MAT 644: Complex Curves and Surfaces
Notes for 04/15/20

Last time: $S = \mathbb{C}$-surface, $F \subset S$ discrete

Prp: (a) If $C \subset S - F$ is a $\mathbb{C}$-curve, then $\overline{C} \subset S$ is also a $\mathbb{C}$-curve

(b) If $f: S - F \to \mathbb{P}^n$ is holomor., then

$f^*\mathcal{O}_{\mathbb{P}^n}(1) \to S - F$ extends to holomor. l.b. $L \to S$

and f induces homom. $f^*: H^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(S; L)$

But f may not extend to $S \to \mathbb{P}^n$

E.g. $f: Bl_p S \hookrightarrow \mathbb{P}^n \Rightarrow f: S - \{pt\} = Bl_p S - E \to \mathbb{P}^n$ does not extend over p

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Rational Maps

Dfn 1: A rational map $f: S \dashrightarrow M$ is a holomor. map $f: S - F \to M$, where $F \subset S$ discrete

Dfn 2: $L \to S$ holomor. l.b., $V \subset H^0(S; L)$ linear subspace

The base locus of V is $BL(V) \equiv \{p \in S : s(p) = 0 \ \forall s \in V\}$

$$\equiv \bigcap_{s \in V} s^{-1}(0) \subset S$$

MAT 545:
$$\begin{pmatrix} V \subset H^0(S; L) \\ BL(V) = \emptyset \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{nondegenerate} \\ f: S \to \mathbb{P}^n \end{pmatrix} \Big/ PSL_{n+1}$$

f nondegen. $\Leftrightarrow$ $\mathrm{Im}\, f \not\subset$ any hyperplane $H \subset \mathbb{P}^n$

Extension ($S = \mathbb{C}$-surface)
$$\begin{pmatrix} V \subset H^0(S; L) \\ \dim BL(V) = 0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{nondegenerate} \\ f: S \dashrightarrow \mathbb{P}^n \end{pmatrix} \Big/ PSL_{n+1}$$

Pf of Extension: (1) $L \to S$, $V \subset H^0(L) \rightsquigarrow f_V: S - BL(V) \to \mathbb{P}V^*$

$$p \to \{s \in V : s(p) = 0\} \subset V \text{ hyperplane } \in \mathbb{P}V^*$$

$\Rightarrow f_V^* \mathcal{O}_{\mathbb{P}V^*}(1) = L|_{S - BL(V)}$ extends to $L \to S$

$f_V^*: H^0(\mathcal{O}_{\mathbb{P}V^*}(1)) \to H^0(S - BL(V); L)$, $H^0(S; L)$ injective; $\mathrm{Im}\, f_V = V$

(2) $f: S \dashrightarrow \mathbb{P}^n \Rightarrow f: S - F \to \mathbb{P}^n$ holomor., $F \subset S$ discrete

Prp $\Rightarrow f^*\mathcal{O}_{\mathbb{P}V^*}(1) \to S - F$ extends to some holom. l.b. $L \to S$

get $f^*: H^0(\mathcal{O}_{\mathbb{P}V^*}(1)) \to H^0(L)$, $V \equiv \mathrm{Im}\, f^*$, $BL(V) \subset F$

Examples of Rational Maps

(0) $f: S \to M$ holomor.

(1) $p \in S \rightsquigarrow f: S - \{p\} \hookrightarrow Bl_p S \rightsquigarrow f: S \dashrightarrow Bl_p S$

(2) $p_1, p_2, p_3 \in \mathbb{P}^2$ non-collinear, $S =$ blowup of $\mathbb{P}^2$ at p_1, p_2, p_3

$E_1, E_2, E_3 =$ exceptional divisors

$L_{12}, L_{13}, L_{23} =$ proper transforms of $\overline{p_1p_2}, \overline{p_1p_3}, \overline{p_2p_3}$

$E_i \cdot E_i = -1$, $E_i \cap E_j = \emptyset$ if $i \neq j$

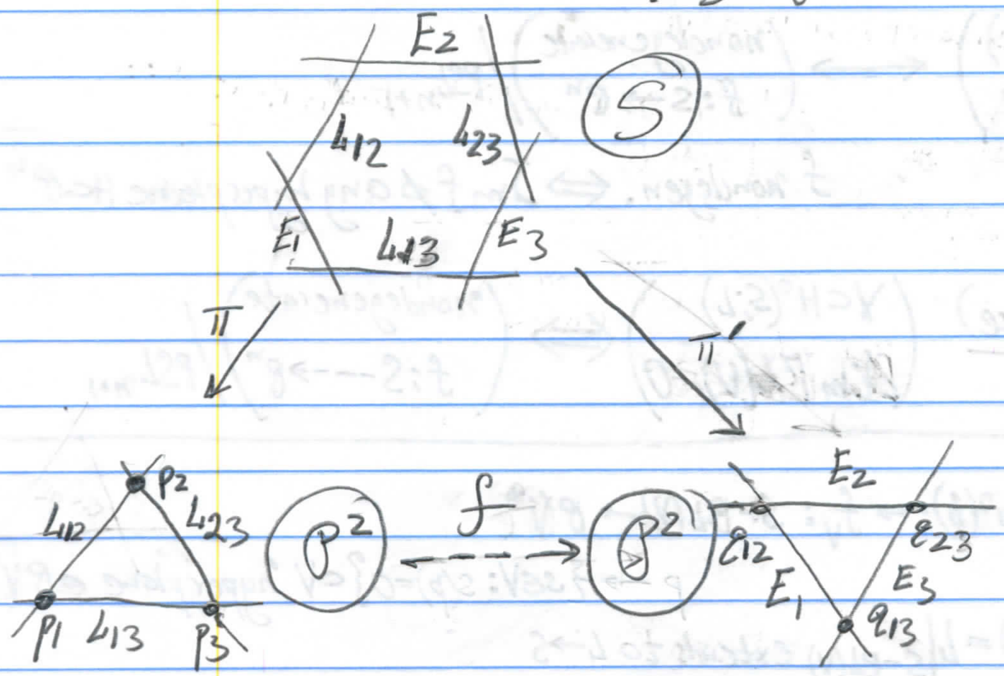
$L_{ij} \approx \mathbb{P}^1$, $L_{ij} \sim \pi^* H - E_i - E_j \Rightarrow L_{ij} \cdot L_{ij} = 1 - 1 - 1 = -1$

$\overline{p_ip_j} \cap \overline{p_jp_k} = \{p_j\}$, $T_{p_j}\overline{p_ip_j} \neq T_{p_j}\overline{p_jp_k}$ if $\{i,j,k\} = \{1,2,3\} \Rightarrow L_{ij} \cap L_{jk} = \emptyset$

$\therefore L_{12}, L_{13}, L_{23}$ disjoint (-1)-curves, $\approx \mathbb{P}^1$

Castelnuovo-Enriques Criterion $\Rightarrow$ can blow down L_{12}, L_{13}, L_{23} to get a new surface S'

$$\left\{\begin{array}{l} \text{Hodge } \Diamond \text{ of } S' = \text{Hodge } \Diamond \text{ of } \mathbb{P}^2 \\ K_{S'} \text{ is not positive: } K_{S'} \cdot \pi'(L_{12}) = -3 \end{array}\right. \Rightarrow S' \approx \mathbb{P}^2$$



$f: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ corresponds to $V = \{s \in H^2(\mathcal{O}_{\mathbb{P}^2}(2)) : s(p_1), s(p_2), s(p_3) = 0\} = \{$conics thr. $p_1, p_2, p_3\}$

f defined on $\mathbb{P}^2 - \{p_1, p_2, p_3\}$, 1:1 on $\mathbb{P}^2 - L_{12} \cup L_{13} \cup L_{23}$ contracts these to pts

Example of Cremona Transform $\mathbb{P}^2 \dashrightarrow$

Lemma 1: if S = $\mathbb{C}$(ompt) $\mathbb{C}$-surface and $f: S \dashrightarrow \mathbb{P}^n$ rational
then $\exists$ $\mathbb{C}$-surface $\tilde{S}$ obtained by a sequence of blowups from S
and holomor. $\tilde{f}: \tilde{S} \to \mathbb{P}^n$ s.t. $\tilde{f} = \pi \circ f$
(wherever RHS is defined)

$$\begin{tikzcd} & \tilde{S} \arrow[dl, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ S \arrow[rr, dashed, "f"] & & \mathbb{P}^n \end{tikzcd}$$

Pf: (0) Can assume f nondegenerate
$\Rightarrow$ corresponds to some linear $V \subset H^0(S;L)$, $BL(V) \subset S$ finite

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(1) Given $p_1 \in BL(V) \rightsquigarrow \operatorname{ord}_{p_1} V \equiv \min\{\operatorname{ord}_{p_1}(s^{-1}(0)) : s \in V\} \in \mathbb{Z}^+$
$S_1 \equiv Bl_{p_1} S$, $E_1 \subset S_1$ exceptional divisor,
$E_1 = (s_1)$ for $s_1 \in H^0(S_1;[E_1])$
$L_1 \equiv \pi_1^* L\left(-(\operatorname{ord}_{p_1} V)E_1\right)$
$V_1 \equiv \{\underbrace{\pi_1^* s}_{\text{vanishes to order} \geq \operatorname{ord}_{p_1} V \text{ along } E_1} / s_1^{\operatorname{ord}_{p_1} V} : s \in V\} \subset H^0(S_1;L_1)$
$E_1 \not\subset BL(V_1) \Rightarrow BL(V_1) \subset S_1$ finite

If $D \in |V| \equiv \mathbb{P}V$ and $D_1 \in |V_1|$,
then $D_1 \sim \pi^* D - (\operatorname{ord}_{p_1} V)E_1 \Rightarrow D_1 \cdot D_1 = D \cdot D - (\operatorname{ord}_{p_1} V)^2 < D \cdot D$ (1)
$f_{V_1} = f_V$ on $S - BL(V) \subset S, S_1$

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(2) Keep blowing up at $p_{r+1} \in Bl(V_r) \subset S_r$
(1) $\Rightarrow$ $D_r \cdot D_r' < D \cdot D - r$ $\forall D_r, D_r' \in |V_r|$ $\boxed{\text{see page 6}}$
$\Rightarrow$ process must terminate
o/w $D_r \cdot D_r' < 0$ $\forall D_r, D_r' \in |V_r| \Rightarrow \exists$ curve $C \subset D$ $\forall D \in |V_r|$
$\Rightarrow BL(V_k) \supset C$ not finite set

explain

Lemma 2: If S, S' = conn. projective surfaces and $\pi: S \to S'$ is 1:1 outside finite $F' \subset S'$, then π is a sequence of blowups

Claim 1: $\forall p' \in S'$, $\pi^{-1}(p') \subset S$ is connected.

O/w not 1:1 just outside p'

or π is constant on an open set $\Rightarrow$ on a connected comp. of S

disjoint neighborhoods of components of $\pi^{-1}(p')$

Claim 2: If $p' \in S'$ and $|\pi^{-1}(p')| \neq 1$, $\pi^{-1}(p') \subset S$ is a curve and contains irred. component C s.t. $C \cdot C, C \cdot K_S < 0$

Pf: $C_1, \dots, C_m \equiv$ irred. components of $\pi^{-1}(p') \subset S$

Pick positive/ample divisor H' on S' s.t. $p' \notin H'$

$\Rightarrow \pi^* H' \cdot \pi^* H' = H' \cdot H' > 0$, $\pi^* H' \cdot C_i = 0 \ \forall i$

"Index Thm" (below) $\Rightarrow (\sum a_i C_i) \cdot (\sum a_i C_i) < 0 \quad \forall a_i \in \mathbb{Z}$ not all zero

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Pick w' = merom. 2-form on S', regular at $p' \in S'$, $w|_{p'} \neq 0$

$\Rightarrow w \equiv \pi^* w'$ merom. 2-form on S, vanishes on $\pi^{-1}(p')$

$\Rightarrow K_S = \sum a_i C_i + D \quad a_i \in \mathbb{Z}^+$, D = divisor on S, disjoint from C_i

$\Rightarrow (\sum a_i C_i) \cdot K_S = (\sum a_i C_i)^2 < 0$

$\Rightarrow C \cdot K_S < 0$ for some $C = C_i$

$C^2 < 0$ by "Index Thm" $\Rightarrow$ Claim 2

Castelnuovo-Enriques $\Rightarrow$ can blow down C to a point p_1 to get $\pi_1: S_1 \to S'$, 1:1 outside of finite subset of S

$\Rightarrow$ get sequence
$$\begin{tikzcd} S \arrow[r] \arrow[dr, "\pi"'] & S_1 \arrow[r] \arrow[d, "\pi_1"] & S_2 \arrow[r] \arrow[dl, "\pi_2"] & \dots \\ & S' & & \end{tikzcd}$$

must terminate b/c $h^2(S_{r+1}) = h^2(S_r) - 1 \Rightarrow$ Lemma 2

On the Index Thm for Connected Cmpt Kähler Surface S ($\dim_{\mathbb{C}} S=2$)

(1) **Hard Lefschetz Thm (p122, proved in MAT 545 in general):**

$$H^{1,1}(S) = \mathbb{C}\omega \oplus P^{1,1}(S)$$

$$P^{1,1}(S) = \text{primitive cohomology of } S \equiv \ker(\omega\wedge\cdot : H^{1,1}(S)\to H^{2,2}(S))$$

(2) **Special case of Hodge-Riemann bilinear relations (top 3rd of p125):**

$$\int_M \eta\wedge\eta < 0 \quad \forall \eta\in P^{1,1}(S)$$

Crl 1: The bilinear form $H^{1,1}(S)\otimes H^{1,1}(S)\to\mathbb{C}$, $(\mu,\eta)\to\int_M \mu\wedge\eta$ has precisely 1 positive eigenvalue.

Crl 2: If $\mu,\eta\in H^{1,1}(S)$ with $\int_M \mu\wedge\mu>0$, $\int_M \mu\wedge\eta=0$, and $\eta\neq0$, then $\int_M\eta\wedge\eta<0$

"Index Thm"

Crl (of Lemmas 1, 2) If S, S' = projective surfaces and $S \underset{g}{\overset{f}{\rightleftarrows}} S'$ are rational maps s.t. $g\circ f$ and $f\circ g$ are defined outside of curves and $= id_S, id_{S'}$, then $\exists$

$$\begin{tikzcd}
 & \tilde{S} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\
S \arrow[rr, dashed, bend left, "f"] & & S' \arrow[ll, dashed, bend left, "g"]
\end{tikzcd}$$

sequences of blowups

Lemma: If S is a cmpt $\mathbb{C}$-surface, $L \to S$ is holomorphic line with $L \cdot L < 0$, and $V \subset H^0(L)$ is a linear subspace, then $Bl(V)$ contains a curve C.

Pf: $\dim V = 0 \Rightarrow Bl(V) = S$ (this case should really be excluded)
$\dim V = 1 \Rightarrow Bl(V) = s^{-1}(0)$ for $s \in V - \{0\}$
Suppose $\dim V \geq 2$ and the statement is true in smaller dimensions.
Let $V' \subset V$ be a hyperplane and $D \in |V| - |V'|$.
By assumption, the curve part C of $Bl(V')$ is nonempty
($Bl(V') - C$ is finitely many points)
Let $|W| = \{D' - C : D' \in V'\} \subset |L(-C)|$. If $C \notin |V'|$, then $\dim |W| \geq 1$.
If D contains no component of C, $D \cdot C \geq 0$
$D \cdot C' < 0$ for $C' \in |W|$.
$\Rightarrow D$ and C' have a component in component $\forall C' \in |W|$
Since $\dim |W| \geq 1$, it follows that the curve D contains
a 2-dim subspace spanned by
at least one component C' $\times$
$\Rightarrow D$ contains a component of C.