Math53: Ordinary Differential Equations
Winter 2004

Problem Set 6 Solutions

PS6-Problem 1 (30pts)

(a; **6pts**) *Sketch the graph, in the $(y, f(y))$-plane, of the function*

$$f(y) = (y+3)^3(y-1)^2(y-3).$$

Find the equilibrium solutions of and sketch the phase line, i.e. the y-line, for the one-dimensional autonomous ODE:

$$y' = (y+3)^3(y-1)^2(y-3).$$

Determine whether each equilibrium point is stable or unstable.
The graph of f is shown in the first sketch of Figure 1. The y-intercepts are -3, 1, and 3. Thus, the equilibrium, or constant, solutions of the ODE are $y=-3$, $y=1$, and $y=3$. They are indicated with dots on the phase line for the ODE; see the middle sketch in Figure 1. If $y=y(t)$ is a solution of the ODE, $y(t)$ increases to the closest equilibrium solution above if $f(y)=y'(t)$ is positive and decreases to the closest equilibrium solution below if $f(y)=y'(t)$ is negative. This information is encoded by arrows on the phase-line. If y_i is an equilibrium point for the ODE, it is asymptotically stable if solution curves on both sides of y_i approach y_i, i.e. the arrows on both sides point toward y_i. It is unstable if solutions on one of the two sides move away, i.e. if one of the two arrows points away from y_i. Thus, $y=-3$ is an asymptotically stable equilibrium point, while $y=1$ and $y=3$ are unstable equilibrium points. The ty-plane sketch of solution curves, i.e. graphs of solutions of the ODE, is shown in the last plot of Figure 1.

For (b)-(d), suppose that $\mathbf{y}_i$ is an equilibrium point for the system of ODEs

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}), \qquad \mathbf{y} = \mathbf{y}(t), \tag{1}$$

and V is a smooth function defined near $\mathbf{y}_i$ such that $V(\mathbf{y}_i)=0$.

(b; **4pts**) *If $V(\mathbf{x})>0$ and $\vec{\nabla}V|_{\mathbf{x}} \cdot \mathbf{f}(x) \le 0$ for all $\mathbf{x}\neq\mathbf{y}_i$ near $\mathbf{y}_i$, show that $\mathbf{y}_i$ is stable.*
We need to show that solution curves that start sufficiently close to $\mathbf{y}_i$ stay arbitrary close to $\mathbf{y}_i$. More precisely, for any $\epsilon>0$, we need to find $\delta>0$ such that if $\mathbf{y}=\mathbf{y}(t)$ is a solution of (1) and $|\mathbf{y}(0)-\mathbf{y}_i|<\delta$, then $|\mathbf{y}(t)-\mathbf{y}_i|<\epsilon$ for all $t\ge 0$; see Figure 1 on p512. It is sufficient to find such a $\delta=\delta(\epsilon)$ for very small values of $\epsilon>0$. Thus, we assume that $\epsilon>0$ is such that $V(\mathbf{x})>0$ and $\vec{\nabla}V|_{\mathbf{x}}\cdot\mathbf{f}(x)\le 0$ for all $\mathbf{x}\neq\mathbf{y}_i$ with $|\mathbf{x}-\mathbf{y}_i|\le\epsilon$. Thus, we can choose $m_\epsilon>0$ such that $V(\mathbf{x})>m_\epsilon$ for all $\mathbf{x}$ with $|\mathbf{x}-\mathbf{y}_i|=\epsilon$. Since $V(0)=0$, we can also choose $\delta\in(0,\epsilon)$ such that $V(\mathbf{x})<m_\epsilon$ for all $\mathbf{x}$ with $|\mathbf{x}-\mathbf{y}_i|\le\delta$. Now suppose that $\mathbf{y}=\mathbf{y}(t)$ is a solution of (1) and $|\mathbf{y}(0)-\mathbf{y}_i|<\delta$; then $V(\mathbf{y}(0))<m_\epsilon$. If $|\mathbf{y}(t)-\mathbf{y}_i|>\epsilon$ for some $t>0$, then $|\mathbf{y}(s)-\mathbf{y}_i|=\epsilon$ for some $s>0$, since $|\mathbf{y}(0)-\mathbf{y}_i|<\epsilon$. On the other hand, by the multivariable chain rule and our assumptions on V:

$$\frac{d}{dt}V(\mathbf{y}(t)) = \vec{\nabla}V|_{\mathbf{y}(t)}\cdot\mathbf{y}'(t) = \vec{\nabla}V|_{\mathbf{y}(t)}\cdot\mathbf{f}(\mathbf{y}(t)) \le 0 \qquad \text{if} \qquad |\mathbf{y}(t)-\mathbf{y}_i|\le\epsilon.$$

Thus, if $s>0$ is the smallest value of s such that $|\mathbf{y}(s)-\mathbf{y}_i|=\epsilon$

$$m_\epsilon < V(\mathbf{y}(s)) \le V(\mathbf{y}(0)) < m_\epsilon.$$

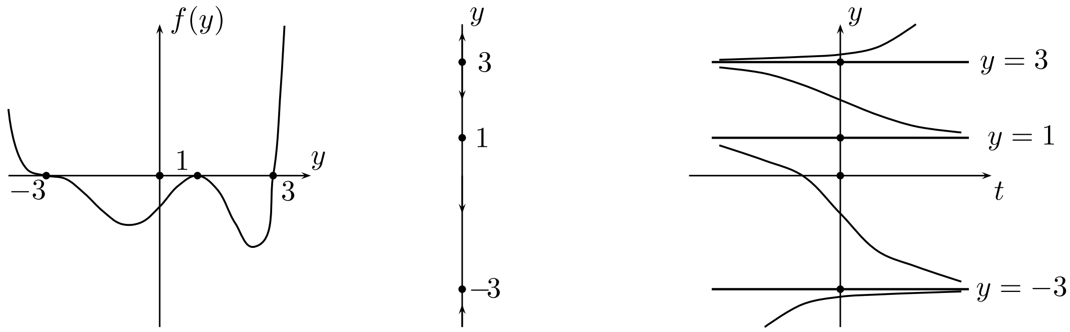


Figure 1: Plots for ODE $\quad y' = f(y) = (y+3)^3(y-1)^2(y-3)$

In other words, $m_\epsilon < m_\epsilon$, which is a contradiction. Thus, $|\mathbf{y}(t) - \mathbf{y}_i| < \epsilon$ for all t, as needed.

(c; **4pts**) *If $V(\mathbf{x}) > 0$ and $\vec{\nabla} V|_{\mathbf{x}} \cdot \mathbf{f}(x) < 0$ for all $\mathbf{x} \neq \mathbf{y}_i$ near $\mathbf{y}_i$, show that $\mathbf{y}_i$ is asymptotically stable.*
We need to show that solution curves that start sufficiently close to $\mathbf{y}_i$ approach $\mathbf{y}_i$ as $t \longrightarrow \infty$. More precisely, we need to find $\delta > 0$ with the following property. If $\mathbf{y} = \mathbf{y}(t)$ is a solution of (1) and $|\mathbf{y}(0) - \mathbf{y}_i| < \delta$, then $\lim_{t \longrightarrow \infty} \mathbf{y}(t) = \mathbf{y}_i$. We first pick $\epsilon > 0$ such that $V(\mathbf{x}) > 0$ and $\vec{\nabla} V|_{\mathbf{x}} \cdot \mathbf{f}(x) < 0$ for all $\mathbf{x} \neq \mathbf{y}_i$ with $|\mathbf{x} - \mathbf{y}_i| \leq \epsilon$. Similarly to (b) above, we can then choose $m_\epsilon > 0$ such that $V(\mathbf{x}) > m_\epsilon$ for all $\mathbf{x}$ with $|\mathbf{x} - \mathbf{y}_i| = \epsilon$ and $\delta \in (0, \epsilon)$ such that $V(\mathbf{x}) < m_\epsilon$ for all $\mathbf{x}$ with $|\mathbf{x} - \mathbf{y}_i| \leq \delta$. Now suppose that $\mathbf{y} = \mathbf{y}(t)$ is a solution of (1), different from $\mathbf{y}(t) = \mathbf{y}_i$, and $|\mathbf{y}(0) - \mathbf{y}_i| < \delta$. By the multivariable chain rule and our assumptions on V:

$$\frac{d}{dt} V(\mathbf{y}(t)) = \vec{\nabla} V|_{\mathbf{y}(t)} \cdot \mathbf{y}'(t) = \vec{\nabla} V|_{\mathbf{y}(t)} \cdot \mathbf{f}(\mathbf{y}(t)) < 0 \qquad \text{if} \qquad |\mathbf{y}(t) - \mathbf{y}_i| \leq \epsilon.$$

Since by the same argument as in (b) above $|\mathbf{y}(t) - \mathbf{y}_i| < \epsilon$ for all $t > 0$, it follows that

$$\frac{d}{dt} V(\mathbf{y}(t)) < 0 \qquad \text{for all} \qquad t > 0.$$

Since $V(\mathbf{y}(t)) \geq 0$ for $t \geq 0$ by our assumptions on V, we conclude that

$$\vec{\nabla} V|_{\mathbf{y}(t)} \cdot \mathbf{f}(\mathbf{y}(t)) = \frac{d}{dt} V(\mathbf{y}(t)) \longrightarrow 0 \qquad \text{as} \qquad t \longrightarrow \infty.$$

Since $\vec{\nabla} V|_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \neq \mathbf{y}_i$ with $|\mathbf{x} - \mathbf{y}_i| \leq \epsilon$, it follows that $\mathbf{y}(t) \longrightarrow \mathbf{y}_i$ as $t \longrightarrow \infty$.

(d; **4pts**) *If $\vec{\nabla} V|_{\mathbf{x}} \cdot \mathbf{f}(x) > 0$ for all $\mathbf{x} \neq \mathbf{y}_i$ near $\mathbf{y}_i$ and there exists a sequence $\mathbf{x}_k \longrightarrow \mathbf{y}_i$ such that $V(\mathbf{x}_k) > 0$ for all k, show that $\mathbf{y}_i$ is unstable.*
We need to show that $\mathbf{y}_i$ is not a stable equilibrium, i.e. the stability condition of (b) is not satisfied. In other words, we need to find $\epsilon > 0$ with the following property. For every $\delta > 0$, there exists a solution $\mathbf{y} = \mathbf{y}(t)$ of (1) such that $|\mathbf{y}(0) - \mathbf{y}_i| < \delta$ and $|\mathbf{y}(t) - \mathbf{y}_i| > \epsilon$ for some $t > 0$. Let $\epsilon > 0$ be such that $\vec{\nabla} V|_{\mathbf{x}} \cdot \mathbf{f}(x) > 0$ for all $\mathbf{x} \neq \mathbf{y}_i$ with $|\mathbf{x} - \mathbf{y}_i| \leq \epsilon$. We can then choose $M_\epsilon > 0$ such that $V(\mathbf{x}) < M_\epsilon$ for all $\mathbf{x}$ with $|\mathbf{x} - \mathbf{y}_i| \leq \epsilon$. If $\delta \in (0, \epsilon)$, by our assumptions we can choose $\mathbf{x}_0$ such that $|\mathbf{x}_0 - \mathbf{y}_i| < \delta$ and $V(\mathbf{x}_0) > 0$. Let $\mathbf{y} = \mathbf{y}(t)$ be the solution of (1) such that $\mathbf{y}(0) = \mathbf{x}_0$. By the multivariable chain rule and our assumptions on V:

$$\frac{d}{dt} V(\mathbf{y}(t)) = \vec{\nabla} V|_{\mathbf{y}(t)} \cdot \mathbf{y}'(t) = \vec{\nabla} V|_{\mathbf{y}(t)} \cdot \mathbf{f}(\mathbf{y}(t)) > 0 \qquad \text{if} \qquad |\mathbf{y}(t) - \mathbf{y}_i| \leq \epsilon.$$

Thus, if $|\mathbf{y}(t)-\mathbf{y}_i|\le\epsilon$ for all $t>0$, $V(t)<M_\epsilon$ for all $t>0$ and

$$\vec{\nabla}V|_{\mathbf{y}(t)}\cdot\mathbf{f}(\mathbf{y}(t))=\frac{d}{dt}V(\mathbf{y}(t))\longrightarrow 0 \qquad \text{as} \qquad t\longrightarrow\infty.$$

Since $\vec{\nabla}V|_{\mathbf{x}}\cdot\mathbf{f}(\mathbf{x})\neq 0$ for all $\mathbf{x}\neq\mathbf{y}_i$ with $|\mathbf{x}-\mathbf{y}_i|\le\epsilon$, we conclude that $\mathbf{y}(t)\longrightarrow\mathbf{y}_i$ as $t\longrightarrow\infty$. However, this is impossible, since

$$V(\mathbf{y}_i)=0<V(\mathbf{x}_0)=V(\mathbf{y}(0))<V(\mathbf{y}(t)) \qquad \text{for all} \qquad t>0.$$

(e; **6pts**) *Find an appropriate function $V=V(y)$ for each of the three equilibrium points of the ODE in (a).*
Since $y=-3$ is asymptotically stable, we are looking for a smooth function $V=V(y)$ such that

$$V(-3)=0, \quad V(y)>0 \;\text{ and }\; V'(y)\cdot f(y)=V'(y)\cdot(y+3)^3(y-1)^2(y-3)>0 \;\text{ for all }\; y\neq 3 \text{ close to } -3.$$

The simplest function that satisfies the first two conditions is $V(y)=(y+3)^2$. It also satisfies the third condition. By (c), the existence of such a function V confirms that $y=-3$ is an asymptotically stable equilibrium point. Since $y=1$ is unstable, we next would like to find a smooth function $V=V(y)$ such that

$$V(1)=0, \qquad V'(y)\cdot f(y)=V'(y)\cdot(y+3)^3(y-1)^2(y-3)>0 \;\text{ for all }\; y\neq 1 \text{ close to } 1,$$

and $V(x_k)>0$ for a sequence $x_k\longrightarrow 1$. The simplest functions that satisfy the first and the last conditions are $y-1$ and $1-y$. Since $f(y)<0$ for all $y\neq 1$ near 1, the function $V(y)=1-y$ also satisfies the middle condition. By (d), the existence of such a function V confirms that $y=1$ is an unstable equilibrium point. Finally, $y=3$ is unstable, and we expect to be able to find a smooth function $V=V(y)$ such that

$$V(3)=0, \qquad V'(y)\cdot f(y)=V'(y)\cdot(y+3)^3(y-1)^2(y-3)>0 \;\text{ for all }\; y\neq 3 \text{ close to } 3,$$

and $V(x_k)>0$ for a sequence $x_k\longrightarrow 3$. Similarly to the above, the simplest functions that satisfy the first and the last conditions are $y-3$ and $3-y$. However, the derivatives of these functions do not change sign at 3, while $f(y)$ does. Thus, neither one can satisfy the middle condition above. The next simplest function that satisfies the first and the last conditions is $V(y)=(y-3)^2$. It satisfies the middle condition as well. By (d), the existence of such a function V confirms that $y=3$ is an unstable equilibrium point.

(f; **6pts**) *Determine whether the origin is an asymptotically stable, stable, or unstable equilibrium for the following systems of ODEs:*

$$\begin{cases} x'=-y+x^3 \\ y'=x+y^3 \end{cases} \qquad \text{and} \qquad \begin{cases} x'=-y-x^3 \\ y'=x-y^3 \end{cases}$$

Based on Note 1 in the statement of the problem and the similarity in the systems, we expect that the origin is a stable, perhaps even asymptotically stable, equilibrium point for one of the two systems, but not the other. We will know that one of the two systems is stable if we can find a smooth function $V=V(x,y)$ such that $V(0,0)=0$, and

$$V(x,y)>0 \;\text{ and }\; \vec{\nabla}V|_{(x,y)}\cdot\mathbf{f}(x,y)>0 \;\text{ for all }\; (x,y)\neq(0,0) \;\text{ near }\; (0,0),$$

where $\mathbf{f}=\mathbf{f}(x,y)$ is the vector field corresponding either to the first or the second system, i.e.

$$\mathbf{f}=\mathbf{f}_1(x,y)=\big(-y+x^3,x+y^3\big) \qquad \text{or} \qquad \mathbf{f}=\mathbf{f}_2(x,y)=\big(-y-x^3,x-y^3\big).$$

The simplest function that satisfies the first conditions is $V(x,y)=x^2+y^2$. For this function,

$$\vec{\nabla}V|_{(x,y)}=(2x,2y) \quad \Longrightarrow \quad \begin{array}{l}\vec{\nabla}V|_{(x,y)}\cdot\mathbf{f}_1(x,y)=2x(-y+x^3)+2y(x+y^3)=2(x^4+y^4)>0;\\ \vec{\nabla}V|_{(x,y)}\cdot\mathbf{f}_2(x,y)=2x(-y-x^3)+2y(x-y^3)=-2(x^4+y^4)<0.\end{array}$$

Thus, the origin is an asymptotically stable equilibrium point for the second system by (c) and is an unstable equilibrium point for the first system by (d).

Remarks: The conditions (b) and (c) on stability are if and only if conditions. In other words, a function V with the required properties must exist if the given equilibrium point $\mathbf{y}_i$ is stable or asymptotically stable. In the one-dimensional case such a function can be obtained just by looking at the graph of f near each equilibrium point. On the other hand, in two and more dimensions, i.e. when these two criteria are actually useful for determining stability, such functions V may be much harder to find. Furthermore, (d) is not a necessary conditions for instability. *Can you find a one-dimensional example such that y_i is unstable, but $V=V(y)$ as in (d) does not exist?*

Section 10.3: 2,16 (36pts)

10.3:2; 4pts: *Show that the x- and y-axes and each of the four quadrants are invariant sets for the system*

$$\begin{cases} x'=4x(1-x)-xy \\ y'=y(3-y)-xy \end{cases}$$

If $(x(t),y(t))$ lies on the x-axis, i.e. $y(t)=0$, for some t, then by the second equation $y'(t)=0$ and thus $y(t)$ does not change, i.e. $y(t)$ stays on the x-axis as t increases. Thus, the x-axis is an invariant set for the system. Similarly, if $x(t)=0$ for some t, $x'(t)=0$ by the first equation. Thus, the y-axis is also an invariant set for the system. It follows that the x- and y-axes are made up of solution curves for the system. Since no two solution curves of the system can cross, any curve that starts in a quadrant cannot cross either axis and thus must stay in the same quadrant. It follows that each of the four quadrants is an invariant set for the system.

10.3:16 (32pts) *Find the equilibrium points of the system*

$$\begin{cases} x'=1-x^2-y^2 \\ y'=x-y \end{cases}$$

and determine their type. Sketch the nullclines and indicate the flow directions across each nullcline. Sketch the phase-plane portrait of the system.
(i; **16pts**) The equilibrium points are the solutions (x,y) of the system

$$\begin{cases} x'=0 \\ y'=0 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} 1-x^2-y^2=0 \\ x-y=0 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} y=x \\ 2x^2=1 \end{cases}$$

Thus, the equilibrium points are $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. In order to determine their type, we compute the Jacobian of $\mathbf{f}$:

$$J\mathbf{f}(x,y) = \begin{pmatrix} -2x & -2y \\ 1 & -1 \end{pmatrix}.$$

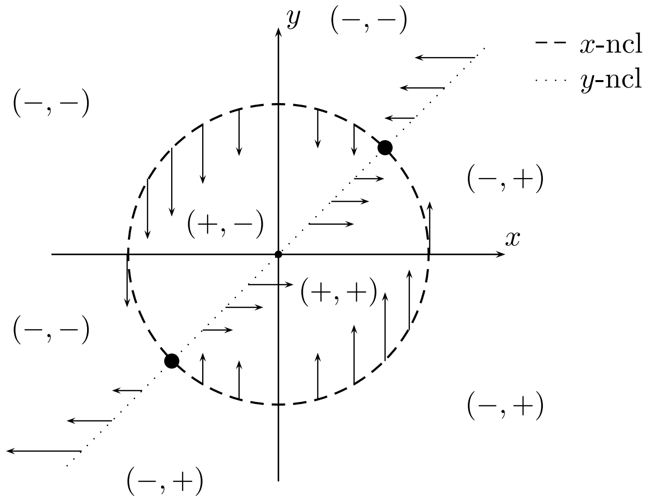
We then evaluate it at the equilibrium points and compute the corresponding eigenvalues:

$$\begin{aligned}
J\mathbf{f}(1/\sqrt{2}, 1/\sqrt{2}) = \begin{pmatrix} -\sqrt{2} & -\sqrt{2} \\ 1 & -1 \end{pmatrix} \quad &\Longrightarrow \quad \lambda^2 + (\sqrt{2}+1)\lambda + 2\sqrt{2} = 0 \\
&\Longrightarrow \quad \lambda_1, \lambda_2 = \frac{-(\sqrt{2}+1) \pm \sqrt{3-6\sqrt{2}}}{2} \\
J\mathbf{f}(-1/\sqrt{2}, -1/\sqrt{2}) = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 1 & -1 \end{pmatrix} \quad &\Longrightarrow \quad \lambda^2 - (\sqrt{2}-1)\lambda - 2\sqrt{2} = 0 \\
&\Longrightarrow \quad \lambda_1, \lambda_2 = \frac{(\sqrt{2}-1) \pm \sqrt{3+6\sqrt{2}}}{2}
\end{aligned}$$

Thus, $J\mathbf{f}(1/\sqrt{2}, 1/\sqrt{2})$ has complex eigenvalues with negative real part. It follows that $(1/\sqrt{2}, 1/\sqrt{2})$ is a spiral sink for the system; the direction of rotation is counterclockwise, since the bottom-left entry in the matrix is positive. Since $J\mathbf{f}(-1/\sqrt{2}, -1/\sqrt{2})$ has a real negative eigenvalue and a real positive eigenvalue, $(-1/\sqrt{2}, -1/\sqrt{2})$ is a saddle point for the system. In order to completely describe the local structure, we *should* also compute the half-line slopes, i.e. find the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for λ_1 and λ_2. However, given the multiple square roots involved in the expressions for λ_1 and λ_2, the two slopes will involve too complicated an expression to be very informative.

(ii; **8pts**) The x-nullcline is defined by $x' = 0$, or $x^2 + y^2 = 1$. The y-nullcline is defined by $y' = 0$, or $y = x$; see the first sketch in Figure 2. Their intersections are the two equilibrium points. Along the y-nullcline, i.e. the dotted line $y = x$, the y-component of the vector field $\mathbf{f}$ for the system is 0. Thus, the flow is horizontal across the line $y = x$. It goes left if $x' < 0$ and right if $x' > 0$. Inside of the unit circle, $x^2 + y^2 < 1$; thus, by the first equation in the system $x' > 0$ inside of the circle, and $x' < 0$ outside of the circle, as indicated on the first sketch in Figure 2. Along the x-nullcline, i.e. the dashed circle $x^2+y^2=1$, the x-component of the vector field $\mathbf{f}$ is 0. Thus, the flow is vertical across this circle. Above the line $y = x$, $y > x$ and thus $y' < 0$ by the second equation in the system, i.e. the flow is downward. Similarly, below the line $y = x$, $y < x$ and thus $y' > 0$, i.e. the flow is upward, as indicated on the sketch. From this we see that in the region outside of the circle and below the line, the flow is up and to the left, and we indicate this with $(-, +)$ on the first sketch and arrows in the second sketch in Figure 2. Every time we cross, the x-nullcline, the x-sign will change; every time we cross the y-nullcline, the y-sign will change. In this way, we label every region in plane cut out by the nullclines with a pair $(\pm, \pm)$ to indicate the flow direction and show it with arrows on the second sketch. We can also get all the signs by looking at the two equations in the system.

(iii; **4pts**) The next step is to draw the pair of the incoming solution curves and the pair of the outgoing solution curves for the saddle point $(-1/\sqrt{2}, -1/\sqrt{2})$. Based on the second sketch in Figure 2, the incoming solutions must approach from below, and *very* slightly from the left, and from above, and *very* slightly from the right, of the saddle point. Using the the second sketch in Figure 2, we can trace these solution curves backwards. The lower curve, traced backwards, descends and goes to the left. The upper curve, traced backwards, at first rises, then swings to the left and continues to rise. Eventually, it must cross the line $y = x$, because the x-component grows much faster than the



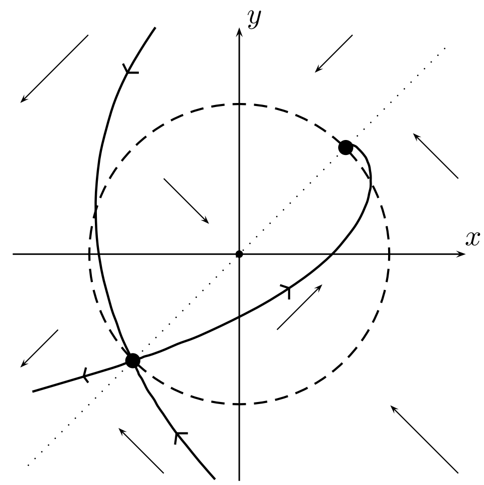


Figure 2: Sketches for Problem 10.3:16

y-component. Once it crosses the line $y=x$, the curve will descend. On the other hand, the outgoing solution curves must leave to the right, and *very* slightly below, and to the left, and *very* slightly above. The latter curve heads toward the sink at $(1/\sqrt{2}, 1/\sqrt{2})$ and spirals down toward it. The other outgoing curve will always be heading down and to the left. It will never cross the line $y=x$. These four distinguished solution curves are shown in the second sketch of Figure 2.

(iv; **4pts**) Finally, we use the flow directions to sketch more solution curves and show the possible behavior in various regions of the plane; see Figure 3. Every solution curve, other than the two outgoing solution curves for the saddle point, rises from below the line $y=x$. The ones that begin between the two incoming solution curves for the saddle point end up going to the sink. The ones that begin to the "right" of the "upper" incoming solution pass over the unit circle and then head left and downward, above the line $y=x$. The ones that begin to the "left" of the "lower" incoming solution rise below the unit circle to the line $y=x$ and then descend to the left above it. The slopes of all solution curves that go off to infinity, either as $t \longrightarrow \infty$ or $t \longrightarrow -\infty$, approach zero, i.e. the curves flatten, because the square terms in the expression for x' dominate the linear terms in the expression for y'. However, none of them is asymptotic to any horizontal line, and in fact every solution curve going off to infinity ends up below every up horizontal line. All solution curves cross the line $y=x$ horizontally and the circle $x^2+y^2=1$ vertically, it they cross at all.

Note: Before drawing solution curves away from the equilibrium points, we should normally check that there are no limit cycles or oriented polygons. In this case, this can be seen from the flow directions. Any cycle would have to circle the point $(1/\sqrt{2}, 1/\sqrt{2})$, leaving the unit circle somewhere on the shorter arc between $(1,0)$ and $(1/\sqrt{2}, 1/\sqrt{2})$. If we start at any point on this arc, other than the endpoints, go up vertically to the line $y=x$, then horizontally to the left to the circle, then vertically down to the line $y=x$, and then back horizontally to the right to the unit circle, we'll end up at a point which is strictly closer to $(1/\sqrt{2}, 1/\sqrt{2})$ than the one we started with. If we start at $(1,0)$, we'll end up back at $(1,0)$. In either case, solution curves move strictly closer to $(1/\sqrt{2}, 1/\sqrt{2})$ than the vertical and horizontal path described. For example, after leaving the unit circle somewhere between $(1,0)$ and $(1/\sqrt{2}, 1/\sqrt{2})$, a solution curve does not move straight, but instead swings to the left at least a little bit. Thus, no matter where on the arc between $(1,0)$ and $(1/\sqrt{2}, 1/\sqrt{2})$ a solution curve

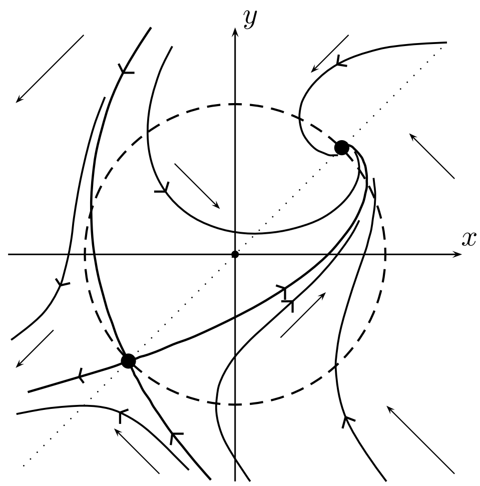


Figure 3: Phase-Plane Portrait for Problem 10.3:16

leaves the unit circle, it will end up closer to the sink after spinning around the sink. It follows that the system has no cycles.

Section 10.4: 2,6 (16pts)

10.4:2; 7pts: *Use the polar coordinates transformation to find the limit cycles for the system:*

$$\begin{cases} x' = -y + x\big(\sqrt{x^2+y^2} - 3\big) \\ y' = x + y\big(\sqrt{x^2+y^2} - 3\big) \end{cases}$$

Determine the stability type of the limit cycles and sketch the phase-plane portrait.
By 10.1:19a, the derivatives of the coordinates r and θ are given by

$$rr' = xx' + yy' = x\big(-y+x(\sqrt{x^2+y^2}-3)\big) + y\big(x+y(\sqrt{x^2+y^2}-3)\big) = (x^2+y^2)(r-3) = r^2(r-3);$$
$$r^2\theta' = xy' - yx' = x\big(x+y(\sqrt{x^2+y^2}-3)\big) - y\big(-y+x(\sqrt{x^2+y^2}-3)\big) = x^2+y^2 = r^2.$$

Thus, we obtain

$$\begin{cases} x' = -y + x\big(\sqrt{x^2+y^2} - 3\big) \\ y' = x + y\big(\sqrt{x^2+y^2} - 3\big) \end{cases} \iff \begin{cases} r' = r(r-3) \\ \theta' = 1 \end{cases}$$

By the first equation, $r=3$ is the only limit cycle for the system, as the equilibrium point $r=0$ for the first equation is an equilibrium point for the entire system. If $0<r(t)<3$, i.e. the point $(r(t),\theta(t))$ is inside of the cycle, $r'(t)<0$. Thus, all solution curves inside of the cycle move toward the origin and *away* from the cycle. Similarly, if $r(t)>3$, i.e. the point $(r(t),\theta(t))$ is outside of the cycle, $r'(t)>0$. Thus, all solution curves outside of the cycle move *away* from it. Since points on both sides of the cycle move away from it, $r=3$ is a repelling cycle for the system. In order to draw the phase-plane portrait, as in the first sketch of Figure 4, we observe that all solution spin counterclockwise, since $\theta'>0$.

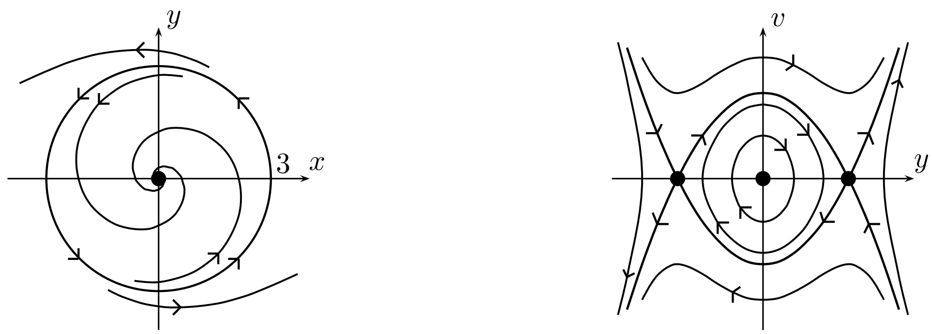


Figure 4: Sketches for Problems 10.4:2 and 10.5:6

10.4:6; 9pts: *Use the polar coordinates transformation to find the limit cycles for the system:*

$$\begin{cases} x' = -y + x(x^2+y^2)\sin\left(\pi/\sqrt{x^2+y^2}\right) \\ y' = x + y(x^2+y^2)\sin\left(\pi/\sqrt{x^2+y^2}\right) \end{cases}$$

Determine the stability type of the limit cycles.
Similarly to 10.4:2,

$$\begin{aligned} rr' = xx' + yy' &= x\big(-y+x(x^2+y^2)\sin(\pi/\sqrt{x^2+y^2})\big) + y\big(x+y(x^2+y^2)\sin(\pi/\sqrt{x^2+y^2})\big) \\ &= (x^2+y^2)r^2\sin(\pi/r) = r^4\sin(\pi/r); \\ r^2\theta' = xy' - yx' &= x\big(x+y(x^2+y^2)\sin(\pi/\sqrt{x^2+y^2})\big) - y\big(-y+x(x^2+y^2)\sin(\pi/\sqrt{x^2+y^2})\big) \\ &= x^2+y^2 = r^2. \end{aligned}$$

Thus, we obtain

$$\begin{cases} x' = -y + x(x^2+y^2)\sin\left(\pi/\sqrt{x^2+y^2}\right) \\ y' = x + y(x^2+y^2)\sin\left(\pi/\sqrt{x^2+y^2}\right) \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} r' = r^3\sin(\pi/r) \\ \theta' = 1 \end{cases}$$

Since the last two equations are separate, the limit cycles are the circles of radius $r > 0$ such that

$$r' = r^3\sin(\pi/r) = 0 \qquad \Longleftrightarrow \qquad \pi/r = \pi n, \quad n \in \mathbb{N} \qquad \Longleftrightarrow \qquad r = 1/n, \quad n \in \mathbb{N}.$$

The limit cycle $r = 1/n$ is attracting if points on both sides of the cycle move toward the cycle, i.e. if $\sin(\pi/r) < 0$ for all r slightly larger than $1/n$ and $\sin(\pi/r) > 0$ for all r slightly smaller than $1/n$. Since

$$\begin{aligned} \sin(\pi/r) > 0 \quad &\text{if} \quad 2k\pi < \pi/r < (2k+1)\pi \iff 1/(2k+1) < r < 1/2k \qquad \text{and} \\ \sin(\pi/r) < 0 \quad &\text{if} \quad (2k+1)\pi < \pi/r < (2k+2)\pi \iff 1/(2k+2) < r < 1/(2k+1), \end{aligned}$$

the limit cycle $r = 1/n$ is attracting if $1/n = 1/2k$, i.e. if n is even. Similarly, the limit cycle $r = 1/n$ is repelling if points on both sides of the cycle move away from the cycle, i.e. if $\sin(\pi/r) > 0$ for all r slightly larger than $1/n$ and $\sin(\pi/r) < 0$ for all r slightly smaller than $1/n$. By the above, this happens precisely when n is odd.

Section 10.5:6 (10pts)

Find a conserved quantity for the system

$$\begin{cases} y' = v \\ v' = -2y + y^3 \end{cases}$$

Verify directly that the quantity you find is actually conserved along the trajectories. Sketch the phase-plane portrait.

Writing $y' = dy/dt$ and $v' = dv/dt$ and dividing the second equation by the first, we obtain

$$\frac{dv}{dy} = \frac{-2y + y^3}{v} \quad \Longleftrightarrow \quad vdv = (-2y+y^3)dy \quad \Longleftrightarrow \quad v^2 = -2y^2 + \frac{1}{2}y^4 + C.$$

Thus, a conserved quantity is $E(y, v) = v^2+2y^2 - \frac{1}{2}y^4$. If $(y, v) = (y(t), v(t))$ is a solution of the system,

$$\frac{d}{dt}E\big(y(t), v(t)\big) = 2vv' + 4yy' - \frac{1}{2}4y^3y' = 2v(-2y + y^3) + 4yv - 2y^3v = 0,$$

as expected. In order to sketch the phase-plane portrait, we first find the equilibrium points:

$$\begin{cases} y' = 0 \\ v' = 0 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} v = 0 \\ -2y + y^3 = 0 \end{cases}$$

Thus, the equilibrium points are $(0, 0)$ and $(\pm\sqrt{2}, 0)$. Normally we would next try to determine the type of each equilibrium point. However, in this case we already know that solution curves for the system lie on the level curves

$$v^2 + 2y^2 - \frac{1}{2}y^4 = C,$$

for various constants C. In order to sketch the level curves, we solve for v and complete the square:

$$v = \pm\frac{1}{\sqrt{2}}\sqrt{(y^2-2)^2+A}, \qquad \text{where} \qquad A = 2C - 4.$$

The level curves are symmetric about the x- and y-axes. Thus, we will concentrate on the first quadrant. If $A=0$, $v=|y^2-2|/\sqrt{2}$. The graph of this function drops from $(0, \sqrt{2})$ to $(\sqrt{2}, 0)$ and then rises as a parabola. The second sketch in Figure 4 shows this graph along with its reflections. If $A>0$, $v=v(y)$ is defined for all y and reaches its minimum, in the absolute value, at $y=\pm\sqrt{2}$. The positive branch lies strictly above $v=|y^2-2|/\sqrt{2}$, as shown in the sketch. If $A<0$, $v=v(y)$ is only defined for y such that $(y^2-2)^2+A\geq 0$. If $A\in(-4, 0)$, this gives us closed curves inside of the region contained by the graphs of $v=\pm|y^2-2|/\sqrt{2}$, as well curves rising and descending from the y-axis to the sides of the graphs of $v=\pm|y^2-2|/\sqrt{2}$. If $A<-4$, we get only side curves. Since each level curve with $A\neq 0$ contains no equilibrium points, every component of it must be a solution curve and thus is smooth. It remains to determine the flow directions. Since $y'=v$, the flow moves right if $v>0$, i.e. above the y-axis, and left if $v<0$, i.e. below the y-axis. From the conserved quantity E and the sketch, we find that the origin is a center; this fact cannot be obtained from the jacobian.

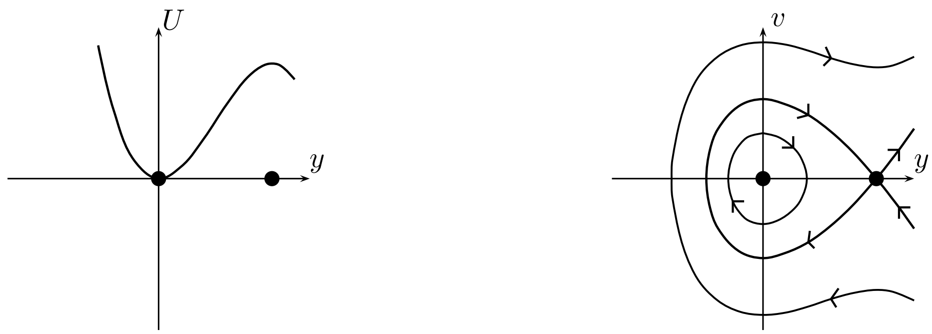


Figure 5: Sketches for Problem 10.6:10

Section 10.6:10 (8pts)

Find the equilibrium points and a potential function $U = U(y)$ for the system

$$\begin{cases} y' = v \\ v' = y^2 - 9y \end{cases}$$

Sketch the graph of U and the phase-plane portrait for the system.
Writing $y' = dy/dt$ and $v' = dv/dt$ and dividing the second equation by the first, we obtain

$$\frac{dv}{dy} = \frac{y^2 - 9y}{v} \quad \Longleftrightarrow \quad vdv = (y^2 - 9y)dy \quad \Longleftrightarrow \quad \frac{1}{2}v^2 = \frac{1}{3}y^3 - \frac{9}{2}y^2 + C.$$

Thus, a conserved quantity for the system is

$$E(y, v) = \frac{1}{2}v^2 + U(y), \qquad \text{where} \qquad U(y) = -\frac{1}{3}y^3 + \frac{9}{2}y^2.$$

We next find the equilibrium points:

$$\begin{cases} y' = 0 \\ v' = 0 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} v = 0 \\ y^2 - 9y = 0 \end{cases}$$

Thus, the equilibrium points are $(0, 0)$ and $(9, 0)$. As in 10.5:6, we do not need to find their type. In order to sketch the level curves of E and thus solution curves for the system, we can again solve for v in terms y:

$$v(y) = \sqrt{\frac{2}{3}y^3 - 9y^2 + A}, \qquad \text{where} \qquad A = 2C.$$

The function $\frac{2}{3}y^3 – 9y^2$, which is twice the negative of $U(y)$, reaches a local maximum at $y = 0$ and a local minimum at $y = 9$. The corresponding critical values are 0 and -243. The level curve corresponding to $A = 243$ contains an oriented polygon, with one vertex. The components of all other level curves correspond to solution curves. Just as in 10.5:6, the flow direction is to the right in the upper-half plane and to the left in the lower half-plane.