MAT 644: Complex Curves and Surfaces

Problem Set 6

Written Solutions (if any) due by Thursday, 05/07, 2pm

Please figure out all of the problems below and discuss them with others.

If you have not passed the orals yet, *you are encouraged to write up concise solutions to problems worth 10 points.*

Problem 1 (5 pts)

Let S be a complex surface such that $K_S \longrightarrow S$ is a negative line bundle.
(a) Using Kodaira Vanishing Theorem, show that $q(S)=0$. Conclude that S is rational.
(b) If in addition S is minimal, show that S is either $\mathbb{P}^2$ or $\mathbb{P}^1\times\mathbb{P}^1$.

Problem 2 (5 pts)

Let α be a primitive 5th root unity (so $\alpha^5=1$, but $\alpha\neq 1$) and

$$\tilde{S}=\big\{[X_0,X_1,X_2,X_3]\in\mathbb{P}^3:\ X_0^5+X_1^5+X_2^5+X_3^5=0\big\}.$$

Then $\mathbb{Z}_5$ acts on $\tilde{S}$ by

$$\alpha\cdot[X_0,X_1,X_2,X_3]=[X_0,\alpha X_1,\alpha^2X_2,\alpha^3X_3].$$

Show that
(a) $\tilde{S}$ and $S\equiv\tilde{S}/\mathbb{Z}_5$ are smooth projective surfaces;
(b) $q(S)=p_g(S)=0$, but $K_S\longrightarrow S$ is a positive line bundle, and thus S is of general type (and in particular not rational).

Problem 3 (5 pts)

Let $\pi: S\longrightarrow\Sigma$ be an irrational ruled surface. Show that every irreducible rational curve is contained in a fiber of π and thus S is minimal.

Problem 4 (5 pts)

Let S be a projective surface containing infinitely many exceptional curves (such surfaces exist by PS4 #6). Show that S is rational.

Problem 5 (10 pts)

If S is a compact Kahler surface, each element $\gamma\in H_1(S;\mathbb{Z})$ defines a homomorphism

$$\int_\gamma \cdot\colon H^{1,0}(S)\longrightarrow\mathbb{C},\qquad \omega\longrightarrow\int_\gamma\omega,$$

and thus an element of $H^{1,0}(S)^*$. Let

$$\Lambda_S=\Big\{\int_\gamma\cdot\colon\ \gamma\in H_1(S;\mathbb{Z})\Big\}\subset H^{1,0}(S)^*.$$

(a) Show that $\Lambda_S\subset H^{1,0}(S)^*$ is a lattice.
(b) If $\alpha:[0,1]\longrightarrow S$ is a path, show that the element

$$\int_\alpha\cdot\in\mathrm{Alb}(S)\equiv H^{1,0}(S)^*/\Lambda_S,\qquad \omega\longrightarrow\int_\alpha\omega\in\mathbb{C},$$

depends only on $\alpha(0)$ and $\alpha(1)$.
(c) Thus, for each $p\in S$, there is a well-defined map

$$\mu_p\colon S\longrightarrow\mathrm{Alb}(S),\qquad q\longrightarrow\int_p^q\cdot\,.$$

Show that the image of this map has dimension at least one, unless $q(S)=0$.
(d) If $q(S)\ge 1$ and $p_g(S)=0$, show that the image of μ_p is one-dimensional.

Problem 6 (10 pts)

Let S be a smooth compact complex surface, $L\longrightarrow S$ a holomorphic line bundle, and $C\subset S$ a curve, possibly singular. Let

$$\check{H}^i(C;L)=\check{H}^i(S;L|_C),\quad i\in\mathbb{Z},\qquad K_C=K_S(C)|_C\,;$$

the latter is a sheaf on S, as is $L|_C$. If C is smooth, these definitions agree with the usual ones. Similarly, let

$$\chi(C,L)=\sum_{i\in\mathbb{Z}}(-1)^i\dim\check{H}^i(C;L)\,,\qquad \deg_C L=L\cdot C\equiv\langle c_1(L),[C]\rangle.$$

(a) Using Serre duality for line bundles on S and the Five Lemma, show that

$$\check{H}^i(C;L)\approx\check{H}^{1-i}(C;K_C(-L))^*\,.$$

Conclude that $\check{H}^i(C;L)=0$ unless $i=0,1$.
(b) Using Riemann-Roch for line bundles on S, show that

$$\chi(C,L)=1-a(C)+\deg_C L\,.$$

(c) Conclude that if C is connected, then

$$\dim\check{H}^0(C;K_C)=a(C).$$

Problem 7 (10 pts)

Let $\pi\colon S \longrightarrow \Sigma$ be a holomorphic map from a smooth compact connected complex surface to a smooth compact connected curve of genus g. Show that
(a) the fiber $F_\lambda = \pi^{-1}(\lambda)$ is smooth for all but finitely many $\lambda \in \Sigma$ (*hint:* get a pencil from Σ to use Bertini);
(b) the fibers F_λ and $F_{\lambda'}$ are not linearly equivalent if $\lambda \neq \lambda'$ and $g \geq 1$;
(c) if $g \geq 1$, every irreducible curve C such that the normalization $\tilde{C}$ of C is $\mathbb{P}^1$ is contained in a fiber;
(d) if in addition a generic fiber F of π is connected and of genus $a(F) \geq 1$, there are finitely many such curves.

Problem 8 (10 pts)

With $\pi\colon S \longrightarrow \Sigma$ and $F \subset S$ as in Exercise 7, assume also that $q(S)=1$, $g(\Sigma)=1$, $a(F) \geq 1$, and S contains a smooth irreducible curve C such that C is not contained in any fiber of π,

$$g(C) = 1, \qquad C^2 > 0, \qquad h^0(K_S + C) = 0.$$

Fixing $\lambda_0 \in \Sigma$, for any $\lambda \in \Sigma$ set

$$L_{C,\lambda} = C + F_\lambda - F_{\lambda_0}\,.$$

(a) Show that no element of the linear system $|L_{C,\lambda}|$ is contained in a finite union of fibers of π.
(b) Suppose mC', where C' is an irreducible curve and $m \in \mathbb{Z}^+$, is an element of $|L_{C,\lambda}|$. Show that $m=1$ and C' is smooth of genus 1.
(c) Suppose $\sum_{i=1}^{i=k} m_i C_i$, where $k \geq 2$, $C_i \subset S$ irreducible, $C_i \neq C_j$ for $i \neq j$, and $m_i \in \mathbb{Z}^+$, is an element of $|L_{C,\lambda}|$. Show that

$$C \cdot (K_S + C_i) < 0, \qquad h^0(K_S + C_i) = 0, \qquad a(C_i) \leq 1$$

for every i. If in addition C_i is not contained in any fiber, then C_i is smooth of genus 1. Furthermore, $C_i \cdot K_S < 0$ for some i; if S is minimal and F is connected, this C_i is not contained in any fiber.

Problem 9 (10 pts)

Let $\pi\colon S\longrightarrow\Sigma$ and C be as in Exercise 8, with S minimal and F connected.
(a) Show that there exists an irreducible curve D on S such that for every $\lambda\in\Sigma$ and every element D' of the linear system $|L_{D,\lambda}|$

$$a(D')=1, \qquad D'^2=D^2\equiv d>0, \qquad h^0(K_S+D')=0;$$

furthermore, D' is smooth of genus 1 and is not contained in any fiber of π.
(b) Show that for all $\lambda,\lambda'\in\Sigma$ with $\lambda\neq\lambda'$ and $D'\in|L_{D,\lambda}|$, the restriction map

$$H^0(S;L_{D,\lambda'})\longrightarrow H^0(D';L_{D,\lambda'})$$

is injective. Conclude that $h^0(S,L_{D,\lambda})=d$ for every $\lambda\in\Sigma$.
(c) Let $p_1,\ldots,p_{d-1}\in S$ be general points. Show that for each $\lambda\in\Sigma$, there is an element D_λ of $|L_{D,\lambda}|$ passing through the points $p_1,\ldots,p_{d-1}$; if λ is generic, this element is unique. If $\lambda,\lambda'\in\Sigma$ and $\lambda\neq\lambda'$, the set $D_\lambda\cap D_{\lambda'}$ consists of a single point which is independent of the choice of D_λ and $D_{\lambda'}$ (if there is a choice). Show that the map

$$\Sigma-\lambda\longrightarrow D_\lambda, \qquad \lambda'\longrightarrow D_\lambda\cap D_{\lambda'}\,,$$

extends to a surjective map f_λ over Σ. Conclude that the choice of λ is in fact unique.
(d) Identify Σ with $\mathbb{C}/\Lambda$, where Λ is a lattice in $\mathbb{C}$. Show that the map

$$h\colon\Sigma\times\Sigma\longrightarrow S, \qquad (\mu,\lambda)\longrightarrow f_\lambda(\mu-\lambda),$$

is holomorphic surjective and induces a family of non-constant holomorphic maps

$$h_\mu\colon\mathbb{P}^1\longrightarrow S, \qquad \lambda\longrightarrow h(\mu,\lambda),$$

which cover S. However, this contradicts Exercise 7d.

The purpose of Exercises 7-9 is to add details for G&H pp561-563, especially for p563. More precisely, suppose S is a minimal surface with $q(S)\geq 1$, which contains an irreducible curve C with $C\cdot K_S<0$ and $h^0(K_S+C)=0$, Σ is a smooth curve with $g(\Sigma)=q(S)$, and $\pi: S\longrightarrow\Sigma$ is a holomorphic map with generic fiber F irreducible and $\pi(C)=\Sigma$. By the top of p562, $g(C)=g(\Sigma)$ so that $\pi: C\longrightarrow\Sigma$ is either an isomorphism or $q(S)=1$. By the middle of p562, $a(F)=0$ if $q(S)\geq 2$. By a more involved argument on p563 and in Exercises 7-9, the same conclusion holds for $q(S)=1$.

Problem 10 (10 pts)

Let $V\subset H^0(\mathbb{P}^2;\mathcal{O}(3))$ be the two-dimensional vector space spanned by two general cubic polynomials. Show that
(a) the base locus B of the linear system $\mathbb{P}V$ on $\mathbb{P}^2$ consists of 9 points;
(b) all of the cubics in $\mathbb{P}V$ are smooth, except for 12 that have exactly one node each;
(c) the proper transform $\tilde{V}$ of V in the blowup S of $\mathbb{P}^2$ at B induces a morphism

$$\pi: S\longrightarrow\mathbb{P}\tilde{V}^*,$$

expressing S as an elliptic surface over $\mathbb{P}^1$ with no multiple fibers and 12 nodal fibers. What do the 9 exceptional divisors have to do with π?