# Math53: Ordinary Differential Equations
# Winter 2004

## Problem Set 4 Solutions

### Section 7.2: 8,14 (6pts)

**7.2:8; 2pts:** *Consider the line $L$ in $\mathbf{R}^2$ with the parametric equation*

$$\mathbf{y} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ -3 \end{pmatrix}.$$

*Is $L$ the solution set for a system of linear equations?*

$$\mathbf{y} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ -3 \end{pmatrix} \quad \Longrightarrow \quad x = -2, \quad y = 3 - 3t.$$

Thus, $L$ is the solution set for the one-equation system $\boxed{x = -2, \ \ (x, y) \in \mathbb{R}^2}$ in two variables.

**7.2:14; 4pts:** *Find a parametric representation for the solution set of the system of equations*

$$\begin{cases} x_1 + 2x_2 - 2x_3 + x_4 = 2 \\ x_2 - 3x_3 - x_4 = 3 \\ x_3 - x_4 = 0 \end{cases}$$

*What is its dimension? How would you describe the solution set?*

$$\begin{cases} x_1 + 2x_2 - 2x_3 + x_4 = 2 \\ x_2 - 3x_3 - x_4 = 3 \\ x_3 - x_4 = 0 \end{cases} \iff \begin{cases} x_1 + 2x_2 - x_4 = 2 \\ x_2 - 4x_4 = 3 \\ x_3 = x_4 \end{cases} \iff \begin{cases} x_1 = -4 - 7x_4 \\ x_2 = 3 + 4x_4 \\ x_3 = x_4 \end{cases}$$

$$\Longrightarrow \quad \boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 - 7x_4 \\ 3 + 4x_4 \\ x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -7 \\ 4 \\ 1 \\ 1 \end{pmatrix}}$$

The solution is a line in $\mathbb{R}^4$, and it has dimension 1.

### Section 7.4: 16,20 (7pts)

*Determine whether each of the matrices*

$$A = \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

*is singular. If not, find its inverse.*
Since the determinant of $A$ is zero, $\boxed{A \text{ is singular}}$ Since the matrix $B$ is upper-triangular, $\det B$

is the product of the diagonal entries. Thus, $\det B = 1 \neq 0$ and $\boxed{B \text{ is nonsingular}}$ The inverse of any non-singular matrix can be computed in three steps, which for a $3\times3$ matrix are:

$$B \longrightarrow \begin{pmatrix} \det B_{11} & \det B_{12} & \det B_{13} \\ \det B_{21} & \det B_{22} & \det B_{23} \\ \det B_{31} & \det B_{32} & \det B_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} \det B_{11} & -\det B_{21} & \det B_{31} \\ -\det B_{12} & \det B_{22} & -\det B_{32} \\ \det B_{13} & -\det B_{23} & \det B_{33} \end{pmatrix}$$
$$\longrightarrow \frac{1}{\det B} \begin{pmatrix} \det B_{11} & -\det B_{21} & \det B_{31} \\ -\det B_{12} & \det B_{22} & -\det B_{32} \\ \det B_{13} & -\det B_{23} & \det B_{33} \end{pmatrix}$$

where $B_{ij}$ is the square matrix obtained from $B$ by removing $i$th row and $j$th column of $B$. In this case, we get

$$B \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \boxed{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}$$

### Section 7.5: 14,26 (9pts)

**7.5:14; 7pts:** *Check whether the vectors*

$$\mathbf{v}_1 = \begin{pmatrix} -8 \\ 9 \\ -6 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 8 \\ -18 \\ 40 \end{pmatrix}$$

*are linearly independent. If not, find a nontrivial linear combination that equals zero.*
Since there are *three three*-vectors, we can simply check whether the determinant of the corresponding matrix is nonzero:

$$\det \begin{pmatrix} -8 & -2 & 8 \\ 9 & 0 & -18 \\ -6 & 7 & 40 \end{pmatrix} = (-2)\cdot 9 \cdot \det \begin{pmatrix} 4 & 1 & -4 \\ 1 & 0 & -2 \\ -6 & 7 & 40 \end{pmatrix}$$
$$= -18\big(4\cdot 0\cdot 40 + (-6)\cdot 1\cdot(-2) + (-4)\cdot 1\cdot 7 - (-4)\cdot 0\cdot(-6) - 4\cdot 7\cdot(-2) - 40\cdot 1\cdot 1\big) = 0.$$

Thus, the three vectors are $\boxed{\text{linearly dependent}}$. We now need to find a solution to the linear system

$$\begin{pmatrix} 4 & 1 & -4 \\ 1 & 0 & -2 \\ -6 & 7 & 40 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} 4c_1 + c_2 - 4c_3 = 0 \\ c_1 - 2c_3 = 0 \\ -6c_1 + 7c_2 + 40c_3 = 0 \end{cases} \iff \begin{cases} c_1 = 2c_3 \\ 4c_3 + c_2 = 0 \\ 28c_3 + 7c_2 = 0 \end{cases}$$
$$\iff \begin{cases} c_1 = 2c_3 \\ c_2 = -4c_3 \end{cases} \implies \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \iff \boxed{2\mathbf{v}_1 - 4\mathbf{v}_2 + \mathbf{v}_3 = 0}$$

**7.5:26; 2pts:** *Find a basis for the nullspace of the matrix* $A = \begin{pmatrix} -3 & 5 \end{pmatrix}$.
We need to find a solution to

$$A\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \iff -3c_1 + 5c_2 = 0 \iff c_2 = 3c_1/5 \implies \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Thus, the single-element set $\boxed{\left\{\mathbf{v}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}\right\}}$ is a basis for the nullspace of $A$.

## Section 7.6: 14,28,44 (21pts)

**7.6:14** (a; **5pts**) *Let $A$ be an $n\times n$-matrix. If row $i$ is a linear combination of the preceding rows, prove that the determinant of $A$ is zero. State and prove a similar statement about the columns of $A$.*
We first state and prove the analogous statement for the columns of $A$:

*If column $j$ is a linear combination of the preceding columns, then* $\det A = 0$.

Let $A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$, where $\mathbf{v}_j$ denotes the $j$th column. Suppose that the $j$th column is a linear combination of the columns that precede it, i.e.

$$\mathbf{v}_j = c_1\mathbf{v}_1 + \ldots + c_{j-1}\mathbf{v}_{j-1} \implies A\begin{pmatrix} c_1 \\ \vdots \\ c_{j-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, the equation $A\mathbf{x} = \mathbf{0}$ has a nonzero solution, and $\det A = 0$ by Corollary 6.3 on p385. On the other hand, if the $i$th row of $A$ is a linear combination of its preceding rows, then the $i$th column of the transpose $A^T$ of $A$ is a linear combination of the preceding columns of $A^T$, and thus

$$\det A = \det A^T = 0.$$

(b; **3pts**) *Explain why the determinant of each of the following matrices is zero:*

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 3 \\ -1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 5 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

In the first matrix, the third row is the sum of its first two rows. In the second matrix, the third column is the sum of its first two columns. In the third matrix, the third row equals twice its first row added to its second row. In the fourth matrix, the third column is the sum of twice its first column and three times its second column. By part (a) of the problem, it follows that all four matrices have zero determinant.

**7.6:28; 8pts:** *Compute the determinant of the matrix*

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

*If the nullspace of this matrix is nontrivial, find a basis for it. Determine if the column vectors in the matrix are linearly independent.*
To calculate the determinant, expand along the first row:

$$\det \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} = (-1)\det\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + (-1)\det\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = -1 + 1 = 0.$$

Since $\det A = 0$, the nullspace of $A$ is nontrivial, and the column vectors of $A$ are linearly dependent. We now find a basis for $\mathrm{null}(A)$:

$$\begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} -c_1 - c_3 = 0 \\ c_1 + c_2 + 2c_3 = 0 \\ 2c_1 + c_2 + 3c_3 = 0 \end{cases} \iff \begin{cases} c_3 = -c_1 \\ c_2 - c_1 = 0 \\ c_2 - c_1 = 0 \end{cases}$$

$$\iff \begin{cases} c_2 = c_1 \\ c_3 = -c_1 \end{cases} \iff \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} c_1.$$

Thus, $\{(1, 1, -1)^T\}$ is a basis for $\mathrm{null}(A)$.

**7.6:44; 5pts:** *Compute the determinant of the matrix*

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 6 & -2 \\ -2 & 3 & 2 \end{pmatrix}$$

*Is there a nonzero vector in the nullspace?*
To compute the determinant, expand along the first column:

$$\det \begin{pmatrix} 1 & 2 & -3 \\ 0 & 6 & -2 \\ -2 & 3 & 2 \end{pmatrix} = 1 \cdot \det\begin{pmatrix} 6 & -2 \\ 3 & 2 \end{pmatrix} - 2 \cdot \det\begin{pmatrix} 2 & -3 \\ 6 & -2 \end{pmatrix}$$
$$= \big(6 \cdot 2 - (-2)3\big) - 2\big(2(-2) - (-3)6\big) = 18 - 28 = \boxed{-10}$$

Since $\det A \neq 0$, $\boxed{\text{the nullspace is trivial}}$ it consists only of the zero vector.

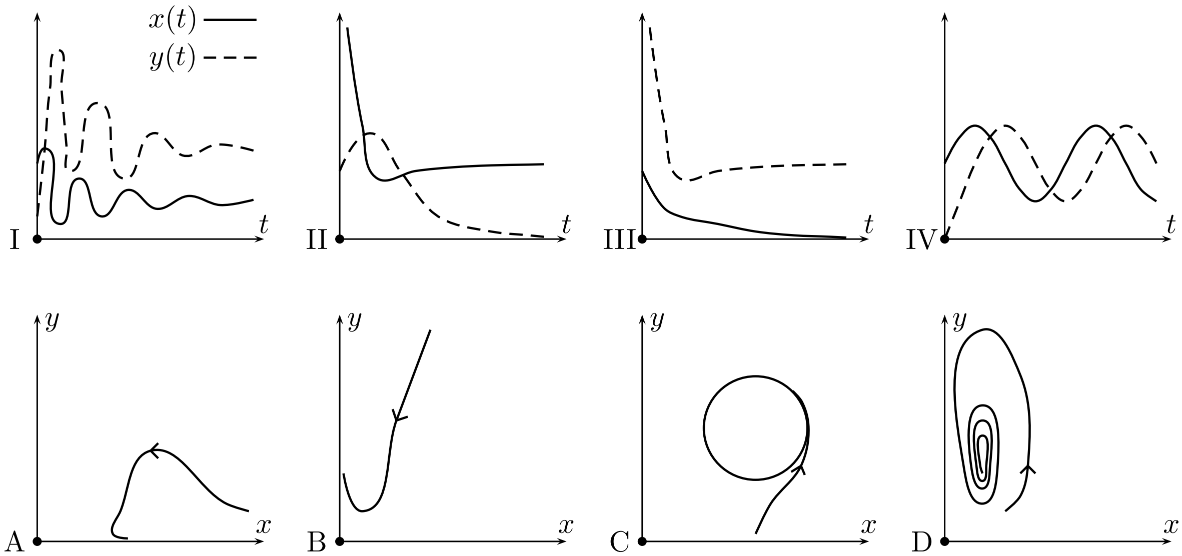


Figure 1: Component and Phase-Plane Plots for Problem 8.2:13

**Section 8.1: 10 (4pts)**

*Show that the functions* $x(t)\!=\!e^t$ *and* $y(t)\!=\!e^{-t}$ *are solutions to the initial value problem*

$$\begin{cases} x' = x^2 y \\ y' = -xy^2 \end{cases} \qquad x(0) = 1, \quad y(0) = 1.$$

The initial conditions are satisfied, since $x(0) = e^0 = 1$ and $y(0) = e^{-0} = 1$. Both equations hold, since:

$$x'(t) = e^t = (e^t)^2 e^{-t} = (x(t))^2 y(t), \qquad y'(t) = -e^{-t} = -e^t(e^{-t})^2 = -x(t)(y(t))^2.$$

**Section 8.2: 13 (7pts)**

*Match each component plot in the first row of Figure 1 with its phase-plane plot in the second row. Explain your reasoning.*

I. $x$ and $y$ start, at $t\!=\!0$, at a small nonzero value. Thus, the corresponding phase-plane sketch is D. Note also that $x$ and $y$ exhibit an oscillatory behavior that dies out as $t$ becomes large. Thus, $(x(t), y(t))$ must spiral down to a point as $t \longrightarrow \infty$. This is the case only in D.

II. $x$ starts out very large, while $y$ starts at a small nonzero value. Thus, the corresponding phase-plane sketch is A. Note also that for $t$ large, $x$ ascends to a small nonzero value, while $y$ decays to zero. In other words, $(x(t), y(t))$ approaches a nonzero point on the $x$-axis from the left and above. This is the case only in *A*.

III. $x$ starts at a small nonzero value, while $y$ starts out very large. Thus, the corresponding phase-plane sketch is B. Note also that for $t$ large, $x$ decays to zero, while $y$ ascends to a small nonzero

value. In other words, $(x(t), y(t))$ approaches a nonzero point on the $y$-axis from the right and below. This is the case only in $B$.
IV. $x$ starts out at a small nonzero value, while $y$ starts at zero. Thus, the corresponding phase-plane sketch is C. Note also that $x$ and $y$ oscillate around finite values without decay. Thus, $(x(t), y(t))$ must move along a curve around some region in the plane, except at the beginning. This is the case only in $C$.
In brief, $\boxed{\mathrm{I}\longleftrightarrow\mathrm{D},\quad \mathrm{II}\longleftrightarrow\mathrm{A},\quad \mathrm{III}\longleftrightarrow\mathrm{B},\quad \mathrm{IV}\longleftrightarrow\mathrm{C}}$

## Section 9.1: 6,54 (10pts)

**9.1:6; 3pts:** *Find the characteristic polynomial and eigenvalues for the matrix* $A=\begin{pmatrix}-2 & 5\\ 0 & 2\end{pmatrix}$.

$$p(\lambda)=\det(A-\lambda I)=\det\begin{pmatrix}-2-\lambda & 5\\ 0 & 2-\lambda\end{pmatrix}=(-2-\lambda)(2-\lambda)=\boxed{\lambda^2-4}$$

The eigenvalues are the zeros of $p(\lambda)$: $\boxed{\lambda_1=-2 \text{ and } \lambda_2=2}$

**9.1:54; 7pts:** *Diagonalize the matrix* $A=\begin{pmatrix}6 & 0\\ 8 & -2\end{pmatrix}$ *by first finding its eigenvalues and the corresponding eigenvectors.*
Since this matrix is lower-triangular, the eigenvalues are the diagonal entries: $\lambda_1=6$ and $\lambda_2=-2$. Furthermore, an eigenvector for $\lambda_2$ is $\mathbf{v}_2=\begin{pmatrix}0\\1\end{pmatrix}$. We next find an eigenvector for $\lambda_1$:

$$(A-\lambda_1 I)\mathbf{v}_1=0 \iff \begin{pmatrix}0 & 0\\ 8 & -8\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \iff 8c_1-8c_2=0 \iff c_2=c_1 \implies \mathbf{v}_1=\begin{pmatrix}1\\1\end{pmatrix}.$$

We have $A=VDV^{-1}$, where

$$D=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}=\begin{pmatrix}6 & 0\\ 0 & -2\end{pmatrix},\quad V=(\mathbf{v}_1\ \mathbf{v}_2)=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\quad\implies\quad V^{-1}=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

## Section 9.2: 1,4,10,24,26,30,38,40,44 (71pts)

**9.2:1; 9pts:** *Find the general solution to the system of linear ODEs*

$$\mathbf{y}'=\begin{pmatrix}2 & -6\\ 0 & -1\end{pmatrix}\mathbf{y},\qquad \mathbf{y}=\mathbf{y}(t).$$

*Sketch the phase-plane portrait of solution curves.*
Since the matrix $A$ is upper-triangular in this case, the eigenvalues of $A$ are the two diagonal entries, $\lambda_1=2$ and $\lambda_2=-1$. Furthermore, $\mathbf{v}_1=\begin{pmatrix}1\\0\end{pmatrix}$ is an eigenvector for $A$ with eigenvalue $\lambda_1=2$. We next find an eigenvector $\mathbf{v}_2$ corresponding to $\lambda_2=-1$:

$$\begin{pmatrix}2-\lambda_2 & -6\\ 0 & -1-\lambda_2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \iff \begin{cases}3c_1-6c_2=0\\ 0=0\end{cases} \iff c_1=2c_2 \implies \mathbf{v}_2=\begin{pmatrix}2\\1\end{pmatrix}.$$

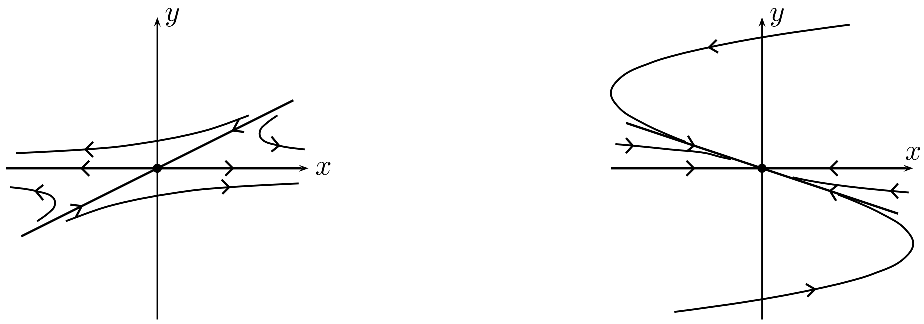


Figure 2: Phase-Plane Plots for Problems 9.2:1 and 9.2:4

The general solution to the ODE is thus given by

$$\boxed{\mathbf{y}(t) = C_1 e^{2t}\begin{pmatrix}1\\0\end{pmatrix} + C_2 e^{-t}\begin{pmatrix}2\\1\end{pmatrix} \quad \text{or} \quad x(t) = C_1 e^{2t} + 2C_2 e^{-t},\ y(t) = C_2 e^{-t}}$$

A phase-plane sketch is the first plot in Figure 2. The origin is a *saddle point*. The solutions move away from the origin along the two half-lines generated by the vectors $\mathbf{v}_1$ and $-\mathbf{v}_1$, since $\lambda_1 > 0$, and approach the origin along the two half-lines generated by the vectors $\mathbf{v}_2$ and $-\mathbf{v}_2$, since $\lambda_2 < 0$. Other solution curves approach one of the first two half-lines as $t \longrightarrow \infty$ and one of the latter two half-lines as $t \longrightarrow -\infty$.

**9.2:4; 9pts:** *Find the general solution to the system of linear ODEs*

$$\mathbf{y}' = \begin{pmatrix}-3 & -6\\ 0 & -1\end{pmatrix}\mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

*Sketch the phase-plane portrait of solution curves.*
Since the matrix $A$ is upper-triangular in this case, the eigenvalues of $A$ are the two diagonal entries, $\lambda_1 = -3$ and $\lambda_2 = -1$. Furthermore, $\mathbf{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ is an eigenvector for $A$ with eigenvalue $\lambda_1 = -3$. We next find an eigenvector $\mathbf{v}_2$ corresponding to $\lambda_2 = -1$:

$$\begin{pmatrix}-3-\lambda_2 & -6\\ 0 & -1-\lambda_2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \iff \begin{cases}-2c_1 - 6c_2 = 0\\ 0 = 0\end{cases} \iff c_1 = -3c_2 \implies \mathbf{v}_2 = \begin{pmatrix}-3\\1\end{pmatrix}.$$

The general solution to the ODE is thus given by

$$\boxed{\mathbf{y}(t) = C_1 e^{-3t}\begin{pmatrix}1\\0\end{pmatrix} + C_2 e^{-t}\begin{pmatrix}-3\\1\end{pmatrix} \quad \text{or} \quad x(t) = C_1 e^{-3t} - 3C_2 e^{-t},\ y(t) = C_2 e^{-t}}$$

A phase-plane sketch is the second plot in Figure 2. The origin is a *nodal sink*. The solutions approach the origin along the four half-lines generated by the vectors $\pm\mathbf{v}_1$ and $\pm\mathbf{v}_2$, since $\lambda_1, \lambda_2 < 0$. All other solution curves must also approach the origin as $t \longrightarrow \infty$. Their slope approaches that of the half-lines generated by $\pm\mathbf{v}_2$ as $t \longrightarrow \infty$ and that of the half-lines generated by $\pm\mathbf{v}_1$ as $t \longrightarrow -\infty$, since $\lambda_2 > \lambda_1$. However, none of these solution curves approaches a horizontal line as $t \longrightarrow -\infty$.

**9.2:10; 5pts:** *Find the solution to the initial value problem*

$$\mathbf{y}' = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By 9.2:4, it remains to find $C_1$ and $C_2$ such that

$$\mathbf{y}(0) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{cases} C_1 - 3C_2 = 1 \\ C_2 = 1 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} C_1 = 4 \\ C_2 = 1 \end{cases}$$

Thus, the solution to the IVP is

$$\boxed{\mathbf{y}(t) = 4e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \text{or} \quad x(t) = 4e^{-3t} - 3e^{-t}, \ y(t) = e^{-t}}$$

**9.2:24; 9pts:** *Find the general solution to the system of linear ODEs*

$$\mathbf{y}' = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

*Sketch the phase-plane portrait of solution curves.*
The characteristic polynomial for this system is

$$\lambda^2 - (-1+3)\lambda + \big((-1)\cdot 3 - (-2)\cdot 4\big) = \lambda^2 - 2\lambda + 5.$$

Thus, the two eigenvalues are $\lambda_1, \lambda_2 = 1 \pm 2i$. We next find an eigenvector $\mathbf{v}_1$ corresponding to $\lambda_1 = 1+2i$:

$$\begin{pmatrix} -1-\lambda_1 & -2 \\ 4 & 3-\lambda_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{cases} -(2+2i)c_1 - 2c_2 = 0 \\ 4c_1 + (2-2i)c_2 = 0 \end{cases}$$

$$\Longleftrightarrow \qquad c_2 = -(1+i)c_1 \quad \Longrightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}.$$

Since our matrix is real, while $\mathbf{v}_1$ is complex, its complex conjugate

$$\mathbf{v}_2 = \bar{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$

must be an eigenvector with eigenvalue $\lambda_2 = \bar{\lambda}_1 = 1-2i$. Thus, the general solution to the ODE is

$$\begin{aligned} y(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 &= e^t \begin{pmatrix} C_1 e^{2it} + C_2 e^{-2it} \\ -(C_1 e^{2it} + C_2 e^{-2it}) - i(C_1 e^{2it} - C_2 e^{-2it}) \end{pmatrix} \\ &= \boxed{e^t \begin{pmatrix} A_1 \cos 2t + A_2 \sin 2t \\ -(A_1 + A_2)\cos 2t + (A_1 - A_2)\sin 2t \end{pmatrix}} \end{aligned}$$

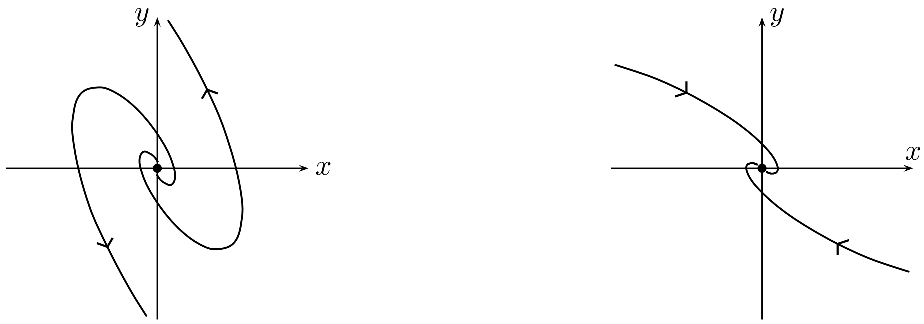


Figure 3: Phase-Plane Plots for Problems 9.2:24 and 9.2:26

A phase-plane sketch is the first plot in Figure 3. The origin is a *spiral source*. The solutions spiral away from the origin as $t$ increases. They spiral out counterclockwise, since the entry in the lower-left corner of the matrix is positive.

**9.2:26; 9pts:** *Find the general solution to the system of linear ODEs*

$$\mathbf{y}' = \begin{pmatrix} 0 & 4 \\ -2 & -4 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

*Sketch the phase-plane portrait of solution curves.*
The characteristic polynomial for this system is

$$\lambda^2 - (0-4)\lambda + \big((-4)\cdot 0 - 4\cdot(-2)\big) = \lambda^2 + 4\lambda + 8.$$

Thus, the two eigenvalues are $\lambda_1, \lambda_2 = 2(-1\pm i)$. We next find an eigenvector $\mathbf{v}_1$ corresponding to $\lambda_1 = 2(-1+i)$:

$$\begin{pmatrix} 0-\lambda_1 & 4 \\ -2 & -4-\lambda_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{cases} 2(1-i)c_1 + 4c_2 = 0 \\ -2c_1 - 2(1+i)c_2 = 0 \end{cases}$$

$$\Longleftrightarrow \qquad c_1 = -(1+i)c_2 \quad \Longrightarrow \quad \mathbf{v}_1 = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}.$$

Since our matrix is real, while $\mathbf{v}_1$ is complex, its complex conjugate

$$\mathbf{v}_2 = \bar{\mathbf{v}}_1 = \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$$

must be an eigenvector with eigenvalue $\lambda_2 = \bar{\lambda}_1 = 2(-1-i)$. Thus, the general solution to the ODE is

$$y(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 = e^{-2t} \begin{pmatrix} -(C_1 e^{2it} + C_2 e^{-2it}) - i(C_1 e^{2it} - C_2 e^{-2it}) \\ C_1 e^{2it} + C_2 e^{-2it} \end{pmatrix}$$

$$= \boxed{e^{-2t} \begin{pmatrix} -(A_1+A_2)\cos 2t + (A_1-A_2)\sin 2t \\ A_1\cos 2t + A_2 \sin 2t \end{pmatrix}}$$

A phase-plane sketch is the second plot in Figure 3. The origin is a *spiral sink*. The solutions spiral into the origin as $t$ increases. They spin clockwise, since the entry in the lower-left corner of the matrix is negative.

**9.2:30; 5pts:** *Find the solution to the initial value problem*

$$\mathbf{y}' = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By 9.2:24, it remains to find $A_1$ and $A_2$ such that

$$\mathbf{y}(0) = \begin{pmatrix} A_1 \\ -(A_1+A_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \iff \begin{cases} A_1 = 0 \\ A_1 + A_2 = -1 \end{cases} \iff \begin{cases} A_1 = 0 \\ A_2 = -1 \end{cases}$$

Thus, the solution to the IVP is

$$\boxed{\mathbf{y}(t) = e^t \begin{pmatrix} -\sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}}$$

**9.2:38; 10pts:** *Find the general solution to the system of linear ODEs*

$$\mathbf{y}' = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

*Sketch the phase-plane portrait of solution curves.*
The characteristic polynomial for this system is

$$\lambda^2 - (-3-1)\lambda + \big((-3)\cdot(-1) - 1\cdot(-1)\big) = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2.$$

Thus, there is only one eigenvalue, $\lambda = -2$. We next find an eigenvector $\mathbf{v}_1$ for $\lambda = -2$:

$$\begin{pmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} -c_1 + c_2 = 0 \\ -c_1 + c_2 = 0 \end{cases} \iff c_1 = c_2 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We now pick a simple vector $\mathbf{v}_2$, express $A\mathbf{v}_2 - \lambda\mathbf{v}_2$ in terms of $\mathbf{v}_1$, and then compute $e^{tA}\mathbf{v}_2$:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \begin{pmatrix} -3 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \end{pmatrix} = (-1)\cdot\mathbf{v}_1$$
$$\implies \quad tA\mathbf{v}_2 = (-t)\mathbf{v}_1 + (-2t)\mathbf{v}_2 \implies e^{tA}\mathbf{v}_2 = -te^{-2t}\mathbf{v}_1 + e^{-2t}\mathbf{v}_2.$$

The general solution to the ODE is thus given by

$$\mathbf{y}(t) = C_1 e^{-2t}\mathbf{v}_1 + C_2\big(-te^{-2t}\mathbf{v}_1 + e^{-2t}\mathbf{v}_2\big) = \boxed{e^{-2t}\begin{pmatrix} C_1 + C_2 - C_2 t \\ C_1 - C_2 t \end{pmatrix}}$$

A phase-plane sketch is the first plot in Figure 4. The origin is a *degenerate nodal sink*. Each solution curve descends to the origin as $t \longrightarrow \infty$, and its slope approaches 1 as $t \longrightarrow \pm\infty$. In order

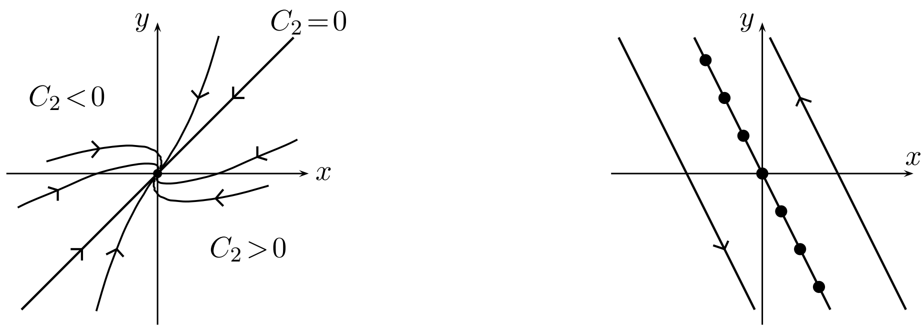


Figure 4: Phase-Plane Plots for Problems 9.2:38 and 9.2:40

to see which way the solution curves move on the two sides of the line $\mathbb{R}\mathbf{v}_1$, we need to determine whether $C_2\!>\!0$ or $C_2\!<\!0$ on each of the two sides of this line. The line itself corresponds to $C_2\!=\!0$. We know that if $C_2\!>\!0$, the point $\mathbf{y}(t)$ corresponding to $C_1$ and $t$ will lie either to the left or to the right of the line, with left or right being the same for all $C_1$ and $t$. Thus, we can test this using $C_1\!=\!0$ and $t\!=\!0$. In this case, $\mathbf{y}(t)\!=\!(1,0)$ lies to the right of the line. Thus, $C_2$ is positive to the right of the line. By looking at $\mathbf{y}(t)$, we see that if $C_2\!>\!0$, the $x$- and $y$-coordinates of $\mathbf{y}(t)$ become very large and positive as $t\longrightarrow-\infty$, and become negative as $t\longrightarrow\infty$. Thus, the solution curves on the right of the line $\mathbb{R}\mathbf{v}_1$ rise up in the direction of $+\mathbf{v}_1$ as $t\longrightarrow-\infty$ and approach the origin from below left as $t\longrightarrow\infty$. The picture on the left side of the line $\mathbb{R}\mathbf{v}_1$ is just a reflection about the origin.

**9.2:40; 10pts:** *Find the general solution to the system of linear ODEs*

$$\mathbf{y}'=\begin{pmatrix}-2&-1\\4&2\end{pmatrix}\mathbf{y},\qquad \mathbf{y}=\mathbf{y}(t).$$

*Sketch the phase-plane portrait of solution curves.*
The characteristic polynomial for this system is

$$\lambda^2-(-2+2)\lambda+\big((-2)\cdot 2-(-1)\cdot 4\big)=\lambda^2.$$

Thus, there is only one eigenvalue, $\lambda\!=\!0$. We next find an eigenvector $\mathbf{v}_1$ for $\lambda\!=\!0$:

$$\begin{pmatrix}-2-\lambda&-1\\4&2-\lambda\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\iff\begin{cases}-2c_1-c_2=0\\4c_1+2c_2=0\end{cases}\iff c_2=-2c_1\implies\mathbf{v}_1=\begin{pmatrix}1\\-2\end{pmatrix}.$$

We now pick a simple vector $\mathbf{v}_2$, express $A\mathbf{v}_2\!-\!\lambda\mathbf{v}_2$ in terms of $\mathbf{v}_1$, and then compute $e^{tA}\mathbf{v}_2$:

$$\mathbf{v}_2=\begin{pmatrix}1\\0\end{pmatrix}\implies A\mathbf{v}_2-\lambda\mathbf{v}_2=\begin{pmatrix}-2\\4\end{pmatrix}-\begin{pmatrix}0\\0\end{pmatrix}=(-2)\cdot\mathbf{v}_1$$
$$\implies tA\mathbf{v}_2=(-2t)\mathbf{v}_1+(0t)\mathbf{v}_2\implies e^{tA}\mathbf{v}_2=(-2t)e^{-0t}\mathbf{v}_1+e^{-0t}\mathbf{v}_2.$$

The general solution to the ODE is thus given by

$$\mathbf{y}(t)=C_1e^{-0t}\mathbf{v}_1+C_2\big(-2te^{-0t}\mathbf{v}_1+e^{-0t}\mathbf{v}_2\big)=\boxed{\begin{pmatrix}C_1\!+\!C_2\!-\!2C_2t\\-2C_1+4C_2t\end{pmatrix}}$$

A phase-plane sketch is the second plot in Figure 4. The origin is an unstable equilibrium, and so is every point on the line $y=-2x$.

**9.2:44; 5pts:** *Find the solution to the initial value problem*

$$\mathbf{y}' = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}\mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 0 \\ -3 \end{pmatrix}.$$

By 9.2:38, it remains to find $C_1$ and $C_2$ such that

$$\mathbf{y}(0) = \begin{pmatrix} C_1+C_2 \\ C_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \quad\Longleftrightarrow\quad \begin{cases} C_1+C_2=0 \\ C_1=-3 \end{cases} \quad\Longleftrightarrow\quad \begin{cases} C_1=-3 \\ C_2=3 \end{cases}$$

Thus, the solution to the IVP is $\boxed{\mathbf{y}(t) = -3e^{-2t}\begin{pmatrix} t \\ 1+t \end{pmatrix}}$

**Section 9.4: 14 (10pts)**

*Solve the initial value problem*

$$\mathbf{y}' = \begin{pmatrix} -3 & 0 & -1 \\ 3 & 2 & 3 \\ 2 & 0 & 0 \end{pmatrix}\mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

The characteristic polynomial $p(\lambda)$ for the matrix is:

$$\det(A-\lambda I) = \det\begin{pmatrix} -3-\lambda & 0 & -1 \\ 3 & 2-\lambda & 3 \\ 2 & 0 & -\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} -3-\lambda & -1 \\ 2 & -\lambda \end{pmatrix}$$
$$= -(\lambda-2)(\lambda^2+3\lambda+2) = -(\lambda-2)(\lambda+1)(\lambda+2).$$

The eigenvalues are $\lambda_1=-2$, $\lambda_2=-1$, $\lambda_3=2$. For each of these, we find an eigenvector:

$$\lambda_1=-2: \quad \begin{pmatrix} -3-\lambda_1 & 0 & -1 \\ 3 & 2-\lambda_1 & 3 \\ 2 & 0 & -\lambda_1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad\Longleftrightarrow\quad \begin{cases} -c_1-c_3=0 \\ 3c_1+4c_2+3c_3=0 \\ 2c_1+2c_3=0 \end{cases}$$

$$\Longleftrightarrow\quad \begin{cases} c_3=-c_1 \\ c_2=0 \end{cases} \quad\Longrightarrow\quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_2=-1: \quad \begin{pmatrix} -3-\lambda_2 & 0 & -1 \\ 3 & 2-\lambda_2 & 3 \\ 2 & 0 & -\lambda_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad\Longleftrightarrow\quad \begin{cases} -2c_1-c_3=0 \\ 3c_1+3c_2+3c_3=0 \\ 2c_1+c_3=0 \end{cases}$$

$$\Longleftrightarrow\quad \begin{cases} c_3=-2c_1 \\ c_2=c_1 \end{cases} \quad\Longrightarrow\quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda_3 = 2: \quad \begin{pmatrix} -3-\lambda_3 & 0 & -1 \\ 3 & 2-\lambda_3 & 3 \\ 2 & 0 & -\lambda_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} -5c_1 - c_3 = 0 \\ 3c_1 + 3c_3 = 0 \\ 2c_1 - 2c_3 = 0 \end{cases}$$

$$\iff \begin{cases} c_1 = 0 \\ c_3 = 0 \end{cases} \implies \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Thus, the general solution is:

$$\mathbf{y}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + C_3 e^{\lambda_3 t}\mathbf{v}_3 = C_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

From the initial condition, we obtain

$$\mathbf{y}(0) = C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \iff \begin{cases} C_1 + C_2 = 1 \\ C_2 + C_3 = -1 \\ -C_1 - 2C_2 = 2 \end{cases}$$

$$\iff \begin{cases} C_1 = 1 - C_2 \\ C_3 = -1 - C_2 \\ -1 - C_2 = 2 \end{cases} \iff \begin{cases} C_2 = -3 \\ C_1 = 4 \\ C_3 = 2 \end{cases}$$

Plugging these constants into the general solution gives

$$\boxed{\mathbf{y}(t) = 4e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - 3e^{-t} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + 2e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4e^{-2t} - 3e^{-t} \\ -3e^{-t} + 2e^{2t} \\ -4e^{-2t} + 6e^{-t} \end{pmatrix}}$$

## Section 9.5: 8,12,14 (18pts)

**9.5:8; 5pts:** *Find* $e^{tA}$ *for* $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.

We split $tA$ as

$$tA = \begin{pmatrix} ta & tb \\ 0 & ta \end{pmatrix} = at \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + bt \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = atI + btB.$$

Since $B^2$ is the zero matrix and $atI$ is diagonal,

$$e^{atI} = e^{at} I \qquad \text{and} \qquad e^{btB} = I + btB = \begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix}.$$

Since $(atI)(btB) = (btB)(atI)$,

$$e^{tA} = e^{atI + btB} = e^{atI} e^{btB} = e^{at} I \begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix} = \boxed{e^{at} \begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{at} & bt\, e^{at} \\ 0 & e^{at} \end{pmatrix}}$$

**9.5:12; 8pts:** *Compute* $e^{tA}$ *for* $A=\begin{pmatrix}-2&0\\-3&-3\end{pmatrix}$.

Since $A$ is lower-triangular, the eigenvalues of $A$ are its diagonal entries, $\lambda_1=-2$ and $\lambda_2=-3$. Furthermore, an eigenvector corresponding to $\lambda_2$ is $\mathbf{v}_2=\begin{pmatrix}0\\1\end{pmatrix}$. We next find an eigenvector for $\lambda_1$:

$$\begin{pmatrix}-2-\lambda_1&0\\-3&-3-\lambda_1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\iff -3c_1-c_2=0\iff c_2=-3c_1\implies \mathbf{v}_1=\begin{pmatrix}1\\-3\end{pmatrix}.$$

These give us a diagonalization of $A$, $A=PDP^{-1}$, where:

$$D=\begin{pmatrix}-2&0\\0&-3\end{pmatrix}\quad P=\begin{pmatrix}1&0\\-3&1\end{pmatrix}\implies P^{-1}=\begin{pmatrix}1&0\\3&1\end{pmatrix}$$

$$\implies\quad e^{tA}=Pe^{tD}P^{-1}=\begin{pmatrix}1&0\\-3&1\end{pmatrix}\begin{pmatrix}e^{-2t}&0\\0&e^{-3t}\end{pmatrix}\begin{pmatrix}1&0\\3&1\end{pmatrix}$$

$$=\begin{pmatrix}1&0\\-3&1\end{pmatrix}\begin{pmatrix}e^{-2t}&0\\3e^{-3t}&e^{-3t}\end{pmatrix}=\boxed{\begin{pmatrix}e^{-2t}&0\\-3e^{-2t}+3e^{-3t}&e^{-3t}\end{pmatrix}}$$

**9.5:14; 5pts:** *Compute* $e^{tA}$ *for* $A=\begin{pmatrix}-1&0\\1&-1\end{pmatrix}$.

We split $tA$ as

$$tA=\begin{pmatrix}-t&0\\t&-t\end{pmatrix}=-t\begin{pmatrix}1&0\\0&1\end{pmatrix}+t\begin{pmatrix}0&0\\1&0\end{pmatrix}=(-t)I+tB.$$

Since $B^2$ is the zero-matrix and $(-t)I$ is diagonal,

$$e^{-tI}=e^{-t}I\qquad\text{and}\qquad e^{tB}=I+tB=\begin{pmatrix}1&0\\t&1\end{pmatrix}.$$

Since $(-tI)(tB)=(tB)(-tI)$,

$$e^{tA}=e^{(-t)I+tB}=e^{-tI}e^{btB}=e^{-t}I\begin{pmatrix}1&0\\t&1\end{pmatrix}=\boxed{e^{-t}\begin{pmatrix}1&0\\t&1\end{pmatrix}=\begin{pmatrix}e^{-t}&0\\t\,e^{-t}&e^{-t}\end{pmatrix}}$$

## Section 9.8: 6,18,29 (27pts)

**9.8:6; 15pts:** *Find the general solution of the system* $\mathbf{y}'=A\mathbf{y}+\mathbf{f}$, *where*

$$A=\begin{pmatrix}4&2\\-1&2\end{pmatrix}\qquad\textit{and}\qquad \mathbf{f}=\begin{pmatrix}t\\e^{3t}\end{pmatrix}.$$

The characteristic polynomial for $A$ is

$$\det(A-\lambda I) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 6\lambda + 10.$$

The eigenvalues of $A$ are the roots of this polynomial: $\lambda_1, \lambda_2 = 3\pm i$. We next find an eigenvector for $\lambda_1$:

$$\begin{pmatrix} 4-\lambda_1 & 2 \\ -1 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} (1-i)c_1 + 2c_2 = 0 \\ -c_1 - (1+i)c_2 = 0 \end{cases}$$
$$\iff \quad c_1 = -(1+i)c_2 \implies \mathbf{v}_1 = \begin{pmatrix} 1+i \\ -1 \end{pmatrix}.$$

The complex conjugate of $\mathbf{v}_1$, i.e. $\mathbf{v}_2 = \begin{pmatrix} 1-i \\ -1 \end{pmatrix}$, is an eigenvector for $\lambda_2 = \bar{\lambda}_1$. Thus, the general solution to the homogeneous system $\mathbf{y}' = A\mathbf{y}$ is

$$\begin{aligned} \mathbf{y}_h(t) &= C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 = C_1 e^{(3+i)t}\begin{pmatrix} 1+i \\ -1 \end{pmatrix} + C_2 e^{(3-i)t}\begin{pmatrix} 1-i \\ -1 \end{pmatrix} \\ &= (A_1\cos t + A_2 \sin t)e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + (A_2\cos t - A_1\sin t)e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{1}$$

The next step is to find a particular solution $\mathbf{y}_p$ to the inhomogeneous system, using

$$\mathbf{y}_p(t) = Y(t)\int_0^t Y(s)^{-1}\mathbf{f}(s)\,ds,$$

where $Y(t) = (\mathbf{y}_1(t)\;\mathbf{y}_2(t))$ is a fundamental matrix and $\{\mathbf{y}_1(t), \mathbf{y}_2(t)\}$ is a fundamental set of solutions for the homogeneous system. We can use either complex or real solutions:

$$Y(t) = e^{3t}\begin{pmatrix} (1+i)e^{it} & (1-i)e^{-it} \\ -e^{it} & -e^{-it} \end{pmatrix} \qquad \text{or} \qquad Y(t) = e^{3t}\begin{pmatrix} \cos t - \sin t & \cos t + \sin t \\ -\cos t & -\sin t \end{pmatrix}. \tag{2}$$

In the first case, the fundamental solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ of the homogeneous system correspond to the $(C_1=1, C_2=0)$ and $(C_1=0, C_2=1)$ cases of (eq1). In the second case, they correspond to the $(A_1=1, A_2=0)$ and $(A_1=0, A_2=1)$ cases of (eq1). As (eq2) might suggest, it is easier to use the complex solutions. In the complex case:

$$Y(t)^{-1} = e^{-3t}\cdot\frac{1}{-2i}\begin{pmatrix} -e^{-it} & -(1-i)e^{-it} \\ e^{it} & (1+i)e^{it} \end{pmatrix} \implies Y^{-1}(s)\mathbf{f}(s) = \frac{i}{2}\begin{pmatrix} -se^{-(3+i)s} - (1-i)e^{-is} \\ se^{-(3-i)s} + (1+i)e^{is} \end{pmatrix}.$$

We next compute

$$\begin{aligned} &\int_0^t (1+i)e^{is}ds = \frac{1+i}{i}e^{is}\Big|_{s=0}^{s=t} = (1-i)\big(e^{it}-1\big) \implies \int_0^t (1-i)e^{-is}ds = (1+i)\big(e^{-it}-1\big); \\ &\int se^{-(3+i)s}ds = \frac{1}{-(3+i)}\Big(se^{-(3+i)s} - \int e^{-(3+i)s}ds\Big) = -\frac{3-i}{10}se^{-(3+i)s} - \frac{4-3i}{50}e^{-(3+i)s} \\ &\qquad\implies \int_0^t se^{-(3+i)s}ds = -\frac{3-i}{10}te^{-(3+i)t} - \frac{4-3i}{50}\big(e^{-(3+i)t}-1\big) \\ &\qquad\implies \int_0^t se^{-(3-i)s}ds = -\frac{3+i}{10}te^{-(3-i)t} + \frac{4+3i}{50}\big(e^{-(3-i)t}-1\big). \end{aligned}$$

Putting everything together, we obtain

$$\begin{aligned}
\mathbf{y}_p(t) &= Y(t)\int_0^t Y(s)^{-1}\mathbf{f}(s)\,ds \\
&= e^{3t}\begin{pmatrix} (1+i)e^{it} & (1-i)e^{-it} \\ -e^{it} & -e^{-it} \end{pmatrix}\cdot\frac{i}{2}\begin{pmatrix} \frac{3-i}{10}te^{-(3+i)t}+\frac{4-3i}{50}e^{-(3+i)t}-(1+i)e^{-it} \\ -\frac{3+i}{10}te^{-(3-i)t}-\frac{4+3i}{50}e^{-(3-i)t}+(1-i)e^{it} \end{pmatrix}+Y(t)\mathbf{v} \\
&= \frac{ie^{3t}}{2}\begin{pmatrix} \frac{2i}{5}te^{-3t}+\frac{i}{25}e^{-3t}-4i \\ \frac{i}{5}te^{-3t}+\frac{3i}{25}e^{-3t}+2i \end{pmatrix}+Y(t)\mathbf{v} = -\frac{1}{50}\begin{pmatrix} 10t+1-100e^{3t} \\ 5t+3+50e^{3t} \end{pmatrix}+Y(t)\mathbf{v},
\end{aligned}$$

for some $\mathbf{v}\in\mathbb{C}$. Since $Y(t)\mathbf{v}$ is a solution of the homogeneous system, the last expression is still a solution of the inhomogeneous system even if we drop the last term. Thus, the general solution of the inhomogeneous system is

$$\begin{aligned}
\mathbf{y}(t) &= \mathbf{y}_h(t)+\mathbf{y}_p(t) \\
&= \boxed{(A_1\cos t+A_2\sin t)e^{3t}\begin{pmatrix}1\\-1\end{pmatrix}+(A_2\cos t-A_1\sin t)e^{3t}\begin{pmatrix}1\\0\end{pmatrix}-\frac{1}{50}\begin{pmatrix}10t+1-100e^{3t}\\5t+3+50e^{3t}\end{pmatrix}}
\end{aligned}$$

Another way of finding $\mathbf{y}_p$ is to use the method of undetermined coefficients. In this case, this would be mean finding $a_1, b_1, c_1$ and $a_2, b_2, c_2$, such that

$$\mathbf{y}_p' = A\mathbf{y}_p+\mathbf{f} \qquad \text{for} \qquad \mathbf{y}_p(t)=\begin{pmatrix} a_1e^{3t}+b_1t+c_1 \\ a_2e^{3t}+b_2t+c_2 \end{pmatrix}.$$

**9.8:18; 8pts:** *Solve the initial value problem*

$$\mathbf{y}'=\begin{pmatrix}-7 & -3\\ 6 & 2\end{pmatrix}\mathbf{y}, \qquad \mathbf{y}(0)=\begin{pmatrix}1\\0\end{pmatrix}.$$

The characteristic polynomial for $A$ is

$$\lambda^2+(\mathrm{tr}\,A)\lambda+\det A=\lambda^2+5\lambda+4=(\lambda+1)(\lambda+4).$$

The eigenvalues of $A$ are the roots of this polynomial: $\lambda_1,\lambda_2=-1,-4$. We next find the corresponding eigenvectors:

$$\lambda_1=-1: \qquad \begin{pmatrix}-7-\lambda_1 & -3\\ 6 & 2-\lambda_1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \iff \begin{cases}-6c_1-3c_2=0\\ 6c_1+3c_2=0\end{cases}$$

$$\iff \quad c_2=-2c_1 \implies \mathbf{v}_1=\begin{pmatrix}1\\-2\end{pmatrix}.$$

$$\lambda_2=-4: \qquad \begin{pmatrix}-7-\lambda_2 & -3\\ 6 & 2-\lambda_2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \iff \begin{cases}-3c_1-3c_2=0\\ 6c_1+6c_2=0\end{cases}$$

$$\iff \quad c_2=-c_1 \implies \mathbf{v}_2=\begin{pmatrix}1\\-1\end{pmatrix}.$$

Thus, a fundamental matrix for this system is

$$Y(t) = \big(e^{\lambda_1 t}\mathbf{v}_1 \; e^{\lambda_2 t}\mathbf{v}_2\big) = \begin{pmatrix} e^{-t} & e^{-4t} \\ -2e^{-t} & -e^{-4t} \end{pmatrix} \Longrightarrow Y(0) = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \Longrightarrow Y(0)^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad e^{tA} = Y(t)Y(0)^{-1} = \begin{pmatrix} e^{-t} & e^{-4t} \\ -2e^{-t} & -e^{-4t} \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 2e^{-4t}-e^{-t} & e^{-4t}-e^{-t} \\ 2e^{-t}-2e^{-4t} & 2e^{-t}-e^{-4t} \end{pmatrix}}$$

Finally,

$$\mathbf{y}(t) = e^{tA}\mathbf{y}(0) = \begin{pmatrix} 2e^{-4t}-e^{-t} & e^{-4t}-e^{-t} \\ 2e^{-t}-2e^{-4t} & 2e^{-t}-e^{-4t} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 2e^{-4t}-e^{-t} \\ 2e^{-t}-2e^{-4t} \end{pmatrix}}$$

**9.8:29; 4pts:** *Show that if $A$ is an $n\times n$ matrix, the function*

$$\mathbf{y}(t) = e^{tA}\mathbf{y}_0 + \int_0^t e^{(t-s)A}\mathbf{f}(s)\,ds$$

*solves the initial value problem* $\mathbf{y}' = A\mathbf{y} + \mathbf{f}$, $\mathbf{y}(0) = \mathbf{y}_0$.
We first check that the initial condition is satisfied:

$$\mathbf{y}(0) = e^{0A}\left(\mathbf{y}_0 + \int_0^0 e^{-s}\mathbf{f}(s)ds\right) = I\mathbf{y}_0 = \mathbf{y}_0,$$

as required. We next use the product rule to check that the ODE is satisfied

$$\mathbf{y}(t) = e^{tA}\left(\mathbf{y}_0 + \int_0^t e^{-sA}\mathbf{f}(s)ds\right)$$

$$\Longrightarrow \quad \mathbf{y}'(t) = Ae^{tA}\left(\mathbf{y}_0 + \int_0^t e^{-sA}\mathbf{f}(s)ds\right) + e^{tA}\big(e^{-tA}\mathbf{f}(t)\big) = A\mathbf{y}(t) + \mathbf{f}(t).$$

## PS4-Problem 30 (20pts)

(a) *Find simple conditions on smooth functions $P = P(t)$, $Q = Q(t)$, and $a = a(t)$ that are equivalent to*

$$\big(Q(y'+ay)\big)' = P(y'' + py' + qy), \qquad p = p(t),\; q = q(t), \tag{3}$$

*for every smooth function $y = y(t)$.*
Expand LHS and compare with RHS:

$$\big(Q(y'+ay)\big)' = Qy'' + (Q'+Qa)y' + (Qa)'y = Py'' + Ppy' + Pqy \qquad \Longrightarrow$$

$$P = Q, \;\; Q' + Qa = pP, \;\; (Qa)' = qP \quad \Longleftrightarrow \quad \boxed{P = Q, \;\; P' + Pa = pP, \;\; (Pa)' = qP}$$

(b) *Find an integrating factor for the second-order ODE (eq3) with constant $p$ and $q$. Use it to find $\tilde{P} = \tilde{P}(t)$ and $R = R(t)$ such that*

$$\big(\tilde{P}(Ry)'\big)' = P(y'' + py' + qy), \qquad p, q = const.$$

By (a), we need to find a nonzero solution to the system

$$\begin{pmatrix} P \\ (Pa) \end{pmatrix}' = \begin{pmatrix} p & -1 \\ q & 0 \end{pmatrix} \begin{pmatrix} P \\ (Pa) \end{pmatrix} \qquad\qquad P{=}P(t),\ a{=}a(t). \tag{4}$$

The characteristic polynomial for the constant-coefficient matrix in (eq4) is $\lambda^2{-}p\lambda{+}q{=}0$. Let $\tilde\lambda_1$ and $\tilde\lambda_2$ be the two roots of this quadratic equation. Note that $\lambda_1{=}{-}\tilde\lambda_1$ and $\lambda_2{=}{-}\tilde\lambda_2$ must then be the roots of $\lambda^2{+}p\lambda{+}q{=}0$, i.e. the characteristic polynomial for the second-order ODE. The reason is that

$$\lambda_1+\lambda_2 = -(\tilde\lambda_1{+}\tilde\lambda_2) = -p \quad \text{and} \quad \lambda_1\cdot\lambda_2 = (-\tilde\lambda_1)\cdot(-\tilde\lambda_2) = \tilde\lambda_1\cdot\tilde\lambda_2 = q.$$

We next find an eigenvector for the eigenvalue $\tilde\lambda_1$:

$$\begin{pmatrix} p{-}\tilde\lambda_1 & -1 \\ q & -\tilde\lambda_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{cases} \tilde\lambda_2 c_1 - c_2 = 0 \\ qc_1 - \tilde\lambda_1 c_2 = 0 \end{cases} \implies \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \tilde\lambda_2 \end{pmatrix}$$

$$\implies \begin{pmatrix} P \\ (Pa) \end{pmatrix} = e^{\tilde\lambda_1 t}\begin{pmatrix} 1 \\ \tilde\lambda_2 \end{pmatrix} = \begin{pmatrix} e^{-\lambda_1 t} \\ -\lambda_2 e^{-\lambda_1 t} \end{pmatrix}$$

Thus, we can take $\boxed{P(t) = e^{-\lambda_1 t},\quad a(t) = -\lambda_2}$ By the above,

$$e^{-\lambda_1 t}(y''{+}py'{+}qy) = \big(e^{-\lambda_1 t}(y'{-}\lambda_2 y)\big)' = \big(e^{-\lambda_1 t}e^{\lambda_2 t}(e^{-\lambda_2 t}y)'\big)' = \big(e^{(\lambda_2-\lambda_1)t}(e^{-\lambda_2 t}y)'\big)', \tag{5}$$

where $\lambda_1$ and $\lambda_2$ are the roots of the characteristic polynomial associated to the ODE (eq3). The middle equality above is obtained from our knowledge of an integrating factor for a first-order ODE, especially one with a constant coefficient. The equality of the first and last terms in (eq5) recovers the formula used in the integrating-factor approach to solving any linear second-order ODE with constant coefficients.

(c) *If $p,q,r{=}const$, find functions $P{=}P(t){\neq}0$, $\tilde P{=}\tilde P(t)$, $Q{=}Q(t)$, and $R{=}R(t)$, such that*

$$\big(\tilde P\big(Q(Ry)'\big)'\big)' = P(y''' + py'' + qy' + ry)$$

*for all smooth function $y{=}y(t)$.*
We first find functions $P{=}P(t)$, $Q{=}Q(t)$, $a{=}a(t)$, and $b{=}b(t)$, such that

$$\begin{aligned} P(y'''{+}py''{+}qy'{+}ry) &= \big(Q(y''{+}ay'{+}by)\big)' = Qy''' + (Q'{+}Qa)y'' + \big((Qa)'{+}Qb\big)y' + (Qb)'y \\ \iff \qquad & P = Q,\quad P'{+}Pa = pP,\quad (Pa)'{+}(Pb) = qP,\ (Pb)' = rP. \end{aligned}$$

Thus, we need a nonzero solution to the ODE

$$\begin{pmatrix} P \\ (Pa) \\ (Pb) \end{pmatrix}' = \begin{pmatrix} p & -1 & 0 \\ q & 0 & -1 \\ r & 0 & 0 \end{pmatrix} \begin{pmatrix} P \\ (Pa) \\ (Pb) \end{pmatrix} \qquad\qquad P{=}P(t),\ a{=}a(t),\ b{=}b(t). \tag{6}$$

The characteristic polynomial for this equation is

$$\begin{aligned} \det\left(\begin{pmatrix} p & -1 & 0 \\ q & 0 & -1 \\ r & 0 & 0 \end{pmatrix} - \lambda I\right) &= \det\begin{pmatrix} p{-}\lambda & -1 & 0 \\ q & -\lambda & -1 \\ r & 0 & -\lambda \end{pmatrix} \\ &= (p{-}\lambda)(-\lambda)(-\lambda) + r(-1)(-1) - (-\lambda)q(-1) = -(\lambda^3 - p\lambda^2 + q\lambda - r). \end{aligned}$$

Let $\tilde\lambda_1$, $\tilde\lambda_2$, and $\tilde\lambda_3$ be the roots of this cubic polynomial. Note that $\lambda_1=-\tilde\lambda_1$, $\lambda_2=-\tilde\lambda_2$, and $\lambda_3=-\tilde\lambda_3$ must then be the roots of

$$\lambda^3+p\lambda^2+q\lambda+r=0,$$

i.e. the characteristic polynomial for the third-order ODE $y'''+py''+qy'+ry=f$, since

$$\begin{aligned}\lambda_1+\lambda_2+\lambda_3 &= -(\tilde\lambda_1+\tilde\lambda_2+\tilde\lambda_3)=-p, \quad \lambda_1\lambda_2\lambda_3=(-\tilde\lambda_1)(-\tilde\lambda_2)(-\tilde\lambda_3)=-\tilde\lambda_1\tilde\lambda_2\tilde\lambda_3=-r,\\ \text{and}\qquad \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3 &= (-\tilde\lambda_1)(-\tilde\lambda_2)+(-\tilde\lambda_1)(-\tilde\lambda_3)+(-\tilde\lambda_2)(-\tilde\lambda_3)=q.\end{aligned}$$

We next find an eigenvector for the eigenvalue $\tilde\lambda_1$ of the matrix in (eq6):

$$\begin{pmatrix} p-\tilde\lambda_1 & -1 & 0\\ q & -\tilde\lambda_1 & -1\\ r & 0 & -\tilde\lambda_1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \implies \begin{cases}(\tilde\lambda_2+\tilde\lambda_3)c_1-c_2=0\\ qc_1-\tilde\lambda_1c_2-c_3=0\\ rc_1-\tilde\lambda_1c_3=0\end{cases} \implies \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}1\\ \tilde\lambda_2+\tilde\lambda_3\\ \tilde\lambda_2\tilde\lambda_3\end{pmatrix}$$

$$\implies \begin{pmatrix}P\\(Pa)\\(Pb)\end{pmatrix}=e^{\tilde\lambda_1 t}\begin{pmatrix}1\\ \tilde\lambda_2+\tilde\lambda_3\\ \tilde\lambda_2\tilde\lambda_3\end{pmatrix}=\begin{pmatrix}e^{-\lambda_1 t}\\ -(\lambda_2+\lambda_3)e^{-\lambda_1 t}\\ \lambda_2\lambda_3e^{-\lambda_1 t}\end{pmatrix}$$

Thus, we can take $P(t)=e^{-\lambda_1 t}$, $a(t)=-(\lambda_2+\lambda_3)$, and $b(t)=\lambda_2\lambda_3$. By the above,

$$\begin{aligned} e^{-\lambda_1 t}(y'''+py''+qy'+ry) &= \big(e^{-\lambda_1 t}(y''-(\lambda_2+\lambda_3)y'+\lambda_2\lambda_3 y)\big)'\\ &= \big(e^{-\lambda_1 t}e^{\lambda_2 t}\big(e^{(\lambda_3-\lambda_2)t}(e^{-\lambda_3 t}y)'\big)'\big)'\\ &= \big(e^{(\lambda_2-\lambda_1)t}\big(e^{(\lambda_3-\lambda_2)t}(e^{-\lambda_3 t}y)'\big)'\big)',\end{aligned}\tag{7}$$

where $\lambda_1$, $\lambda_2$, and $\lambda_3$ are the roots of the characteristic polynomial associated to the ODE

$$y'''+py''+qy'+ry=f.$$

The middle equality in (eq7) is obtained from (eq5) with $\lambda_1$ and $\lambda_2$ replaced by $\lambda_2$ and $\lambda_3$, respectively. The equality of the first and last terms in (eq7) can be used to solve any linear third-order ODE with constant coefficients.

*Can you guess and prove the analogue of (eq7) for linear ODEs with constant coefficients of any order?*

## Section 9.6: 7,9

**9.6:7** *Determine whether the origin is an unstable, stable, or asymptotically stable equilibrium for the system*

$$\mathbf{y}'=\begin{pmatrix}1 & -4\\ 1 & -3\end{pmatrix}\mathbf{y}, \qquad \mathbf{y}=\mathbf{y}(t).$$

*Sketch the phase-plane portrait of solution curves.*

The characteristic polynomial for this system is

$$\lambda^2-(1+(-3))\lambda+\big(1\cdot(-3)-(-4)\cdot 1\big)=\lambda^2+2\lambda+1=(\lambda+1)^2.$$

Thus, the matrix has only one eigenvalue $\lambda = \lambda_1 = -1$. Since this eigenvalue is negative and it is the only eigenvalue, the origin is an $\boxed{\text{asymptotically stable}}$ point. It is a degenerate sink. The phase-plane portrait is similar to that in the first sketch of Figure 4, except the half-lines have slope .5, instead of 1.

**9.6:9** *Determine whether the origin is an unstable, stable, or asymptotically stable equilibrium for the system*

$$\mathbf{y}' = \begin{pmatrix} -3 & -4 & 2 \\ -2 & -7 & 4 \\ -3 & -8 & 4 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

The characteristic polynomial for this system is

$$\det \begin{pmatrix} -3-\lambda & -4 & 2 \\ -2 & -7-\lambda & 4 \\ -3 & -8 & 4-\lambda \end{pmatrix} = -\big(\lambda^3 + 6\lambda^2 + 11\lambda + 6) = -(\lambda+1)(\lambda+2)(\lambda+3).$$

All three eigenvalues $\lambda_1, \lambda_2, \lambda_3 = -1, -2, -3$ are negative. Thus, the origin is a nodal sink and an $\boxed{\text{asymptotically stable}}$ equilibrium point.

## Section 9.7: 17

*Find the general solution of the equation* $y^{(4)} + 36y = 13y''$.
The characteristic polynomial for $y^{(4)} - 13y'' + 36y = 0$ is:

$$\lambda^4 - 13\lambda^2 + 36 = (\lambda^2 - 4)(\lambda^2 - 9) = (\lambda+2)(\lambda-2)(\lambda+3)(\lambda-3).$$

It has four distinct roots: $\pm 2$, $\pm 3$. Thus, the general solution is:

$$\boxed{y(t) = C_1 e^{-3t} + C_2 e^{-2t} + C_3 e^{2t} + C_4 e^{3t}}$$