Math53: Ordinary Differential Equations
Autumn 2004

Homework Assignment 6

Problem Set 6 is due by 2:15p.m. on Monday, 11/15, in MuddChem 101

Problem Set 6:

9.2: 38*,40*,44;   9.4: 14;   9.6: 7,9;   9.7: 17;   9.8: 6,18,29;   Problem E (see next page)

Note: In 9.2:38,40, sketch phase-plane portraits, as in Section 9.3.

Daily Assignments:

Date	*Read*	*Exercises*
11/8 M	9.2 (pp459-463), 9.5	9.2:38*,40*,44
11/9 T	9.4	9.4:14; Problem E
11/10 W	9.8	9.8:6,18,29
11/11 R	8.4,9.6,9.7	9.6:7,9; 9.7:17
11/12 F		

Note: In 9.2:38,40, sketch phase-plane portraits, as in Section 9.3.

Problem E

Recall that we are able to reduce the general *first*-order linear ODE

$$y' + a(t)y = f(t), \qquad y = y(t),$$

to a ready-to-integrate equation $(Py)' = Pf$ by finding an integrating factor $P = P(t)$ such that

$$P' = aP \qquad \Longrightarrow \qquad (Py) = Py' + aPy.$$

Similarly, we can reduce a *second*-order linear ODE *with constant coefficients*

$$y'' + py' + qy = f, \quad y = y(t), \qquad p, q = const, \tag{1}$$

to a first-order linear ODE by multiplying by an integrating factor such that

$$\big(P(y' + ay)\big)' = P(y'' + py' + q),$$

for some function $a = a(t)$. This integrating factor is $P(t) = e^{-\lambda_2 t}$, where λ_2 is one of the roots of the corresponding characteristic polynomial $\lambda^2 + p\lambda + q = 0$. We cannot adapt this approach to an arbitrary *second*-order linear ODE. Here is why.

(a) Suppose we would like to find smooth nonzero functions $P = P(t)$ and $Q = Q(t)$ such that

$$\big(Q(y' + ay)\big)' = P(y'' + py' + qy), \qquad p = p(t),\ q = q(t), \tag{2}$$

for some smooth function $a = a(t)$ and for every smooth function $y = y(t)$. Show that we must have

$$P = Q, \qquad P' + Pa = Pp, \quad \text{and} \quad (Pa)' = qP.$$

(b) Thus, the functions P and a can be found by finding a nonzero solution to

$$\begin{pmatrix} P \\ (Pa) \end{pmatrix}' = \begin{pmatrix} p & -1 \\ q & 0 \end{pmatrix} \begin{pmatrix} P \\ (Pa) \end{pmatrix} \qquad P = P(t),\ a = a(t).$$

Find a nonzero solution to this ODE if p and q are constant, obtaining an integrating factor for second-order ODEs with constant coefficients. Use it to find $R_1 = R_1(t)$ and $R_2 = R_2(t)$ such that

$$\big(R_2(R_1 y)'\big)' = P(y'' + py' + qy), \qquad p, q = const.$$

Express your final answer in terms of the roots λ_1 and λ_2 of the characteristic polynomial associated to the ODE (1).

(c) Apply the same approach to third-order ODEs. In other words, if $p, q, r = const$, find functions $P = P(t) \neq 0$, $R_1 = R_1(t)$, $R_2 = R_2(t)$, and $R_3 = R_3(t)$ such that

$$\big(R_3\big(R_2(R_1 y)'\big)'\big)' = P(y''' + py'' + qy' + ry).$$

Express your final answer in terms of the roots λ_1, λ_2, and λ_3 of

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0.$$