Math53: Ordinary Differential Equations
Winter 2004

Unit 3 Summary
Laplace Transform and ODEs

Definitions, Key Properties, and Applications to ODEs

(1) The *Laplace Transform* $\mathcal{L}(f)$ of a function $f = f(t)$ is the function $F = F(s)$ defined by

$$F(s) = \int_0^\infty f(t)e^{-st}dt, \qquad \text{for all} \quad s > a,$$

where a is a nonnegative number, dependent on f. The Laplace Transform $\mathcal{L}(f)$ is *not* defined for all functions f. However, if there exist a and C such that

$$|f| < Ce^{at} \qquad \text{for all} \quad t \geq 0,$$

the Laplace Transform $F = F(s)$ is defined for $s > a$. The Laplace Transforms $F = F(s)$ of a few functions $f = f(t)$ can be computed directly from the definition. The transforms of three functions are shown in the first table on the next page.
Note: The two tables that appear on the next page will be put on the second midterm and on the final exam, if they are needed for any problem. However, you need to know how to obtain all eight statements contained in the two tables. You also need to know how to use the two tables.

(2) The linearity of the integral implies that for all real numbers α and β and functions $f = f(t)$ and $g = g(t)$,

$$\boxed{\{\mathcal{L}(\alpha f + \beta g)\}(s) = \alpha\{\mathcal{L}f\}(s) + \beta\{\mathcal{L}g\}(s) \ \text{ for all } \ s \quad \Longleftrightarrow \quad \mathcal{L}(\alpha f + \beta g) = \alpha\{\mathcal{L}f\} + \beta\{\mathcal{L}g\}}$$

One of the two key properties that make the Laplace Transform useful in solving ODEs is that the transform of the derivative of a function does *not* involve any derivative:

$$\boxed{\{\mathcal{L}f'\}(s) = s \cdot \{\mathcal{L}f\}(s) - f(0) \qquad \text{for all} \quad s} \tag{1}$$

This identity can be used to express the Laplace Transform $\mathcal{L}f^{\langle k \rangle}$ of the kth derivative of f in terms of the Laplace Transform $\mathcal{L}f$ of f. *Please make sure you know how to do so.* Due to (eq1), the Laplace Transform takes a kth-order linear ODE *with constant coefficients* to a kth-order algebraic equation, which is very easy to solve. For example,

$$\begin{aligned} y'' + py' + qy = f, \;\; y(0) = y_0, \;\; y'(0) = y_1 \quad &\longleftrightarrow \quad (s^2 Y - sy_0 - y_1) + p(sY - y_0) + qY = F \\ &\longleftrightarrow \quad Y = \frac{F}{s^2 + ps + q} + \frac{(s+p)y_0 + y_1}{s^2 + ps + q} \end{aligned} \tag{2}$$

Note that the denominator in each of the two fractions in the last expression is the *characteristic polynomial* for the ODE.

$f(t)$	$F(s)=\{\mathcal{L}f\}(s)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, \quad s>a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s>a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s>a$
δ	1

$f(t)$	$F(s)=\{\mathcal{L}f\}(s)$
f'	$s\cdot F(s)-f(0)$
$t\cdot f(t)$	$-F'(s)$
$e^{at}f(t)$	$F(s-a)$
$H(t-a)f(t-a)$	$e^{-as}F(s)$

Laplace Transforms

(3) The other key property of the Laplace Transform is that it is an isomorphism to the extent possible:

$$\boxed{f,g \ \text{ cont. on } \ [0,\infty) \ \text{ and } \ \{\mathcal{L}f\}(s)=\{\mathcal{L}g\}(s) \ \text{ for all } \ s\ge a \quad\Longrightarrow\quad f(t)=g(t) \ \text{ for all } \ t\ge 0}$$

Thus, if the function $Y=Y(s)$ is defined by the last identity in (eq2) and $y=y(t)$ is a continuous function such that $\mathcal{L}(y)=Y$, then y is the solution to the initial value problem on the left-hand side of (eq2). Given a function $Y=Y(s)$, our approach to finding its *Inverse Laplace Transform* $y=y(t)$ will be to write Y as a linear combination of the fractions in the middle three rows of the first table, if possible. Then, y will be the corresponding linear combination of the functions in the left column of the table. For example,

$$Y(s)=\frac{1}{s^2-2s}=\frac{1}{s(s-2)}=\frac{1}{2}\cdot\Big(\frac{1}{s-2}-\frac{1}{s}\Big) \quad\Longrightarrow\quad y(t)=\frac{1}{2}\big(e^{2t}-1\big) \ \text{ for all } \ t\ge 0.$$

By convention, if y is the Inverse Laplace Transform of a function, $y(t)=0$ for all $t<0$.
Note: Not every function Y can be split into a linear combination of the fractions in the middle three rows of the first table. For example, by the last row of this table, the Inverse Laplace Transform of the function $F=1$ is the *delta function* δ, which in fact is not a function at all.

(4) By (2) and (3), the Laplace Transform can be used to solve linear ODEs with constant coefficients. It is especially well-suited to solving initial value problems,

$$y^{\langle k\rangle}+p_1y^{\langle k-1\rangle}+\ldots+p_{k-1}y'+p_ky=f, \quad y(0)=y_0,\ y'(0)=y_1,\ldots,\ y^{\langle k-1\rangle}(0)=y_{k-1}, \tag{3}$$

with constant coefficients, i.e. $p_1,\ p_2,\ldots,p_k$:
Step 1: take the Laplace Transform of both sides of (eq3);
Step 2: solve the resulting algebraic equation, which is linear in Y, for Y;
Step 3: use the *method of partial fractions* to rewrite Y as a linear combination of
the Laplace Transforms of known functions, if possible;
Step 4: take the Inverse Laplace Transform.
This method works well if f is a sum of products of polynomials, exponentials, cosines, and sines, since in such a case the Laplace Transform of (eq3) can be computed quickly with the help of the two tables above and the resulting function Y will be decomposable as a linear combination of the Laplace Transforms of known functions, based on the two tables. In these cases, we could also use the approach of Section 4:
Step 1: find the general solution of the homogeneous ODE;
Step 2: use the method of undetermined coefficients to find a particular solution
of the inhomogeneous ODE;
Step 3: form the general solution of the inhomogeneous ODE and use the initial conditions

to solve for the constant $C_1, C_2, \ldots, C_k$.

Which of the two approaches is faster will depend on the ODE. The Laplace Transform approach is likely to be faster for higher-order ODEs and for more complicated forcing terms f. For example, for the IVP

$$y''' + 3y'' + 4y' + 2y = te^{-t}\sin t, \quad y(0) = y_0,\ y'(0) = y_1,$$

the Laplace Transform approach is probably faster, especially if you do not use the complex method to find a particular solution of the ODE. In addition, the Laplace Transform approach also works well for forcing terms that are as above, but defined piecewise. Such forcing terms arise in applications.

(5) The *delta "function" centered at* p, or δ_p, is the "limit" as $\epsilon \longrightarrow 0$ of the functions

$$\delta_p^\epsilon(t) = \epsilon^{-1}\big(H_\epsilon(t) - H_{p+\epsilon}(t)\big) = \begin{cases} \epsilon^{-1}, & \text{if } p \le t < p+\epsilon; \\ 0, & \text{otherwise.} \end{cases}$$

The key property of the delta function δ_p, for $p \ge 0$, is that

$$\int_0^\infty \delta_p(t) f(t)\, dt = f(p), \tag{4}$$

for every continuous function f. In fact, δ_p is not a function, but just a symbol. Instead, (eq4) should be interpreted as *the definition* of the entire expression $\int_0^\infty \delta_p(t) f(t)\, dt$. In this sense, δ_p is a *generalized function*; see PS3-Problem 16. It is the "limit" of the functions δ_p^ϵ in the sense that

$$\lim_{\epsilon \longrightarrow 0} \int_0^\infty \delta_p^\epsilon(t) f(t)\, dt = \int_0^\infty \delta_p(t) f(t)\, dt, \tag{5}$$

for every continuous function f. Equations (eq4) and (eq5) imply that

$$\boxed{\{\mathcal{L}\delta_p\}(s) = e^{-ps}} \qquad \text{and} \qquad \boxed{\lim_{\epsilon \longrightarrow 0}\{\mathcal{L}\delta_p^\epsilon\}(s) = \{\mathcal{L}\delta_p\}(s)}$$

(6) The *unit impulse response function*, or *Green's function*, for linear operator $y \longrightarrow y'' + py' + qy$ is the solution $e = e(t)$ to the initial value problem

$$y'' + py' + qy = \delta_0, \qquad y(0) = 0, \quad y'(0) = 0.$$

Since δ_0 is not a function, this ODE does not quite make sense. However, its Laplace Transform does:

$$s^2E + psE + qE = 1, \quad \text{if} \quad p, q = const,$$

where E is the Laplace Transform of e. In other words, $e = e(t)$ is *defined* by

$$\boxed{\mathcal{L}e = E = \frac{1}{s^2+ps+q} \qquad \text{if} \qquad p, q = const}$$

In PS3-Problem 16, you obtain an explicit expression for $e = e(t)$ in terms of p and q, or equivalently the eigenvalues λ_1 or λ_2 of the characteristic polynomial for the linear operator $y \longrightarrow y'' + py' + qy$.

(7) The *convolution* $f*g$ of two functions f and g is the function defined by

$$\{f*g\}(t) = \int_0^t f(u)g(t-u)\,du.$$

Among the important properties of the convolution product are

$$\boxed{f*(\alpha g+\beta h) = \alpha f*g + \beta f*h, \quad f*g = g*f, \quad (f*g)*h = f*(g*h);}$$

$$\boxed{f*\delta_0 = f, \quad \mathcal{L}(f*g) = \mathcal{L}f\cdot\mathcal{L}g, \quad \{f*g\}'(t) = \{f'*g\}(t) + f(0)g(t)}$$

Note that $f*1$ is *not* f. The convolution product appears in applications, even without ODEs. Perhaps, the most important relation to ODEs is that

$$\boxed{y''+py'+qy = f, \quad y(0)=0, \quad y'(0)=0 \qquad \Longrightarrow \qquad y = e*f} \tag{6}$$

where $e=e(t)$ is the unit impulse response function for linear operator $y \longrightarrow y''+py'+qy$. The conclusion in (eq6) holds for linear ODEs with constant coefficients of any order, with the corresponding number of initial conditions. The solution to the general initial value problem is given by

$$\boxed{y''+py'+qy = f, \quad y(0)=y_0, \quad y'(0)=y_1 \qquad \Longrightarrow \qquad y = e*f + y_0 e' + (y_1+py_0)e}$$

Review of Partial Functions

(1) The *method of partial fractions* is used to split a fraction P/Q, of polynomials P and Q, into a sum of a polynomial P_0 and fractions P_i/Q_i, where Q_i is a polynomial factor of Q and P_i is a polynomial of degree less than the degree of Q_i:

$$\frac{P}{Q} = P_0 + \frac{P_1}{Q_1} + \ldots + \frac{P_n}{Q_n}. \tag{7}$$

The method of partial fractions is useful in evaluating integrals, in finding Inverse Laplace Transforms, and thus in solving ODEs. For these purposes, we need all the denominators Q_i to be a power of either a linear polynomial, $Q_i = (t+b_i)^{p_i}$, or of a quadratic polynomial of the form $Q_i = ((t+a_i)^2+b_i^2)^{p_i}$. If the degree of Q is at most 3, this is always possible to achieve. Otherwise, it may be not be possible to split P/Q in this way, unless we allow the coefficients to be complex. If the coefficients of P_i and Q_i are allowed to be complex, every polynomial fraction P/Q can be split as in (eq7), with each Q_i of the form $(t+a_i)^{p_i}$. For the purposes of evaluating integrals and of finding Inverse Laplace Transforms, a decomposition with complex coefficients is perfectly acceptable.

(2) There are three steps in obtaining a partial fraction decomposition for P/Q:
Step 1: use the polynomial division with reminder to find P_0 and R such that $P = P_0Q+R$ and the degree of R is less than the degree of Q. This is usually the easiest step. For example,

$$\begin{aligned} t^5 &= (t-4)\cdot(t^4+4t^3+7t^2+6t+2) + (9t^3+22t^2+22t+8) \\ \Longrightarrow \quad \frac{t^5}{t^4+4t^3+7t^2+6t+2} &= t-4 + \frac{9t^3+22t^2+22t+8}{t^4+4t^3+7t^2+6t+2}. \end{aligned}$$

Step 2: decompose the polynomial Q as much as possible. This is usually by far the hardest step. One way of trying to find a linear factor of Q is by trying to guess an integer root of Q. In order to do so, plug in all positive and negative integer factors of the constant term of Q into Q. For example, if

$$Q = t^4+4t^3+7t^2+6t+2,$$

try $t = \pm 1, \pm 2$. Since $Q(-1)=0$, $t+1$ divides Q. Using the polynomial division, we find that

$$Q_1 = Q/(t+1) = t^3 + 3t^2 + 4t + 2.$$

We then apply the same procedure to Q_1. Since $Q_1(-1)=0$, $t+1$ divides Q_1, and

$$Q_2 = Q_1/(t+1) = t^2 + 2t + 2 = (t+1)^2 + 1 \implies Q = (t+1)^2\big((t+1)^2+1\big).$$

We have now decomposed the polynomial Q into factors. There will be three denominators in this case $Q_1=(t+1)$, $Q_2=(t+1)^2$, and $Q_3=((t+1)^2+1)$
Step 3: find $P_1, \dots, P_n$ such that the degree of P_i is less than the degree of Q_i and

$$\frac{R}{Q} = \frac{P_1}{Q_1} + \dots + \frac{P_n}{Q_n}.$$

In order to do so, we solve for the coefficients of $P_1, \dots, P_n$ by putting the right-hand side above under a common denominator, which must be Q, and comparing the coefficients in front of $t^{k-1}, \dots, t, 1$ in the resulting numerator with the coefficients in R. The number of coefficients and the number of resulting equations should be the degree of Q. For example, we would write

$$\begin{aligned}\frac{9t^3+22t^2+22t+8}{t^4+4t^3+7t^2+6t+2} &= \frac{A}{t+1} + \frac{B}{(t+1)^2} + \frac{Cx+D}{t^2+2t+2} \\ &= \frac{(A+C)t^3 + (3A+B+2C+D)t^2 + (4A+2B+C+2D)t + (2A+2B+D)}{t^4 + 4t^3 + 7t^2 + 6t + 2}\end{aligned}$$

Equating the coefficients on the two sides, we obtain

$$\begin{cases} A + C = 9 \\ 3A+B+2C+D = 22 \\ 4A+2B+C+2D = 22 \\ 2A+2B+D = 8 \end{cases} \iff \begin{cases} A = 5 \\ B = -1 \\ C = 4 \\ D = 0 \end{cases}$$

Putting the three steps together, we conclude that

$$\frac{t^5}{t^4+4t^3+7t^2+6t+2} = t-4 + \frac{5}{t+1} - \frac{1}{(t+1)^2} + \frac{4t}{(t+1)^2+1}.$$

Note: Given that Step 2 of the method of partial fractions may get very complicated if the degree of Q is high, it may appear that the Laplace Transform approach to initial value problems is nearly always slower than that of Section 4. However, in the Laplace Transform approach, the polynomial Q will be the characteristic polynomial for the given ODE. In order to use the approach of Section 4, one would need to find its roots. This procedure is essentially equivalent to Step 2 of the method of partial fractions for Q. Step 3 of this method is analogous to the last step of the Section 4 approach, which involves solving a system of k equations with k unknowns for the constants $C_1, \dots, C_k$, if k is the order of the ODE and the degree of Q.