Math53: Ordinary Differential Equations
Winter 2004

Homework Assignment 6

Problem Set 6 is due by 2:15p.m. on Friday, 3/12, in 380Y

Problem Set 6:

PS6-Problem 1 (see next page); 10.3: 2,16*; 10.4: 2*,6*; 10.5: 6*; 10.6:10

**Note:* In 10.3:16, you do not need to use graph paper; sketch the nullclines as dashed/dotted curves; either make your plot very large or sketch solution curves on a separate plot, without the nullclines. In 10.5:6, sketch the phase-plane portrait. You do not have to use a numerical solver in any of the problems.

Daily Assignments:

Date	*Read*	*Exercises*
3/5 F	10.3,10.4	PS6-Problem 1; 10.3:2,16*; 10.4:2*,6*
3/8 M	10.5	10.5:6*
3/9 T	10.6	10.6:10
3/10 W		
3/11 R		
3/12 F		

**Note:* In 10.3:16, you do not need to use graph paper; sketch the nullclines as dashed/dotted curves; either make your plot very large or sketch solution curves on a separate plot, without the nullclines. In 10.5:6, sketch the phase-plane portrait. You do not have to use a numerical solver in any of the problems.

About the Last Few Days of Class

A portion of Tuesday may be devoted to reviewing Chapter 10. On Wednesday, I will probably go over Problems 3 and 4 from the second midterm, including some possible ways of avoiding mistakes on similar problems, and over related things. On Friday, I am planning to review all possible phase-plane portraits for planar linear systems with constant coefficients. Please let me know what you'd like to review on Thursday and/or if you'd to go over different topics on Wednesday or Friday. There will also be office hours and a discussion-style review session over the weekend preceding the final exam.

PS6-Problem 1

(a) Sketch the graph, in the $(y, f(y))$-plane, of the function

$$f(y) = (y+3)^3(y-1)^2(y-3).$$

Find the equilibrium solutions of and sketch the phase line, i.e. the y-line, for the one-dimensional autonomous ODE:

$$y' = (y+3)^3(y-1)^2(y-3).$$

Determine whether each equilibrium point is stable or unstable.

An n-dimensional autonomous ODE, or an autonomous system of n ODEs, has the form

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}), \tag{1}$$

where $\mathbf{y}=\mathbf{y}(t)$ is a column vector of n functions and $\mathbf{f}(\mathbf{y})$ is an n-vector for each n-vector $\mathbf{y}$. Similarly to the one-dimensional case, the equilibrium points $\mathbf{y}_i \in \mathbb{R}^n$ for (1), i.e. the constant solutions $\mathbf{y}(t)=\mathbf{y}_i$ of (1), are the zeros of $\mathbf{f}$. By Section 10.2, we can determine whether each point equilibrium $\mathbf{y}_i$ of (1) is asymptotically stable or unstable *if* the real part of every eigenvalue of the Jacobian $\mathcal{J}\mathbf{f}(\mathbf{y}_i) = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}\Big|_{\mathbf{y}_i}$ of $\mathbf{f}$ at $\mathbf{y}_i$ is nonzero. The one-dimensional version of this condition is that $f'(y_i) \neq 0$, in which case the stability situation is resolved by the derivate test of Section 1.9. In a one-dimensional case of (1), such as part (a), we can, however, always determine whether an equilibrium point y_i is asymptotically stable, stable, or unstable. All we need to do is to see whether f is positive, zero, or negative just to the left and just to the right of y_i. If $n \geq 2$, there is no left and right for a point $\mathbf{y}_i \in \mathbb{R}^n$, but there are still stability tests that always apply, even when the jacobian test does not.

Suppose $\mathbf{y}_i$ is an equilibrium point for (1) and V is a smooth function defined near $\mathbf{y}_i$ such that $V(\mathbf{y}_i)=0$. Show that
(b) if $V(\mathbf{x})>0$ and $\vec{\nabla} V|_{\mathbf{x}} \cdot \mathbf{f}(x) \leq 0$ for all $\mathbf{x} \neq \mathbf{y}_i$ near $\mathbf{y}_i$, then $\mathbf{y}_i$ is stable;
(c) if $V(\mathbf{x})>0$ and $\vec{\nabla} V|_{\mathbf{x}} \cdot \mathbf{f}(x) < 0$ for all $\mathbf{x} \neq \mathbf{y}_i$ near $\mathbf{y}_i$, then $\mathbf{y}_i$ is asymptotically stable;
(d) if $\vec{\nabla} V|_{\mathbf{x}} \cdot \mathbf{f}(x) > 0$ for all $\mathbf{x} \neq \mathbf{y}_i$ near $\mathbf{y}_i$ and there exists a sequence $\mathbf{x}_k \longrightarrow \mathbf{y}_i$ such that $V(\mathbf{x}_k)>0$ for all k, then $\mathbf{y}_i$ is unstable.

Hint: How does the function $V(\mathbf{y}(t))$ change with t if $\mathbf{y}=\mathbf{y}(t)$ is a solution to (1)?

(e) Find an appropriate function $V=V(y)$ for each of the three equilibrium points of the ODE in (a).
(f) Determine whether the origin is an asymptotically stable, stable, or unstable equilibrium for the following systems of ODEs:

$$\begin{cases} x' = -y + x^3 \\ y' = x + y^3 \end{cases} \qquad \text{and} \qquad \begin{cases} x' = -y - x^3 \\ y' = x - y^3 \end{cases}$$

Note 1: While the two systems in (f) have the same linearization at the origin, they have the opposite stability properties. This can happen only because the real part of an eigenvalue of the jacobian at the origin is zero.

Note 2: Parts (b)-(d) are closely related to Section 10.7, though it is neither necessary or sufficient for doing them.