Math53: Ordinary Differential Equations
Autumn 2004

Solutions to Problem Set 2

Note: Even if you have done every problem, you are encouraged to look over these solutions, especially 2.9:20,26,28, 4.3:26, B-(b). In the last two problems, complex numbers are used to simplify computations.

Section 2.6, Problems 10,14,26,36 (25pts)

2.6: 10; 6pts: *Determine whether the equation*

$$(1 - y\sin x)dx + (\cos x)dy = 0$$

is exact. If it is, solve it.
With $P(x,y) = 1 - y\sin x$ and $Q(x,y) = \cos x$, we get:

$$\frac{\partial P}{\partial y} = -\sin x = \frac{\partial Q}{\partial x}.$$

Thus, the equation is exact. We solve it by setting

$$F(x,y) = \int P(x)dx = \int (1 - y\sin x)dx = x + y\cos x + \phi(y);$$

$$\Longrightarrow \quad \cos x = Q(x,y) = \frac{\partial F}{\partial y} = \cos x + \phi'(y) \quad \Longrightarrow \quad \phi'(y) = 0 \quad \Longrightarrow \quad \phi(y) = C;$$

$$\Longrightarrow \quad F(x,y) = x + y\cos x + C.$$

Thus, the solution is $\boxed{F(x,y) = x + y\cos x + C = 0}$ or $\boxed{F(x,y) = x + y\cos x = C}$

2.6: 14; 3pts: *Determine whether the equation $dy/dx = x/(x-y)$ is exact. If it is, solve it.*
Rewrite the equation as $(x)dx + (y-x)dy = 0$. Then, $P(x,y) = x$, $Q(x,y) = y-x$, and

$$\frac{\partial P}{\partial y} = 0 \neq -1 = \frac{\partial Q}{\partial x}.$$

Thus, the equation is $\boxed{\text{not exact}}$

2.6: 26; 8pts: *The equation*

$$y\,dx + (x^2y - x)\,dy = 0$$

is not exact. Suppose it has an integrating factor that is a function of x alone. Find the integrating factor and use it to solve the equation.
Let $\mu(x)$ be the integrating factor, so the equation becomes

$$\mu(x)y\,dx + \mu(x)(x^2y - x)\,dy = 0.$$

In order for this equation to be exact, we need:

$$
\begin{aligned}
&\frac{\partial}{\partial y}(\mu(x)y) = \frac{\partial}{\partial x}(\mu(x)(x^2y-x))\\
\Longrightarrow\ &\mu(x) = \mu'(x)(x^2y-x) + \mu(x)(2xy-1)\\
\Longrightarrow\ &2\mu(x)(1-xy) = x\mu'(x)(xy-1) \quad \Longrightarrow\ \mu(x) = -\frac{1}{2}x\mu'(x).
\end{aligned}
$$

This is a separable equation on $\mu = \mu(x)$:

$$
\begin{aligned}
\frac{d\mu}{dx} = -\frac{2}{x}\mu \ \Longrightarrow\ \frac{d\mu}{\mu} = -\frac{2dx}{x} \ &\Longrightarrow\ \int\frac{d\mu}{\mu} = -\int\frac{2dx}{x}\\
&\Longrightarrow\ \ln\mu = -2\ln x \ \Longrightarrow\ \mu(x) = x^{-2}.
\end{aligned}
$$

Note that we need to find only one integrating factor. After multiplying the original equation by $\mu(x)$, we get the exact equation

$$
\frac{y}{x^2}\,dx + \left(y - \frac{1}{x}\right)dy = 0 \quad \Longrightarrow \quad F(x,y) = \int\frac{y}{x^2}dx = -\frac{y}{x} + \phi(y).
$$

To find ϕ, differentiate F with respect to y:

$$
\begin{aligned}
y - \frac{1}{x} = \frac{\partial F}{\partial y}(x,y) = -\frac{1}{x} + \phi'(y) \quad &\Longrightarrow \quad \phi'(y) = y \quad \Longrightarrow \quad \phi(y) = \frac{y^2}{2} + C\\
&\Longrightarrow \quad F(x,y) = -\frac{y}{x} + \frac{y^2}{2} \quad \Longrightarrow \quad \boxed{-\tfrac{y}{x} + \tfrac{y^2}{2} + C = 0}
\end{aligned}
$$

2.6: 36; 8pts: *Solve the homogeneous equation* $(x+y)dx + (y-x)dy = 0$.
After making the substitution $y = xv$, we get

$$
\begin{aligned}
(x+y)dx + (y-x)dy = 0 \ &\Longleftrightarrow\ (1+v)x\,dx + (v-1)x\,(v\,dx + x\,dv) = 0\\
&\Longleftrightarrow\ (1+v^2)x\,dx + (v-1)x^2\,dv = 0 \ \Longleftrightarrow\ \frac{dx}{x} = \frac{(1-v)dv}{1+v^2}\\
&\Longleftrightarrow\ \int\frac{dx}{x} = \int\frac{dv}{1+v^2} - \int\frac{v\,dv}{1+v^2}\\
&\Longrightarrow\ \ln|x| = \arctan v - \frac{1}{2}\ln|1+v^2| + C\\
&\Longrightarrow\ \ln|x| + \frac{1}{2}\ln\left(\frac{x^2+y^2}{x^2}\right) - \arctan\left(\frac{y}{x}\right) = C\\
&\Longrightarrow\ \boxed{\ln(x^2+y^2) - 2\arctan(y/x) = C}
\end{aligned}
$$

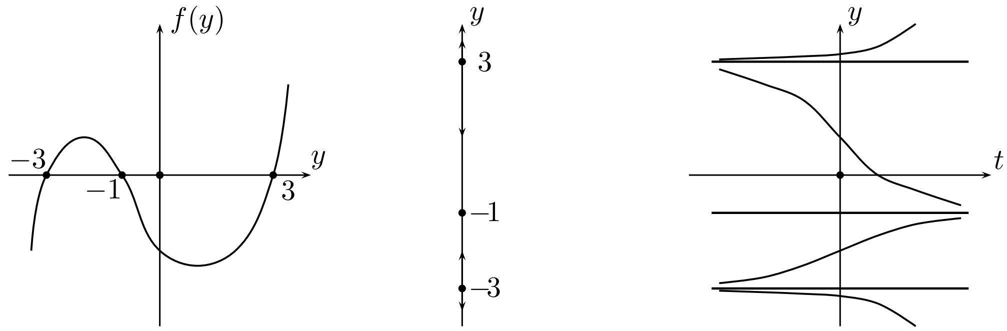

Figure 1: Plots for Problem 2.9:20: (i),(ii),(iii)

Section 2.9, Problems 20,26,28 (23pts)

2.9: 20; 9pts: *For the autonomous differential equation*

$$y' = f(y) = (y+1)(y^2-9),$$

sketch the graph of $f(y)$ and use it to develop a phase line and to classify each equilibrium point as either unstable or asymptotically stable. Sketch the equilibrium solutions in the (t, y)-plane and at least one solution trajectory in each plane region bounded by these equilibrium solutions.
Since $f(y) = (y+3)(y+1)(y-3)$, the equilibrium solutions are $y(t) = -3$, $y(t) = -1$, and $y(t) = 3$. The corresponding solution curves are horizontal lines. Other solution curves cannot cross these lines. Thus, a solution curve that starts in one of the four bands must stay there. The graph of $f(y)$ in Figure 1 shows the sign of $f(y)$ and the sign of $y'(t)$ in each of the four regions, i.e. whether solution curves rise or descend in each region. For example, if $-3 < y(t) < -1$, then $y' = f(y)$ is positive, and $y(t)$ increases toward $y = -1$. It descends toward $y = -3$ as $t \longrightarrow -\infty$. This information about what happens to $y(t)$ is indicated on the vertical phase line in the middle of Figure 1. Since both nearby arrows point toward $y=-1$, this equilibrium point is asymptotically stable. This means that if a solution of the ODE starts near -1, it will approach -1 as $t \longrightarrow \infty$. This is not the case for the other two equilibrium points. In fact, for both of them, if a solution starts nearby, it may (in fact, will) move away. Thus, the equilibria $y=-3$ and $y=3$ are unstable.

2.9: 26; 9pts: *Solve the initial value problem*

$$y' = (3+y)(1-y), \qquad y(0) = 2,$$

and describe the behavior of the solution when $t \longrightarrow \infty$.
This equation is autonomous and thus separable:

$$\frac{dy}{dt} = (3+y)(1-y) \iff \frac{dy}{(3+y)(1-y)} = dt \iff \frac{1}{4}\left(\frac{1}{3+y} + \frac{1}{1-y}\right) dy = dt$$
$$\iff \ln|3+y| - \ln|1-y| = 4t + C \iff \ln\frac{|3+y|}{|1-y|} = 4t + C.$$

Since $y(0) = 2$, ln(5/1)=C, $C = \ln 5$, and

$$\frac{|3+y|}{|1-y|} = e^{4t+\ln 5} = 5e^{4t} \qquad \Longrightarrow \qquad \frac{3+y}{1-y} = -5e^{4t}.$$

Due to the initial condition, we must use the negative sign here. We solve this equation for y:

$$3+y = -5(1-y)e^{4t} \implies 3+5e^{4t} = (5e^{4t}-1)y \implies y = \frac{3+5e^{4t}}{5e^{4t}-1} = \frac{3e^{-4t}+5}{5-e^{-4t}}.$$

It follows that $\lim_{t\to\infty} y = (0+5)/(5-0) = 1$. This conclusion can also be obtained by sketching the graph of $f(y)=(3+y)(1-y)$ and the phase line for the ODE.

2.9: 28; 5pts: *Determine the stability of the equilibrium solutions of $x' = x(x-1)(x+2)$.*
The equilibrium points for

$$x' = f(x) = x(x-1)(x+2)$$

are the zeros of f. Thus, they are -2, 0 and 1. By the derivative test for stability, we have:

$$\begin{aligned} f'(-2) = 6 > 0 &\implies x = -2 \text{ is unstable.} \\ f'(0) = -2 < 0 &\implies x = 0 \text{ is asymptotically stable.} \\ f'(1) = 3 > 0 &\implies x = 1 \text{ is unstable.} \end{aligned}$$

This conclusion can also be obtained by sketching the graph of $f(y)$ and the phase line for the ODE. Unlike the derivative test, this latter method will always work.

Section 4.3, Problems 4,10,14,26 (24pts)

4.3:4; 5pts: *Find the general solution of the ODE*

$$2y'' - y' - y = 0.$$

The characteristic polynomial for this equation is

$$2\lambda^2 - \lambda - 1 = (2\lambda+1)(\lambda-1).$$

Thus, the two characteristic roots are $\lambda_1 = -1/2$ and $\lambda_2 = 1$. Since they are real and distinct, and the general solution of the ODE is $\boxed{y(t) = C_1 e^t + C_2 e^{-t/2}}$

4.3:10; 6pts: *Find the general solution of the ODE*

$$y'' + 2y' + 17y = 0.$$

The characteristic polynomial for this equation is

$$\lambda^2 + 2\lambda + 17 = (\lambda - \lambda_1)(\lambda - \lambda_2), \qquad \lambda_1, \lambda_2 = -1 \pm \sqrt{1-17} = -1 \pm 4i.$$

Thus, the two characteristic roots are complex, and so is the general solution of the ODE

$$y(t) = C_1 e^{(-1+4i)t} + C_2 e^{(-1-4i)t}.$$

The corresponding general real solution is given by $\boxed{y(t) = C_1 e^{-t}\cos 4t + C_2 e^{-t}\sin 4t}$

4.3:14; 5pts: *Find the general solution of the ODE*

$$y'' - 6y' + 9y = 0.$$

The characteristic polynomial for this equation is

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

Thus, this equation has a repeated root, $\lambda=3$, and the general solution of the ODE is

$$\boxed{y(t) = C_1 e^{3t} + C_2 t e^{3t}}$$

4.3:26; 8pts: *Find the solution to the initial value problem*

$$4y'' + y = 0, \qquad y(1) = 0, \quad y'(1) = -2.$$

The characteristic polynomial for this equation is

$$4\lambda^2 + 1 = (2\lambda + i)(2\lambda - i).$$

Thus, the two roots, $\lambda_1 = i/2$ and $\lambda = -i/2$, are distinct, and the general (complex) solution is

$$y(t) = C_1 e^{it/2} + C_2 e^{-it/2}.$$

The initial conditions $y(1)=0$ and $y'(1)=-2$ give

$$0 = y(1) = C_1 e^{i/2} + C_2 e^{-i/2} \qquad \text{and} \qquad -2 = y'(1) = C_1 \frac{i}{2} e^{i/2} - C_2 \frac{i}{2} e^{-i/2}.$$

Thus, $C_1 = 2ie^{-i/2}$ and $C_2 = -2ie^{i/2}$, and

$$\begin{aligned} y(t) &= 2ie^{-i/2} e^{it/2} - 2ie^{i/2} e^{-it/2} = 2i\big(e^{i(t-1)/2} - e^{-i(t-1)/2}\big) \\ &= 2i \cdot 2i \sin((t-1)/2) = -4\sin((t-1)/2). \end{aligned}$$

Thus, the solution to the initial value problem is $\boxed{y(t) = -4\sin((t-1)/2)}$ *Please check that this function indeed satisfies the ODE and the initial conditions.*

Section 4.4, Problem 17 (8pts)

Prove that an overdamped solution of $my''+\mu y'+ky=0$ can cross the time axis no more than once.
Rewrite the given equation as

$$y'' + \frac{\mu}{m} y' + \frac{k}{m} = 0 \quad \Longrightarrow \quad y'' + 2cy' + \omega_0^2 y = 0,$$

where $2c = \mu/m$ and $\omega_0^2 = k/m$. The characteristic equation is $\lambda^2 + 2c\lambda + \omega_0^2 = 0$. Its roots are

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2} \qquad \text{and} \qquad \lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

Since the system is overdamped, $c^2-\omega_0^2>0$. Thus, λ_1 and λ_2 are real and $\lambda_1\neq\lambda_2<0$. The general solution is of the form

$$y(t)=C_1e^{\lambda_1 t}+C_2^{\lambda_2 t}.$$

The number of times any such curve crosses the t-axis is the number of values of t for which

$$C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}=e^{\lambda_1 t}(C_1+C_2e^{(\lambda_2-\lambda_1)t})=0.$$

Since $e^{\lambda_1 t}$ is never zero, the point $(t,y(t))$ will lie on the t-axis if and only if

$$C_1+C_2e^{(\lambda_2-\lambda_1)t}=0 \quad\Longrightarrow\quad e^{(\lambda_2-\lambda_1)t}=-\frac{C_1}{C_2}$$

If $C_1/C_2\geq 0$, the right hand side is negative or zero. This equation has then has no solutions in t, since the exponential of a real number is always positive. Thus, if $C_1/C_2\geq 0$, $y(t)$ is never zero. If $C_1/C_2<0$, the solution curve intersects the t-axis only at the time

$$t=\frac{1}{\lambda_2-\lambda_1}\ln\Big(-\frac{C_1}{C_2}\Big).$$

Note that $\lambda_1\neq\lambda_2$. Thus, if $C_1/C_2<0$, then the solution curve intersects the t-axis exactly once.

Problem B (20pts)

(a; **10pts**) *Use the second-order integrating factor method to find the real general solution of*

$$y''+5y'+4y=t\cdot e^{-t}. \tag{1}$$

In this case, the characteristic polynomial is

$$\lambda^2+5\lambda+4=(\lambda+1)(\lambda+4).$$

Thus, the two characteristic roots are $\lambda_1=-4$ and $\lambda_2=-1$, and

$$\big(e^{((-4)-(-1))t}(e^{-(-4)t}y)'\big)'=e^{-(-1)t}(y''+5y'+4y). \tag{2}$$

Multiplying both sides of (1) by e^t and using (2), we obtain

$$y''+5y'+4y=t\cdot e^{-t} \quad\Longrightarrow\quad e^t(y''+5y'+4y)=t \quad\Longrightarrow\quad \big(e^{-3t}(e^{4t}y)'\big)'=t.$$

Integrating twice, we obtain

$$\begin{aligned} & e^{-3t}(e^{4t}y)'=\int t\,dt=\frac{1}{2}t^2+C_1 \quad\Longrightarrow\quad (e^{4t}y)'=\frac{1}{2}t^2e^{3t}+C_1e^{3t}\\ \Longrightarrow\qquad & e^{4t}y(t)=\frac{1}{2}\int t^2e^{3t}dt+C_1\int e^{3t}dt=\frac{1}{6}\big(t^2e^{3t}-\int 2te^{3t}dt\big)+\frac{C_1}{3}e^{3t}\\ & =\frac{1}{6}t^2e^{3t}-\frac{1}{9}\big(te^{3t}-\int e^{3t}dt\big)+\frac{C_1}{3}e^{3t}=\frac{1}{6}t^2e^{3t}-\frac{1}{9}te^{3t}+\frac{1}{27}e^{3t}+\frac{C_1}{3}e^{3t}+C_2. \end{aligned}$$

Since we can replace $(1/27)+(C_1/3)$ with C_1, the general solution of (1) is

$$\boxed{y(t)=\tfrac{1}{6}t^2e^{-t}-\tfrac{1}{9}te^{-t}+C_1e^{-t}+C_2e^{-4t}}$$

(b; **10pts**) *Use the second-order integrating factor method to find the real general solution of*

$$y''+4y=4\cos 2t. \tag{3}$$

Here is one approach. The general real solution $y=y(t)$ of this equation is given by $y=\mathrm{Re}\,z$, where $z=z(t)$ is the complex general solution of

$$z''+4z=4e^{2it}. \tag{4}$$

The characteristic polynomial for this equation is

$$\lambda^2+0\cdot\lambda+4=(\lambda+2i)(\lambda-2i).$$

Thus, the two characteristic roots are $\lambda_1=-2i$ and $\lambda_2=2i$, and

$$\big(e^{((-2i)-(2i))t}(e^{-(-2i)t}z)'\big)'=e^{-(2i)t}(z''+4z). \tag{5}$$

Multiplying both sides of (4) by e^{-2it} and using (5), we obtain

$$z''+4z=4e^{2it} \quad\Longrightarrow\quad e^{-2it}(z''+4z)=4 \quad\Longrightarrow\quad \big(e^{-4it}(e^{2it}z)'\big)'=4.$$

Integrating twice, we obtain

$$\begin{aligned}\big(e^{-4it}(e^{2it}z)'\big)'=4 \quad&\Longrightarrow\quad e^{-4it}(e^{2it}z)'=4t+C_1 \quad\Longrightarrow\quad (e^{2it}z)'=4te^{4it}+C_1e^{4it}\\ &\Longrightarrow\quad e^{2it}z=\int(4te^{4it}+C_1e^{4it})dt=\frac{4}{4i}\Big(te^{4it}-\int e^{4it}dt\Big)+\frac{C_1}{4i}e^{4it}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad =\frac{1}{i}te^{4it}+\frac{1}{4}e^{4it}+\frac{C_1}{4i}e^{4it}+C_2.\end{aligned}$$

Since we can replace $(1/4)+(C_1/4i)$ with C_1, the general solution of (4) is

$$z(t)=\frac{1}{i}te^{2it}+C_1e^{2it}+C_2e^{-2it}.$$

Taking the real part of this equation and modifying the constants, we obtain

$$\boxed{y(t)=\mathrm{Re}\,z(t)=t\sin 2t+C_1\cos 2t+C_2\sin 2t}$$

Here is another approach. The characteristic polynomial and roots for the original equation are the same as for its complex version. Thus, (5) holds with z replaced by y, and

$$y''+4y=4\cos 2t \quad\Longrightarrow\quad e^{-2it}(y''+4y)=4e^{-2it}\cos 2t \quad\Longrightarrow\quad \big(e^{-4it}(e^{2it}y)'\big)'=4e^{-2it}\cos 2t.$$

Integrating the last expression once, we obtain

$$\begin{aligned}
e^{-4it}(e^{2it}y)' &= \int 4e^{-2it}\cos 2t\,dt = 4\int\cos^2 2t\,dt - 4i\int\cos 2t\,\sin 2t\,dt \\
&= 2\int(\cos 4t+1)dt - 2i\int\sin 4t\,dt = \frac{1}{2}\sin 4t + 2t + \frac{i}{2}\cos 4t + C_1 = \frac{i}{2}e^{-4it} + 2t + C_1.
\end{aligned}$$

The second and last equalities above follow from Euler's formula, applied in opposite directions. The third inequality uses the half-angle trigonometric formulas. Finally, proceeding as in the second integration step of the first approach, we obtain

$$\begin{aligned}
e^{2it}y &= \int\big(2te^{4it} + C_1e^{4it} + \frac{i}{2}\big)dt = \frac{1}{2i}te^{4it} + \frac{1}{8}e^{4it} + \frac{C_1}{4i}e^{4it} + \frac{it}{2} + C_2 \\
\Longrightarrow \quad y(t) &= \frac{t}{2i}\big(e^{2it} - e^{-2it}\big) + C_1e^{2it} + C_2e^{-2it} = t\sin 2t + C_1e^{2it} + C_2e^{-2it}.
\end{aligned}$$

As before, the complex form $C_1e^{2it}+C_2e^{-2it}$ is equivalent to the real form $A_1\cos 2t+A_2\sin 2t$.

Remarks: (1) When the nonhomogeneous term, i.e. RHS in (3), is $\cos\omega t$ or $\sin\omega t$, the first approach, i.e. complexifying the ODE, is generally faster, but riskier if you are not used to complex numbers. This is the case whether you use the second-order integrating factor approach or the method of undetermined coefficients. Note that if the the forcing term is $\sin\omega t$, you would need to take the imaginary part of the complex solution.
(2) The complex form $C_1e^{at+ibt}+C_2e^{at-ibt}$ of the general solution of an ODE is always equivalent to the real form $A_1e^{at}\cos bt+A_2e^{at}\sin bt$.

Remark: In these two cases, i.e. (a) and (b), the second-order integrating factor approach is not any easier and perhaps a bit harder than the method of undetermined coefficients, which is described in Section 4.5. In general, the method of undetermined coefficients will be faster whenever it is applicable, i.e. you know what form a solution should have. On the other hand, the integrating factor approach works for all forcing terms.