

# Math53: Ordinary Differential Equations Winter 2004

## Midterm I Solutions

### Problem 1 (20pts)

(a; 15pts) Find the general solution  $y=y(t)$  to the ODE

$$y' = \cos t - y \cos t.$$

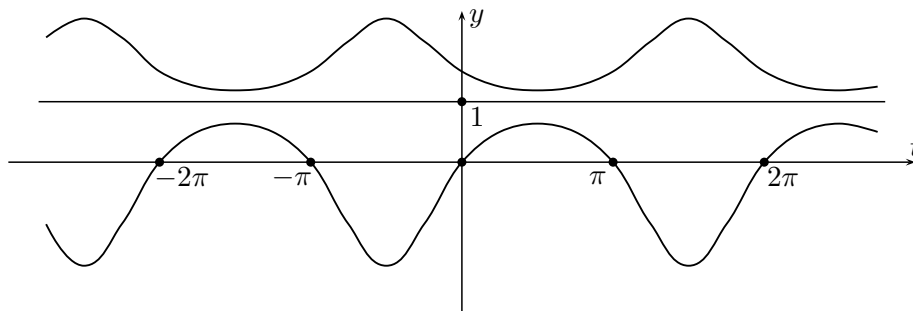
Sketch three solution curves.

This ODE, which is as in 2.4:13, can be treated as either linear or separable. If treated as linear:

$$\begin{aligned} y' + (\cos t)y &= \cos t \implies P(t) = e^{\int \cos t dt} = e^{\sin t}; \quad e^{\sin t}(y' + (\cos t)y) = e^{\sin t} \cos t \\ &\implies (e^{\sin t}y)' = e^{\sin t} \cos t \implies e^{\sin t}y = \int e^{\sin t} \cos t dt \\ &\implies e^{\sin t}y = e^{\sin t} + C \implies \boxed{y(t) = 1 + Ce^{-\sin t}} \end{aligned}$$

If treated as separable:

$$\begin{aligned} \frac{dy}{dt} &= (1-y) \cos t \implies \frac{dy}{1-y} = \cos t dt \implies \int \frac{dy}{1-y} = \int \cos t dt \implies -\ln|1-y| = \sin t + C \\ &\implies |1-y| = e^{-\sin t - C} \implies 1-y = \pm e^{-C} e^{-\sin t} \implies \boxed{y(t) = 1 + Ae^{-\sin t}} \end{aligned}$$



(b; 5pts) Find the solution  $y=y(t)$ , including the interval of existence, to the initial value problem

$$y' = \cos t - y \cos t, \quad y(\pi) = 3.$$

We need to find  $C$  (or  $A$ ) such that  $y(\pi) = 1 + Ce^{-\sin \pi} = 3$ . The corresponding (as well as any other) solution of the ODE is defined for all  $t$ . Thus,  $\boxed{y(t) = 1 + 2e^{-\sin t}, t \in (-\infty, \infty)}$

## Problem 2 (25pts)

(a; 15pts) Show that the ODE

$$2t - y^2 + (y^3 - 2ty)y' = 0$$

is exact and solve it for  $y=y(t)$ , implicitly or explicitly.

Since  $(2t - y^2)_y = -2y$  and  $(y^3 - 2ty)_t = -2y$ , these two partial derivatives are equal. Since  $2t - y^2$  and  $y^3 - 2ty$  are defined for all  $t$  and  $y$ , it follows that the ODE is exact. In order to solve it, we need to find  $H = H(t, y)$  such that  $H_t = (2t - y^2)$  and  $H_y = (y^3 - 2ty)$ :

$$\begin{aligned} H_t(t, y) = 2t - y^2 &\implies H(t, y) = \int (2t - y^2) dt = t^2 - ty^2 + \phi(y) \\ H_y(t, y) = y^3 - 2ty &\implies 0 - 2ty + \phi'(y) = y^3 - 2ty \implies \phi'(y) = y^3 \\ &\implies \phi(y) = \int y^3 dy = \frac{1}{4}y^4 \implies H(t, y) = \frac{1}{4}y^4 - ty^2 + t^2. \end{aligned}$$

Thus, the general solution  $y=y(t)$  of the above ODE is implicitly defined by

$$\boxed{\frac{1}{4}y^4 - ty^2 + t^2 = C} \quad \text{or} \quad \boxed{y^4 - 4ty^2 + 4t^2 = C}$$

(b; 10pts) Find an explicit solution  $y=y(t)$ , including the interval of existence, to the initial value problem

$$2t - y^2 + (y^3 - 2ty)y' = 0, \quad y(1) = 1.$$

We first need to find  $C$  such that  $(t, y) = (1, 1)$  solves  $y^4 - 4ty^2 + 4t^2 = C$ :

$$C = 1^4 - 4 \cdot 1 \cdot 1^2 + 4 \cdot 1^2 = 1.$$

We next solve the resulting equation for  $y$ :

$$y^4 - 4ty^2 + 4t^2 = 1 \implies (y^2 - 2t)^2 = 1 \implies y^2 - 2t = -1 \implies y(t) = \sqrt{2t - 1}.$$

Note that at the first stage we need to take the negative square root of RHS, since  $1^2 - 2 \cdot 1 = -1$ , while at the second stage we need to take the positive square root. Since  $y = y(t)$  is defined for  $t \geq 1/2$ , we conclude that

$$\boxed{y(t) = \sqrt{2t - 1}, \quad t \in (1/2, \infty)}$$

**Problem 3 (30pts)**

(a; 7pts) Find the general solution  $y=y(t)$  to the ODE

$$y'' - 4y' + 4y = 0.$$

The characteristic polynomial for this ODE is

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

Since the two roots  $\lambda_1, \lambda_2 = 2$ , the general solution of this ODE is given by

$$y(t) = C_1 e^{2t} + C_2 t e^{2t}$$

(b; 15pts) Find a solution  $y=y(t)$  to the ODE

$$y'' - 4y' + 4y = 16 \sin 2t.$$

One way to find a solution  $y_p$  for this ODE is to find a solution  $z_p$  of

$$z'' - 4z' + 4z = 16e^{2it}$$

and then take  $y_p = \text{Im } z_p$ . We try  $z_p = ae^{2it}$ :

$$\begin{aligned} z'_p &= 2iae^{2it}, \quad z''_p = (2i)^2 ae^{2it} = -4ae^{2it} &\implies & -4ae^{2it} - 4 \cdot 2iae^{2it} + 4 \cdot ae^{2it} = 16e^{2it} \\ \implies & -8ia = 16 \implies a = 2i \implies z_p = 2ie^{2it} = 2i(\cos 2t + i \sin 2t) = -2 \sin 2t + 2i \cos 2t \\ & \implies y_p = \text{Im } z_p = 2 \cos 2t \end{aligned}$$

Another way is to try  $y_p = a \cos 2t + b \sin 2t$ :

$$\begin{aligned} y'_p &= -2a \sin 2t + 2b \cos 2t, \quad y''_p = -4a \cos 2t - 4b \sin 2t \\ \implies & (-4a \cos 2t - 4b \sin 2t) - 4(-2a \sin 2t + 2b \cos 2t) + 4(a \cos 2t + b \sin 2t) = 16 \sin 2t \\ \implies & 8a = 16, \quad -8b = 0 \implies a = 2, \quad b = 0 \implies y_p = 2 \cos 2t \end{aligned}$$

Yet another way is to observe that

$$(\cos 2t)'' - 4(\cos 2t)' + 4 \cos 2t = 8 \sin 2t, \quad (\sin 2t)'' - 4(\sin 2t)' + 4 \sin 2t = -8 \cos 2t \implies y_p = 2 \cos 2t$$

(c; 8pts) Find the solution  $y=y(t)$  to the initial value problem

$$y'' - 4y' + 4y = 24 \sin 2t, \quad y(0) = 0, \quad y'(0) = 0.$$

Since  $y = 2 \cos 2t$  is a solution of  $y'' - 4y' + 4y = 16 \sin 2t$ , a solution of

$$y'' - 4y' + 4y = 24 \sin 2t$$

is given by  $y_p = \frac{24}{16} \cdot 2 \cos 2t = 3 \cos 2t$ . Thus, the general solution of  $y'' - 4y' + 4y = 24 \sin 2t$  is

$$y = C_1 e^{2t} + C_2 t e^{2t} + 3 \cos 2t,$$

using part (a). We need to find  $C_1$  and  $C_2$  such that

$$\begin{aligned} y(0) &= C_1 + 0 + 3 = 0, \quad y'(0) = 2C_1 + C_2 - 6 \cdot 0 = 0 \implies C_1 = -3, \quad C_2 = 6 \\ \implies & y(t) = -3e^{2t} + 6te^{2t} + 3 \cos 2t, \quad t \in (-\infty, \infty) \end{aligned}$$

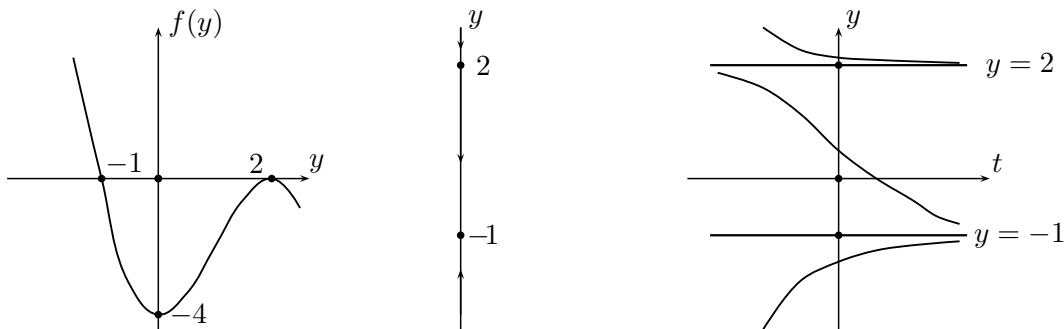


Figure 1: Plots for Problem 4

**Problem 4 (25pts)**

(a; 5pts) *Sketch the graph of the function*

$$f(y) = -(y + 1)(y - 2)^2.$$

*Label all its intercepts with the  $y$ -axis and the  $f(y)$ -axis.*

See the first plot in Figure 1.

(b; 5pts) *Find the equilibrium solutions of the ODE*

$$y' = -(y + 1)(y - 2)^2$$

*and sketch their graphs in the  $ty$ -plane.*

The equilibrium, or constant solutions, are  $y = y^*$  such that  $f(y^*) = 0$ . In this case, the equilibrium solutions are  $y = -1, y = 2$ . Their graphs are the horizontal lines  $y = -1$  and  $y = 2$ , shown in the last plot of Figure 1.

(c; 15pts) *On the same plot, sketch at least one solution curve of the ODE*

$$y' = -(y + 1)(y - 2)^2$$

*in each region of the  $ty$ -plane cut out by the graphs of the equilibrium solutions. Indicate their asymptotic behavior, i.e. as  $t \rightarrow \pm\infty$ . Explain your reasoning. Determine whether each of the equilibrium solutions is asymptotically stable or unstable. Draw the phase line.*

Since no two solution curves can cross, no solution curve can cross the horizontal lines  $y = -1$  and  $y = 2$ . Thus, if  $y(t_0) < -1$  for some  $t_0$ ,  $y(t) < -1$  for all  $t$ . It follows that in this case  $y'(t) > 0$  for all  $t$ , as can be seen either from the graph of  $f$  or directly from its definition. Thus, the solution curves in the bottom region ascend and approach the horizontal line  $y = -1$  as  $t \rightarrow \infty$  and drop to  $-\infty$  as  $t \rightarrow -\infty$ . By the same reasoning, if  $-1 < y(t_0) < 2$  for some  $t_0$ ,  $-1 < y(t) < 2$  and  $y'(t) < 0$  for all  $t$ , and the solution curves in the middle region descend. They approach the horizontal lines  $y = -1$  and  $y = 2$  as  $t \rightarrow \infty$  and  $t \rightarrow -\infty$ , respectively. Finally, if  $y(t_0) > 2$  for some  $t_0$ ,  $y(t) > 2$  and  $y'(t) < 0$  for all  $t$ , and the solution curves in the top region also descend. They approach the horizontal line  $y = 2$  as  $t \rightarrow \infty$  and rise to  $\infty$  as  $t \rightarrow -\infty$ ; see the third plot of Figure 1. The phase line, i.e. the middle plot of Figure 1, encodes what happens to the solution curves in each region by arrows. The equilibrium solution  $y = -1$  is stable since both arrows around the point  $y = -1$  on the phase line point toward it. Since this is not the case for  $y = 2$ , the equilibrium solution  $y = 2$  is unstable.