MAT 644: Complex Curves and Surfaces
Notes for 03/30/20

Last time: (1) a compactification $\overline{\mathcal{M}}_{1,1}$ of $\mathcal{M}_{1,1}\equiv(H,\mathrm{SL}_2\mathbb{Z})$ by adding the disk

$$\mathbb{D}\equiv\big\{q\in\mathbb{C}\colon |q|<\mathrm{e}^{-2\pi}\big\} \quad \text{with} \quad q=\mathrm{e}^{2\pi\mathrm{i}\tau}\in\mathbb{D}^*, \quad [\tau]\in\big\{z\in H\colon \operatorname{Im} z>1\big\}\Big/\begin{pmatrix}1&\mathbb{Z}\\0&1\end{pmatrix}.$$

(2) line bundles $\mathcal{L}_k=\mathcal{L}_1^{\otimes k}\longrightarrow\overline{\mathcal{M}}_{1,1}$
(3) sections of $\mathcal{L}_k$ are the modular forms (on H) of weight k:

$$f\colon H\longrightarrow\mathbb{C} \qquad \text{s.t.} \quad f\bigg(\frac{a\tau+b}{c\tau+d}\bigg)=(c\tau+d)^k f(\tau) \quad \forall\, \tau\in H,\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2\mathbb{Z} \tag{1}$$

with $\widetilde{f}(q)\equiv f\bigg(\frac{1}{2\pi\mathrm{i}}\ln q\bigg)$ extending over $q=0$

Example of modular form of weight $2k$:

$$G_k\colon H\longrightarrow\mathbb{C}, \qquad G_k(\tau)=\sum_{(m,n)\in\mathbb{Z}^2-\{0\}}\frac{1}{(m\tau+n)^{2k}}\,.$$

Last time: G_k satisfies (1) with k replaced by $2k$ (easy)

$\Longrightarrow$ get $\widetilde{G}_k\colon\mathbb{D}^*\longrightarrow\mathbb{C}$, $\widetilde{G}_k(q)\equiv G_k\bigg(\frac{1}{2\pi\mathrm{i}}\ln q\bigg)$

Prp 1: $\widetilde{G}_k(q)=2\zeta(2k)+2\frac{(2\pi\mathrm{i})^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n$, where

$$\zeta(s)\equiv\sum_{n=1}^{\infty}\frac{1}{n^s} \ \text{ if } \operatorname{Re} s>1, \qquad \sigma_k(n)=\sum_{d\in\mathbb{Z}^+,\, d|n}d^k \ \text{ if } n\in\mathbb{Z}^+.$$

Prp $\Longrightarrow$ $\widetilde{G}_k$ extends over $q=0$ $\Longrightarrow$ G_k is a modular form of weight $2k$

Last time: deduced Prp 1 from

Lemma 1: if $k\ge 2$ and $q=\mathrm{e}^{2\pi\mathrm{i}\tau}\in H$, then $\sum_{n\in\mathbb{Z}}\frac{1}{(\tau+n)^k}=\frac{(-2\pi\mathrm{i})^k}{(k-1)!}\sum_{d=1}^{\infty}d^{k-1}q^d$.

Claim: $\pi\mathrm{i}\frac{q+1}{q-1}=\pi\frac{\cos\pi\tau}{\sin\pi\tau}=\frac{1}{\tau}+\sum_{n=1}^{\infty}\bigg(\frac{1}{\tau+n}+\frac{1}{\tau-n}\bigg)$ (converges if $\tau\notin\mathbb{Z}$)

1st equality follows from $2\cos\pi\tau=\mathrm{e}^{\pi\mathrm{i}\tau}+\mathrm{e}^{-\pi\mathrm{i}\tau}$, $2\mathrm{i}\sin\pi\tau=\mathrm{e}^{\pi\mathrm{i}\tau}-\mathrm{e}^{-\pi\mathrm{i}\tau}$

2nd equality: LHS and RHS have same poles (all simple, at $\tau\in\mathbb{Z}$) and residues (all 1)

$\Longrightarrow$ LHS$-$RHS holomorphic on $\mathbb{C}$

Since it is bounded along $\tau=\mathrm{i}t$ as $t\longrightarrow\infty$ and vanishes at $\tau=1/2$, LHS$-$RHS$=0$

Claim $\Longrightarrow$ $\frac{1}{\tau}+\sum_{n=1}^{\infty}\bigg(\frac{1}{\tau+n}+\frac{1}{\tau-n}\bigg)=\pi\mathrm{i}-2\pi\mathrm{i}\sum_{d=1}^{\infty}q^d$

Differentiate $k-1$ times w.r.t. τ:

$$(-1)^{k-1}(k-1)!\sum_{n\in\mathbb{Z}}\frac{1}{(\tau+n)^k}=-2\pi\mathrm{i}(2\pi\mathrm{i})^{k-1}\sum_{d=1}^{\infty}d^{k-1}q^d$$

This gives Lemma 1.

Note: $G_k(\tau\!=\!\infty)\!\equiv\!\widetilde{G}_k(q\!=\!0)\!=\!2\zeta(2k)\!\neq\!0$

Lemma 2: if $k\!\in\!\mathbb{Z}^+$, $\zeta(2k)=\dfrac{(2\pi)^{2k}}{2(2k)!}B_k$, where B_k is the k-th Bernoulli number:

$$\frac{x}{\mathrm{e}^x-1}\equiv 1-\frac{x}{2}-\sum_{k=1}^{\infty}(-1)^k B_k\frac{x^{2k}}{(2k)!}\qquad\Longleftrightarrow\qquad z\frac{\cos z}{\sin z}\equiv 1-\sum_{k=1}^{\infty}\frac{2^{2k}B_k}{(2k)!}z^{2k}\,;$$

the equivalence of the two definitions is obtained by moving $\frac{x}{2}$ to LHS and taking $z\!=\!\frac{x}{2\mathrm{i}}$

Crl 1: $G_k(\tau\!=\!\infty)\!=\!\dfrac{(2\pi)^{2k}}{(2k)!}B_k$, e.g. $G_2(\tau\!=\!\infty)\!=\!\dfrac{\pi^4}{45}$, $G_3(\tau\!=\!\infty)\!=\!\dfrac{2\pi^6}{45\!\cdot\!21}$

Pf of Lemma 2: Claim with $z\!=\!\pi\tau$ gives

$$\begin{aligned} z\frac{\cos z}{\sin z} &= 1\!+\!2\sum_{n=1}^{\infty}\frac{z^2}{z^2\!-\!(\pi n)^2}=1-2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{z^{2k}}{\pi^{2k}n^{2k}}\\ &= 1-\sum_{k=1}^{\infty}\left(2\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\right)\frac{z^{2k}}{\pi^{2k}}\equiv 1-\sum_{k=1}^{\infty}2\zeta(2k)\frac{z^{2k}}{\pi^{2k}}\,. \end{aligned}$$

This gives Lemma 2.

Remaining goal for Part II: $\overline{\mathcal{M}}_{1,1}$ is the moduli space of
stable nodal genus 1 curves with 1 marked point

(1) extend the universal family from $\mathcal{M}_{1,1}$ to $\overline{\mathcal{M}}_{1,1}$

(2) prove compactness

Do (1) by relating modular forms to cubic curves in $\mathbb{P}^2$

Define modular forms $g_2,g_3\colon H\longrightarrow\mathbb{C}$ of weights 4 and 6 by

$$g_2(\tau)=60G_2(\tau),\qquad g_3(\tau)=140G_3(\tau)\,.$$

Crl 2: $g_2(\tau\!=\!\infty)^3\!-\!27g_3(\tau\!=\!\infty)^2\!=\!0$

For $\tau\!\in\!H$, let $\Lambda_\tau\!\equiv\!\mathbb{Z}\!\oplus\!\mathbb{Z}\tau$ be the corresponding lattice.

Prp 2: $\forall\,\tau\!\in\!H$, $\exists$ embedding $\widetilde{\Phi}_\tau\colon S_\tau\!\equiv\!\mathbb{C}/\Lambda_\tau\hookrightarrow\mathbb{P}^2$ such that

$$\widetilde{\Phi}_\tau(0)=[0,1,0]\qquad\text{and}\qquad \mathrm{Im}\,\widetilde{\Phi}_\tau=\big\{[X,Y,Z]\!\in\!\mathbb{P}^2\colon Y^2Z=4X^3\!-\!g_2(\tau)XZ^2\!-\!g_3(\tau)Z^3\big\}.$$

Previously: (S,p) = elliptic curves $\Longrightarrow\mathcal{O}(3p)\longrightarrow S$ induces $\varphi\colon S\hookrightarrow\mathbb{P}^2$ as a cubic

Explicitly: take x = meromorphic function on S with $(x)_\infty\!=\!2p$
y = meromorphic function on S with $(y)_\infty\!=\!3p$

$$\Longrightarrow\qquad \varphi\colon S\longrightarrow\mathbb{P}^2,\quad \varphi(z)=\big[x(z),y(z),1\big],\quad \varphi(p)=[0,1,0].$$

$(S,p)=(S_\tau, z=0)$: take x = Weierstrass $\mathcal{P}$-function, $y=x'\equiv\frac{\mathrm{d}}{\mathrm{d}z}x$:

$$x(z)\equiv\mathcal{P}_\tau(z)\equiv\frac{1}{z^2}+\sum_{\gamma\in\Lambda_\tau-\{0\}}\left(\frac{1}{(z-\gamma)^2}-\frac{1}{\gamma^2}\right)$$

converges if $z\notin\Lambda_\tau$ b/c $\displaystyle\int_{\mathbb{C}-B_\delta(0)}\frac{1}{R^3}$ does

$\mathcal{P}_\tau(z+\gamma)=\mathcal{P}(z)\ \ \forall\, z\in\mathbb{C},\ \gamma\in\Lambda_\tau \implies \mathcal{P}_\tau$ is well-defined on S_τ

Lemma 3: $\mathcal{P}_\tau'(z)^2=4\mathcal{P}_\tau(z)^3-g_2(\tau)\mathcal{P}_\tau(z)-g_3(\tau)$

Proof: $\displaystyle\frac{1}{(\gamma-z)^2}=\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{\gamma-z}\right)=\sum_{m=0}^{\infty}\frac{1}{\gamma^{m+1}}z^{m-1}\cdot m=\frac{1}{\gamma^2}+\sum_{m=1}^{\infty}\frac{1}{\gamma^{m+2}}(m+1)z^m$

$$\implies\qquad \mathcal{P}_\tau(z)=\frac{1}{z^2}+\sum_{m=1}^{\infty}\left(\sum_{\gamma\in\Lambda_\tau-\{0\}}\frac{1}{\gamma^{m+2}}\right)(m+1)z^m=\frac{1}{z^2}+\sum_{k=1}^{\infty}G_{k+1}(\tau)(2k+1)z^{2k}$$

$$=\frac{1}{z^2}+3G_2(\tau)z^2+5G_3(\tau)z^4+\ldots$$

$$\implies\qquad \mathcal{P}_\tau'(z)=-\frac{2}{z^3}+\sum_{k=1}^{\infty}G_{k+1}(\tau)2k(2k+1)z^{2k-1}=-\frac{2}{z^3}+6G_2(\tau)z+20G_3(\tau)z^3+\ldots$$

Thus, $4\mathcal{P}_\tau(z)^3-\mathcal{P}_\tau'(z)^2-60G_2(\tau)\mathcal{P}_\tau(z)-140G_3(\tau)$ is holomorphic on S_τ, $=0$ at $z=0\implies\equiv 0$

Lemma 3 $\implies$ Prp 2: $\widetilde{\Phi}_\tau(z)=\big[\mathcal{P}_\tau(z),\mathcal{P}_\tau'(z),1\big]$

Define $\mathcal{U}'\equiv\big\{(q,[X,Y,Z])\in\mathbb{D}\times\mathbb{P}^2\colon Y^2Z=4X^3-\widetilde{g}_2(q)XZ^2-\widetilde{g}_3(q)Z^3\big\}$

Implicit FT $\implies\mathcal{U}'$ is smooth; $\mathbb{Z}_2$ acts on $\mathcal{U}'$ by $(-1)\cdot(q,[X,-Y,Z])$

Since $\overline{\mathcal{M}}_{1,1}$ is obtained by gluing $\mathcal{M}_{1,1}$ and $(\mathbb{D},\mathbb{Z}_2)$ along $(H',\mathbb{Z}_2\times\mathbb{Z})$ as in the bottom row below, we get a family of curves $\overline{\mathcal{U}}\longrightarrow\overline{\mathcal{M}}_{1,1}$ by gluing $\mathcal{U}$ and $(\mathcal{U}',\mathbb{Z}_2)$ along $(\mathcal{W}|_{H'},\mathbb{Z}_2\times\mathbb{Z})$ as in the top row:

$$\begin{array}{ccccccc}
\mathcal{U}=(\mathcal{W},\mathrm{SL}_2\mathbb{Z}) & \xleftarrow{(\widetilde{\iota},\phi)} & (\mathcal{W}|_{H'},\mathbb{Z}_2\times\mathbb{Z}) & \xrightarrow{(\widetilde{\Phi},\pi_1)} & (\mathcal{U}',\mathbb{Z}_2) & \qquad & \overline{\mathcal{U}}=\mathcal{U}\cup_{(\mathcal{W}|_{H'},\mathbb{Z}_2\times\mathbb{Z})}(\mathcal{U}',\mathbb{Z}_2)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\mathcal{M}_{1,1}=(H,\mathrm{SL}_2\mathbb{Z}) & \xleftarrow{(\iota,\phi)} & (H',\mathbb{Z}_2\times\mathbb{Z}) & \xrightarrow{(\Phi,\pi_1)} & (\mathbb{D},\mathbb{Z}_2) & & \overline{\mathcal{M}}_{1,1}=\mathcal{M}_{1,1}\cup_{(H',\mathbb{Z}_2\times\mathbb{Z})}(\mathbb{D},\mathbb{Z}_2)
\end{array}$$

where $H'\equiv\{\tau\in H\colon\mathrm{Im}\,\tau>1\}$, $\quad\phi(\pm1,k)=\pm\begin{pmatrix}1&k\\0&1\end{pmatrix}$, $\quad\Phi(\tau)=\mathrm{e}^{2\pi\mathrm{i}\tau}$

$\mathcal{W}\equiv(H\times\mathbb{C})/\sim$, $\ (\tau,z)\sim(\tau,z+m\tau+n)\ \forall\,(\tau,z)\in H\times\mathbb{C},\ m,n\in\mathbb{Z}$

$\mathbb{Z}_2\times\mathbb{Z}$ acts on $\mathcal{W}$ by $(\pm1,k)\cdot[\tau,z]=(\tau+k,\pm z)$

The fiber of $\overline{\mathcal{U}}\longrightarrow\overline{\mathcal{M}}_{1,1}$ over $[\tau]\in\mathcal{M}_{1,1}=\overline{\mathcal{M}}_{1,1}-\{q=0\}$ is the smooth elliptic curve $(S_\tau,0)$

The fiber of $\overline{\mathcal{U}}\longrightarrow\overline{\mathcal{M}}_{1,1}$ over $q=0$ (or $\tau=\infty$) is the plane cubic

$$\mathcal{C}\equiv\big\{[X,Y,Z]\in\mathbb{P}^2\colon Y^2Z=4X^3-\widetilde{g}_2(q=0)XZ^2-\widetilde{g}_3(q=0)Z^3\big\}$$

Plane cubic $\mathcal{C}_{a,b}\equiv\big\{[X,Y,Z]\in\mathbb{P}^2\colon Y^2Z=4X^3-aXZ^2-bZ^3\big\}$

contains $[0,1,0]$

near $[0,1,0]$: $Z=4X^3-aXZ^2-bZ^3$

$\Longrightarrow$ $[0,1,0]$ is a smooth point of $\mathcal{C}_{a,b}$

$\mathcal{C}_{a,b}\cap\big\{[X,Y,0]\in\mathbb{P}^2\big\}=\{[0,1,0]\}$

$\Longrightarrow$ $[0,1,0]$ is a flex point of $\mathcal{C}_{a,b}$ (order 3 contact with the line $\{Z=0\}$)

on $Z\neq0$: $Y^2=4X^3-aX-b$

$\Longrightarrow$ $\mathcal{C}_{a,b}$ is smooth iff $4X^3-aX-b$ has simple roots iff the discriminant $D(a,b)\neq0$:

$$D(a,b)\equiv(r_1-r_2)^2(r_1-r_3)^2(r_2-r_3)^2=\frac{1}{16}(a^3-27b^2)$$

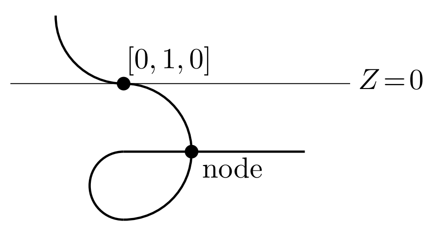
Weight 12 modular form $\Delta\colon H\longrightarrow\mathbb{C}$, $\Delta(\tau)=g_2(\tau)^3-27g_3(\tau)^2$

$\big\{[X,Y,Z]\in\mathbb{P}^2\colon Y^2Z=4X^3-g_2(\tau)XZ^2-g_3(\tau)Z^3\big\}\approx S_\tau\equiv\mathbb{C}/\Lambda_\tau$

smooth cubic in $\mathbb{P}^2\Longrightarrow\Delta(\tau)\neq0\ \forall\,\tau\in H$

Crl 2 $\Longrightarrow\Delta(q=0)=\Delta(\tau=\infty)=0$

$\Longrightarrow$ the fiber of $\overline{\mathcal{U}}\longrightarrow\overline{\mathcal{M}}_{1,1}$ over $q=0$ is a plane cubic with one simple node, not $[0,1,0]$



Added after discussion: Δ is a modular form of weight 12 and thus a section of $\mathcal{L}_{12}=\mathcal{L}_1^{\otimes12}$. Its only zero is the point $q=0$ in $\mathbb{D}$; the stabilizer of this point is $\mathbb{Z}_2$. By Prp 1, this zero is transverse. Thus,

$$\int_{\overline{\mathcal{M}}_{1,1}}c_1\big(\mathcal{L}_1^{\otimes12}\big)=\frac{1}{2}\,. \tag{2}$$

By Problems 1 and 2d on HW4,

$$\int_{\overline{\mathcal{M}}_{1,1}}c_1\big(\mathcal{L}_1\big)=\frac{1}{2}\cdot\frac{1}{n_3}\,, \tag{3}$$

where n_3 is the number of plane rational cubics that pass through 8 general points in $\mathbb{P}^2$. By (2) and (3), $n_3=12$. A direct way of computing this number is suggested by the hint for Problem 2c on HW4; see also Section 2.3 in math/0507105. Another approach is explained in

https://www.math.tamu.edu/~frank.sottile/research/pages/shapiro/real_cubics.html