Math53: Ordinary Differential Equations
Autumn 2004

Problem Set 7 Solutions

Note: Even if you have done every problem, you are encouraged to look over these solutions, especially 6.1:2,18 and 6.2:2, where the computations are arranged into tables. In the second part of 6.1:18, the IVP is solved using only the complex form of the general solution.

Section 6.1: 2,18 (22pts)

6.1:2; 8pts: *For the initial value problem*

$$y' = y, \qquad y(0) = 1,$$

compute the first five iterations of Euler's method with step size $h = 0.1$. Then solve the initial value problem exactly and compare the obtained estimate for $y(0.5)$ with its exact value.
We start with $t_0 = 0$, $y_0 = 1$ and $f(t, y) = y$.
In the first iteration, we get that $t_1 = t_0 + h = 0.1$, $y_1 = y_0 + y_0 h = 1.1$.
In the second iteration we get that $y_2 = y_1 + y_1 h = 1.21$ and $t_2 = t_1 + h = 0.2$ and so on.
The first five iterations are given in the following table:

k	t_k	y_k	$f(t_k, y_k) = y_k$	$f(t_k, y_k)h$
0	0.0	1.0000	1.0000	0.1000
1	0.1	1.1000	1.1000	0.1100
2	0.2	1.2100	1.2100	0.1210
3	0.3	1.3310	1.3310	0.1331
4	0.4	1.4641	1.4641	0.1464
5	0.5	1.6105	–	–

The exact value of the solution $y(t) = e^t$ at .5 is $e^{1/2} \approx 1.6487$.

6.1:18; 14pts: *For the initial value problem*

$$x' = y, \quad y' = -x, \qquad x(0) = 1, \quad y(0) = -1,$$

compute the first five iterations of Euler's method with step size $h = 0.1$. Then solve the initial value problem exactly and compare the obtained estimates for $x(0.5)$ and $y(0.5)$ with their exact values.
We start with $t_0 = 0$, $x_0 = 1$, and $y_0 = -1$. We also have that $f(t, x, y) = y$ and $g(t, x, y) = -x$, so from here, the iteration proceeds with

$$y_{k+1} = x_k + y_k h \qquad \text{and} \qquad x_{k+1} = y_k - x_k h.$$

The first five iterations are arranged in the following table:

t_k	x_k	y_k	$f(t_k,x_k,y_k)h=y_kh$	$g(t_k,x_k,y_k)h=-x_kh$
0.0	1.0000	-1.0000	-0.1000	-0.1000
0.1	0.9000	-1.1000	-0.1100	-0.0900
0.2	0.7900	-1.1900	-0.1190	-0.0790
0.3	0.6710	-1.2690	-0.1269	-0.0671
0.4	0.5441	-1.3361	-0.1336	-0.0544
0.5	0.4105	-1.3905	$-$	$-$

In order to solve this problem exactly, we re-write the IVP as

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The characteristic polynomial for this equation is $\lambda^2+1=0$. Its roots are $\lambda_1,\lambda_2=\pm i$. We first find an eigenvector for λ_1:

$$\begin{pmatrix} 0-i & 1 \\ -1 & 0-i \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} -ic_1 + c_2 = 0 \\ -c_1 - ic_2 = 0 \end{cases} \iff c_2 = ic_1 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

The complex conjugate of $\mathbf{v}_1$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, must then be an eigenvector with eigenvalue $\lambda_2=\bar{\lambda}_1$. Thus, the general solution to the system of ODEs is

$$\mathbf{y}(t) = C_1e^{\lambda_1 t}\mathbf{v}_1 + C_2e^{\lambda_2 t}\mathbf{v}_2 = C_1e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} + C_2e^{-it}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Plugging in the initial condition, we obtain

$$\mathbf{y}(0) = C_1\begin{pmatrix} 1 \\ i \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \iff \begin{cases} C_1 + C_2 = 1 \\ iC_1 - iC_2 = -1 \end{cases} \iff \begin{cases} C_1 + C_2 = 1 \\ C_1 - C_2 = i \end{cases} \iff \begin{cases} C_1 = \frac{1+i}{2} \\ C_2 = \frac{1-i}{2} \end{cases}$$

$$\implies \mathbf{y}(t) = \frac{1+i}{2}e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1-i}{2}e^{-it}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{e^{it}+e^{-it}}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{e^{it}-e^{-it}}{2}i\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \cos t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + (i\sin t)i\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos t - \sin t \\ -\cos t - \sin t \end{pmatrix}.$$

The value of the last expression at .5 Radians is $\mathbf{y}(.5) \approx \begin{pmatrix} .398 \\ -1.357 \end{pmatrix}$.

Note that in the above IVP we never needed to use the real form of the general solution. We found the two constants C_1 and C_2 for the complex form. With these constants, the corresponding complex expression automatically reduces to a real one. The key formulas to remember are

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i};$$

they follow from $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$.

Section 6.2:2 (8pts)

For the initial value problem

$$y' = y, \qquad y(0) = 1,$$

compute the first five iterations of the second-order Runge-Kutta method with step size $h=0.1$ and compare the obtained estimate for $y(0.5)$ with its exact value.

We begin with $t_0=0$, $y_0=1$, and $f(t,y)=y$. Thus, the initial slopes are

$$s_{0,1} = f(0,1) = 1 \qquad \text{and} \qquad s_{0,2} = f\big(t_0+h, y_0+s_{0,1}h\big) = f(0.1, 1.1) = 1.1.$$

From here, we iterate using:

$$s_{k,1} = f(t_k, y_k) = y_k, \qquad s_{k,2} = f\big(t_k+h, y_k+s_{k,1}h\big) = y_k + s_{k,1}h,$$
$$y_{k+1} = y_k + \frac{s_{k,1}+s_{k,2}}{2}h, \qquad t_{k+1} = t_k + h.$$

The first five iterations are presented in the following table:

t_k	y_k	$s_{k,1}$	$s_{k,2}$	$\frac{s_{k,1}+s_{k,2}}{2}h$
0.0	1.0000	1.0000	1.1000	0.1050
0.1	1.1050	1.1050	1.2155	0.1160
0.2	1.2210	1.2210	1.3431	0.1282
0.3	1.3492	1.3492	1.4842	0.1417
0.4	1.4909	1.4909	1.6400	0.1565
0.5	1.6474	–	–	–

Just as in 6.1:2, the exact value of $y(.5)$ is $e^{1/2}\approx 1.6487$. So, the approximation obtained after just five iterations, 1.6474, is quite good. Compare this with Euler's method!

Problem F (20pts)

(a; **7pts**) *Suppose y and $\tilde{y}$ are smooth functions on the interval $[c,d]$ and M is a positive number such that*

$$\big|y''(t)\big|, \big|\tilde{y}''(t)\big| \le M \qquad \text{for all} \qquad t \in [c,d].$$

Show that

$$\big|y(d) - \tilde{y}(d)\big| \le \big|y(c) - \tilde{y}(c)\big| + \big|y'(c) - \tilde{y}'(c)\big||d-c| + M|d-c|^2.$$

We will apply FTC to the function

$$z(t) = y(t) - \tilde{y}(t)$$

and its derivative to estimate the change in $z(t)$ from $t=c$ to $t=d$. We first note

$$|z''(s)| = \big|y''(s) - \tilde{y}''(s)\big| \le |y''(s)| + |\tilde{y}''(s)| \le M + M = 2M \quad \text{for all} \quad s \in [c,d],$$

by our assumption on y and $\tilde{y}$. On the other hand, by FTC, for all $t\in[c,d]$.

$$z'(t) = z'(c) + \int_c^t z''(s)\,ds \quad\Longrightarrow\quad \begin{aligned} \big|z'(t)\big| &\le \big|z'(c)\big| + \Big|\int_c^t z''(s)\,ds\Big| \le \big|z'(c)\big| + \int_c^t |z''(s)|\,ds \\ &\le \big|z'(c)\big| + 2M|t-c| = \big|z'(c)\big| + 2M(t-c). \end{aligned} \tag{1}$$

Similarly, by FTC,

$$\begin{aligned} z(d) = z(c) + \int_c^d z'(t)\,dt \qquad &\Longrightarrow \\ |z(d)| \le |z(c)| + \Big|\int_c^d z'(t)\,dt\Big| &\le |z(c)| + \int_c^d |z'(t)|\,dt \\ &\le |z(c)| + \int_c^d \big(|z'(c)|+2M(t-c)\big)\,dt = |z(c)| + |z'(c)||d-c| + M|d-c|^2, \end{aligned}$$

by (1). Since $z(t)=y(t)-\tilde{y}(t)$, we conclude that

$$\big|y(d) - \tilde{y}(d)\big| \le \big|y(c) - \tilde{y}(c)\big| + \big|y'(c) - \tilde{y}'(c)\big||d-c| + M|d-c|^2.$$

Suppose now that $f=f(t,y)$ is a smooth function and M_0, M_t, and M_y are positive numbers such that

$$\big|f(t,y)\big| \le M_0, \quad \big|f_t(t,y)\big| \le M_t, \quad \big|f_y(t,y)\big| \le M_y \qquad \text{for all} \qquad t\in[a,b],\ y\in(-\infty,\infty).$$

Let $y=y(t)$ be the solution to the initial value problem

$$y' = f(t,y), \qquad y(a) = y_0. \tag{2}$$

Given a positive integer N, let

$$\begin{gathered} h = \frac{b-a}{N}, \quad t_0 = a, \quad t_{i+1} = t_i + h = h\cdot(i+1), \quad s_i = f(t_i,y_i), \quad y_{i+1} = y_i + s_i h; \\ \epsilon_i = \big|y(t_i) - y_i\big|, \qquad \tilde{y}_i(t) = y_i + s_i(t-t_i). \end{gathered}$$

Note that

$$\epsilon_0 = 0, \quad \epsilon_N = y(b) - y_N, \quad \tilde{y}_i(t_i) = y_i, \quad \tilde{y}_i(t_{i+1}) = y_{i+1}, \quad \tilde{y}_i'(t_i) = s_i, \quad \tilde{y}_i''(t) = 0.$$

(b; **6pts**) *Use the ODE and the assumptions on f to show that*

$$|y''(t)| \le M_t + M_0 M_y \qquad \text{and} \qquad \big|y'(t_i) - \tilde{y}_i'(t_i)\big| \le M_y \epsilon_i.$$

Since $y'(t)=f(t,y(t))$, by the chain rule

$$\begin{aligned} y''(t) &= \frac{d}{dt} f\big(t,y(t)\big) = f_t\big(t,y(t)\big) + f_y\big(t,y(t)\big)\cdot y'(t) = f_t\big(t,y(t)\big) + f_y\big(t,y(t)\big)\cdot f\big(t,y(t)\big) \\ &\Longrightarrow \qquad \big|y''(t)\big| \le \big|f_t(t,y(t))\big| + \big|f(t,y(t))\big|\big|f_y(t,y(t))\big| \le M_t + M_0 M_y, \end{aligned}$$

by our assumptions on f. On the other hand, by the same argument as in the first part of (a),

$$\big|y'(t_i) - \tilde{y}_i'(t_i)\big| = \big|f(t_i, y(t_i)) - f(t_i, y_i)\big| \le M_y\big|y(t_i) - y_i\big| = M_y\epsilon_i.$$

(c; **3pts**) *Use part (a) to show that*

$$\epsilon_{i+1} \le \epsilon_i + M_y\epsilon_i h + (M_t + M_0 M_y)h^2.$$

By parts (a) and (b),

$$\begin{aligned}\epsilon_{i+1} = \big|y(t_{i+1}) - y_{i+1}\big| &= \big|y(t_{i+1}) - \tilde{y}_i(t_{i+1})\big| \\ &\le \big|y(t_i) - \tilde{y}_i(t_i)\big| + \big|y'(t_i) - \tilde{y}_i'(t_i)\big||t_{i+1} - t_i| + (M_t + M_0 M_y)|t_{i+1} - t_i|^2 \\ &\le \epsilon_i + M_y\epsilon_i h + (M_t + M_0 M_y)h^2.\end{aligned}$$

(d; **4pts**) *Conclude that*

$$\epsilon_N \le \big(M_t + M_0 M_y\big)\frac{(1 + M_y h)^N - 1}{M_y}h \le \frac{M_t + M_0 M_y}{M_y}\big(e^{M_y(b-a)} - 1\big)h.$$

By part (c),

$$\begin{aligned}\epsilon_N &\le (M_t + M_0 M_y)h^2 + (1 + M_y h)\epsilon_{N-1} \\ &\le (M_t + M_0 M_y)h^2 + (1 + M_y h)(M_t + M_0 M_y)h^2 + (1 + M_y h)^2\epsilon_{N-2} \le \ldots \\ &\le (M_t + M_0 M_y)h^2 + (1 + M_y h)(M_t + M_0 M_y)h^2 + \ldots + (1 + M_y h)^{N-1}(M_t + M_0 M_y)h^2 + (1 + M_y h)^N\epsilon_0.\end{aligned}$$

Since $\epsilon_0 = 0$, it follows that

$$\begin{aligned}\epsilon_N &\le (M_t + M_0 M_y)h^2\big(1 + (1 + M_y h) + \ldots + (1 + M_y h)^{N-1}\big) \\ &\le (M_t + M_0 M_y)h^2\frac{(1 + M_y h)^N - 1}{(1 + M_y h) - 1} = (M_t + M_0 M_y)\frac{(1 + M_y h)^N - 1}{M_y}h.\end{aligned} \tag{3}$$

In order to obtain the final statement, recall that one definition of the number e is

$$e = \lim_{N \longrightarrow \infty}\Big(1 + \frac{1}{N}\Big)^N \qquad \Longrightarrow \qquad \lim_{N \longrightarrow \infty}\Big(1 + \frac{c}{N}\Big)^N = e^c \quad \text{for all} \quad c.$$

Furthermore, the sequence $(1 + c/N)^N$ is increasing with N, if $c > 0$. Since $h = (b-a)/N$, it follows from (3) that

$$\epsilon_N \le \frac{M_t + M_0 M_y}{M_y}\bigg(\Big(1 + \frac{M_y(b-a)}{N}\Big)^N - 1\bigg)h \le \frac{M_t + M_0 M_y}{M_y}\big(e^{M_y(b-a)} - 1\big)h.$$