $C_N\in\mathbb{R}$ such that

$$\Bigg| n_d - e^{-dx_0}\Bigg(-a_{-1}+\frac{1}{\pi\mathrm{i}}\sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ -1<k<N-1}} a_k\Gamma\big(k+1\big)d^{-k-1}\Bigg)\Bigg| \le C_N e^{-dx_0}d^{-N-\frac{1}{2}} \quad \forall\, d\in\mathbb{Z}^+\,. \tag{3.21}$$

Proof [32, pp8-11]. For $d\in\mathbb{Z}^+$ and $k,\delta\in\mathbb{R}^+$, let

$$\Gamma_{d;\delta}(k) \equiv \int_0^{\delta} t^{k-1}e^{-dt}\mathrm{d}t = d^{-k}\int_0^{d\delta} t^{k-1}e^{-t}\mathrm{d}t < d^{-k}\Gamma(k)\,. \tag{3.22}$$

If $0<N\le k$, then

$$0<\Gamma_{d;\delta}(k) \le \delta^{k-N}\Gamma_{d;\delta}(N) < \delta^{k-N}d^{-N}\Gamma(N). \tag{3.23}$$

If $N\ge k$, then

$$0<\Gamma(k)-d^k\Gamma_{d;\delta}(k) = \int_{d\delta}^{\infty} t^{k-1}e^{-t}\mathrm{d}t \le (d\delta)^{k-N}\int_{d\delta}^{\infty} t^{N-1}e^{-t}\mathrm{d}t < (d\delta)^{k-N}\Gamma(N). \tag{3.24}$$

The assumptions on F imply that there exists $\delta\in(0,\delta_0)$ so that F extends analytically over the region

$$\big\{x+\mathrm{i}y\colon x\in[x_0,x_0+2\delta],\ y\in[0,2\pi]\big\} - \big\{x_0,x_0+2\pi\mathrm{i}\big\} \subset \mathbb{C}$$

with

$$\begin{aligned} F\big(x_0+re^{\mathrm{i}\theta}\big) &= \sum_{\substack{2k\in\mathbb{Z}\\ -1\le k}} a_k r^k e^{\mathrm{i}k\theta} && \forall\ (r,\theta)\in(0,2\delta)\times[0,\pi],\\ F\big(x_0+2\pi\mathrm{i}+re^{\mathrm{i}\theta}\big) &= \sum_{\substack{2k\in\mathbb{Z}\\ -1\le k}} a_k r^k e^{\mathrm{i}k\theta} && \forall\ (r,\theta)\in(0,2\delta)\times[\pi,2\pi]. \end{aligned} \tag{3.25}$$

For $\epsilon\in(0,\delta)$, define oriented curves in $\mathbb{C}$ by

$$\begin{aligned} &\mathcal{C}^{\mathrm{v}}_{\epsilon;-} = \big\{x_0-\epsilon+\mathrm{i}t\colon t\in[0,2\pi]\big\}, && \mathcal{C}^{\mathrm{h}}_{\epsilon;-} = \big\{x_0+t\colon t\in[\epsilon,\delta]\big\}, && \mathcal{C}^{\circ}_{\epsilon;-} = \big\{x_0+\epsilon e^{\mathrm{i}t}\colon t\in[0,\pi]\big\},\\ &\mathcal{C}^{\mathrm{v}}_{+} = \big\{x_0+\delta+\mathrm{i}t\colon t\in[0,2\pi]\big\}, && \mathcal{C}^{\mathrm{h}}_{\epsilon;+} = \big\{x_0+t+2\pi\mathrm{i}\colon t\in[\epsilon,\delta]\big\}, && \mathcal{C}^{\circ}_{\epsilon;+} = \big\{x_0+2\pi\mathrm{i}+\epsilon e^{\mathrm{i}t}\colon t\in[\pi,2\pi]\big\}; \end{aligned}$$

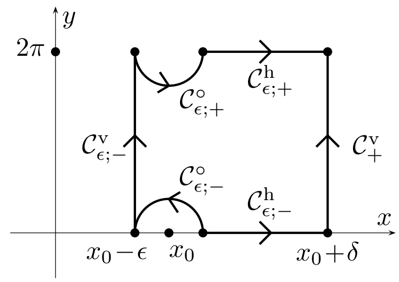

Figure 1: The curves in the proof of Lemma 3.6.

see Figure 1. By (3.25), there exist $C^h_\delta, C^\circ_\delta \in \mathbb{R}$ such that

$$\sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ -1<k}} |a_k| t^k \le C^h_\delta t^{-1/2} \quad \forall\, t\in(0,\delta], \tag{3.26}$$

$$\big|F(z)e^{-dz} - a_{-1}\big(z-x_0-(1\pm1)\pi\mathfrak{i}\big)^{-1}e^{-dx_0}\big| \le C^\circ_\delta e^{-dx_0}\epsilon^{-1/2} \quad \forall\, z\in\mathcal{C}^\circ_{\epsilon;\pm},\ \epsilon\in(0,\delta),\ d\in\mathbb{Z}^+. \tag{3.27}$$

For each $d\in\mathbb{Z}^+$,

$$\begin{aligned} 2\pi\mathfrak{i}\, n_d = \int_{\mathcal{C}^{\mathrm{v}}_{\epsilon;-}} F(z)e^{-dz}\mathrm{d}z = \int_{\mathcal{C}^{\mathrm{v}}_{+}} F(z)e^{-dz}\mathrm{d}z - \bigg(\int_{\mathcal{C}^{\circ}_{\epsilon;-}} F(z)e^{-dz}\mathrm{d}z + \int_{\mathcal{C}^{\circ}_{\epsilon;+}} F(z)e^{-dz}\mathrm{d}z\bigg) \\ + \bigg(\int_{\mathcal{C}^{h}_{\epsilon;-}} F(z)e^{-dz}\mathrm{d}z - \int_{\mathcal{C}^{h}_{\epsilon;+}} F(z)e^{-dz}\mathrm{d}z\bigg). \end{aligned} \tag{3.28}$$

By the compactness of $\mathcal{C}^{\mathrm{v}}_+$, there exists $C^{\mathrm{v}}_\delta\in\mathbb{R}$ such that

$$\bigg|\int_{\mathcal{C}^{\mathrm{v}}_{+}} F(z)e^{-dz}\mathrm{d}z\bigg| \le C^{\mathrm{v}}_\delta\, e^{-d(x_0+\delta)} \qquad \forall\, d\in\mathbb{Z}^+. \tag{3.29}$$

By (3.27),

$$\bigg|\bigg(\int_{\mathcal{C}^{\circ}_{\epsilon;-}} F(z)e^{-dz}\mathrm{d}z + \int_{\mathcal{C}^{\circ}_{\epsilon;+}} F(z)e^{-dz}\mathrm{d}z\bigg) - 2\pi\mathfrak{i}\, a_{-1}e^{-dx_0}\bigg| \le 2\pi C^\circ_\delta e^{-dx_0}\epsilon^{1/2} \quad \forall\, \epsilon\in(0,\delta),\ d\in\mathbb{Z}^+. \tag{3.30}$$

By (3.25),

$$\begin{aligned} \int_{\mathcal{C}^{h}_{\epsilon;-}} F(z)e^{-dz}\mathrm{d}z - \int_{\mathcal{C}^{h}_{\epsilon;+}} F(z)e^{-dz}\mathrm{d}z &= 2\sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ -1<k}} a_k \int_\epsilon^\delta t^k e^{-d(x_0+t)}\mathrm{d}t \\ &= 2e^{-dx_0}\sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ -1\le k}} a_k\Gamma_{d;\delta}(k+1) - 2e^{-dx_0}\sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ -1\le k}} a_k\int_0^\epsilon t^k e^{-dt}\mathrm{d}t\,. \end{aligned}$$

Along with (3.26), this gives

$$\bigg|\bigg(\int_{\mathcal{C}^{h}_{\epsilon;-}} F(z)e^{-dz}\mathrm{d}z - \int_{\mathcal{C}^{h}_{\epsilon;+}} F(z)e^{-dz}\mathrm{d}z\bigg) - 2e^{-dx_0}\sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ -1\le k}} a_k\Gamma_{d;\delta}(k+1)\bigg| \le 4C^h_\delta e^{-dx_0}\epsilon^{1/2}. \tag{3.31}$$

By (3.24), (3.23), and (3.26),

$$\begin{aligned}
&\Big|\sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ -1\le k}} a_k\Gamma_{d;\delta}(k+1)-\sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ -1<k<N-1}} a_k\Gamma\big(k+1\big)d^{-k-1}\Big|\\
&\qquad\le \sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ -1<k<N-1}}|a_k|\big|\Gamma_{d;\delta}\big(k+1\big)-\Gamma\big(k+1\big)d^{-k-1}\big|+\sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ N-1<k}}|a_k|\Gamma_{d;\delta}\big(k+1\big)\\
&\qquad\le \sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ -1<k<N-1}}|a_k|(d\delta)^{k-N+\frac12}\Gamma\big(N+1/2\big)d^{-k-1}+\sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ N-1<k}}|a_k|\delta^{k-N+\frac12}\Gamma\big(N+1/2\big)d^{-N-\frac12}\\
&\qquad=\Gamma\big(N+1/2\big)(d\delta)^{-N-\frac12}\sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ -1<k}}|a_k|\delta^{k+1}\le C_\delta^h\delta^{1/2}\Gamma\big(N+1/2\big)(d\delta)^{-N-\frac12}.
\end{aligned}\tag{3.32}$$

By (3.28)-(3.32), there exists $C_N\in\mathbb{R}$ such that

$$\Bigg|n_d+e^{-dx_0}\Bigg(a_{-1}-\frac{1}{\pi\mathfrak{i}}\sum_{\substack{2k\in\mathbb{Z}-2\mathbb{Z}\\ -1<k<N-1}} a_k\Gamma\big(k+1\big)d^{-k-1}\Bigg)\Bigg|\le C_Ne^{-dx_0}\big(e^{-d\delta}+\epsilon^{1/2}+d^{-N-\frac12}\big)$$

for all $\epsilon\in(0,\delta)$ and $d\in\mathbb{Z}^+$. Sending ϵ to 0, we obtain the claim. □

Remark 3.7. Lemma 3.6 can be used to obtain asymptotics similar to (3.21) for power series $F(z)$ as in its statement satisfying

$$F(x_0+z)=\sum_{\substack{2k\in\mathbb{Z}\\ k_0\le k}} a_kz^k \tag{3.33}$$

for some $k_0\in\mathbb{Z}^-$. The coefficients a_k with $k\in\mathbb{Z}^-$ can be eliminated by adding appropriate multiples of the power series

$$F_k(z)=\sum_{d=1}^{\infty}d^{1-k}e^{-dx_0}e^{dz}=\bigg\{\frac{\mathrm{d}}{\mathrm{d}z}\bigg\}^{1-k}\bigg(\frac{e^{z-x_0}}{1-e^{z-x_0}}\bigg).$$

The coefficients a_k with $k\in\mathbb{Z}^- -2\mathbb{Z}$ can then be eliminated by integrating $F(z)$ enough times. These two modifications do not break the remaining requirements imposed on F by Lemma 3.6 and reduce any expansion as in (3.33) to one with $k_0=0$. In such a case, the $\epsilon=0$ contour of [32, pp8,9] suffices.

3.4 Proof of Proposition 3.3

It remains to establish the last statement of Proposition 3.3.

Lemma 3.8. *Let $x_0\in\mathbb{R}$, $\delta\in\mathbb{R}^+$, $x^*\in(x_0-\delta,x_0)$, and*

$$F,G\colon(x_0-\delta,x_0)\longrightarrow\mathbb{R} \tag{3.34}$$

be solutions of (3.9) satisfying (3.10) and

$$F(x^*)=G(x^*),\qquad F'(x^*)=G'(x^*),\qquad F''(x^*)<G''(x^*)\,. \tag{3.35}$$

Then $F''(x)<G''(x)$ for all $x\in[x^*,x_0]$.

Proof. We can assume that $\delta<1$. Suppose $x'\in(x^*,x_0)$ and $F'''(x)<G'''(x)$ for all $x\in(x^*,x')$. Then,

$$0\le G'(x')-F'(x')=\int_{x^*}^{x'}\big(G''(x)-F''(x)\big)\mathrm{d}x\le\big(G''(x')-F''(x')\big)(x'-x^*)\le G''(x')-F''(x').$$

Thus,

$$F(x')\le G(x'),\ F'(x')\le G'(x'),\ F''(x')<G''(x'),\ F''(x')-F'(x')\le G''(x')-G'(x').$$

Along with (3.9), this implies that $F'''(x')<G'''(x')$. The claim now follows. □

Corollary 3.9. *Let F,G be solutions of (3.9) as in (3.34) satisfying (3.10) and $x'\in(x^*,x_0)$ be such that*

$$F(x')=G(x^*),\quad F'(x')=G'(x^*),\qquad \lim_{x\longrightarrow^- x_0}F'''(x),\ \lim_{x\longrightarrow^- x_0}G'''(x)=\infty. \tag{3.36}$$

Then $F''(x')\ge G''(x^*)$.

Proof. Suppose $F''(x')<G''(x^*)$. Let $y=x'-x^*>0$. Since (3.9) is a homogeneous differential equation, the function

$$\widetilde{F}\colon\big(x_0-y-\delta,x_0-y\big)\longrightarrow\mathbb{R},\qquad \widetilde{F}(x)=F(x+y),$$

is a solution of (3.9) satisfying (3.10) and

$$\widetilde{F}(x^*)=G(x^*),\quad \widetilde{F}'(x^*)=G'(x^*),\quad \widetilde{F}''(x^*)<G''(x^*),\qquad \lim_{x\longrightarrow^- x_0-y}\widetilde{F}'''(x)=\infty\,.$$

This contradicts the conclusion of Lemma 3.8, since $G'''(x_0-y)$ is finite. □

Lemma 3.10. *Let $a_0,a_2\in\mathbb{R}$ be such that*

$$4a_2^2+45a_2+18a_0+567>0. \tag{3.37}$$

Then there exist $\delta\in\mathbb{R}^+$ and unique

$$a_4\in\mathbb{R},\quad a_5\in\mathrm{i}\mathbb{R}^-,\quad a_{2d}\in\mathbb{R},\ a_{2d+1}\in\mathrm{i}\mathbb{R}\ \ \forall\ d\in\mathbb{Z},\ d\ge3, \tag{3.38}$$

such that the power series in (3.8) converges uniformly for $z\in B_\delta(0)$ with $\mathrm{Re}\,(z)\le0$ to a solution of (3.9). The number $\delta=\delta(a_0,a_2)$ can be chosen to depend continuously on $(a_0,a_2)\in\mathbb{R}^2$ satisfying (3.37).

Proof. For arbitrary $a_d\in\mathbb{C}$, let

$$F_0(x_0+z)=\sum_{d=0}^{\infty}a_dz^{d/2}. \tag{3.39}$$

The differential equation (3.9) is then equivalent to

$$\begin{aligned}
&a_1,a_3=0\,,\qquad (9+2a_2-6a_4)a_5=0\,,\\
&(9+2a_2-6a_4)\frac{(d+2)(d+4)(d+6)}{8}a_{d+6}=2a_d-\frac{11(d+2)}{2}a_{d+2}+\frac{9(d+2)(d+4)}{2}a_{d+4}\\
&\qquad-\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge0}}\frac{(d_1+1)(d_1+3)(d_1+5)(d_2+3)}{32}a_{d_1+5}\big(4a_{d_2+3}-3(d_2+5)a_{d_2+5}\big)\\
&\qquad+\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge0}}\frac{(d_1+2)(d_1+4)(d_2+2)(d_2+4)}{16}a_{d_1+4}a_{d_2+4}\,.
\end{aligned}$$

The last equation holds for all $d \geq 0$.

If $a_5 \neq 0$, the last two conditions above are equivalent to

$$9+2a_2-6a_4=0\,, \qquad -\frac{675}{32}a_5^2=2a_0-11a_2+36a_4+4a_4^2\,, \tag{3.40}$$

$$\begin{aligned}\frac{45(d+2)(d+3)(d+5)}{32}a_5a_{d+5} &= -2a_d+\frac{11(d+2)}{2}a_{d+2}+\frac{(d+2)(d+4)}{2}\big((d-2)a_4-9\big)a_{d+4}\\ &\quad -\sum_{\substack{d_1+d_2=d\\ d_1,d_2\geq 1}}\frac{3(d+2)(d_1+3)(d_1+5)(d_2+3)(d_2+5)}{64}a_{d_1+5}a_{d_2+5}\\ &\quad +\sum_{\substack{d_1+d_2=d\\ d_1,d_2\geq 1}}\frac{(d_1^2+d_2^2-d_1d_2-4)(d_1+4)(d_2+4)}{16}a_{d_1+4}a_{d_2+4}\,;\end{aligned} \tag{3.41}$$

the last equation is valid for $d \geq 1$. By (3.40),

$$a_4=\frac{3}{2}+\frac{1}{3}a_2, \qquad -\frac{6075}{32}a_5^2=4a_2^2+45a_2+18a_0+567\,.$$

By (3.37), these equations determine $a_d \in \mathbb{C}$ with $d \geq 4$. By induction, the coefficients a_d satisfy (3.38).

It remains to show that the power series (3.39) with a_d given by (3.41) converges uniformly on a small disk. Let $\widetilde{a}_d$ be the sequence recursively defined by

$$\begin{aligned}\widetilde{a}_d &= 1+\sum_{k=1}^{6}|a_k| \qquad \forall\, d\leq 6,\\ d^3\,\widetilde{a}_5\widetilde{a}_{d+5} &= 2\widetilde{a}_d+20d\,\widetilde{a}_{d+2}+\big(64+16da_4\big)d^2\,\widetilde{a}_{d+4}\\ &\quad +36\sum_{\substack{d_1+d_2=d\\ d_1,d_2\geq 1}}d_1^3d_2^3\,\widetilde{a}_{d_1+5}\widetilde{a}_{d_2+5}+5\sum_{\substack{d_1+d_2=d\\ d_1,d_2\geq 1}}d_1^3d_2^3\,\widetilde{a}_{d_1+4}\widetilde{a}_{d_2+4} \quad \forall\, d\geq 2.\end{aligned}$$

Let n_d be the sequence recursively defined by

$$n_1=\widetilde{a}_6, \qquad n_d=150\big(1+|a_5|^{-1}\big)\sum_{\substack{d_1+d_2=d\\ d_1,d_2\geq 1}}\frac{d_1^3d_2^3}{d^3}n_{d_1}n_{d_2} \quad \forall\, d\geq 2.$$

By (3.41) and the sequence $\widetilde{a}_d$ being positive and non-decreasing,

$$|a_d|\leq \widetilde{a}_d\leq n_{d-5} \qquad \forall\, d\geq 6\,.$$

By Lemma 3.1 and (3.6), there thus exists $C\in\mathbb{R}^+$ such that

$$|a_d|\leq C^d\big(1+|a_5|^{-1}\big)^d \qquad \forall\, d\in\mathbb{Z}^+\,.$$

It follows that (3.39) defines a solution of (3.9) around $z=x_0$ for any choice of a_0 and a_2 such that (3.37) is satisfied. □

We define

$$W = \big\{(a_0,a_2)\in\mathbb{R}^2 : 0<a_0<a_2<9\big\}\,. \tag{3.42}$$

For each $(a_0,a_2)\in W$, let

$$F_{a_0,a_2}\colon \big\{z\in\mathbb{C}^{\le}_{x_0} : |x_0-z|<\delta(a_0,a_2)\big\} \longrightarrow \mathbb{C}$$

be the solution (3.39) of (3.9) provided by Lemma 3.10. In particular,

$$F_{a_0,a_2}(x_0)=a_0 < F'_{a_0,a_2}(x_0)=a_2 < F''_{a_0,a_2}(x_0)=3+\frac{2}{3}a_2\,. \tag{3.43}$$

Reducing the continuous function $\delta=\delta(a_0,a_2)$ if necessary, we can assume that

$$0 < F_{a_0,a_2}(z) < F'_{a_0,a_2}(z) < F''_{a_0,a_2}(z) < F'''_{a_0,a_2}(z) \qquad \forall\ z\in\big(x_0-\delta(a_0,a_2),x_0\big).$$

We define

$$\Phi\colon \widetilde{W}\equiv\{(z,a_0,a_2)\in(-\infty,x_0]\times W : x_0-z<\delta(a_0,a_2)\big\} \longrightarrow (-\infty,x_0]\times\mathbb{R}^2,$$
$$\Phi(z,a_0,a_2) = \big(z,F_{a_0,a_2}(z),F'_{a_0,a_2}(z)\big)\,.$$

Lemma 3.11. *Let $(a_0,a_2)\in W$. There exist neighborhoods $W_{(a_0,a_2)}$ and $V_{(a_0,a_2)}$ of (x_0,a_0,a_2) in $(-\infty,x_0]\times\mathbb{R}^2$ such that the map*

$$\Phi\colon W_{(a_0,a_2)} \longrightarrow V_{(a_0,a_2)}$$

is a homeomorphism.

Proof. By (3.43),

$$\Phi(x_0,a'_0,a'_2) = (x_0,a'_0,a'_2) \qquad \forall\ (a'_0,a'_2)\in W.$$

Thus, Φ extends to a continuous map

$$\widetilde{\Phi}\colon \widetilde{W}\cup\big([x_0,\infty)\times W\big) \longrightarrow \mathbb{R}^3, \qquad \widetilde{\Phi}(z,a'_0,a'_2) = \begin{cases} \Phi(z,a'_0,a'_2), & \text{if } z\le x_0;\\ (z,a'_0,a'_2), & \text{if } z\ge x_0.\end{cases}$$

This map is injective on a neighborhood of (x_0,a_0,a_2). By [25, Theorem 36.5], there thus exist neighborhoods $W'_{(a_0,a_2)}$ and $V'_{(a_0,a_2)}$ of (x_0,a_0,a_2) in $\mathbb{R}^3$ such that $\widetilde{\Phi}$ takes $W'_{(a_0,a_2)}$ homeomorphically onto $V'_{(a_0,a_2)}$. Taking

$$W_{(a_0,a_2)} = W'_{(a_0,a_2)}\cap\big((-\infty,x_0]\times\mathbb{R}^2\big) \quad\text{and}\quad V_{(a_0,a_2)} = V'_{(a_0,a_2)}\cap\big((-\infty,x_0]\times\mathbb{R}^2\big),$$

we complete the proof. □

Corollary 3.12. *Let $(a_0,a_2)\in\mathbb{R}^2$ and G be a solution of (3.9) as in (3.34) satisfying (3.10) such that*

$$G(x_0)=a_0, \qquad \lim_{x\longrightarrow^- x_0} G'(x)=a_2, \qquad \lim_{x\longrightarrow^- x_0} G'''(x)=\infty. \tag{3.44}$$

Then $(a_0,a_2)\in W$ and there exists $\delta'\in(0,\delta)$ such that $G=F_{a_0,a_2}$ on $(x_0-\delta',x_0)$.

Proof. By the reasoning in Section 3.2, the assumptions on G imply that $(a_0,a_2)\in W$. Let Φ be as in Lemma 3.11. By (3.44), there exist $x^*\in\mathbb{R}$ and $(a_0^*,a_2^*)\in\mathbb{R}^2$ such that

$$x^*\in(x_0-\delta,x_0),\quad (x^*,a_0^*,a_2^*)\in W_{(a_0,a_2)},\quad \big(x^*,G(x^*),G'(x^*)\big)=\Phi(x^*,a_0^*,a_2^*)\,.$$

If $F''_{a_0^*,a_2^*}(x^*)<G''(x^*)$, there exist

$$\begin{gathered}x'\in(x^*,x_0),\quad (x',a_0',a_2')\in W_{(a_0,a_2)}\qquad\text{s.t.}\\ \Phi(x',a_0',a_2')=\big(x',G(x^*),G'(x^*)\big),\quad F''_{a_0',a_2'}(x')<G''(x^*).\end{gathered}$$

This contradicts Corollary 3.9. If $F''_{a_0^*,a_2^*}(x^*)>G''(x^*)$, there exist

$$\begin{gathered}x'\in\big(x_0-\delta,x^*\big),\quad (x',a_0',a_2')\in W_{(a_0,a_2)}\qquad\text{s.t.}\\ \Phi(x',a_0',a_2')=\big(x',G(x^*),G'(x^*)\big),\quad F''_{a_0',a_2'}(x')>G''(x^*).\end{gathered}$$

This contradicts Corollary 3.9 with x^* and x' interchanged.

We conclude that $F''_{a_0^*,a_2^*}(x^*)=G''(x^*)$. Since

$$9+2F'_{a_0^*,a_2^*}(x^*)-3F''_{a_0^*,a_2^*}(x^*)>0,$$

the uniqueness of solutions of differential equations implies that $G=F_{a_0^*,a_2^*}$ on $[x^*,x_0]$. By (3.44) and (3.43), $(a_0^*,a_2^*)=(a_0,a_2)$. □

4 Other observations

We conclude by elaborating two separate points brought up earlier in this note.

4.1 Counts of curves in $\mathbb{P}^3$

We first apply the reasoning of [6] described in Section 3.1 to obtain a coarse upper bound for the numbers of Conjecture 2.4.

For $d\in\mathbb{Z}^+$ and $p\in\mathbb{Z}^{\ge0}$, let

$$n_{0,d}(p)=\frac{N_{0,d}(p)}{(2d+p)!}\,.$$

For $d_1,d_2\in\mathbb{Z}^+$ and $p_1,p_2\in\mathbb{Z}^{\ge0}$ with

$$p_1\le 2d_1,\qquad p_2\le 2d_2,\qquad 1<p\equiv p_1+p_2<2d\equiv 2(d_1+d_2),\tag{4.1}$$

define

$$f(d_1,d_2,p_1,p_2)=\frac{(2d_1+p_1)!(2d_2+p_2)!}{(2d+p)!}d_2\binom{2d-p-1}{2d_1-p_1-1}\left(d_1^2\binom{2p-2}{2p_1}-d_2^2\binom{2p-2}{2p_2}\right).$$

Since

$$\frac{(2d_1+p_1)(2d_2+p_2)(2d_2+p_2-1)}{(2d+p)(2d+p-1)(2d+p-2)} d_1^2 d_2 \binom{2d-p-1}{2d_1-p_1-1}\binom{2p-2}{2p_1} \le 8\frac{d_1^3 d_2^3}{d^3}\binom{2d+p-3}{2d_1+p_1-1},$$
$$\frac{(2d_1+p_1)(2d_1+p_1-1)(2d_1+p_1-2)}{(2d+p)(2d+p-1)(2d+p-2)} d_2^3 \binom{2d-p-1}{2d_1-p_1-1}\binom{2p-2}{2p_2} \le 8\frac{d_1^3 d_2^3}{d^3}\binom{2d+p-3}{2d_1+p_1-3},$$

we find that

$$\big|f(d_1,d_2,p_1,p_2)\big| \le 8\frac{d_1^3 d_2^3}{d^3} \tag{4.2}$$

under the assumptions (4.1).

By the recursion of [29, Theorem 10.4],

$$\begin{aligned} n_{0,d}(0) &= \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}} \frac{(d_2-d_1)d_1^2 d_2(2d_2+1)}{d(d-1)(2d-1)} n_{0,d_1}(0)n_{0,d_2}(1),\\ n_{0,d}(2d) &= \frac{1}{2}n_{0,d}(2d-1) + \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}} \frac{d_1d_2(4dd_1d_2-d^2+2d_1d_2)}{2d(2d-1)(4d-1)} n_{0,d_1}(2d_1)n_{0,d_2}(2d_2) \end{aligned} \tag{4.3}$$

for $d\ge2$ and $d\ge1$, respectively. For $d\ge1$ and $1\le p\le 2d-1$, this recursion with $j_1=3$ and $j_2=j_3=2$ gives

$$n_{0,d}(p) = \frac{d}{2d+p}n_{0,d}(p-1) + \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}}\sum_{\substack{p_1+p_2=p\\ 0\le p_i\le 2d_i}} f(d_1,d_2,p_1,p_2)\, n_{0,d_1}(p_1)n_{0,d_2}(p_2); \tag{4.4}$$

the assumption $j_i\ge j_{i+1}$ in [29] is not essential. For $p=1$, (4.4) simplifies to

$$n_{0,d}(1) = \frac{d}{2d+1}n_{0,d}(0) + \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}} \frac{d_1^3 d_2(2d_2+1)}{d(2d-1)(2d+1)} n_{0,d_1}(0)n_{0,d_2}(1). \tag{4.5}$$

Let $\widetilde{n}_1(0)=1/2$ and define

$$\widetilde{n}_d(0) = 4\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}} \frac{d_1^3d_2^3}{d^3}\widetilde{n}_{d_1}(0)\widetilde{n}_{d_2}(1) \qquad \forall\ d\ge2,$$

$$\widetilde{n}_d(1) = \frac{1}{2}\widetilde{n}_d(0) + \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}} \frac{d_1^3d_2^3}{d^3}\widetilde{n}_{d_1}(0)\widetilde{n}_{d_2}(1) \qquad \forall\ d\ge1,$$

$$\widetilde{n}_d(p) = \frac{1}{2}\widetilde{n}_d(p-1) + 8\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}}\sum_{\substack{p_1+p_2=p\\ 0\le p_i\le 2d_i}} \frac{d_1^3d_2^3}{d^3}\widetilde{n}_{d_1}(p_1)\widetilde{n}_{d_2}(p_2) \qquad \forall\ d\ge1,\ p\ge2.$$

We also define $\widetilde{n}'_1(0)=1/2$ and

$$\begin{aligned}\widetilde{n}'_d(0) &= 8\sum_{\substack{d_1+d_2=d\\ d_1,d_2\geq 1}}\widetilde{n}'_{d_1}(0)\widetilde{n}'_{d_2}(0) && \forall\ d\geq 2,\\ \widetilde{n}'_d(p) &= \frac{1}{2}\widetilde{n}'_d(p-1)+8\sum_{\substack{d_1+d_2=d\\ d_1,d_2\geq 1}}\sum_{\substack{p_1+p_2=p\\ 0\leq p_i\leq 2d_i}}\widetilde{n}'_{d_1}(p_1)\widetilde{n}'_{d_2}(p_2) && \forall\ d\geq 1,\ p\geq 1\,.\end{aligned}\tag{4.6}$$

By (4.2)-(4.5),

$$n_{0,d}(p)\leq\widetilde{n}_d(p)\leq\widetilde{n}'_d(p)/d^3 \qquad \forall\ d\in\mathbb{Z}^+,\ p\in\mathbb{Z}^{\geq 0}. \tag{4.7}$$

By (4.6), the power series

$$f(x)\equiv\sum_{d=1}^{\infty}\tilde{n}'_d(0)x^d \qquad\text{and}\qquad g(x,y)\equiv\sum_{d=1}^{\infty}\sum_{p=0}^{\infty}\tilde{n}'_d(p)x^dy^p$$

satisfy

$$f(x)=\frac{1}{2}x+8f(x)^2, \qquad g(x,y)-f(x)=\frac{1}{2}yg(x,y)+8\big(g(x,y)^2-f(x)^2\big).$$

Combining these two equations, we obtain

$$8g(x,y)^2-\big(1-y/2\big)g(x,y)+\frac{1}{2}x=0.$$

Solving the last equation, we find that

$$\begin{aligned}16g(x,y) &= (1-y/2)-\sqrt{(1-y/2)^2-32x}\Big) = \big(1-y/2\big)\Big(1-\sqrt{1-16x/(1-y/2)^2}\Big)\\ &= 2\big(1-y/2\big)\sum_{d=1}^{\infty}\frac{(2d-2)!}{d!(d-1)!}\bigg(\frac{4x}{(1-y/2)^2}\bigg)^d\\ &= 2\big(1-y/2\big)\sum_{d=1}^{\infty}\sum_{p=0}^{\infty}\frac{(2d-2)!}{d!(d-1)!}4^d2^{-p}\binom{2d-\,1+p}{p}x^dy^p\,.\end{aligned}$$

Since the binomial coefficient above increases with p,

$$\tilde{n}'_d(p)\leq\frac{1}{8d}\binom{2d-2}{d-1}4^d\binom{4d-1}{2d}\leq\frac{1}{8d}2^{2d-2}2^{2d}2^{4d-1}\leq\frac{1}{d}2^{8d} \qquad \forall\,p\leq 2d.$$

Combining with (4.7), we conclude that

$$n_{0,d}(p)\leq 2^{8d}d^{-4} \qquad \forall\ d,p\in\mathbb{Z}^+,\ p\in\mathbb{Z}^{\geq 0}\,. \tag{4.8}$$

This is an analogue (and a very rough one) of the upper bound in (3.3). A lower bound for the sequences of Conjecture 2.4 is more elusive because the recursions (4.3) and (4.4) involve negative coefficients.

4.2 On recursively defined sequences

As indicated above Corollary 3.4 in Section 3.2, the asymptotics for Gromov-Witten invariants in some basic cases may reflect the behavior of more general recursively defined sequences of the form (3.1).

Conjecture 4.1. *Suppose* $f\in\mathbb{Q}(d_1,d_2)$ *is a rational function defined on* $\mathbb{Z}^+\times\mathbb{Z}^+$ *and satisfying*

$$f(d_1,d_2)>0 \quad \forall\, d_1,d_2\in\mathbb{Z}^+, \qquad \lim_{d_1,d_2\longrightarrow\infty}\left(f(d_1,d_2)\Big/\left(\frac{d_1d_2}{d}\right)^k\right)\in\mathbb{R}^+$$

for some $k\in\mathbb{R}^+$*. If* n_d *is a sequence recursively defined by (3.1) and* $n_1>0$*, then the sequence* $\sqrt[d]{n_d}$ *is eventually increasing, i.e. there exists* $d^*\in\mathbb{Z}^+$ *such that*

$$\sqrt[d]{n_d}\leq\sqrt[d+1]{n_{d+1}} \qquad \forall d\geq d^*. \tag{4.9}$$

The conjectural dependence of the asymptotic behavior of n_d only on the asymptotic behavior of $f(d_1,d_2)$ may be related to the following property. Let $p(q)\in q\mathbb{R}[q]$ be a polynomial with *positive* coefficients and vanishing constant term. Define the numbers n_d by

$$\sum_{d=1}^{\infty}n_dq^d=\sum_{d=1}^{\infty}\frac{(2d-2)!}{d!(d-1)!}\frac{1}{ad^k}q^d\big(1+p(q)\big)^d.$$

It appears that the numbers $\sqrt[d]{n_d}$ are eventually increasing. In other words, this property is invariant under the change of variables

$$q\longrightarrow\big(1+p(q)\big)q$$

if $p(q)$ is a polynomial with *positive* coefficients and vanishing constant term. For the asymptotic behavior conclusion, $p(q)$ would perhaps need to be a power series with coefficients declining sufficiently quickly.

We now confirm Conjecture 4.1 in the model cases, i.e. for

$$f(d_1,d_2)=a\frac{d_1^kd_2^k}{d^k} \qquad \text{with} \quad a\in\mathbb{R}^+,\ k\in\mathbb{R}^{\geq 0}.$$

The crucial problem is how to reduce more general cases of this conjecture to the model ones.

By Lemma 3.1,

$$n_d=\frac{(2d-2)!}{d!(d-1)!}\frac{1}{ad^k}\big(n_1a\big)^d \qquad \forall\, d\in\mathbb{Z}^+.$$

Thus, the eventually increasing property in this case is equivalent to the existence of $d^*\in\mathbb{Z}^+$ such that

$$\frac{\sum\limits_{r=d+1}^{2d-2}\ln r-\sum\limits_{r=1}^{d-1}\ln r-\ln a-k\ln d}{d}<\frac{\sum\limits_{r=d+2}^{2d}\ln r-\sum\limits_{r=1}^{d}\ln r-\ln a-k\ln(d+1)}{d+1} \qquad \forall\, d\geq d^*.$$

This is equivalent to

$$d\ln(d{+}1)+\sum_{r=d+1}^{2d-2}\ln r-\sum_{r=2}^{d-1}\ln r-\ln a+k\big(d\ln(1+\frac{1}{d})-\ln d\big)<d\ln(2d-1)+d\ln 2. \tag{4.10}$$

Since $\ln x$ is an increasing function,

$$\sum_{r=d+1}^{2d-2}\ln r<\int_{d+1}^{2d-1}\ln x\,\mathrm{d}x=(x\ln x-x)\Big|_{d+1}^{2d-1}=(2d{-}1)\ln(2d{-}1)-(d{+}1)\ln(d{+}1)-(d{-}2),$$
$$\sum_{r=2}^{d-1}\ln r>\int_{1}^{d-1}\ln x\,\mathrm{d}x=(x\ln x-x)\Big|_{1}^{d-1}=(d{-}1)\ln(d{-}1)-(d{-}2).$$

Thus, the left-hand side of (4.10) is bounded by

$$\begin{aligned}&(2d{-}1)\ln(2d{-}1)-(d{-}1)\ln(d{-}1)-\ln(d{+}1)-\ln a-k\big(\ln d-1\big)\\&\qquad\le d\ln(2d{-}1)+(d{-}1)\ln 2+(d{-}1)\ln\left(1+\frac{1}{2(d{-}1)}\right)-\ln(d{+}1)-\ln a\\&\qquad\le d\ln(2d{-}1)+(d{-}1)\ln 2+\frac{1}{2}-\ln(d{+}1)-\ln a\le d\ln(2d{-}1)+d\ln 2-\ln(d{+}1)-\ln a.\end{aligned}$$

For d sufficiently large, the combination of the last two terms above is negative, which establishes (4.9) in this case.

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