

Math53: Ordinary Differential Equations Autumn 2004

Solutions to Problem Set 1

Note: Even if you have done every problem, you are encouraged to look over these solutions, especially 2.2:14,18, 2.4:18, 2.5:4, and 3.1:12. In these problems, the constant C is found as soon as it appears. In the last two problems, the answers are found from implicitly defined solutions.

Problem 1 (20pts)

(a; **2pts**) *State the two Fundamental Theorems of Calculus.*

(a-i) If F is a continuously differentiable function on the interval (a, b) and $t_0 \in (a, b)$, then

$$F(t) = F(t_0) + \int_{t_0}^t F'(s) ds \quad \text{for all } t \in (a, b). \quad (1)$$

(a-ii) If f is a continuous function on the interval (a, b) , $t_0 \in (a, b)$,

$$\begin{aligned} \text{and} \quad F(t) &\equiv \int_{t_0}^t f(s) ds \quad \text{for all } t \in (a, b), \\ \text{then} \quad F'(t) &= f(t) \quad \text{for all } t \in (a, b). \end{aligned}$$

(b; **2pts**) *State the chain rule of one-variable differentiation.*

If f and g are continuously differentiable functions on (a, b) and (c, d) , respectively, and $a < g(t) < b$ for all $t \in (c, d)$, then the function

$$h(t) \equiv f(g(t)), \quad t \in (c, d),$$

is defined and continuously differentiable on (c, d) and

$$h'(t) = f'(g(t)) \cdot g'(t) \quad \text{for all } t \in (c, d).$$

(c; **2pts**) *State the product rule of one-variable differentiation.*

If f and g are continuously differentiable functions on (a, b) , then the function

$$h(t) \equiv f(t) \cdot g(t), \quad t \in (a, b),$$

is also continuously differentiable and

$$h'(t) = f'(t) \cdot g(t) + f(t) \cdot g'(t) \quad \text{for all } t \in (a, b). \quad (2)$$

(d; **1pt**) *If a is a real number and $f(x) = x^a$, what is $f'(x)$? (no proof necessary)*

$$f'(x) = a \cdot x^{a-1}.$$

(e; **1pt**) *If $f(x) = e^x$, what is $f'(x)$? (no proof necessary)*

$$f'(x) = e^x.$$

(f; **3pts**) State the quotient rule of one-variable differentiation. Deduce it from (b)-(d).

If f and g are continuously differentiable functions on (a, b) and $g(t) \neq 0$ for all $t \in (a, b)$, then the function

$$h(t) \equiv f(t)/g(t), \quad t \in (a, b),$$

is also continuously differentiable and

$$h'(t) = \frac{f'(t)}{g(t)} - \frac{f(t) \cdot g'(t)}{g(t)^2} \quad \text{for all } t \in (a, b). \quad (3)$$

In order to prove (3), we apply (c) to the functions f and $G(t) = 1/g(t)$. Since $h = f \cdot G$,

$$h'(t) = f'(t) \cdot G(t) + f(t) \cdot G'(t) = \frac{f'(t)}{g(t)} + f(t) \cdot G'(t). \quad (4)$$

In order to compute $G'(t)$, we apply (b) to the functions $y(x) = x^{-1}$ and g and use (d) with $a = -1$. Since $G(t) = y(g(t))$,

$$G'(t) = y'(g(t)) \cdot g'(t) = (-1) \cdot g(t)^{-2} \cdot g'(t) = -\frac{g'(t)}{g(t)^2}. \quad (5)$$

The quotient rule, i.e. (3), is obtained by plugging (5) into (4).

(g; **3pts**) State the change-of-variables formula of one-variable integration. Deduce it from (a) and (b).

If f is a continuous function on (a, b) , g is a continuously differentiable function on (c, d) such that $a < g(t) < b$ for all $t \in (c, d)$, $t_0 \in (c, d)$,

$$\text{and} \quad F(x) \equiv \int_{g(t_0)}^x f(y) dy \quad \text{for all } x \in (a, b), \quad (6)$$

$$\text{then} \quad \int_{t_0}^t f(g(s)) \cdot g'(s) ds = F(g(t)) \quad \text{for all } t \in (c, d). \quad (7)$$

By (c) applied to F , g , and $h(t) = F(g(t))$,

$$h'(t) = F'(g(t)) \cdot g'(t). \quad (8)$$

Since $h(t_0) = 0$ by (6), the change-of-variables formula, i.e. (7), follows from (1) and (8).

(h; **2pts**) State the integration-by-parts formula of one-variable integration. Deduce it from (a) and (c).

If f and g are continuously differentiable functions on (a, b) and $t_0 \in (c, d)$,

$$\int_{t_0}^t f(s) \cdot g'(s) ds = (f(t)g(t) - f(t_0)g(t_0)) - \int_{t_0}^t f'(s) \cdot g(s) ds \quad \text{for all } t \in (a, b). \quad (9)$$

Rearranging (2) with $h(s) = f(s)g(s)$, we obtain

$$f(s) \cdot g'(s) = h'(s) - f'(s) \cdot g(s) \quad \text{for all } s \in (a, b). \quad (10)$$

The integration-by-parts formula, i.e. (9), is obtained by integrating both sides of (10) and applying (1) to the middle term.

(i; **3pts**) Suppose $a=a(t)$ is a smooth function, c is a real number,

$$f(t) = \int_c^t a(s)ds, \quad \text{and} \quad h(t) = e^{f(t)}.$$

Compute $h'(t)$, using (a), (b), and (e).

We apply (b) to the functions $F(x)=e^x$ and $G(t)=f(t)$. Since $h(t)=F(G(t))$,

$$h'(t) = F'(G(t)) \cdot G'(t) = e^{G(t)} \cdot G'(t) = e^{f(t)} \cdot f'(t) = f'(t) \cdot e^{f(t)}. \quad (11)$$

The second equality in (11) is a consequence of (e). Since $f'(t)=a(t)$ by (a-ii),

$$h'(t) = a(t) \cdot e^{f(t)}. \quad (12)$$

(j; **1pt**) Find a nontrivial first-order differential equation which is solved by the function $h = h(t)$ of (i).

Since $h=e^f$, by (12),

$$h'(t) = a(t) \cdot h(t) \quad \text{or} \quad h' = a \cdot h, \quad h=h(t).$$

This is the simplest possible nontrivial ODE satisfied by h .

Note: There are a number of ways of phrasing (a)-(c) and (f)-(h).

Section 1.3, Problems 4 and 23 (12pts)

1.3: 4; 5pts: Find the general solution of the differential equation

$$y' = 2 \sin 3t - \cos 5t.$$

Indicate the interval of existence and sketch at least two members of the family of solution curves. By FTC, e.g. (a-i) of Problem 1,

$$y(t) = \int (2 \sin 3t - \cos 5t) dt = \boxed{-\frac{2}{3} \cos 3t - \frac{1}{5} \sin 5t + C}$$

Since y is defined for all t , the interval of existence is $\boxed{(-\infty, \infty)}$. Three solution curves are shown in Figure 1. The most important feature here is that the three graphs differ by vertical shifts.

1.3: 23; 7pts: Find the solution the initial value problem

$$y' = \frac{t+1}{t(t+4)}, \quad y(-1) = 0.$$

Indicate the interval of existence and sketch the solution.

By FTC, e.g. (a-i) of Problem 1,

$$\begin{aligned} y(t) &= y(-1) + \int_{-1}^t \frac{s+1}{s(s+4)} ds = 0 + \int_{-1}^t \frac{1}{4} \left(\frac{1}{s} + \frac{3}{s+4} \right) ds \\ &= \frac{1}{4} (\ln |s| + 3 \ln |s+4|) \Big|_{s=-1}^{s=t} = \frac{1}{4} \ln |t| + \frac{3}{4} \ln |t+4| - \frac{3}{4} \ln 3. \end{aligned}$$

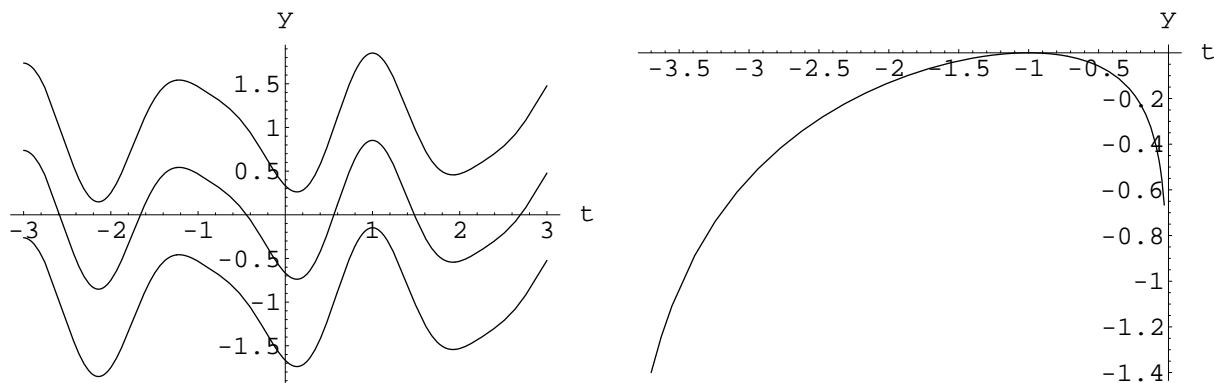


Figure 1: Plots for Problems 1.3:4 and 1.3:23

The last expression is defined on the intervals $(-\infty, -4)$, $(-4, 0)$, and $(0, -\infty)$. Since the initial value of the parameter lies in the middle interval, the solution to the initial value problem is

$$y(t) = \frac{1}{4} \ln |t| + \frac{3}{4} \ln(t+4) - \frac{3}{4} \ln 3, \quad t \in (-4, 0)$$

The solution curve is shown in Figure 1. Note that y approaches $-\infty$ as t tends to -4 and 0 , and the curve passes through the point $(-1, 0)$, as required by the initial condition.

Section 2.1, Problems 8 and 18 (12pts)

2.1: 8 (a; 2pts) Use implicit differentiation to show that $t^2 + y^2 = C^2$ implicitly defines solutions of the differential equation $t + yy' = 0$.

We differentiate both sides of $t^2 + y^2 = C^2$ with respect to t . The derivative of RHS is 0. On the other hand, by the chain rule, i.e. (b) of Problem 1,

$$\frac{d}{dt}(t^2 + y^2) = 2t + 2y \cdot y' = 2(t + yy').$$

Comparing the derivatives of the two sides, we obtain $t + yy' = 0$, as needed.

(b; 2pts) Solve $t^2 + y^2 = C^2$ for y in terms of t to find explicit solutions. Show that these functions are also solutions of $t + yy' = 0$.

Solving for y , we obtain $y(t) = \pm\sqrt{C^2 - t^2}$. By the chain rule,

$$\begin{aligned} y'(t) &= \pm \frac{d}{dt}(C^2 - t^2)^{1/2} = \pm \frac{1}{2}(C^2 - t^2)^{1/2-1} \cdot (-2t) = \pm \frac{-t}{\sqrt{C^2 - t^2}} \\ \implies t + yy' &= t + (\pm\sqrt{C^2 - t^2}) \left(\pm \frac{-t}{\sqrt{C^2 - t^2}} \right) = 0. \end{aligned}$$

(c; 2pts) Discuss the interval of existence of each solution in part (b).

We need $C^2 - t^2 \geq 0$. Thus, $t \in (-C, C)$ for $C > 0$.

(d; 1pt) Sketch the solutions in part (b) for $C = 1, 2, 3, 4$.

The solution curves are the upper and lower circles of radii 1, 2, 3, 4 centered at the origin as in the first diagram in Figure 2.

2.1: 18; 5pts: Plot the direction field for the ODE $y' = y^2 - t$ by drawing short lines of the appropriate slope centered at the integer valued coordinates (t, y) , where $-2 \leq t \leq 2$ and $-1 \leq y \leq 1$.



Figure 2: Plots for Problems 2.1:8 and 2.1:18

The second plot in Figure 2 shows short lines of the slope $y' = y^2 - t$ at the fifteen points. Notice that the top and bottom row look the same.

Section 2.2, Problems 4,12,14,18 (26pts)

2.2: 4; 5pts: Find the general solution of the equation $y' = (1+y^2)e^x$.

Write $y' = \frac{dy}{dx}$ and split the variables:

$$\begin{aligned}
 y' = (1+y^2)e^x &\iff \frac{dy}{1+y^2} = e^x dx \iff \int \frac{dy}{1+y^2} = \int e^x dx \\
 &\iff \tan^{-1} y = e^x + C \iff \boxed{y = \tan(e^x + C)}
 \end{aligned}$$

2.2: 12; 5pts: Find the general solution of the equation $y' = (2xy+2x)/(x^2-1)$.

Write $y' = \frac{dy}{dx}$ and split the variables:

$$\begin{aligned}
 y' = \frac{2x(y+1)}{x^2-1} &\iff \frac{dy}{y+1} = \frac{2xdx}{x^2-1} \iff \int \frac{dy}{y+1} = \int \frac{2x}{x^2-1} dx \\
 &\iff \ln|y+1| = \ln|x^2-1| + C \\
 &\iff |y+1| = e^C|x^2-1| \iff \boxed{y = -1 + C(x^2-1), x \neq \pm 1}
 \end{aligned}$$

2.2: 14; 8pts: Find the exact solution, including the interval of existence, to the initial value problem

$$y' = -\frac{2t(1+y^2)}{y}, \quad y(0) = 1.$$

Write $y' = \frac{dy}{dx}$ and split the variables:

$$\begin{aligned}
 y' = -\frac{2t(1+y^2)}{y} &\iff \frac{ydy}{1+y^2} = -2tdt \iff \int \frac{ydy}{1+y^2} = -\int 2tdt \\
 &\iff \frac{1}{2} \ln(1+y^2) = -t^2 + C \iff 1+y^2 = e^{2C} e^{-2t^2}
 \end{aligned}$$

Since $y(0)=1$, $1+1=e^{2C} \cdot 1$ and $e^{2C}=2$. Thus,

$$\boxed{y = \sqrt{2e^{-2t^2} - 1}, \quad t \in (-\sqrt{(\ln 2)/2}, \sqrt{(\ln 2)/2})}$$

We must take the positive square root in order to satisfy the initial condition.

2.2: 18; 8pts: Find the exact solution, including the interval of existence, to the initial value problem

$$y' = \frac{x}{1+2y}, \quad y(-1) = 0.$$

Write $y' = \frac{dy}{dx}$ and split the variables:

$$\begin{aligned} y' = \frac{x}{1+2y} &\iff (1+2y)dy = xdx \iff \int(1+2y)dy = \int x dx \\ &\iff y + y^2 = \frac{1}{2}x^2 + C. \end{aligned}$$

Since $y(-1)=0$, $0+0=(1/2)+C$ and $C=-1/2$. Thus,

$$\boxed{y = \frac{1}{2}(-1 + \sqrt{2x^2 - 1}), \quad x \in (-\infty, -1/\sqrt{2})}$$

In order to satisfy the initial condition, we must take the positive square root.

Section 2.3, Problem 4 (8pts)

A rocket ascends vertically with constant acceleration $a = 100\text{m/s}^2$ for $t_1 = 1$ min. The rocket motor is then shut-off and the rocket continues upward under the influence of gravity. Find the maximum altitude y_m reached by the rocket and the total time T elapsed from the take-off until the rocket returns to the ground.

The upward velocity $v = v(t)$ is described by

$$v'(t) = a \text{ if } t \in (0, t_1), \quad v'(t) = -g \text{ if } t \in (t_1, T).$$

Integrating the two equations, we obtain

$$\begin{aligned} v(t) &= v(0) + \int_0^t a ds = at && \text{if } t \in (0, t_1), \\ v(t) &= v(t_1) + \int_{t_1}^t (-g) ds = at_1 - g \cdot (t - t_1) = (a+g)t_1 - gt && \text{if } t \in (t_1, T). \end{aligned}$$

Since $y'(t) = v(t)$, integrating again, we obtain $y(t) = \frac{1}{2}at^2$ if $t \in (0, t_1)$, and

$$\begin{aligned} y(t) &= y(t_1) + \int_{t_1}^t v(t_1) ds = \frac{1}{2}at_1^2 + (a+g)t_1 \cdot (t - t_1) - \frac{1}{2}g \cdot (t^2 - t_1^2) \\ &= -\frac{1}{2}(a+g)t_1^2 + (a+g)t_1t - \frac{1}{2}gt^2 && \text{if } t \in (t_1, T). \end{aligned}$$

The maximum altitude is reached at the time $t_2 \in (t_1, T)$ such that

$$v(t_2) = 0 \iff (a+g)t_1 - gt_2 = 0 \iff t_2 = \frac{a+g}{g}t_1 \implies y_m = y(t_2) = \frac{a^2+ag}{2g}t_1^2.$$

The rocket returns to the ground at the time $T > t_1$ such that

$$y(T) = -\frac{1}{2}(a+g)t_1^2 + (a+g)t_1T - \frac{1}{2}gT^2 = 0 \implies T = \frac{a+g + \sqrt{a(a+g)}}{g}t_1.$$

In order to satisfy the condition $T > t_1$, we must take the positive square root. *Can you check that the smaller root does not satisfy this inequality, if $a > 0$?* Plugging in $a = 100$, $g = 9.8$, and $t_1 = 60$ into the above expressions for y_m and T , we conclude that $y_m \approx 2,016,735 \text{ m.}$, $T \approx 1,314 \text{ sec.}$

Section 2.4, Problems 2,6,13,14,18 (32pts)

2.4: 2; 5pts: Find the general solution of the first-order linear ODE $y' - 3y = 5$.

The integrating factor $P(t)$ is given by

$$\begin{aligned} P(t) = e^{\int(-3)dt} = e^{-3t} &\implies y' - 3y = 5 \iff e^{-3t}y' - 3e^{-3t}y = 5e^{-3t} \iff (e^{-3t}y)' = 5e^{-3t} \\ &\iff e^{-3t}y(t) = 5 \int e^{-3t}dt = -\frac{5}{3}e^{-3t} + C \iff \boxed{y(t) = -\frac{5}{3} + Ce^{3t}} \end{aligned}$$

2.4: 6; 5pts: Find the general solution of the first-order linear ODE $tx' = 4x + t^4$.

First, rewrite this equation as $x' - 4t^{-1}x = t^3$. The integrating factor $P(t)$ is given by

$$\begin{aligned} P(t) = e^{\int(-4t^{-1})dt} = e^{-4\ln t} = t^{-4} &\implies x' - 4t^{-1}x = t^3 \iff t^{-4}x' - 4t^{-5}x = t^{-1} \iff (t^{-4}x)' = t^{-1} \\ &\iff t^{-4}x(t) = \int t^{-1}dt = \ln|t| + C \iff \boxed{x(t) = t^4 \ln|t| + Ct^4} \end{aligned}$$

2.4: 13 (a; 4pts) Solve the ODE $y' + y \cos t = \cos t$, using the integrating factor approach.

The integrating factor $P(t)$ is given by

$$\begin{aligned} P(t) = e^{\int \cos t dt} = e^{\sin t} &\implies y' + y \cos t = \cos t \iff e^{\sin t}(y' + y \cos t) = e^{\sin t} \cos t \\ &\iff (e^{\sin t}y)' = e^{\sin t} \cos t \iff e^{\sin t}y = \int e^{\sin t} \cos t dt \iff \boxed{y(t) = 1 + Ce^{-\sin t}} \end{aligned}$$

(b; 4pts) Solve the ODE $y' + y \cos t = \cos t$ using the separation of variables approach. Discuss any discrepancies between this solution and the solution found in part (a).

$$\begin{aligned} y' + y \cos t = \cos t &\iff \frac{dy}{dt} = (1 - y) \cos t \iff \frac{dy}{1-y} = \cos x dx \\ &\iff -\ln|1-y| = \sin x + C \iff |1-y| = e^{-\sin x - C} \iff 1-y = \pm e^{-C} e^{-\sin x} \end{aligned}$$

The constant $A = \pm e^{-C}$ varies over all real numbers except zero. On the other hand, we divided by $y-1$ and thus might have missed the constant solution $y=1$. We then let A vary over all real numbers, including zero, to get all solutions:

$$\boxed{y(t) = 1 - Ae^{-\sin t}}$$

This is the same result as the one in part (a), since A and C vary over the set of all reals.

2.4: 14; 6pts: Find the solution to the initial value problem $y' = y + 2te^{2t}$, $y(0) = 3$.

First, rewrite the ODE as $y' - y = 2te^{2t}$. The integrating factor $P(t)$ is given by

$$\begin{aligned} P(t) = e^{\int(-1)dt} = e^{-t} &\implies y' - y = 2te^{2t} \iff e^{-t}y' - e^{-t}y = 2te^t \iff (e^{-t}y)' = 2t \\ &\iff e^{-t}y(t) = 2 \int te^t dt = 2(te^t - \int e^t dt) = 2te^t - 2e^t + C. \end{aligned}$$

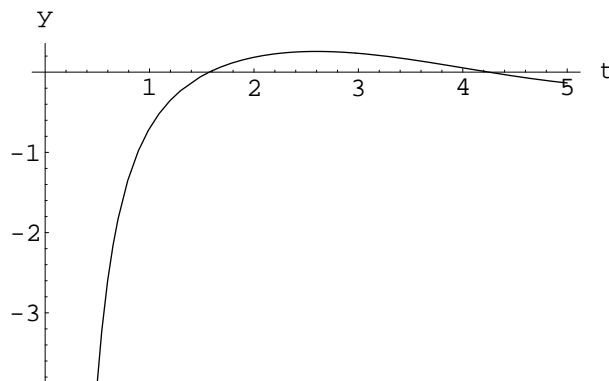


Figure 3: Solution Curve for Problem 2.4:18

Since $y(0)=3$, $1 \cdot 3 = -2+C$, and $C=5$. Thus, $y(t) = 2te^{2t} - 2e^{2t} + 5e^t$

2.4: 18; 8pts: Find the solution, including the interval of existence, to the initial value problem $ty' + 2y = \sin t$, $y(\pi/2)=0$, and sketch it.

Rewrite the ODE as $y' + 2t^{-1}y = t^{-1} \sin t$. The integrating factor $P(t)$ is given by

$$P(t) = e^{\int 2t^{-1} dt} = e^{2 \ln |t|} = t^2 \implies y' + 2t^{-1}y = t^{-1} \sin t \iff t^2 y' + 2ty = t^2 \sin t \iff (t^2 y)' = t \sin t$$

$$\iff t^2 y(t) = \int t \sin t dt = -t \cos t + \int \cos t dt = -t \cos t + \sin t + C.$$

Since $y(\pi/2)=0$, $(-\pi/2) \cdot 0 + 1 + C = 0$, and $C = -1$. Thus,

$$y(t) = \frac{\sin t - t \cos t - 1}{t^2}$$

Since the solution cannot be extended to zero, the interval of existence is $(0, \infty)$. Figure 3 shows the solution curve. Its main features are that the curve approaches $-\infty$ as t tends to 0, passes through the point $(\pi/2, 0)$, as required by the initial condition, and is close to the x -axis for large values of t .

Section 2.5, Problem 4 (8pts)

A tank contains $V = 500$ Gl of a salt-water solution at the concentration of $\rho_0 = .05$ lbs/Gl. Pure water is poured into the tank, and a drain at the bottom is adjusted so as to keep the volume of solution constant. At what rate r should the water be poured into the tank to lower the concentration to $\rho_1 = .01$ lbs/Gl in $t_1 = 1$ Hr.

Let $y(t)$ be the amount of salt at time t and $\rho(t) = y(t)/V$ be the salt concentration. Then,

$$y'(t) = 0 - \rho(t) \cdot r,$$

since no salt is coming in, while it is leaving at the rate of $\rho(t) \cdot r$. Thus,

$$\rho' = -\frac{r}{V}\rho.$$

Since this equation is separable and r/V is constant,

$$\frac{d\rho}{\rho} = -\frac{r}{V} dt \implies \int \frac{d\rho}{\rho} = -\int \frac{r}{V} dt \implies \ln \rho(t) = -\frac{r}{V}t + C.$$

We could now solve the last equation for $\rho(t)$, then find C using the initial condition, and then find r using the final condition. However, we can do this without solving for $\rho(t)$. Plugging in $\rho(0) = \rho_0$, we get

$$\ln \rho_0 = -\frac{r}{V} \cdot 0 + C \implies C = \ln \rho_0.$$

Plugging in $\rho(t_1) = \rho_1$, we get

$$\ln \rho_1 = -\frac{r}{V} \cdot t_1 + \ln \rho_0 \implies r = \frac{V}{t_1} (\ln \rho_0 - \ln \rho_1) = \frac{V}{t_1} \ln(\rho_0/\rho_1).$$

Plugging in $t_1 = 60$, $\rho_0 = .05$, $V = 500$, we obtain $r \approx 13.4 \text{ Gls/min}$

Section 2.7, Problems 2,4,6,26 (16pts)

2.7: 2; 4pts: Does the initial value problem $y' = \sqrt{y}$, $y(4) = 0$ satisfy the conditions of the Theorem on the uniqueness of solutions (Theorem 7.16 in the textbook)?

The equation is of the form $y' = f(t, y) = \sqrt{y}$. f is defined only on the half-plane $\{(t, y) : y \geq 0\}$ and it is continuous there. But $\partial f/\partial y = 1/(2\sqrt{y})$ is continuous only on the open half-plane $\{(t, y) : y > 0\}$. Hence, any rectangle in the (y, t) -plane containing the initial point $(4, 0)$ contains points where $\partial f/\partial y$ is discontinuous, so the conditions of the theorem are not satisfied.

2.7: 4; 4pts: Does the initial value problem $\omega' = \omega \sin \omega + s$, $\omega(0) = -1$ satisfy the conditions of the Theorem on the uniqueness of solutions (Theorem 7.16 in the textbook)?

The equation is of the form $\omega' = f(s, \omega) = \omega \sin \omega + s$. Function f is continuous in the whole plane, and so is its partial $\partial f/\partial \omega = \sin \omega + \omega \cos \omega$. In particular, any rectangle around the initial value point will satisfy the conditions of the theorem.

2.7: 6; 4pts: Does the initial value problem $y' = (y/x) + 2$, $y(0) = 1$ satisfy the conditions of the Theorem on the uniqueness of solutions (Theorem 7.16 in the textbook)?

The equation is of the form $y' = f(x, y) = (y/x) + 2$. Function f is continuous outside the line $x = 0$. The initial value point is $(0, 1)$, so there is no rectangle containing it in which f is continuous, and the conditions for uniqueness of solution are not satisfied.

2.7: 26; 4pts: Is it possible to find a function $f(t, x)$ that is continuous and has continuous partial derivatives such that the functions $x_1(t) = \cos t$ and $x_2(t) = 1 - \sin t$ are both solutions to the equation $x' = f(t, x)$ near $t = \pi/2$?

Since f is continuous and has continuous partial derivatives in the entire (t, x) -plane, the equation $x' = f(t, x)$ satisfies the conditions of the uniqueness theorem. Notice that $x_1(\pi/2) = x_2(\pi/2) = 0$, so the curves $x_1(t) = \cos t$ and $x_2(t) = 1 - \sin t$ have a common point $(\pi/2, 0)$, so if they were both solutions of our equation, by the uniqueness theorem they would have to agree on any rectangle containing $(\pi/2, 0)$. Since they do not, they cannot both be solutions of the equation $x' = f(t, x)$.

Section 3.1, Problem 12 (8pts)

A population is observed to obey the logistic equation with eventual population 20,000. The initial population is 1000, and 8 hours later, the observed population is 1200. Find the reproductive rate and the time required for the population to reach three quarters of its carrying capacity.

In the logistic model of population growth, we assume that the death and birth rates vary with population P according to formulas $d = d_0 + aP$ and $b = b_0 - cP$. Then we get:

$$P(t + \Delta t) - P(t) \approx (b - cP(t))P(t)\Delta t - (d + aP(t))P(t)\Delta t = (b - d - (a + c)P(t))P(t)\Delta t$$

$$P'(t) = \lim_{\Delta t \rightarrow 0} \frac{P(t + \Delta t) - P(t)}{\Delta t} = (b - d - (a + c)P(t))P(t)$$

Set $r = b - d$ and $a + c = r/K$ where K is a new constant, to get the logistic equation

$$P' = r(1 - P/K)P$$

This equation is separable:

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right) \implies \frac{K dP}{P(K - P)} = r dt \implies \left(\frac{1}{P} + \frac{1}{K - P}\right) dP = r dt$$

$$\implies \ln|P| - \ln|K - P| = rt + C \implies \ln\left(\frac{P}{K - P}\right) = rt + C \implies P(t) = \frac{KAe^{rt}}{1 + Ae^{rt}}$$

If P_0 is population at time $t_0 = 0$, then

$$C = \ln\left(\frac{P_0}{K - P_0}\right) \implies A = e^C = \frac{P_0}{K - P_0} \implies P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

As $t \rightarrow \infty$, $e^{-rt} \rightarrow 0$, and $\lim_{t \rightarrow \infty} P(t) = K$.

In this particular problem, the initial population is $P_0 = 1000$ and the carrying capacity, or eventual population, is $K = 20,000$. We want to find t_2 such that $P(t_2) = fK$, where $f = 3/4$. We first find the rate r , using $P(t_1) = P_1 = 1200$, where $t_1 = 8$ Hrs and then use it find t_2 :

$$\ln\left(\frac{P_1}{K - P_1}\right) = rt_1 + \ln\left(\frac{P_0}{K - P_0}\right) \implies r = \frac{1}{t_1} \ln\left(\frac{P_1}{K - P_1} : \frac{P_0}{K - P_0}\right)$$

$$\implies \ln\left(\frac{fP_0}{K - fP_0}\right) = rt_2 + \ln\left(\frac{P_0}{K - P_0}\right)$$

$$\implies t_2 = \frac{1}{r} \ln\left(\frac{fK}{K - fK} : \frac{P_0}{K - P_0}\right) = t_1 \ln\left(\frac{f}{1 - f} : \frac{P_0}{K - P_0}\right) / \ln\left(\frac{P_1}{K - P_1} : \frac{P_0}{K - P_0}\right).$$

Plugging in the numbers, we get $t \approx 167.671$ Hrs

Section 3.4, Problem 14 (8pts)

Solve the general IVP modeling the LR circuit,

$$L \frac{dI}{dt} + RI = E, \quad I(0) = I_0,$$

where L , R , and E are constants.

Rewrite the ODE as $I' + (R/L)I = (E/L)$. The integrating factor $P(t)$ is given by

$$P(t) = e^{\int (R/L) dt} = e^{(R/L)t} \implies I' + \frac{R}{L}I = \frac{E}{L} \implies e^{(R/L)t} I' + \frac{R}{L} e^{(R/L)t} I = \frac{E}{L} \cdot e^{(R/L)t}$$

$$\implies (e^{(R/L)t} I)' = \frac{E}{L} \cdot e^{(R/L)t} \implies e^{(R/L)t} I(t) = \frac{E}{L} \int e^{(R/L)t} dt = \frac{E}{R} e^{(R/L)t} + C.$$

Since $I(0) = I_0$, $1 \cdot I_0 = \frac{E}{R} \cdot 1 + C$, and $C = I_0 - \frac{E}{R}$. Thus,

$$I(t) = \frac{E}{R} + \left(I_0 - \frac{E}{R}\right) e^{-(R/L)t}$$