Math53: Ordinary Differential Equations
Winter 2004

Unit 2 Summary
Second- and Higher-Order Ordinary Differential Equations

Finding Solutions of Special Second-Order ODEs

(1) We first recall that the general solution of a first-order *linear homogeneous* equation with a constant coefficient is given by

$$\boxed{y' + p \cdot y = 0, \quad p = const, \quad y = y(t), \qquad \Longrightarrow \qquad y(t) = Ce^{\lambda t}}$$

where $\lambda = -p$ is the root of the corresponding characteristic equation

$$\lambda + p = 0.$$

An *inhomogeneous* linear equation can be solved by observing that

$$\boxed{\left(e^{-\lambda t} y\right)' = e^{-\lambda t}\left(y' + py\right)} \quad \Longrightarrow \quad \boxed{y' + p \cdot y = f(t), \;\; y = y(t), \quad \Longrightarrow \quad \left(e^{-\lambda t} y\right)' = e^{-\lambda t} f(t)} \tag{1}$$

The last equation is solved by integration.
Caution: The above discussion applies only to first-order linear equations *with a constant coefficient*, i.e. p. If p is not constant, the integrating factor is more complicated.

(2) The general solution of a *second-order linear homogeneous* equation with constant coefficients

$$y'' + py' + qy = 0, \quad p, q = const, \quad y = y(t), \tag{2}$$

is determined by the two roots, λ_1 and λ_2, of the associated *characteristic equation*

$$\lambda^2 + p\lambda + q = 0. \tag{3}$$

The general solution can be of two or three different forms, depending on whether one is looking for complex or real solutions:

$$\boxed{y'' + py' + qy = 0, \quad y = y(t), \quad \Longrightarrow \quad y(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t} \qquad \text{if} \quad \lambda_1 = \lambda_2 \quad \Longleftrightarrow \quad p^2 = 4q}$$

$$\boxed{y'' + py' + qy = 0, \quad y = y(t), \quad \Longrightarrow \quad y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \qquad \text{if} \quad \lambda_1 \neq \lambda_2 \quad \Longleftrightarrow \quad p^2 \neq 4q}$$

If the coefficients p and q are real, the roots λ_1 and λ_2 of (eq3) are either real or complex conjugates of each other. In the latter case, Euler's formula,

$$e^{i\theta} = \cos\theta + i \sin\theta,$$

can be used to extract the general real solution from the general complex solution:

$$\boxed{y'' + py' + qy = 0 \quad \Longrightarrow \quad y(t) = C_1 e^{at} \cos bt + C_2 e^{at} \sin bt, \quad a = \tfrac{1}{2}p, \;\; b = \tfrac{1}{2}\sqrt{4q - p^2}, \quad \text{if } p^2 < 4q}$$

The numbers a and b are related to the eigenvalues λ_1 and λ_2 by $\lambda_1, \lambda_2 = a \pm ib$.

(3) We have discussed two very different approaches to solving second-order linear *inhomogeneous* equations with constant coefficients:

$$y'' + py' + qy = f(t), \quad p, q = const, \quad y = y(t). \tag{4}$$

The first approach, described in class and in PS2-Problem 1, can be viewed as the integrating-factor method for second-order linear ODEs with constant coefficients. The analogue of the first box in (eq1) for second-order equations is

$$\boxed{\big(e^{(\lambda_2-\lambda_1)t}(e^{-\lambda_2 t}y)'\big)' = e^{-\lambda_1 t}\big(y'' + py' + qy\big)} \tag{5}$$

if λ_1 and λ_2 are the two roots of the characteristic equation (eq3) associated to the linear equation (eq4). Thus, every ODE (eq4) can be solved by multiplying both sides by $e^{-\lambda_1 t}$ and using (eq5) to compress LHS:

$$\boxed{y''+py'+qy=f(t), \quad p, q = const, \; y=y(t), \quad \Longrightarrow \quad \big(e^{(\lambda_2-\lambda_1)t}(e^{-\lambda_2 t}y)'\big)' = e^{-\lambda_1 t}f(t)}$$

The last equation is solved by integrating twice. If one is looking only for a *particular* solution of (eq4), the two constants of integration can be dropped. If one is looking for the general solution of (eq4), it may in fact be simpler to find a particular solution y_p and then use (eq6) and the knowledge of the general solution y_h of the associated homogeneous equation (eq2), as described in (2) above, to form the general solution of (eq4).

(4) The second approach to finding the general solution of (eq4) is to find a particular solution y_p of (eq4) by using the *method of undetermined coefficients*, described in Section 4.5 of the textbook, and then form the general solution (eq4) using

$$\boxed{y''+py'+qy=f(t), \; y=y(t), \quad \Longrightarrow \quad y(t) = y_p(t) + y_h(t)} \tag{6}$$

where y_h is the general solution of the associated homogeneous equation (eq2). In order to find $y_p=y_p(t)$, one tries plugging into (eq4) functions of the same form as $f(t)$. For example,

if $f(t) =$	try $y_p(t) =$
e^{rt}	ae^{rt}
t^2	at^2+bt+c
$\cos\omega t$ or $\sin\omega t$	$a\cos\omega t+b\sin\omega t$

In this table, r and ω are known constants, while a, b, and c are the coefficients to be determined, by plugging $y_p = y_p(t)$ into (eq4). In the last case of this table, it may in fact be simpler to first *complexify* the ODE (eq4) via Euler's formula,

$$y'' + py' + qy = \cos\omega t \quad \text{or} \quad y'' + py' + qy = \sin\omega t \qquad \Longrightarrow \qquad z'' + pz' + qz = e^{i\omega t},$$

then find a particular complex solution $z_p=z_p(t)$ of the last equation, and take y_p to be its real or imaginary part. This is certainly the easier approach if you use the integrating-factor method to find y_p, as described in (3) above, since you will end up with a much simpler integral, e.g.

$$\int e^{rt}\cos\omega t\,dt \qquad \longrightarrow \qquad \int e^{rt}e^{i\omega t}\,dt = \int e^{(r+i\omega)t}\,dt.$$

In some cases, the trial solution suggested by the above table will be a solution of the associated homogeneous equation (eq2). If so, the suggested trial solution should be multiplied by t. If the resulting function is still a solution of the associated homogeneous equation, then it should be multiplied by t yet again. For example, suppose we would like to find a particular solution of the ODE

$$y'' + 10y' + 25y = 2e^{-5t}, \qquad y = y(t). \tag{7}$$

Since $y = e^{-5t}$ is a solution of the associated homogeneous equation,

$$y'' + 10y' + 25y = 0, \tag{8}$$

$y_p(t) = ae^{-5t}$ cannot be a solution of (eq7) for any constant a. Neither can $y_p(t) = ate^{-5t}$, since it is also a solution of (eq8). Thus, we must try $y_p(t) = at^2e^{-5t}$, and indeed this is a solution of (eq7) for $a = 1$. *Please check all these statements!* Finally, if $f(t)$ is a product of the expressions in the left column of the table, the trial form for $y_p(t)$ will be the corresponding product of the terms in the right column of the table. If $f(t)$ is a sum of various terms, it is useful to observe that

$$\boxed{y''+py'+qy=\alpha f(t)+\beta g(t), \quad y_f''+py_f'+qy_f=f(t), \quad y_g''+py_g'+qy_g=g(t), \quad \Longrightarrow \quad y=\alpha y_f+\beta y_g}$$

for any *constants* α and β.
Caution: Note that the coefficients, p and q, in front of y' and y above are *not* added together.

(5) The method of undetermined coefficients can also be used to find particular solutions of more general linear equations

$$y'' + py' + qy = f(t), \quad p = p(t), \quad q = q(t), \quad y = y(t), \tag{9}$$

for a few special functions p and q. On the other hand, the *variation-of-parameters* method can be used to find a particular solution of any ODE (eq9), provided two *linearly independent* solutions y_1 and y_2 of the associated homogeneous equation

$$y'' + py' + qy = 0, \qquad y = y(t), \tag{10}$$

are known. In this case, we look for a solution $y_p = v_1y_1 + v_2y_2$ of (eq9), for some functions $v_1 = v_1(t)$ and $v_2 = v_2(t)$. Since y_1 and y_2 solve (eq10), y_p solves (eq9) if and only if

$$\big(v_1''y_1 + 2y_1'v_1' + py_1v_1'\big) + \big(v_2''y_2 + 2y_2'v_2' + py_2v_2'\big) = f(t). \tag{11}$$

We simplify this equation by imposing the condition

$$y_1v_1' + y_2v_2' = 0 \qquad \Longrightarrow \qquad y_1'v_1' + y_2'v_2' = f(t).$$

In other words, v_1 and v_2 solve (eq11) if

$$\begin{cases} y_1v_1' + y_2v_2' = 0; \\ y_1'v_1' + y_2'v_2' = f(t). \end{cases} \tag{12}$$

This system of equations can be solved for v_1' and v_2' if the determinant of the coefficient matrix

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1y_2' - y_1'y_2$$

is not zero, as a function of t. This determinant is the *Wronskian* of y_1 and y_2 and thus is never zero, since the solutions y_1 and y_2 are assumed to be independent. Thus, the system (eq12) can be solved for $v_1' = v_1'(t)$ and $v_2' = v_2'(t)$ algebraically. Integrating each of the two expressions, we obtain functions $v_1 = v_1(t)$ and $v_2 = v_2(t)$ that solve (eq11). Thus, $y_p = v_1 y_1 + v_2 y_2$ is a solution (eq9). A somewhat simpler version of this method can be used to find a second independent solution y_2 of (eq10) if one nonzero solution y_1 of (eq10) is known, as illustrated in Exercise 4.1:14.

(6) Analogous approaches can be used to study higher-order linear equations. For example, the general solution of the third-order linear homogeneous ODE with constant coefficients

$$y''' + py'' + qy' + ry = 0, \qquad p, q, r = const, \qquad y = y(t), \tag{13}$$

is described by

$$\boxed{y''' + py'' + qy' + ry = 0 \quad \Longrightarrow \quad y(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t} + C_3 t^2 e^{\lambda_1 t} \qquad \text{if} \quad \lambda_1 = \lambda_2 = \lambda_3}$$

$$\boxed{y''' + py'' + qy' + ry = 0 \quad \Longrightarrow \quad y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + C_3 t e^{\lambda_2 t} \qquad \text{if} \quad \lambda_1 \neq \lambda_2 = \lambda_3}$$

$$\boxed{y''' + py'' + qy' + ry = 0 \quad \Longrightarrow \quad y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + C_3 e^{\lambda_3 t} \qquad \text{if} \quad \lambda_1 \neq \lambda_2 \neq \lambda_3\ , \lambda_1 \neq \lambda_3}$$

Here λ_1, λ_2, and λ_3 are the roots of the characteristic equation for (eq13):

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0.$$

Terminology and Qualitative Descriptions

(1) A *second-order linear ODE* is a relation of the form

$$y'' + py' + qy = f, \quad p = p(t), \quad q = q(t), \quad f = f(t), \quad y = y(t). \tag{14}$$

An *initial value problem* for (eq14) is a set of conditions

$$y'' + py' + qy = f, \quad y = y(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_1. \tag{15}$$

As is the case for first-order linear ODEs, every IVP (eq15) has a unique solution, provided the functions p, q, and f are continuous near t_0. Furthermore, the interval of the existence of the solution to (eq15) is the largest interval on which p, q, and f are defined.
Caution: For second-order equations, IVPs must include an initial requirement on the derivative, e.g. $y'(t_0) = y_1$. Thus, the graphs of solutions of second-order ODEs intersect, but they cannot be tangent to each other.

(2) A *linear combination of two functions* $y_1 = y_1(t)$ and $y_2 = y_2(t)$, defined on the same interval, is a function of the form $\alpha y_1 + \beta y_2$ for any two constants α and β. Two functions $y_1 = y_1(t)$ and $y_2 = y_2(t)$ are called *linearly independent* if neither one of them is a constant multiple of the other. This is the same as saying that no nontrivial linear combination, i.e. $\alpha y_1 + \beta y_2$ with α and β not both zero, is identically zero. The *Wronskian* of y_1 and y_2 is the function

$$W_{y_1,y_2} = W_{y_1,y_2}(t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = y_1(t) y_2'(t) - y_1'(t) y_2(t).$$

If $W_{y_1,y_2}(t_0) \neq 0$ for some t_0, the functions y_1 and y_2 are linearly independent, but the converse need not to be true.

(3) A second-order linear *homogeneous* ODE is a relation of the form

$$y'' + py' + qy = 0, \quad p = p(t), \quad q = q(t), \qquad y = y(t). \tag{16}$$

If y_1 and y_2 are solutions of (eq16), so is any linear combination $C_1y_1+C_2y_2$ of y_1 and y_2. Furthermore, the Wronskian W_{y_1,y_2} of y_1 and y_2 is either identically zero or never zero. If the former is the case, the two solutions y_1 and y_2 of (eq16) are linearly dependent. On the other hand, the ODE (eq16) always has a pair (y_1, y_2) of linearly independent solutions. If y_1 and y_2 are linearly independent solutions of (eq16), the general solution of (eq16) is given by all linear combinations of y_1 and y_2:

$$\boxed{y'' + py' + qy = 0, \quad y = y(t) \qquad \Longrightarrow \qquad y(t) = C_1y_1(t) + C_2y_2(t)}$$

For this reason, (y_1, y_2) is called a *fundamental set of solutions* for (eq16), or a *basis* for the vector space of solutions of (eq16).

(4) The general solution of any linear equation (eq14) has the form $y = y_h + y_p$, where y_p is a fixed *particular* solution of (eq14) and y_h is the general solution of the corresponding homogeneous equation, i.e. (eq16) with the same $p=p(t)$ and $q=q(t)$ as in (eq14). In order to check this claim, you need to show two things. The first one is that if y_p is a solution of (eq14) and y_h is a solution of (eq16), then y_h+y_p is a solution of (eq14). The second statement is that if y_p and y are solutions of (eq14), then $y-y_p$ is a solution of (eq16). We also note if y_f is a particular solutions of (eq14) and y_g is a particular solutions of (eq14) with f replaced by $g=g(t)$, then $y_p=\alpha y_f+\beta y_g$ is a particular solution of

$$y'' + py' + qy = \alpha f + \beta g, \qquad y = y(t),$$

for all constants α and β. *Please check this!* The above properties hold for linear ODEs of any order.

Additions to Unit 1 Summary

(1) The *Existence and Uniqueness Theorem* for first-order ODEs *guarantees* a solution to IVP

$$y' = Q(t, y), \qquad y(t_0) = y_0,$$

if the function Q is continuous near (t_0, y_0). If in addition $\partial Q/\partial y$ is defined and continuous near (t_0, y_0), this theorem *guarantees* that there is only *one* solution to this IVP near t_0. Thus, this theorem's applicability depends on the ODE *and* the initial condition. For example, IVP

$$y' = \sqrt{|y|}/|t|, \qquad y(0) = 0,$$

is not *guaranteed* to have a solution at all, because $\sqrt{|y|}/|t|$ is not continuous near $(0, 0)$, but it *may* have a solution or even lots of solutions. On the other hand, IVP

$$y' = \sqrt{|y|}/|t|, \qquad y(2) = 0,$$

is guaranteed to have a solution, because $\sqrt{|y|}/|t|$ continuous near $(2, 0)$. However, this IVP *may* have many solutions, because $\partial(\sqrt{|y|}/|t|)/\partial y$ is not continuous near $(2, 0)$.

$f(y)$ y y

3 $y=3$

y t

-3 -1 3

-1 $y=-1$

-3 $y=-3$

Figure 1: Plots for ODE $\quad y' = f(y) = (y+3)^2(y+1)(y-3)$

(2) An *autonomous* first-order ODE is an ODE of the form

$$y' = f(y), \qquad y = y(t). \tag{17}$$

It is possible to say a lot about solutions of this equation without solving it. First of all, since RHS of (eq17) does not involve t, the direction field of (eq17) does not change under horizontal shifts. Thus, a *horizontal* shift of a solution curve is again a solution curve. Furthermore, if y^* is a real number such that $f(y^*)=0$, the constant function $y(t)=y^*$ is a solution of (eq17). Such a number y^* is an *equilibrium point* for (eq17) and $y(t)=y^*$ is an *equilibrium solution* of (eq17). The corresponding solution curve is the horizontal line $y=y^*$ in ty-plane. For example, the equilibrium solutions of

$$y' = (y+3)^2(y+1)(y-3) \tag{18}$$

are $y=-3$, $y=-1$, and $y=3$. The graphs of these solutions are the horizontal lines $y=-3$, $y=-1$, and $y=3$, shown in the third plot above. These lines partition the ty-plane into horizontal bands $y_1^* < y < y_2^*$. Since solution curves of the ODE (eq17) do not intersect, no solution curve can cross the graphs of the equilibrium solutions. For example, if $y=y(t)$ is a solution of (eq18) such that $y(t_0) \in (-1,-3)$ for some t_0, then $y(t) \in (-1,-3)$ for all t. In each band, the function $f(y)$ does not change sign. Thus, in each single band, all solution curves of (eq17) either descend or ascend. Furthermore, each solution curve must approach either an equilibrium solution curve or $\pm\infty$ as $t \longrightarrow \pm\infty$. The best way to tell whether the solution curves in the given band descend or ascend is by sketching the graph of the function $f=f(y)$, as done for the ODE (eq18) in the first plot above. Note that the y-intercepts of this graph correspond to the equilibrium solutions of the ODE. The equilibrium point y^* and the equilibrium solution $y=y^*$ are *stable* if the solution curves in the two bands surrounding the horizontal line $y=y^*$ approach $y=y^*$ as t approaches ∞. Otherwise, they are *unstable*. The (*first-*) *derivative test* can often, but not always, be used to determine the type of an equilibrium solution. However, it is usually simpler to first draw the *phase line* for the ODE by looking at the graph of $f=f(y)$, as done in the middle plot above for the ODE (eq18). The phase line shows the equilibrium points for the ODE (eq17), or the y-intercepts of the graph of f. It also indicates, using arrows, whether the solution curves in each band cut out by the horizontal equilibrium-solution lines ascend and descend. The arrow corresponding to a segment of the phase line points up (down) if $f(y)$ is positive (negative) on the this segment. An equilibrium point y^* of (eq17) is *stable* if on the phase line *both* arrows surrounding y^* point toward y^* and is *unstable* otherwise. For example, $y=-1$ is a stable solution of (eq18), while $y=-3$ and $y=3$ are not.