Math53: Ordinary Differential Equations
Winter 2004

Midterm II Solutions

Problem 1 (25pts)

(a; 5pts) *Show that if $Y = Y(s)$ is the Laplace Transform of the function $y = y(t)$, then the Laplace Transforms of y'' and of y'''' are given by*

$$\{\mathcal{L}y''\}(s) = s^2Y(s) - sy(0) - y'(0), \qquad \{\mathcal{L}y''''\}(s) = s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0).$$

Since $\{\mathcal{L}f'\}(s) = s\{\mathcal{L}f\}(s) - f(0)$ by the second table,

$$\begin{aligned}\{\mathcal{L}y''\}(s) &= s\{\mathcal{L}y'\}(s) - y'(0) = s\big(s\{\mathcal{L}y\}(s) - y(0)\big) - y'(0) = s^2Y(s) - sy(0) - y'(0) \quad \Longrightarrow \\ \{\mathcal{L}y''''\}(s) &= s^2\{\mathcal{L}y''\}(s) - sy''(0) - y'''(0) = s^2\big(s^2Y(s) - sy(0) - y'(0)\big) - sy''(0) - y'''(0) \\ &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0).\end{aligned}$$

(b; 5pts) *Show that if $y = y(t)$ is the solution to the initial value problem*

$$y'''' + 2y'' + y = 9\cos 2t, \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -3, \quad y'''(0) = 0,$$

then the Laplace Transform $Y = Y(s)$ of y is given by

$$Y(s) = \frac{9s}{(s^2+4)(s^4+2s^2+1)} - \frac{3s}{s^4+2s^2+1}$$

Taking the Laplace Transform of both sides of the ODE, using (a) and the first table, and applying the initial conditions we obtain,

$$\begin{aligned}(s^4Y + 3s) + 2s^2Y + Y = 9\frac{s}{s^2+4} \quad &\Longrightarrow \quad (s^4+2s^2+1)Y = \frac{9s}{s^2+4} - 3s \\ \Longrightarrow \qquad Y(s) &= \frac{9s}{(s^2+4)(s^4+2s^2+1)} - \frac{3s}{s^4+2s^2+1}\end{aligned}$$

(c; 15pts) *Find the solution $y = y(t)$ to the initial value problem*

$$y'''' + 2y'' + y = 9\cos 2t, \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -3, \quad y'''(0) = 0.$$

By part (b),

$$\begin{aligned}Y = \mathcal{L}y = 3s\frac{3-(s^2+4)}{(s^2+4)(s^2+1)^2} &= 3s\frac{-1}{(s^2+4)(s^2+1)} = \frac{-3s}{4-1}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) = -\frac{s}{s^2+1} + \frac{s}{s^2+4} \\ \Longrightarrow \qquad & \boxed{y(t) = \cos 2t - \cos t}\end{aligned}$$

by the first table.

$f(t)$	$F(s) = \{\mathcal{L}f\}(s)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
δ	1

$f(t)$	$F(s) = \{\mathcal{L}f\}(s)$
f'	$s \cdot F(s) - f(0)$
$t \cdot f(t)$	$-F'(s)$
$e^{at}f(t)$	$F(s-a)$
$H(t-a)f(t-a)$	$e^{-as}F(s)$

Laplace Transforms

Problem 2 (15pts)

(a; 8pts) *Find all values of the constant c such that the origin in the xy-plane is a spiral sink or source for the solutions of the linear system*

$$\mathbf{y}' = \begin{pmatrix} 5 & c \\ -c & 1 \end{pmatrix} \mathbf{y}.$$

Specify whether the origin is a spiral source or a spiral sink for the values of c you find.
The origin is a spiral sink or source if and only if the eigenvalues of the matrix are complex, with nonzero real part. The characteristic polynomial in this case is

$$\lambda^2 - (\operatorname{tr} A)\lambda + (\det A) = \lambda^2 - 6\lambda + (5+c^2) \implies \lambda_1, \lambda_2 = 3 \pm \sqrt{3^2 - (5+c^2)} = 3 \pm \sqrt{4-c^2}.$$

Thus, we need $c^2 > 4$, or $\boxed{c \in (-\infty, -2), (2, \infty)}$ For these values of c, the origin is a $\boxed{\text{spiral source}}$ since the real part of the eigenvalues is positive.

(b; 7pts) *Sketch the phase-plane portraits, in the xy-plane, for the system of ODEs in (a) with the values of c you found. Clearly show all important qualitative information, including the direction of rotation. Each of your sketches should contain at least two solution curves. Explain your reasoning.*
Since the origin is a spiral *source*, the solution curves spiral *out from* the origin. The direction of rotation is determined by the matrix entry in the lower-left corner, i.e. $-c$. If this entry is positive, i.e. $c < 0$, the direction of rotation is positive, and vice versa.

y $\boxed{c < -2}$

x

y $\boxed{c > 2}$

x

Problem 3 (30pts)

(a; 20pts) *Find the general solution* $(x, y) = (x(t), y(t))$ *to the system of ODEs*

$$\begin{cases} x' = 4y - x \\ y' = x + 2y \end{cases}$$

This system of ODEs can be written as

$$\mathbf{y} = \begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{y} = \mathbf{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The characteristic polynomial for this matrix is

$$\lambda^2 - (\operatorname{tr} A)\lambda + (\det A) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

Thus, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$. We first find an eigenvector $\mathbf{v}_1$ for λ_1:

$$\begin{pmatrix} -1-\lambda_1 & 4 \\ 1 & 2-\lambda_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} -4c_1 + 4c_2 = 0 \\ c_1 - c_2 = 0 \end{cases} \iff c_1 = c_2 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We next find an eigenvector $\mathbf{v}_2$ for λ_2:

$$\begin{pmatrix} -1-\lambda_2 & 4 \\ 1 & 2-\lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} c_1 + 4c_2 = 0 \\ c_1 + 4c_2 = 0 \end{cases} \iff c_1 = -4c_2 \implies \mathbf{v}_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

Thus, the general solution to the system of ODEs is

$$\boxed{\mathbf{y}(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} C_1 e^{3t} + 4C_2 e^{-2t} \\ C_1 e^{3t} - C_2 e^{-2t} \end{pmatrix}}$$

(b; 8pts) *Sketch the phase-plane portrait, in the xy-plane, for the system of ODEs in (a).*

y

x

slopes of half-lines: 1 and $-1/4$
other solution curves approach
$y = x$ as $t \longrightarrow \infty$
$y = -x/4$ as $t \longrightarrow -\infty$

(c; 2pts) *Determine whether the origin is a stable, asymptotically stable, or an unstable equilibrium point. Explain why.*
Since one of the eigenvalues, λ_1, is positive, some, and in fact nearly all, solution curves move away from the origin. Thus, the origin is an $\boxed{\text{unstable}}$ equilibrium.

Problem 4 (30pts)

(a; 12pts) *Let A be a square matrix. Write down the power-series definition of e^A and use it to show that*

$$\text{if}\quad B=\begin{pmatrix}0&0\\2&0\end{pmatrix}\quad\text{and}\quad C=\begin{pmatrix}3&0\\0&3\end{pmatrix},\qquad\text{then}\qquad e^{tB}=\begin{pmatrix}1&0\\2t&1\end{pmatrix}\quad\text{and}\quad e^{tC}=\begin{pmatrix}e^{3t}&0\\0&e^{3t}\end{pmatrix}.$$

The power-series definition of e^A is

$$e^A=I+\frac{1}{1!}A+\frac{1}{2!}A^2+\frac{1}{3!}A^3+\ldots=\boxed{\sum_{k=0}^{k=\infty}\tfrac{1}{k!}A^k}$$

Applying this definition to tB and tC, we obtain

$$e^{tB}=I+tB+\frac{t^2}{2!}B^2+\frac{t^3}{3!}B^3+\ldots=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&0\\2t&0\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}+\ldots=\begin{pmatrix}1&0\\2t&1\end{pmatrix}$$

$$e^{tC}=\sum_{k=0}^{k=\infty}\frac{t^k}{k!}\begin{pmatrix}3&0\\0&3\end{pmatrix}^k=\sum_{k=0}^{k=\infty}\frac{t^k}{k!}\begin{pmatrix}3^k&0\\0&3^k\end{pmatrix}=\begin{pmatrix}\sum_{k=0}^{k=\infty}\frac{1}{k!}(3t)^k&0\\0&\sum_{k=0}^{k=\infty}\frac{1}{k!}(3t)^k\end{pmatrix}=\begin{pmatrix}e^{3t}&0\\0&e^{3t}\end{pmatrix}.$$

(b; 6pts) *Use any approach you can justify to show that*

$$\text{if}\qquad A=\begin{pmatrix}3&0\\2&3\end{pmatrix},\qquad\text{then}\qquad e^{tA}=\begin{pmatrix}e^{3t}&0\\2t\,e^{3t}&e^{3t}\end{pmatrix}.$$

Since $tC=(3t)I$, $(tB)(tC)=(tC)(tB)$ and thus

$$e^{tA}=e^{tB+tC}=e^{tB}e^{tC}=\begin{pmatrix}1&0\\2t&1\end{pmatrix}\begin{pmatrix}e^{3t}&0\\0&e^{3t}\end{pmatrix}=\begin{pmatrix}e^{3t}&0\\2t\,e^{3t}&e^{3t}\end{pmatrix}.$$

(c; 12pts) *Find the solution $\mathbf{y}=\mathbf{y}(t)$ to the initial value problem*

$$\mathbf{y}'=\begin{pmatrix}3&0\\2&3\end{pmatrix}\mathbf{y}-\begin{pmatrix}3\\6t\end{pmatrix},\qquad\mathbf{y}(0)=\begin{pmatrix}1\\0\end{pmatrix}.$$

Since $t_0=0$,

$$\begin{aligned}\mathbf{y}(t)&=e^{tA}\big(\mathbf{y}(0)+\int_0^t e^{-sA}\mathbf{f}(s)\,ds\big)=e^{tA}\left\{\begin{pmatrix}1\\0\end{pmatrix}-\int_0^t e^{-3s}\begin{pmatrix}1&0\\-2s&1\end{pmatrix}\begin{pmatrix}3\\6s\end{pmatrix}ds\right\}\\&=e^{tA}\left\{\begin{pmatrix}1\\0\end{pmatrix}-\int_0^t e^{-3s}\begin{pmatrix}3\\0\end{pmatrix}ds\right\}=e^{tA}\left\{\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}e^{-3s}\Big|_{s=0}^{s=t}\right\}\\&=e^{3t}\begin{pmatrix}1&0\\2t&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}e^{-3t}=\boxed{\begin{pmatrix}1\\2t\end{pmatrix}}\end{aligned}$$