# MAT 644: Complex Curves and Surfaces

## Problem Set 4
### Written Solutions (if any) due by Wednesday, 04/15, 10am

*Please figure out all of the problems below and discuss them with others.*

**If you have not passed the orals yet**, *you are encouraged to write up concise solutions to problems worth 10 points.*

### Problem 1 (5 pts)

Let $\mathcal{U} \longrightarrow \overline{\mathcal{M}}_{1,1}$ be the universal family of stable genus 1 1-marked curves with section $s$; see Hain's Section 5.2. Let

$$\mathbb{L}_1 \equiv s^*(T\mathcal{U}^{\mathrm{vert}})^*, \ \mathcal{L}_1 \longrightarrow \overline{\mathcal{M}}_{1,1}$$

be the universal cotangent line bundle at the (first) marked point and the line bundle defined in Hain's Section 4, respectively.
(a) Show that $\mathbb{L}_1$ is indeed a line bundle in the orbifold category (i.e. describe equivariant trivializations over the charts and the isomorphism on the overlap).
(b) Show that $\mathbb{L}_1 \approx \mathcal{L}_1$ in the orbifold category.

### Problem 2 (10 pts)

Fix any 8 general points, $p_1, \ldots, p_8$, in $\mathbb{P}^2$.
(a) Show that the space of cubics passing through the 8 points is a linearly embedded $\mathbb{P}^1$ in $\mathbb{P}H^0(\mathbb{P}^2; \mathcal{O}(3)) \approx \mathbb{P}^9$.
(b) Show that

$$X \equiv \big\{ ([f], q) \in \mathbb{P}^1 \times \mathbb{P}^2 \colon f(q) = 0 \big\}$$

is a smooth submanifold of $\mathbb{P}^1 \times \mathbb{P}^2$. What is its Hodge diamond?
(c) Show that $\pi \colon X \longrightarrow \mathbb{P}^1$ with the holomorphic section

$$s \colon \mathbb{P}^1 \longrightarrow X, \qquad [f] \longrightarrow ([f], p_1),$$

is a family of stable genus 1 1-marked curves (i.e. $(\pi^{-1}(b), s(b))$ is a stable genus 1 1-marked curve for every $b \in \mathbb{P}^1$) and the generic fiber is smooth. Show that the number of singular fibers is 12.
*Hint:* the set of nodes of the fibers is a subset of $X$ which can be written as the zero set of a holomorphic section of a rank-two vector bundle over $X$.
(d) Let $L_1 = s^*(TX^{\mathrm{vert}})^* \longrightarrow \mathbb{P}^1$. Show that $L_1 \approx \mathcal{O}(1) \longrightarrow \mathbb{P}^1$ and $L_1 \approx \Phi^*\mathbb{L}_1$, where $\Phi \colon \mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}_{1,1}$ is the morphism corresponding to the family in (c). Conclude that

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24} \ ,$$

where $\psi_1 = c_1(\mathbb{L}_1) \in H^2(\overline{\mathcal{M}}_{1,1})$.

**Problem 3** (15 pts)

Let $S$ be a compact complex surface ($\dim_{\mathbb{C}} S = 2$). Show that
(a) if $f: S \longrightarrow \mathbb{C}$ is a smooth function s.t. $\bar{\partial}\partial f = 0$, then $f$ is constant;
(b) there are natural injections $\overline{H^{1,0}(S)} \longrightarrow H^{0,1}(S)$ and $H^{1,0}(S) \oplus \overline{H^{1,0}(S)} \longrightarrow H^1(S; \mathbb{C})$;
(c) $b_1(S) \leq h^{1,0}(S) + h^{0,1}(S)$;
(d) there are natural injections $\overline{H^{2,0}(S)} \longrightarrow H^{0,2}(S)$ and $H^{2,0}(S) \oplus \overline{H^{2,0}(S)} \longrightarrow H^2(S; \mathbb{C})$;
(e) $(b_2^+(S) - 2h^{0,2}(S)) + (2h^{0,1}(S) - b_1(S)) = 1$;
(f) $b_1(S) = h^{1,0}(S) + h^{0,1}(S)$ and either $h^{1,0}(S) = h^{0,1}(S)$ and $b_2^+(S) = 2h^{0,2}(S) + 1$
$\qquad\qquad\qquad$ or $h^{1,0}(S) = h^{0,1}(S) - 1$ and $b_2^+(S) = 2h^{0,2}(S)$;
(g) $h^{1,0}(S), h^{0,1}(S), h^{2,1}(S), h^{1,2}(S)$ (resp. $h^{2,0}(S), h^{0,2}(S)$) are topological invariants of the unoriented (resp. oriented) manifold $S$.

*Hints:* (a) show that $f$ is harmonic in each variable in each coordinate chart;

(b) for $\alpha \in H^{1,0}(S)$, show that $\displaystyle\int_S (\mathrm{d}\alpha) \wedge (\mathrm{d}\bar{\alpha}) = 0$;

(c) let $\mathcal{Z}$ denote the sheaf of closed holomorphic 1-forms on $S$; use the short exact sequence

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{O}_S \xrightarrow{\ \mathrm{d}\ } \mathcal{Z} \longrightarrow 0$$

of sheaves on $S$;

(d) for $\beta \in H^{2,0}(S) - \{0\}$, show that $\displaystyle\int_S \beta \wedge \bar{\beta} \neq 0$;

(e) use Noether's and Hirzerbruch's formulas:

$$\chi(\mathcal{O}_S) = \frac{1}{12}\left(K_S^2 + \chi(S)\right) \qquad \text{and} \qquad b_2^+(S) - b_2^-(S) = \frac{1}{3}\left(K_S^2 - 2\chi(S)\right).$$

*Note:* From Serre Duality and Riemann-Roch for $T^*S \longrightarrow S$,

$$2h^{1,0}(S) - h^{1,1}(S) = \chi(T^*S) = \frac{1}{6}\left(K_S^2 - 5\chi(S)\right),$$

we find that $h^{1,1}(S)$ is also a topological invariant of the oriented manifold $S$ and that

$$b_2(S) = h^{2,0}(S) + h^{1,1}(S) + h^{0,2}(S).$$

**Problem 4** (5 pts)

Let $f: C \longrightarrow \mathbb{P}^2$ be an immersion with only simple normal crossing singularities (thus, $|f^{-1}(p)| \leq 2$ for all $p \in \mathbb{P}^2$; if $f^{-1}(p) = \{z_1, z_2\}$ with $z_1 \neq z_2$, $d_{z_1} f(T_{z_1} C) \neq d_{z_2} f(T_{z_2} C)$). Let $S$ be the blowup of $\mathbb{P}^2$ at the double points of $f(C)$ (the singular values of $f$).
(a) Show that $f$ lifts to an embedding $\tilde{f}: C \longrightarrow S$.
(b) Use Adjunction Formula in $S$ to show that if $C \subset \mathbb{P}^2$ is of degree $d$ and has $\delta$ double points, then

$$g(C) = \binom{d-1}{2} - \delta.$$

**Problem 5** (5 pts)

Let $X$ be the blowup of $\mathbb{P}^2$ at one point and $E \subset X$ the exceptional divisor. Show that
(a) $X$ is not biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$;
(b) the blowup of $X$ at a point of $X - E$ is biholomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point;
(c) $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are minimal surfaces (contain no exceptional curves).

**Problem 6** (5 pts)

Let $\pi \colon S \longrightarrow \mathbb{P}^2$ be the blowup at $k$ general points, $p_1, \ldots, p_k \in \mathbb{P}^2$, with the corresponding exceptional divisors $E_1, \ldots, E_k$. Let

$$D = dH - \sum_{i=1}^{i=k} m_i E_i, \qquad m_i \in \mathbb{Z},$$

and $L_{ij}$ be the proper transform of the line $\overline{p_i p_j}$, with $i \neq j$. Suppose $|D| \neq \emptyset$ (i.e. there is an effective divisor linearly equivalent to $D$). Show that
(a) $E_i$ is a fixed component of every curve in $|D|$ iff $m_i < 0$;
(b) $L_{i,j}$ is a fixed component of every curve in $|D|$ iff $d < m_i + m_j$.

**Problem 7** (10 pts)

Let $\pi \colon S \longrightarrow \mathbb{P}^2$ be the blowup at $k$ general points, $p_1, \ldots, p_k \in \mathbb{P}^2$.
(a) Determine the number of exceptional curves in $S$ for $k \leq 8$.
(b) Show that there are infinitely many exceptional curves in $S$ for $k \geq 9$.
*Hint.* The structure of $\mathrm{Pic}(S)$ and the intersection theory on $S$ give necessary numerical conditions for which classes in $\mathrm{Pic}(S)$ *may* be representable by exceptional curves. These conditions readily imply that the number of exceptional curves in $S$ is finite if $k \leq 8$. It then remains to show that all the necessary conditions are sufficient if $k \leq 8$ and that there are infinitely many exceptional curves if $k = 9$. For both purposes, the Cremona transform on pp496,7 can be useful.