Notes on Enumerative Geometry

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September 25, 2013

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0 Introduction

Enumerative geometry of algebraic varieties is a fascinating field of mathematics that dates back to the nineteenth century. The general goal of this subject is to determine the number of geometric objects that satisfy pre-specified geometric conditions. The objects are typically (complex) curves in a smooth algebraic manifold. Such curves are usually required to represent the given homology class, to have certain singularities, and to satisfy various contact conditions with respect to a collection of subvarieties.

Example 0.1. There is precisely 1 real/complex line that passes through two distinct points in $\mathbb{R}^n/\mathbb{C}^n$ (or in $\mathbb{RP}^n/\mathbb{CP}^n$).

Example 0.2. *How many degree 2 curves pass through 5 points in $\mathbb{C}^2$ or in $\mathbb{CP}^2$?* A degree 2 curve (or conic) C in $\mathbb{C}^2$ is the zero set of a nonzero degree 2 polynomial in 2 variables, e.g.

$$C = \big\{(x,y)\in\mathbb{C}^2:\ x^2+2xy-xy=0\big\}.$$

Each degree 2 polynomial on $\mathbb{C}^2$ is determined by 6 complex coefficients (of $x^2, xy, y^2, x, y, 1$). Two nonzero degree 2 polynomials Q_1 and Q_2 determine the same curve if and only if $Q_1=\lambda Q_2$ for some $\lambda\in\mathbb{C}^*$. Thus, the space of degree 2 curves in $\mathbb{C}^2$ can be identified with

$$\big\{(a_{20},a_{11},a_{02},a_{10},a_{01},a_{00})\in(\mathbb{C}^6-0)\big\}/\mathbb{C}^* = \mathbb{CP}^5\,.$$

The curve $C=Q^{-1}(0)$ determined by the degree 2 polynomial

$$Q(x,y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$$

passes through a point (x_i,y_i) if and only if

$$a_{20}x_i^2 + a_{11}x_iy_i + a_{02}y_i^2 + a_{10}x_i + a_{01}y_i + a_{00} = 0. \tag{0.1}$$

Thus, the space of degree 2 curves passing through a fixed point (x_i,y_i) is a hyperplane H_{x_i,y_i} in $\mathbb{C}^5$. If the hyperplanes $H_{x_1,y_1},\ldots,H_{x_5,y_5}$ are sufficiently general, the number of degree 2 curves passing through the points $(x_1,y_1),\ldots,(x_5,y_5)$ is

$$\big|H_{x_1,y_1}\cap\ldots\cap H_{x_5,y_5}\big| = \big\langle(\mathrm{PD}_{\mathbb{CP}^5}H)^5,\mathbb{CP}^5\big\rangle = 1,$$

where $\mathrm{PD}_{\mathbb{CP}^5}H\in H^2(\mathbb{CP}^5;\mathbb{Z})$ is the Poincare dual of the homology class of a hyperplane in $\mathbb{CP}^5$. Alternatively, the 5 points determine 5 linear equations (0.1) on the 6 coefficients of Q. If these equations are linearly independent, the space of the solutions is one complex line $\mathbb{C}Q$ in the space $\mathbb{C}^6$ of the coefficients, which corresponds to one curve passing through the 5 points.

r	0	1	2	3	4	5
#	1	2	4	4	2	1

Table 1: The number of conics passing through $5-r$ general points and tangent to r general lines

Example 0.3. *How many conics pass through $5-r$ points and are tangent to r lines in $\mathbb{C}^2$ or in $\mathbb{CP}^2$?* By Example 0.2, the $r=0$ number is 1. Fairly straightforward topological computations show that the $r=1$ and $r=2$ are 2 and 4, respectively. A direct topological computation for $r\geq 3$ turns out to be much harder, as it involves an *excess intersection* (failure of transversality). Fortunately, there is a simple geometric reason for the r and $5-r$ numbers to be the same; see Table 1 for the list of numbers and [8, Sections 2,3] for a detailed discussion of this problem.

In this course, we will apply various topological methods to enumerative problems in algebraic geometry. We will encounter two flavors of computational setups: a fairly classical one, which involves counting curves fairly directly, and a fairly recent one (Gromov-Witten theory), which involves counting parametrizations of curves. We will focus on counting complex curves in complex manifolds, as this is generally much easier than counting real curves in real manifolds. The basic reasons for this are

(0) the number of complex roots (counted with multiplicity) of a degree d complex polynomial in one variable is d, while the number of real roots (counted with multiplicity) of a degree d real polynomial in one variable is at most d and is of the same parity as d;

(1) complex curves are often parametrized by complex manifolds, which have a canonical orientation, while real curves are often parametrized by real manifolds, which may not be even orientable;

(2) compactifications of spaces parameterizing complex curves often have additional strata (known as boundary) of complex codimension one (which does not cause problems with integration of top de Rham forms), while compactifications of spaces parameterizing real curves often have additional strata of complex real codimension one (which does generally cause problems with integration).

The next statement, which is a standard fact in topology, will lie behind many arguments in this course.

Theorem 0.4. *If M is a compact oriented m-manifold, $V \longrightarrow M$ is a rank k real oriented vector bundle, and $s: M \longrightarrow V$ is a section transverse to the zero set, then $s^{-1}(0) \subset M$ is a smooth oriented submanifold and*

$$\left[s^{-1}(0)\right]_M = \mathrm{PD}_M e(V) \in H_{m-k}(M;\mathbb{Z}), \tag{0.2}$$

where $[s^{-1}(0)]_M$ is the homology class on M determined by $s^{-1}(0)$ and $e(V)\in H^k(M;\mathbb{Z})$ is the euler class of V.

If k is odd, (0.2) may be off by sign, depending on one's sign conventions. However, $2e(V)=0$ then, which makes such cases of little relevance to us. A proof of this theorem is outlined in [12, Exercise 11-C], which uses a rather unusual sign convention for the pairing of homology and cohomology elements.

Exercise 0.5. Let M be a topological space and $V \longrightarrow M$ be an oriented vector bundle of odd rank. Show that $2e(V)=0$.

Exercise 0.6. Verify Theorem 0.4.

Corollary 0.7. *If M is a compact oriented m-manifold, $V \longrightarrow M$ is a rank m real oriented vector bundle, and $s\colon M \longrightarrow V$ is a section transverse to the zero set, then $s^{-1}(0)\subset M$ is a finite set of signed points and*

$$^{\pm}\big|s^{-1}(0)\big| = \big\langle e(V), [M]\big\rangle \in \mathbb{Z}, \tag{0.3}$$

where $^{\pm}|s^{-1}(0)|$ is the signed cardinality of the set $s^{-1}(0)$.

Exercise 0.8. Deduce Corollary 0.7 from Theorem 0.4.

We will often encounter cases when a curve count can be written as $^{\pm}|s^{-1}(0)|$ for some transverse section s of a vector bundle $V \longrightarrow M$. The above immediate corollary of Theorem 0.4 will then allow us to express such a curve in terms of a topological quantity, the euler class, which is often computable.

The proof of Theorem 0.4 also leads to the following statements.

Proposition 0.9. *If M is a compact oriented manifold and $Y, Z \subset M$ are compact oriented submanifolds of M intersecting transversally in M, then $Y\cap Z$ is a compact oriented submanifold of M and*

$$\begin{aligned} &\mathrm{PD}_M\big([Y\cap Z]_M\big) = \mathrm{PD}_M([Y]_M) \cup \mathrm{PD}_M([Z]_M) \in H^*(M;\mathbb{Z}) \qquad \textit{and} \\ &\big(\mathrm{PD}_M([Z]_M)\big)|_Y = \mathrm{PD}_Y([Y\cap Z]_Y) \in H^*(Y;\mathbb{Z}). \end{aligned}$$

Corollary 0.10. *If M is a compact oriented manifold and $Y, Z\subset M$ are compact oriented submanifolds of M intersecting transversally in M such that*

$$\dim Y + \dim Z = \dim M,$$

then $Y\cap Z$ is a finite set of signed points and

$$^{\pm}\big|Y\cap Z\big| = \big\langle \mathrm{PD}_M([Y]_M), [Z]_M\big\rangle.$$

Exercise 0.11. Verify Proposition 0.9 and deduce Corollary 0.10 from it.

We will typically be taking euler classes of complex vector bundles. Every complex vector bundle $V \longrightarrow M$ has a well-defined (total) chern class

$$\begin{aligned} c(V) = 1 + c_1(V) + c_2(V) + \ldots \in H^0(M;\mathbb{Z}) \oplus H^2(M;\mathbb{Z}) \oplus H^4(M;\mathbb{Z}) \oplus \ldots \\ \text{s.t.} \qquad c_r(V) = \begin{cases} e(V), & \text{if } r = \mathrm{rk}_{\mathbb{C}} V, \\ 0, & \text{if } r > \mathrm{rk}_{\mathbb{C}} V, \end{cases} \end{aligned}$$

where $e(V)$ is the euler class of V with respect to the canonical complex orientation (given by the real basis $e_1, \mathrm{i}e_1, \ldots, e_k, \mathrm{i}e_k$ for a complex basis $e_1, \ldots, e_k$ for a fiber). If $V, W \longrightarrow M$ are two complex vector bundles, so is $V\oplus W \longrightarrow M$ and

$$c(V \oplus W) = c(V) \cdot c(W) \in H^{2*}(M;\mathbb{Z}).$$

If $f\colon X \longrightarrow M$ is a continuous map, then

$$c(f^*V) = f^*c(V) \in H^{2*}(M;\mathbb{Z}).$$

A detailed construction of chern classes is contained in [12, Section 14].

The construction of the complex projective space extends to complex vector bundles. If $V \longrightarrow M$ is a complex vector bundle of (complex) rank k, the projectivization of V is the $\mathbb{P}^{k-1}$-fiber bundle

$$\mathbb{P}V \equiv (V-M)/\mathbb{C}^* \longrightarrow M,$$

where $M \subset V$ is the zero section and $\mathbb{C}^*$ acts by the usual multiplication in each fiber. We will view each element of $\mathbb{P}V$ as a complex line ℓ (through the origin) in a fiber V_x of $V \longrightarrow M$. The tautological line bundle over $\mathbb{P}V$ is defined by

$$\gamma_V = \big\{(\ell,v)\in\mathbb{P}V\times V : v\in\ell\subset V\big\} \longrightarrow \mathbb{P}V.$$

Let

$$\lambda_V = c_1(\gamma_V^*) \in H^2(\mathbb{P}V;\mathbb{Z})$$

denote the chern class of the hyperplane line bundle. The restrictions of $1, \lambda_V, \ldots, \lambda_V^{k-1}$ to a fiber generate its cohomology as a $\mathbb{Z}$-module. Thus, by the generalized Thom Isomorphism Theorem [22, Theorem 5.7.9],

$$H^*(\mathbb{P}V;\mathbb{Z}) \approx H^*(M;\mathbb{Z})[\lambda_V]/\big(\lambda_V^k + c_1(V)\lambda^{k-1} + \ldots \lambda_V c_{k-1}(V) + c_k(V)\big) \tag{0.4}$$

as $\mathbb{Z}$-modules.

Exercise 0.12. Show that the isomorphism in (0.4) respects the ring structure.

Example 0.13. If M is a point and $V \longrightarrow M$ is a complex vector bundle of (complex) rank k, then

$$V = \mathbb{C}^k, \quad c(V) = 1, \quad \mathbb{P}V = \mathbb{P}^{k-1}, \quad H^*(\mathbb{P}V;\mathbb{Z}) \approx \mathbb{Z}[\lambda_V]/\lambda_V^k,$$

and $\mathrm{PD}_{\mathbb{P}V}\lambda_V = [\mathbb{P}\mathbb{C}^{k-1}]_{\mathbb{P}V}$ for any $\mathbb{C}^{k-1} \subset V$.

1 Schubert Calculus

1 Lines in affine/projective spaces

A (complex) line in $\mathbb{C}^n$ is a set of points of the form

$$\mathrm{pt} + \mathbb{C}\vec{v} \equiv \{\mathrm{pt} + \lambda\vec{v} \colon \lambda \in \mathbb{C}\}$$

for some point $\mathrm{pt} \in \mathbb{C}^n$ and a nonzero vector $\vec{v} \in \mathbb{C}^n - 0$. A (projective) line in $\mathbb{P}^n$ is the closure in $\mathbb{P}^n$ of a line contained in a chart

$$U_i = \{[Z_0, \ldots, Z_n] \in \mathbb{P}^n \colon Z_i \neq 0\} \approx \mathbb{C}^n$$

as in Section B.1. The number of lines in $\mathbb{C}^n$ passing through a general points, b general lines, etc. is the same as the number of lines in $\mathbb{P}^n$ passing a general points, b general lines, etc., with the bijection given by the inclusion

$$U_0 = \mathbb{C}^n \longrightarrow \mathbb{P}^n, \qquad (z_1, \ldots, z_n) \longrightarrow [1, z_1, \ldots, z_n].$$

Thus, enumerative problems for $\mathbb{C}^n$ and $\mathbb{P}^n$ are generally the same. The latter space has the advantage of being compact and thus is more suitable for topological computations. Instead of thinking of points, lines, etc. in $\mathbb{P}^n$ as closures in $\mathbb{P}^n$ of such objects in $\mathbb{C}^n$, it is often more convenient to think of them as projectivizations of linear subspaces of $\mathbb{C}^n$. For example, a point in $\mathbb{P}^n$ is the projectivization of a one-dimensional linear subspace of $\mathbb{C}^n$, a line in $\mathbb{P}^n$ is the projectivization of a two-dimensional linear subspace of $\mathbb{C}^n$, etc.

There is a unique line passing through any 2 distinct points in $\mathbb{C}^n$ or in $\mathbb{P}^n$. There are no other interesting constraints to be imposed on lines in $\mathbb{P}^2$. There are still two such questions left regarding lines in $\mathbb{P}^3$: how many lines in $\mathbb{P}^3$ pass through

(1) 1 point and 2 general lines;

(2) 4 general lines.

We first observe that the expected answers are finite, i.e. the dimensions of the conditions are the same as the dimension of the space of lines in $\mathbb{P}^3$. Each line in $\mathbb{P}^3$ is determined by two distinct points in $\mathbb{P}^3$; the dimension of the space of pairs of such points is $2\cdot 3 = 6$. However, the space of pairs of points on a fixed line is of dimension $2\cdot 1 = 2$; thus, the dimension of the space of lines in $\mathbb{P}^3$ is $6-2=4$. The dimension of the condition of passing through a point in $\mathbb{P}^3$ is 2, since each point is of codimension 3, but it can be any of the points in the one-dimensional space of points on a line. It follows that the dimension of the condition of passing through a point in $\mathbb{P}^3$ is $2-1=1$, since a line can now pass through any point on a fixed line. Thus, (1) and (2) impose the expected number of conditions on the space of lines in $\mathbb{P}^3$.

Example 1.1. *How many lines in $\mathbb{P}^3$ pass through 1 point and 2 general lines?* The space of lines passing through the point and one of the lines form a plane, which meets the other line in a single point. Along with the original point, the latter determines the unique line passing through the three constraints. Thus, the answer is 1. By the same argument, if a, b, and c are non-negative integers such that $a+b+c=n-1$, then the number of lines in $\mathbb{C}^n$ or $\mathbb{P}^n$ meeting general subspaces of dimensions a, b, and c is again 1.

In the remainder of this section, we will determine the number of lines in $\mathbb{P}^3$ passing through 4 general lines. A line ℓ in $\mathbb{P}^3$ corresponds to a plane (two-dimensional linear subspace) $\pi\subset\mathbb{C}^4$ by $\ell=\mathbb{P}^1\pi$. The space of all lines in $\mathbb{P}^3$ is thus the same as the space of planes in $\mathbb{C}^3$, which is known as the Grassmannian

$$\mathbb{G}(2,4) = \mathrm{GL}_4(\mathbb{C})/H = U(4)/U(2)\times U(2),$$

where $H\subset\mathrm{GL}_4(\mathbb{C})$ is the subgroup of matrices of the form

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

The Grassmannian $\mathbb{G}(2,4)$ is a compact complex manifold of dimension $2(4-2)=4$; see Section 2.

If $\ell,\ell_i\subset\mathbb{P}^3$ are lines,

$$\ell\cap\ell_i\neq\emptyset \qquad\Longleftrightarrow\qquad \dim(\pi\cap\pi_i)\geq 1.$$

Thus, the subspace of lines ℓ in $\mathbb{P}^3$ meeting ℓ_i is

$$\sigma_1(\pi_i)\equiv\big\{\pi\in\mathbb{G}(2,4)\colon\ \dim(\pi\cap\pi_i)\geq 1\big\}.$$

This is a complex subvariety of $\mathbb{G}(2,4)$; it determines a homology/cohomology class

$$\sigma_1\in H_6\big(\mathbb{G}(2,4);\mathbb{Z}\big), H^2\big(\mathbb{G}(2,4);\mathbb{Z}\big),$$

which is independent of the choice of π_i (since the space of the planes π_i is path-connected). The number we are interested in is

$$\big|\sigma_1(\pi_1)\cap\sigma_1(\pi_2)\cap\sigma_1(\pi_3)\cap\sigma_1(\pi_4)\big| = \big\langle\sigma_1^4,\mathbb{G}(2,4)\big\rangle;$$

the last equality holds if the intersection is transverse. Thus, the number of lines in $\mathbb{P}^3$ passing through 4 general lines is determined by the ring $H^*(\mathbb{G}(2,4);\mathbb{Z})$.

If the lines ℓ_1 and ℓ_2 intersect at a point $\mathrm{pt}=\mathbb{P}L$ (which is not generically the case), they form a plane $P=\mathbb{P}V$ in $\mathbb{P}^3$, for some 3-dimensional linear subspace $V\subset\mathbb{C}^4$. In this case, $\sigma_1(\pi_1)\cap\sigma_1(\pi_2)$ is the union of the sets

$$\sigma_2(L) = \big\{\pi\in\mathbb{G}(2,4)\colon\ \dim(\pi\cap L)\geq 1\big\} \qquad\text{and}\qquad \sigma_{1,1}(V) = \big\{\pi\in\mathbb{G}(2,4)\colon\ \dim(\pi\cap V)\geq 2\big\}$$

consisting of the lines passing through pt and of the lines contained in P, respectively. This corresponds to the statement

$$\sigma_1^2 = \sigma_2+\sigma_{11}, \tag{1.1}$$

which is a special case of (2.11) below. Thus,

$$\sigma_1(\pi_1)\cap\sigma_1(\pi_2)\cap\sigma_1(\pi_3)\cap\sigma_1(\pi_4)=\sigma_2(L)\cap\sigma_1(\pi_3)\cap\sigma_1(\pi_4)\ \cup\ \sigma_{1,1}(V)\cap\sigma_1(\pi_3)\cap\sigma_1(\pi_4).$$

The set $\sigma_2(L)\cap\sigma_1(\pi_3)\cap\sigma_1(\pi_4)$ consists of the lines in $\mathbb{P}^3$ passing through the point pt and the lines ℓ_3 and ℓ_4; the number of such lines is 1 by Example 1.1. The set $\sigma_{1,1}(V)\cap\sigma_1(\pi_3)\cap\sigma_1(\pi_4)$, consists of the lines ℓ in $\mathbb{P}^3$ that lie in the plane $P\approx\mathbb{P}^2$ and pass through the lines ℓ_3 and ℓ_4, i.e. of the lines $\ell\subset P$ that pass through the points $P\cap\ell_3$ and $P\cap\ell_4$; the number of such lines is 1 by Example 0.1. Thus, the number of lines in $\mathbb{P}^3$ meeting 4 general lines is 2.

2 Grassmannians of two-planes

The set of two-dimensional linear subspaces of $\mathbb{C}^n$, which we will denote by $\mathbb{G}(2,n)$, admits a natural complex structure which can be constructed as follows. Denote by

$$\mathcal{B}(2,n)\subset(\mathbb{C}^n-0)\times(\mathbb{C}^n-0)$$

the open subspace consisting of pairs of linearly independent vectors in $\mathbb{C}^n$. Let $\mathcal{B}^*(2,n)\subset\mathcal{B}(2,n)$ be the subset of pairs (v_1,v_2) such that v_1 and v_2 are orthonormal (w.r.t. the standard hermitian inner-product on $\mathbb{C}^n$). Since the maps

$$\mathcal{B}(2,n),\mathcal{B}^*(2,n)\longrightarrow\mathbb{G}(2,n),\qquad(v_1,v_2)\longrightarrow\mathbb{C}v_1+\mathbb{C}v_2,$$

are surjective, the topologies of $\mathcal{B}(2,n)$ and $\mathcal{B}^*(2,n)$ induce quotient topologies on $\mathbb{G}(2,n)$; see [15, Section 22]. The maps

$$\mathrm{GL}_n(\mathbb{C}),U(n)\longrightarrow\mathbb{G}(2,n)$$

sending each matrix to the span of the first two columns are also surjective.

Exercise 2.1. Let $n\in\mathbb{Z}^+$ be such that $n\ge2$. Show that

(1) the four quotient topologies on $\mathbb{G}(2,n)$ are the quotient topologies

$$\mathcal{B}(2,n)/\mathrm{GL}_2(\mathbb{C}),\quad\mathcal{B}^*(2,n)/U(2),\quad\mathrm{GL}_n(\mathbb{C})/G_2,\quad U(n)/U(2)\times U(n-2)$$

for certain free group actions and a for a certain subgroup $G_2\subset\mathrm{GL}_n\mathbb{C}$;

(2) the four quotient topologies on $\mathbb{G}(2,n)$ are in fact the same.

Exercise 2.2. Let $n\in\mathbb{Z}^+$ be such that $n\ge2$. Show that $\mathbb{G}(2,n)$ is a compact complex manifold of dimension $2(n-2)$ and the projection maps

$$\mathcal{B}(2,n),\mathrm{GL}_n(\mathbb{C})\longrightarrow\mathbb{G}(2,n)$$

defined above are holomorphic submersions.

We will next describe a stratification of $\mathbb{G}(2,n)$ and its cohomology. A flag $\mathbf{V}$ in $\mathbb{C}^n$ is a strictly increasing sequence of $n+1$ linear subspaces of $\mathbb{C}^n$,

$$\{0\}=V_0\subsetneq V_1\subsetneq\ldots V_{n-1}\subsetneq V_n=\mathbb{C}^n.$$

Given such a flag on $\mathbb{C}^n$ and nonnegative integers a and b, we define

$$\sigma^0_{ab}(\mathbf{V}) = \big\{P\in\mathbb{G}(2,n)\colon\ \dim(P\cap V_{n-1-a})=1,\ \dim(P\cap V_{n-2-a})=0;\\ \dim(P\cap V_{n-b})=2,\ \dim(P\cap V_{n-1-b})=1\big\}.$$

Since P is a linear subspace of $\mathbb{C}^n$ of dimension 2,

$$\sigma^0_{ab}(\mathbf{V}) = \big\{P\in\mathbb{G}(2,n)\colon\ P\cap V_{n-1-a}\neq\{0\},\ P\cap V_{n-2-a}=\{0\};\\ P\subset V_{n-b},\ P\not\subset V_{n-1-b}\big\}.$$

Since a generic element of $\mathbb{G}(2,n)$ is not contained in V_{n-1} and intersects V_{n-2} trivially, the numbers a and b measure the extent of the deviation of the elements of σ^0_{ab} from a generic element of $\mathbb{G}(2,n)$. Note that $\sigma^0_{ab}(\mathbf{V})=\emptyset$ unless $n-2\geq a\geq b$. Furthermore,

$$\mathbb{G}(2,n) = \bigsqcup_{n-2\geq a\geq b\geq 0}\sigma^0_{ab}(\mathbf{V}). \tag{2.1}$$

The closure of $\sigma^0_{ab}(\mathbf{V})$ in $\mathbb{G}(2,n)$ is given by

$$\sigma_{ab}(\mathbf{V}) \equiv \bar{\sigma}^0_{ab}(\mathbf{V}) = \big\{P\in\mathbb{G}(2,n)\colon P\subset V_{n-b},\ P\cap V_{n-1-a}\neq\{0\}\big\}. \tag{2.2}$$

These subspaces of $\mathbb{G}(2,n)$ are called Schubert cells. We will write $\sigma_a(\mathbf{V})$ for $\sigma_{a0}(\mathbf{V})$. If

$$\mathbf{V}^{n-b} \equiv (V_0, V_1, \ldots, V_{n-b})$$

is a flag for $\mathbb{C}^{n-b}$, then

$$\sigma_{ab}(\mathbf{V}) = \sigma_{a-b}(\mathbf{V}^{n-b}) \subset \mathbb{G}(2,n-b). \tag{2.3}$$

Lemma 2.3. *The decomposition (2.1) is a stratification of* $\mathbb{G}(2,n)$ *with*

$$\dim_{\mathbb{C}}\sigma^0_{ab}(\mathbf{V}) = 2(n-2)-(a+b) \qquad \text{if}\quad n-2\geq a\geq b\geq 0. \tag{2.4}$$

Proof. It is sufficient to prove this statement for the standard flag $\mathbf{V}$ given by

$$V_k = \mathbb{C}^k\times\{0\}^{n-k}\subset\mathbb{C}^n.$$

If a,b are as in (2.4), let

$$\mathcal{B}_{ab}(2,n)\subset\mathcal{B}(2,n)$$

denote the subspace of pairs (v_1,v_2) such that $v_1\in\mathbb{C}^{n-1-a}$, $v_2\in\mathbb{C}^{n-b}$, the $n-1-a$ coordinate of v_1 and the $n-b$ coordinate of v_2 are both 1, and the $n-1-a$ coordinate of v_2 is 0. Thus,

$$\mathcal{B}_{ab}(2,n)\approx\mathbb{C}^{n-1-a-1}\times\mathbb{C}^{n-b-2}$$

and the quotient projection map

$$q_{ab}\colon\mathcal{B}_{ab}(2,n)\longrightarrow\sigma^0_{ab}(\mathbf{V})$$

is a bijection. Since q_{ab} is holomorphic (being a composition of holomorphic maps), q_{ab} is bi-holomorphic [5, p19]. This shows that $\sigma^0_{ab}(\mathbf{V})$ is bi-holomorphic to $\mathbb{C}^{2(n-2)-(a+b)}$. Since

$$\sigma_{ab}(\mathbf{V})-\sigma^0_{ab}(\mathbf{V})\subset\bigcup_{a'+b'>a+b}\sigma^0_{ab}(\mathbf{V})$$

by (2.2), it follows that (2.1) is indeed a stratification of $\mathbb{G}(2,n)$. □

Remark 2.4. The decomposition (2.1) in fact presents $\mathbb{G}(2,n)$ as a CW-complex; see Section 6 in [12].

Exercise 2.5. Suppose $\mathbf{V}$ is a flag in $\mathbb{C}^n$ and $n-2 \geq a \geq b$. Using (2.2) and (2.3), show that $\sigma_{ab}(\mathbf{V})$ is a complex variety, which is smooth if and only if $a=b$ or $a=n-2$. Give a geometric description of $\sigma_{ab}(\mathbf{V})$ in these two cases.

Since the space of flags in $\mathbb{C}^n$ is path-connected, the Schubert cycles $\sigma_{ab}(\mathbf{V})$ and $\sigma_{ab}(\mathbf{V}')$ corresponding to two different flags determine the same elements in the homology of $\mathbb{G}(2,n)$ and via the Poincare duality in the cohomology of $\mathbb{G}(2,n)$. Both of these elements will be denoted by σ_{ab}. By Lemma 2.3,

$$\sigma_{ab} \in H_{2(2(n-2)-(a+b))}\big(\mathbb{G}(2,n);\mathbb{Z}\big), H^{2(a+b)}\big(\mathbb{G}(2,n);\mathbb{Z}\big).$$

Furthermore, $H_*(\mathbb{G}(2,n);\mathbb{Z})$ and $H^*(\mathbb{G}(2,n);\mathbb{Z})$ are the free $\mathbb{Z}$-modules generated by σ_{ab} with $n-2 \geq a \geq b \geq 0$. The classes σ_{ab} and $\sigma_{a'b'}$ have complimentary dimensions if and only if

$$a + b + a' + b' = 2(n-2). \tag{2.5}$$

The next lemma describes the Poincare pairing on $\mathbb{G}(2,n)$.

Lemma 2.6. *Suppose n, a, b, a', b' are non-negative integers. If $n-2 \geq a \geq b \geq 0$, then*

$$\big\langle \sigma_{ab}\sigma_{a'b'}, \mathbb{G}(2,n) \big\rangle = \begin{cases} 1, & \textit{if } a'=n-2-b,\ b'=n-2-a; \\ 0, & \textit{otherwise.} \end{cases} \tag{2.6}$$

Proof 1. Let $\mathbf{V}$ and $\mathbf{V}'$ be two generic flags. By (2.2),

$$\sigma_{ab}(\mathbf{V}) \cap \sigma_{a'b'}(\mathbf{V}') = \big\{ P \in \mathbb{G}(2,n) \colon \ P \subset (V_{n-b} \cap V'_{n-b'}), \\ P \cap V_{n-1-a} \neq \{0\},\ P \cap V'_{n-1-a'} \neq \{0\} \big\}. \tag{2.7}$$

Thus, $\sigma_{ab}(\mathbf{V}) \cap \sigma_{a'b'}(\mathbf{V}')$ is empty unless

$$\dim\big(V_{n-b} \cap V'_{n-1-a'}\big) \geq 1, \quad \dim\big(V'_{n-b'} \cap V_{n-1-a}\big) \geq 1.$$

Since the flags $\mathbf{V}$ and $\mathbf{V}'$ are general, it follows that $\sigma_{ab}(\mathbf{V}) \cap \sigma_{a'b'}(\mathbf{V}')$ is empty unless

$$(n-b) + (n-1-a') - n \geq 1, \quad (n-b') + (n-1-a) - n \geq 1.$$

Since we can assume that (2.5) holds, it follows that

$$\sigma_{ab}(\mathbf{V}) \cap \sigma_{a'b'}(\mathbf{V}') \neq \emptyset \qquad \Longrightarrow \qquad a'=n-2-b,\ b'=n-2-a.$$

This implies the second case in (2.6). If $a'=n-2-b$ and $b'=n-2-a$, then

$$\sigma_{ab}(\mathbf{V}) \cap \sigma_{a'b'}(\mathbf{V}') = \sigma^0_{ab}(\mathbf{V}) \cap \sigma^0_{a'b'}(\mathbf{V}')$$

consists of the single element $P \in \mathbb{G}(2,n)$ which is the span of the disjoint one-dimensional linear subspaces $V_{n-b} \cap V'_{n-1-a'}$ and $V'_{n-b'} \cap V_{n-1-a}$ of $\mathbb{C}^n$. This is a transverse point of the intersection of complex sub-manifolds $\sigma^0_{ab}(\mathbf{V})$ and $\sigma^0_{a'b'}(\mathbf{V}')$ in $\mathbb{G}(2,n)$ and thus contributes 1 to the homology intersection of $\sigma_{ab}(\mathbf{V})$ and $\sigma_{a'b'}(\mathbf{V}')$. $\square$

Proof 2. By (2.2) and (2.3),

$$\begin{aligned}\big\langle \sigma_{ab}\sigma_{a'b'}, \mathbb{G}(2,n)\big\rangle &= \big\langle \sigma_{a-b}\sigma_{a'b'}, \mathbb{G}(2,n-b)\big\rangle \\ &= \big\langle \sigma_{a-b}\sigma_{a'-b'}, \mathbb{G}(2,n-b-b')\big\rangle.\end{aligned} \tag{2.8}$$

The last number above is zero unless

$$n-b-b'-2 \ge a-b, a'-b' \qquad \Longleftrightarrow \qquad a+b' \le n-2, \quad a'+b \le n-2.$$

In light of (2.5), the last condition is equivalent to

$$a+b' = n-2, \quad a'+b = n-2.$$

If these equalities hold, by (2.8)

$$\big\langle \sigma_{ab}\sigma_{a'b'}, \mathbb{G}(2,n)\big\rangle = \big\langle \sigma_{a-b}\sigma_{a-b}, \mathbb{G}(2,a-b+2)\big\rangle = 1,$$

since this is the number of two-dimensional linear subspaces of $\mathbb{C}^{a-b+2}$ containing two fixed distinct one-dimensional linear subspaces. □

Exercise 2.7. (a) If $n, a_1, \ldots, a_k, b_1, \ldots, b_k$ are nonnegative integers, show that

$$\big\langle \sigma_{a_1b_1} \cdot \ldots \cdot \sigma_{a_kb_k}, \mathbb{G}(2,n)\big\rangle = \big\langle \sigma_{a_1-b_1} \cdot \ldots \cdot \sigma_{a_k-b_k}, \mathbb{G}(2, n-b_1-\ldots-b_k)\big\rangle. \tag{2.9}$$

(b) Verify the transversality statement at the end of the first proof of Lemma 2.6.

Exercise 2.8. Prove the following identities for Schubert cycles:
(a) if $n, a_1, a_2, a_3 \in \mathbb{Z}^+$ are such that $n-2 \ge a_1, a_2, a_3 \ge 0$, then

$$\big\langle \sigma_{a_1}\sigma_{a_2}\sigma_{a_3}, \mathbb{G}(2,n)\big\rangle = \begin{cases} 1, & \text{if } a_1+a_2+a_3 = 2n-4; \\ 0, & \text{otherwise;} \end{cases} \tag{2.10}$$

(b) if $a_1, a_2 \ge 0$,

$$\sigma_{a_1} \cdot \sigma_{a_2} = \sum_{c \ge a_1, a_2} \sigma_{c, a_1+a_2-c}. \tag{2.11}$$

Hint: Use (a) along with (2.6) and (2.9).

The identity (2.11) is a special case of Pieri's formula for Schubert cycles. Along with (2.6) and (2.9), it suffices to compute the intersection of any collection of Schubert classes on $\mathbb{G}(2,n)$.

Exercise 2.9. (a) The inclusion $\mathbb{C}^n \longrightarrow \mathbb{C}^{n+1}$ induces an embedding

$$\iota\colon \mathbb{G}(2,n) \longrightarrow \mathbb{G}(2,n+1).$$

Show that the cohomology homomorphism induced by the latter is given by

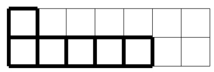
$$\iota^*\colon H^*\big(\mathbb{G}(2,n+1);\mathbb{Z}\big) \longrightarrow H^*\big(\mathbb{G}(2,n);\mathbb{Z}\big), \qquad \sigma_{ab} \longrightarrow \sigma_{ab} \quad \forall\, a,b.$$

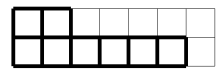
(b) Let $\iota\colon \mathbb{P}^{n-1} \longrightarrow \mathbb{G}(2,n+1)$ be the embedding defined by $L \longrightarrow L \oplus (0^n \times \mathbb{C})$. Show that the induced cohomology homomorphism is given by

$$\iota^*\colon H^*\big(\mathbb{G}(2,n+1);\mathbb{Z}\big) \longrightarrow H^*\big(\mathbb{P}^{n-1};\mathbb{Z}\big), \qquad \sigma_{ab} \longrightarrow \sigma_{ab} \quad \forall\, a,b,$$

with $\sigma_{ab} \in H^*(\mathbb{P}^{n-1};\mathbb{Z})$ defined to be 0 if $b \ne 0$.

It is often convenient to represent Schubert cycles by Young diagrams. The Schubert cycle σ_{ab} then corresponds to the tableaux μ with the first (bottom) row consisting of a boxes and the second of b boxes. In order to indicate that this cycle lies in $\mathbb{G}(2,n)$, we draw this tableaux at the bottom left corner of a $2\times(n-2)$ grid and thus indicate all possibilities for μ. By Lemma 2.6, the tableaux μ^c describing the Schubert cycle σ_{μ^c} dual to σ_μ is the complement of μ in the grid:


$\mu=(5,1)\subset\mathbb{G}(2,9)$


$\mu^c=(6,2)\subset\mathbb{G}(2,9)$

By Exercise 2.5, $\sigma_\mu(\mathbf{V})$ is smooth if and only μ^c is a rectangle.

The projectivization of a 2-dimensional linear subspace of $\mathbb{C}^{n+1}$ is a projective line in $\mathbb{P}^n$ and vice versa. Thus, the set of lines in $\mathbb{P}^n$, which we denote by $\mathrm{Ln}(\mathbb{P}^n)$, can be identified with $\mathbb{G}(2,n{+}1)$. If $\mathbf{V}$ is a flag on $\mathbb{C}^{n+1}$ and a and b are nonnegative integers, under this identification the Schubert cycle $\sigma_{ab}(\mathbf{V})$ is given by

$$\sigma_{ab}(\mathbf{V})=\big\{\ell\in\mathrm{Ln}(\mathbb{P}^n):\,\ell\subset\mathbb{P}V_{n+1-b},\ \ell\cap\mathbb{P}V_{n-a}\neq\emptyset\big\}. \tag{2.12}$$

Thus, σ_{ab} is the space of lines in $\mathbb{P}^n$ that are contained in a linearly embedded projective subspace $\mathbb{P}^{n-b}$ and meet a linearly embedded projective subspace $\mathbb{P}^{n-1-a}$ of $\mathbb{P}^{n-b}$. The identification (2.12) can be used along with (2.6), (2.9), and (2.11) to determine the number of lines in $\mathbb{C}^n$ (or $\mathbb{P}^n$) meeting a specified collection of affine (or linear) subspaces.

Example 2.10. The number of lines through 2 distinct points, in $\mathbb{C}^n$ or $\mathbb{P}^n$, is of course 1. This corresponds to the statement

$$\big\langle\sigma_{n-1}\sigma_{n-1},\mathbb{G}(2,n{+}1)\big\rangle=1,$$

which is a special case of (2.6).

Example 2.11. Let a, b, and c be non-negative integers such that $a{+}b{+}c{=}n{-}1$. By Example 1.1, the number of lines in $\mathbb{C}^n$ or $\mathbb{P}^n$ meeting general subspaces of dimensions a, b, and c is 1. This corresponds to the statement

$$\big\langle\sigma_{n-1-a}\sigma_{n-1-b}\sigma_{n-1-c},\mathbb{G}(2,n{+}1)\big\rangle=1,$$

which is a special case of (2.10).

Example 2.12. By the last part of Section 1, the number of lines meeting 4 general lines in $\mathbb{C}^3$ or $\mathbb{P}^3$ is 2. This corresponds to the statement that

$$\big\langle\sigma_1^4,\mathbb{G}(2,4)\big\rangle=2,$$

which can be deduced from (2.11) and (2.9).

The intersection arguments in Examples 2.10-2.12 rely on the assumption that general representatives for Schubert cycles intersect transversally in $\mathbb{G}(2,n{+}1)$. This follows from Exercise 2.13 below, which also implies that the numbers obtained by intersecting appropriate Schubert cycles on $\mathbb{G}(2,n{+}1)$ are indeed actual counts of lines passing through specified linear constraints.

n	...	$\langle \ldots, \mathbb{G}(2,n)\rangle$	n	...	$\langle \ldots, \mathbb{G}(2,n)\rangle$

Table 2: Top intersections of Schubert classes on $\mathbb{G}(n,5)$. The numbers directly obtainable from (2.6) and (2.11) are not shown.

Exercise 2.13. If $\mathbf{V}^{(1)},\ldots,\mathbf{V}^{(k)}$ are general flags on $\mathbb{C}^{n+1}$, show that for every $l=2,\ldots,k$, then

(1) $\sigma^0_{a_1b_1}(\mathbf{V}^{(1)})\cap\ldots\cap\sigma^0_{a_{l-1}b_{l-1}}(\mathbf{V}^{(l-1)})$ is a smooth complex submanifold of $\mathbb{G}(2,n{+}1)$;

(2) $\sigma^0_{a_1b_1}(\mathbf{V}^{(1)})\cap\ldots\cap\sigma^0_{a_{l-1}b_{l-1}}(\mathbf{V}^{(l-1)})$ and $\sigma_{a_lb_l}(\mathbf{V}^{(l)})$ intersect transversally in $\mathbb{G}(2,n{+}1)$.

Exercise 2.14. Verify the numbers in Table 3.

Analogously to (B.3.1), let

$$\pi\colon \gamma_2=\big\{(P,v)\in\mathbb{G}(2,n)\times\mathbb{C}^n\colon v\in P\subset\mathbb{C}^n\big\}\longrightarrow\mathbb{G}(2,n). \tag{2.13}$$

Exercise 2.15. Show that γ_2 is a complex submanifold of $\mathbb{G}(2,n)\times\mathbb{C}^n$ and (2.13) is a holomorphic vector bundle of rank 2.

The vector bundle $\pi:\gamma_2\longrightarrow\mathbb{G}(2,n)$ is called the tautological two-plane bundle. It turns out to be useful in particular for counting lines on projective hypersurfaces and more generally on projective complete intersections.

Lemma 2.16. *The total chern class of the vector bundle $\gamma_2^*\longrightarrow\mathbb{G}(2,n)$ is given by*

$$c(\gamma_2^*)=1+\sigma_1+\sigma_{11}\in H^*\big(\mathbb{G}(2,n);\mathbb{Z}\big).$$

Proof. Since the Schubert cells provide a CW-decomposition of $\mathbb{G}(2,n)$,

$$c_1(\gamma_2^*)=a\sigma_1,\qquad c_2(\gamma_2^*)=b\sigma_{11}+c\sigma_2,$$

for some $a,b,c\in\mathbb{Z}$. Since the tautological bundle $\gamma_2\longrightarrow\mathbb{G}(2,n+1)$ restricts to the tautological bundle over $\mathbb{G}(2,n)$ under the embedding $\mathbb{G}(2,n)\longrightarrow\mathbb{G}(2,n+1)$ induced by the inclusion $\mathbb{C}^n\longrightarrow\mathbb{C}^{n+1}$, by the naturality of chern classes and Exercise 2.9(a) the numbers a, b, and c are independent of n. Since the pull-back of $\gamma_2\longrightarrow\mathbb{G}(2,n{+}1)$ by the embedding $\iota\colon\mathbb{P}^{n-1}\longrightarrow\mathbb{G}(2,n{+}1)$ of Exercise 2.9(b) is $\gamma_1\oplus\tau_1$, $a=1$ and $c=0$ by the naturality of chern classes and Exercise 2.9(b). Finally, the pull-back of $\gamma_2\longrightarrow\mathbb{G}(2,4)$ by the embedding

$$h\colon\mathbb{P}^1\times\mathbb{P}^1\longrightarrow\mathbb{G}(2,4),\qquad(\ell_1,\ell_2)\longrightarrow\ell_1\times\ell_2\subset\mathbb{C}^2\times\mathbb{C}^2$$

is the bundle $\gamma_1\times\gamma_1\longrightarrow\mathbb{P}^1\times\mathbb{P}^1$. Thus, $b=+1$ or $b=-1$ depending on whether

$$\big\langle h^*\sigma_{11},\mathbb{P}^1\times\mathbb{P}^1\big\rangle=+1\quad\text{or}\quad\big\langle h^*\sigma_{11},\mathbb{P}^1\times\mathbb{P}^1\big\rangle=-1;$$

since h is a holomorphic map, the latter is impossible. □

n	λ	$N(\lambda)$	n	λ	$N(\lambda)$	n	λ	$N(\lambda)$
4	(1,0,3)	1	6	(0,0,1,3,1)	5	7	(0,0,2,0,0,4)	6
4	(0,2,2)	2	6	(0,0,1,2,3)	7	7	(0,0,1,2,1,0)	3
4	(0,1,4)	3	6	(0,0,1,1,5)	10	7	(0,0,1,2,0,2)	4
4	(0,0,6)	5	6	(0,0,1,0,7)	14	7	(0,0,1,1,2,1)	5
5	(1,0,1,2)	1	6	(0,0,0,5,0)	6	7	(0,0,1,1,1,3)	7
5	(1,0,0,4)	1	6	(0,0,0,4,2)	9	7	(0,0,1,1,0,5)	10
5	(0,2,0,2)	2	6	(0,0,0,3,4)	13	7	(0,0,1,0,4,0)	6
5	(0,1,2,1)	2	6	(0,0,0,2,6)	19	7	(0,0,1,0,3,2)	8
5	(0,1,1,3)	3	6	(0,0,0,1,8)	28	7	(0,0,1,0,2,4)	11
5	(0,1,0,5)	4	6	(0,0,0,0,10)	42	7	(0,0,1,0,1,6)	15
5	(0,0,4,0)	3	7	(1,0,0,1,1,1)	1	7	(0,0,1,0,0,8)	20
5	(0,0,3,2)	4	7	(1,0,0,1,0,3)	1	7	(0,0,0,4,0,0)	4
5	(0,0,2,4)	6	7	(1,0,0,0,3,0)	1	7	(0,0,0,3,1,1)	6
5	(0,0,1,6)	9	7	(1,0,0,0,2,2)	1	7	(0,0,0,3,0,3)	8
5	(0,0,0,8)	14	7	(1,0,0,0,1,4)	1	7	(0,0,0,2,3,0)	7
6	(1,0,1,0,2)	1	7	(1,0,0,0,0,6)	1	7	(0,0,0,2,2,2)	10
6	(1,0,0,2,1)	1	7	(0,2,0,0,0,2)	2	7	(0,0,0,2,1,4)	14
6	(1,0,0,1,3)	1	7	(0,1,1,0,1,1)	2	7	(0,0,0,2,0,6)	20
6	(1,0,0,0,5)	1	7	(0,1,1,0,0,3)	3	7	(0,0,0,1,4,1)	12
6	(0,2,0,0,2)	2	7	(0,1,0,2,0,1)	2	7	(0,0,0,1,3,3)	17
6	(0,1,1,1,1)	2	7	(0,1,0,1,2,0)	2	7	(0,0,0,1,2,5)	24
6	(0,1,1,0,3)	3	7	(0,1,0,1,1,2)	3	7	(0,0,0,1,1,7)	34
6	(0,1,0,3,0)	2	7	(0,1,0,1,0,4)	4	7	(0,0,0,1,0,9)	48
6	(0,1,0,2,2)	3	7	(0,1,0,0,3,1)	3	7	(0,0,0,0,6,0)	15
6	(0,1,0,1,4)	4	7	(0,1,0,0,2,3)	4	7	(0,0,0,0,5,2)	21
6	(0,1,0,0,6)	5	7	(0,1,0,0,1,5)	5	7	(0,0,0,0,4,4)	30
6	(0,0,3,0,1)	2	7	(0,1,0,0,0,7)	6	7	(0,0,0,0,3,6)	43
6	(0,0,2,2,0)	3	7	(0,0,2,1,0,1)	2	7	(0,0,0,0,2,8)	62
6	(0,0,2,1,2)	4	7	(0,0,2,0,2,0)	3	7	(0,0,0,0,1,10)	90
6	(0,0,2,0,4)	6	7	(0,0,2,0,1,2)	4	7	(0,0,0,0,0,12)	132

Table 3: The number of lines $N(\lambda)$ in $\mathbb{C}^n$ or $\mathbb{P}^n$ that meet λ_k general affine or linear subspaces of dimension k, with $k=0,2,\ldots,n-2$. The numbers provided by Examples 2.10-2.12 are omitted.

Here is another way to see that $e(\gamma_2^*)=\sigma_{11}$. The map

$$\gamma_2 \longrightarrow \mathbb{C}^n, \qquad \big(P,(c_1,\ldots,c_n)\big) \longrightarrow c_n,$$

induces a holomorphic section $\tilde{s}_{Z_n}$ of $\gamma_2^*\longrightarrow\mathbb{G}(2,n)$. Since

$$\begin{aligned}\tilde{s}_{Z_n}^{-1}(0) &= \big\{\big(P,(c_1,\ldots,c_n)\big)\in\gamma_2:\ c_n=0\ \forall\,(c_1,\ldots,c_n)\in P\big\}\\ &= \big\{P\in\mathbb{G}(2,n):\ P\subset\mathbb{C}^{n-1}\big\} = \sigma_{11}\end{aligned}$$

and this section is transverse to the zero set,

$$e(\gamma_2^*) = \mathrm{PD}_{\mathbb{G}(2,n)}\big([\tilde{s}_Q^{-1}(0)]_{\mathbb{G}(2,n)}\big) = \sigma_{11}\,;$$

the first equality follows from Theorem 0.4.

By Lemma B.3.3, a nonzero degree a homogeneous polynomial Q in $(n+1)$ variables determines a holomorphic section s_Q of the line bundle $\gamma^{*\otimes a}\longrightarrow\mathbb{P}^n$. The zero set of this section,

$$X_{n;a} \equiv X_Q \equiv s_Q^{-1}(0) \equiv \big\{[Z_0,\ldots,Z_n]\in\mathbb{P}^n\colon Q(X_0,\ldots,X_n)=0\big\}.$$

is a degree a hypersurface. By the Lefschetz Hyperplane Theorem [5, p156, p158], the homomorphisms on the fundamental groups

$$\pi_i(X_{n;a}) \longrightarrow \pi_i(\mathbb{P}^n)$$

are isomorphisms for $i<n-1$. Thus, the restriction isomorphisms

$$H^i(\mathbb{P}^n;\mathbb{Z}) \longrightarrow H^i(X_{n;a};\mathbb{Z})$$

are isomorphisms for $i<n-1$ as well.

Exercise 2.17. Let $X_{n;a}\subset\mathbb{P}^n$ be a smooth degree a hypersurface.

(1) Show that $c_1(X_{n;\mathbf{a}})=(n+1-a)c_1(\gamma^*)|_{X_{n;a}}$.

(2) Show that the euler characteristic of $X_{n;a}$ is given by

$$\chi(X_{n;a}) = \frac{(1-a)^{n+1}+(n+1)a-1}{a}\,.$$

(3) If $n=2$, show that the genus of the curve $X_{n;a}$ is $\binom{a-1}{2}$.

(4) Determine the betti numbers of $X_{n;a}$ (the dimensions of $H^i(X_{n;a})$).

(5) For $n\le 4$, determine the Hodge diamond of $X_{n;a}$.

Exercise 2.18. Show that

(1) a degree a homogeneous polynomial Q in $n+1$ variables induces a holomorphic section $\tilde{s}_Q$ of the vector bundle $\mathrm{Sym}^a\gamma^*\longrightarrow\mathbb{G}(2,n+1)$;

(2) a line $\ell=\mathbb{P}^1P\subset\mathbb{P}^n$, with $P\in\mathbb{G}(2,n+1)$, is contained in the hypersurface X_Q if and only if $\tilde{s}_Q(P)=0$;

(3) the bundle section $\tilde{s}_Q$ is transverse to the zero set for a generic choice of Q .

If s_Q ($\tilde{s}_Q$) is transverse to the zero set, does it follow that so is $\tilde{s}_Q$ (s_Q)?

By Exercise 2.18, the dimension of the space of lines lying on a generic degree a hypersurface $X_{n;a}$ in $\mathbb{P}^n$ is given by

$$\begin{aligned}\dim \overline{\mathfrak{M}}_0\big(X_{n;a},\ell\big) &= \dim \mathbb{G}(2,n{+}1) - \operatorname{rk}\operatorname{Sym}^a\gamma_2^* = 2(n{-}1) - (a{+}1)\\ &= 2n-a-3.\end{aligned} \tag{2.14}$$

In particular, there are no lines on $X_{n;a}$ if $a>2n-3$. Using Exercise 2.18, we can determine the number of lines that lie on $X_{n;a}$ and meet a generic collection of linear subspaces of $\mathbb{P}^n$. By (2.14), the total codimension of the subspaces minus the number of subspaces should be $2n-a-3$.

Example 2.19. The number of lines on a generic conic (degree 2) surface in $\mathbb{P}^3$ that meet a general line in $\mathbb{P}^3$ is given by

$$\begin{aligned}\big\langle \sigma_1\cdot e(\operatorname{Sym}^2\gamma_2^*),\mathbb{G}(2,4)\big\rangle &= \big\langle \sigma_1\cdot 4c_1(\gamma_2^*)c_2(\gamma_2^*),\mathbb{G}(2,4)\big\rangle\\ &= 4\big\langle \sigma_1^2\sigma_{11},\mathbb{G}(2,4)\big\rangle = 4;\end{aligned}$$

the last identity follows from (1.1) and (2.10). Since a generic line in $\mathbb{P}^3$ meets $X_{3;2}$ at two points, it follows that the number of lines on $X_{3;2}$ through a fixed point is 2. By [5, p478], every smooth conic surface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$; the two lines through any point p are the horizontal and vertical slices of this cartesian product through p.

Example 2.20. The number of lines on a generic cubic (degree 3) surface in $\mathbb{P}^3$ is given by

$$\begin{aligned}\big\langle e(\operatorname{Sym}^3\gamma_2^*),\mathbb{G}(2,4)\big\rangle &= \big\langle 9c_2(\gamma^*)(2c_1(\gamma_2^*)^2+c_2(\gamma_2^*)),\mathbb{G}(2,4)\big\rangle\\ &= 9\,\big\langle 2\sigma_1^2\sigma_{11}+\sigma_{11}^2,\mathbb{G}(2,4)\big\rangle = 27;\end{aligned}$$

the first equality above follows from Exercise A.3.4 and Lemma 2.16.

Exercise 2.21. Describe the 27 lines on the cubic surface in $\mathbb{P}^3$ given by $X_0^3{+}X_1^2{+}X_2^3{+}X_3^3=0$ explicitly.

If $\mathbf{a}=(a_1,\ldots,a_c)$ is a tuple positive integers, a complete intersection of multi-degree $\mathbf{a}$ in $\mathbb{P}^n$ is the intersection of hypersurfaces of degrees $a_1,\ldots,a_c$ in $\mathbb{P}^n$. A generic complete intersection of multi-degree $\mathbf{a}$ is the intersection of generic hypersurfaces of degrees $a_1,\ldots,a_c$:

$$X_{n;\mathbf{a}} = \bigcap_{i=1}^{i=c} X_{n;a_i}.$$

The above method for counting lines on a generic hypersurface extends to counting lines on $X_{n;\mathbf{a}}$: simply replace the euler class of $\operatorname{Sym}^a\gamma^*$ by the product of the euler classes of $\operatorname{Sym}^{a_i}\gamma_2^*$ with $i=1,\ldots,c$.

Exercise 2.22. Verify the numbers in Table 4.

n	a	λ	$N_a(\lambda)$	n	a	λ	$N_a(\lambda)$
4	2	(1,1)	4	6	2	(0,1,2,0)	8
4	2	(0,3)	8	6	2	(0,1,1,2)	12
4	3	(1,0)	18	6	2	(0,1,0,4)	16
4	3	(0,2)	45	6	2	(0,0,3,1)	16
4	4	(0,1)	320	6	2	(0,0,2,3)	24
4	5	(0,0)	2875	6	2	(0,0,1,5)	36
4	(2,2)	0	16	6	2	(0,0,0,7)	56
5	2	(1,1,0)	4	6	3	(1,0,1,0)	18
5	2	(1,0,2)	4	6	3	(1,0,0,2)	18
5	2	(0,2,1)	8	6	3	(0,2,0,0)	45
5	2	(0,1,3)	12	6	3	(0,1,1,1)	63
5	2	(0,0,5)	20	6	3	(0,0,3,0)	81
5	3	(1,0,1)	18	6	3	(0,0,2,2)	126
5	3	(0,2,0)	45	6	3	(0,0,1,4)	189
5	3	(0,1,2)	63	6	3	(0,0,0,6)	297
5	3	(0,0,4)	108	6	4	(1,0,0,1)	96
5	4	(1,0,0)	96	6	4	(0,1,1,0)	416
5	4	(0,1,1)	416	6	4	(0,1,0,2)	512
5	4	(0,0,3)	736	6	4	(0,0,2,1)	832
5	5	(0,1,0)	3250	6	4	(0,0,1,3)	1248
5	5	(0,0,2)	6125	6	4	(0,0,0,5)	1984
5	6	(0,0,1)	60480	6	5	(1,0,0,0)	600
5	7	(0,0,0)	698005	6	5	(0,1,0,1)	3850
5	(2,2)	(1,0)	16	6	5	(0,0,2,0)	6725
5	(2,2)	(0,2)	32	6	5	(0,0,1,2)	9975
5	(2,3)	(0,1)	180	6	5	(0,0,0,4)	16100
5	(2,4)	(0,0)	1280	6	6	(0,1,0,0)	33264
5	(3,3)	(0,0)	1053	6	6	(0,0,1,1)	93744
6	2	(1,1,0,0)	4	6	6	(0,0,0,3)	154224
6	2	(1,0,1,1)	4	6	7	(0,0,1,0)	1009792
6	2	(1,0,0,3)	4	6	7	(0,0,0,2)	1707797
6	2	(0,2,0,1)	8	6	8	(0,0,0,1)	21518336

Table 4: The number of lines $N_a(\lambda)$ that lie on a generic complete intersection of multi-degree $(a_1, \ldots, a_c)$ in $\mathbb{P}^n$ that meet λ_k general linear subspaces of $\mathbb{P}^n$ of dimension $k = c, c+1, \ldots, n-2$. The numbers provided by Examples 2.19 and 2.20 and those with $a_l = 1$ for some l are omitted.

3 Conic curves in affine/projective spaces

A conic (degree 2) curve $C\subset\mathbb{P}^n$ is a complex curve, possibly with some singularities, such that $[C]=2[\mathbb{P}^1]$, for any linear subspace $\mathbb{P}^1\subset\mathbb{P}^n$. Every conic lies in a $\mathbb{P}^2\subset\mathbb{P}^n$ and every conic, other than a double line, is contained in a unique $\mathbb{P}^2$. Thus, counting conics in $\mathbb{P}^n$ is usually equivalent to counting pairs (π,C), where $\pi\approx\mathbb{P}^2$ is a linear subspace of $\mathbb{P}^n$ and $C\subset\pi$ is a conic lying in π. Since each π equals $\mathbb{P}V$ for a unique 3-dimensional linear subspace V of $\mathbb{C}^{n+1}$, the space of two-dimensional linear subspaces of $\mathbb{P}^n$ is $\mathbb{G}(3,n+1)$, the Grassmannian of 3-dimensional linear subspaces of $\mathbb{C}^{n+1}$. Let

$$\pi\colon\gamma_3=\big\{(V,v)\in\mathbb{G}(3,n+1)\times\mathbb{C}^{n+1}\colon v\in V\subset\mathbb{C}^{n+1}\big\}\longrightarrow\mathbb{G}(3,n) \tag{3.1}$$

be the tautological rank 3 vector bundle. A conic C lying in V corresponds to an element of $\mathbb{P}(\mathrm{Sym}^2V^*)$, i.e. a nonzero degree 2 polynomial on V determined up to multiplication by $\mathbb{C}^*$. Thus, the space of conics in $\mathbb{P}^n$ is essentially $\mathbb{P}(\mathrm{Sym}^2\gamma_3^*)$; see Example A.2.4.

The Grassmannian $\mathbb{G}(3,4)$ parametrizes two-dimensional linear subspaces of $\mathbb{P}^3$ or equivalently three-dimensional linear subspaces of $\mathbb{C}^n$. Every such subspace V can be identified with its annihilator,

$$\mathrm{Ann}(V)\equiv\{\eta\in(\mathbb{C}^4)^*\colon\ \eta(v)=0\ \forall\,v\in V\big\}\subset(\mathbb{C}^4)^*\equiv\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^4,\mathbb{C}),$$

which is a one-dimensional linear subspace of $(\mathbb{C}^4)^*\approx\mathbb{C}^4$. Thus, the map

$$\mathbb{G}(3,4)\longrightarrow\hat{\mathbb{P}}^3\equiv\mathbb{P}^1(\mathbb{C}^4)^*\approx\mathbb{P}^3\,,\qquad V\longrightarrow\mathrm{Ann}(V), \tag{3.2}$$

is a bijection and thus can be used to topologize $\mathbb{G}(3,4)$ and give it a complex structure. With this complex structure,

$$\dim_{\mathbb{C}}\mathbb{G}(3,4)=\dim_{\mathbb{C}}\mathbb{P}^3=3,\qquad \dim_{\mathbb{C}}\mathbb{P}(\mathrm{Sym}^2\gamma_3^*)=\dim_{\mathbb{C}}\mathbb{G}(3,4)+\dim_{\mathbb{C}}\mathrm{Sym}^2\mathbb{C}^3-1=8,$$

where $\gamma_3\longrightarrow\mathbb{G}(3,4)$ is the tautological three-plane bundle.

Every conic curve in $\mathbb{P}^3$ meets a generic hyperplane, i.e. a two-dimensional linear subspace of $\mathbb{P}^3$, in 2 points. Thus, passing through a hyperplane does not impose a condition on the conics. Passing through a line ℓ imposes a one-dimensional condition on the space of conics in $\mathbb{P}^3$, since ℓ is of codimension two in $\mathbb{P}^3$, while a conic is of dimension one. Similarly, passing through a point in $\mathbb{P}^3$ imposes a two-dimensional condition on the space of conics in $\mathbb{P}^3$. By the last two paragraphs, the complex dimension of the space of conics in $\mathbb{P}^3$ is 8. Thus, we expect that there are finitely many conics passing through a generic points and b generic lines in $\mathbb{P}^3$ if $2a+b=8$.

Example 3.1. Four generic points in $\mathbb{P}^3$ do not lie in any two-dimensional linear subspace $\mathbb{P}^2\subset\mathbb{P}^3$. Since every conic lies in such a subspace, the number of conics passing through 4 general points in $\mathbb{P}^3$ is 0.

Example 3.2. Three general points in $\mathbb{P}^3$ determine a $\mathbb{P}^2\subset\mathbb{P}^3$. Every $\mathbb{P}^2$ meets a generic line ℓ in $\mathbb{P}^3$ in a single point. Since every conic lies in such a $\mathbb{P}^2\subset\mathbb{P}^3$, the number of conics passing through 3 general points and 2 general lines in $\mathbb{P}^3$ is the same as the number of conics in the $\mathbb{P}^2$ determined by the 3 points that pass through 5 points: the 3 original points and the 2 points of the intersection of $\mathbb{P}^2$ with the 2 original lines. By Example 0.2, this number is 1.

Exercise 3.3. Let $\hat{\gamma}\longrightarrow\hat{\mathbb{P}}^3$ denote the tautological line bundle and

$$\hat{a}=c_1(\hat{\gamma}^*)\in H^2\big(\hat{\mathbb{P}}^3;\mathbb{Z}\big)=H^2\big(\mathbb{G}(3,4);\mathbb{Z}\big).$$

Show that

(1) $\gamma_3\oplus\hat{\gamma}^*\approx\mathbb{G}(3,4)\times\mathbb{C}^4$ as complex vector bundles;

(2) $c(\gamma_3^*)=1+\hat{a}+\hat{a}^2+\hat{a}^3$;

(3) $c(\mathrm{Sym}^2\gamma_3^*)=1+4\hat{a}+10\hat{a}^2+20\hat{a}^3$.

We identify the space of conics in $\mathbb{P}^3$ with the elements $([\eta],[s])$ of $\mathbb{P}(\mathrm{Sym}^2\gamma_3^*)$, consisting of a one-dimensional linear subspace $\mathbb{C}\eta\subset(\mathbb{C}^4)^*$ and a one-dimensional linear subspace $[s]$ in the space of degree 2 homogeneous polynomials on $V=\eta^{-1}(0)\subset\mathbb{C}^4$ (or equivalently of a two-dimensional linear subspace $\mathbb{P}V\subset\mathbb{P}^3$ and a conic $s^{-1}(0)\subset\mathbb{P}V$). Let

$$\tilde{\gamma}\longrightarrow\mathbb{P}(\mathrm{Sym}^2\gamma_3^*)\qquad\text{and}\qquad\tilde{\lambda}=c_1(\tilde{\gamma}^*)\in H^2\big(\mathrm{Sym}^2\gamma_3^*;\mathbb{Z}\big)$$

denote the tautological line bundle as in Example A.2.4 and the chern class of its dual, respectively. A conic $([\eta],[s])$ passes through a point $[p]\in\mathbb{P}^3$ if and only if $p\in\eta^{-1}(0)$ and $p\in s^{-1}(0)$. The first requirement imposes a linear condition on η, defining a section $\hat{\varphi}_p$ of $\hat{\gamma}^*\longrightarrow\hat{\mathbb{P}}^3$ dependent on the choice of the representative p for $[p]$; the zero set of this transverse section is a two-dimensional linear subspace $\hat{\mathbb{P}}^2\subset\hat{\mathbb{P}}^3$. The second requirement similarly imposes a linear condition on s and defines a section $\tilde{\varphi}_p$ of $\tilde{\gamma}^*\longrightarrow\mathbb{P}(\mathrm{Sym}^2\gamma_3^*)$, dependent on p:

$$\big\{\tilde{\varphi}_p([s])\big\}(s)=s(p)\qquad\forall\ s\in\tilde{\gamma}\big|_{\mathbb{P}(\mathrm{Sym}^2\gamma_3^*)|_{\hat{\mathbb{P}}^2}}\,.$$

Thus, the subspace $M_{[p]}\subset\mathbb{P}(\mathrm{Sym}^2\gamma_3^*)$ of conics passing through a point $[p]$ in $\mathbb{P}^3$ represents the Poincare dual of $\tilde{\lambda}$ in $\mathbb{P}(\mathrm{Sym}^2\gamma_3^*)|_{\hat{\mathbb{P}}^2}$.

Exercise 3.4. Let V be a vector space over $\mathbb{C}$ of dimension 3, $\alpha,\beta\in V^*$, and $s\colon V\longrightarrow\mathbb{C}$ be a homogeneous function of order 2. Show that the linear map

$$\alpha^2\wedge\beta^2\wedge s\colon(\Lambda^3_{\mathbb{C}}V)^{\otimes2}\longrightarrow\mathbb{C},$$
$$(u\wedge v\wedge w)^{\otimes2}\longrightarrow\big(\alpha(u)\beta(v)-\alpha(v)\beta(u)\big)^2s(w)\quad\forall\ u,v\in V,\ w\in\ker\alpha\cap\ker\beta,$$

is well-defined and is identically zero if and only if s vanishes somewhere on $\ker\alpha\cap\ker\beta-0$.

We next describe the space of conics in $\mathbb{P}^3$ meeting a line $\ell=\mathbb{P}\pi$ in $\mathbb{P}^3$, where $\pi\subset\mathbb{C}^4$ is a two-dimensional linear subspace of $\mathbb{C}^4$. In particular,

$$\pi=\ker\alpha\cap\ker\beta$$

for some $\alpha,\beta\in(\mathbb{C}^4)^*$. By Exercise 3.4, the subspace $M_\ell\subset\mathbb{P}(\mathrm{Sym}^2\gamma_3^*)$ of conics passing through the line ℓ in $\mathbb{P}^3$ is described by

$$M_\ell=\big\{([\eta],[s])\in\mathbb{P}^1(\mathrm{Sym}^2\gamma_3^*)\colon\ \alpha^2\wedge\beta^2\wedge s=0\in(\Lambda_{\mathbb{C}}\eta^{-1}(0))^{*\otimes2}\big\}.$$

Thus, M_ℓ is the zero set of the section $\varphi_{\alpha,\beta}$ of the line bundle

$$\tilde{\gamma}^* \otimes \pi^*(\Lambda_{\mathbb{C}}\gamma_3)^{*\otimes 2} \longrightarrow \mathbb{P}(\mathrm{Sym}^2\gamma_3^*),$$

where $\pi\colon \mathbb{P}(\mathrm{Sym}^2\gamma_3^*) \longrightarrow \mathbb{G}(3,4)$ is the projection map, given by

$$\big\{\varphi_{\alpha,\beta}([\eta],[s])\big\}(s) \longrightarrow \alpha|^2_{\eta^{-1}(0)} \wedge \beta|^2_{\eta^{-1}(0)} \wedge s$$

and so represents the Poincare dual of

$$e\big(\tilde{\gamma}^* \otimes \pi^*(\Lambda_{\mathbb{C}}\gamma_3)^{*\otimes 2}\big) = c_1(\tilde{\gamma}^*) + 2\pi_1^* c_1\big(\Lambda_{\mathbb{C}}\gamma_3^*\big) = \tilde{\lambda} + 2\pi^*\hat{a}.$$

Example 3.5. By the above, the set of conics in $\mathbb{P}^3$ passing through 2 general points and 4 general lines is the zero set of a section of the vector bundle

$$2\,\tilde{\gamma}^* \oplus 4\,\tilde{\gamma}^* \otimes \pi^*(\Lambda_{\mathbb{C}}\gamma_3)^{*\otimes 2} \longrightarrow \mathbb{P}(\mathrm{Sym}^2\gamma_3^*)\big|_{\hat{\mathbb{P}}^1},$$

where $\hat{\mathbb{P}}^1$ is the intersection of the two $\hat{\mathbb{P}}^2 \subset \hat{\mathbb{P}}^3$ corresponding to the two points. As this section is transverse to the zero set, the cardinality of this set is

$$\begin{aligned}\big\langle e\big((2\tilde{\gamma}^* \oplus 4\,\tilde{\gamma}^* \otimes \pi^*(\Lambda_{\mathbb{C}}\gamma_3)^{*\otimes 2}\big), \mathbb{P}(\mathrm{Sym}^2\gamma_3^*)|_{\hat{\mathbb{P}}^1}\big\rangle &= \big\langle \tilde{\lambda}^2(\tilde{\lambda}+2\hat{a})^4, \mathbb{P}(\mathrm{Sym}^2\gamma_3^*)|_{\hat{\mathbb{P}}^1}\big\rangle \\ &= \big\langle \tilde{\lambda}^6 + 8\hat{a}\tilde{\lambda}^5, \mathbb{P}(\mathrm{Sym}^2\gamma_3^*)|_{\hat{\mathbb{P}}^1}\big\rangle.\end{aligned}$$

By Exercises A.2.6(2) and 3.3(3),

$$\tilde{\lambda}^6 = -c_1(\mathrm{Sym}^2\gamma_3^*)\lambda^5 = -4\hat{a}\tilde{\lambda}^5 \in H^*\big(\mathbb{P}(\mathrm{Sym}^2\gamma_3^*)|_{\hat{\mathbb{P}}^1};\mathbb{Z}\big).$$

We conclude that the number of conics in $\mathbb{P}^3$ passing through 2 general points and 4 general lines is

$$\big\langle e\big((2\tilde{\gamma}^* \oplus 4\,\tilde{\gamma}^* \otimes \pi^*(\Lambda_{\mathbb{C}}\gamma_3)^{*\otimes 2}\big), \mathbb{P}(\mathrm{Sym}^2\gamma_3^*)|_{\hat{\mathbb{P}}^1}\big\rangle = \big\langle -4\hat{a}\tilde{\lambda}^5 + 8\hat{a}\tilde{\lambda}^5, \mathbb{P}(\mathrm{Sym}^2\gamma_3^*)|_{\hat{\mathbb{P}}^1}\big\rangle = 4.$$

Exercise 3.6. Show that the number of conics in $\mathbb{P}^3$ passing through

(1) 1 point and 6 general lines is 18;

(2) 8 general lines is 92.

4 Grassmannians of three-planes

In order to count conics in $\mathbb{P}^n$ with $n>3$, we need to describe a topology and a complex structure on $\mathbb{G}(3,n+1)$, the space of three linear subspace of $\mathbb{C}^n$. Denote by

$$\mathcal{B}(3,n) \subset (\mathbb{C}^n-0) \times (\mathbb{C}^n-0) \times (\mathbb{C}^n-0)$$

the open subspace consisting of triples of linearly independent vectors in $\mathbb{C}^n$. Let $\mathcal{B}^*(3,n) \subset \mathcal{B}(3,n)$ be the subset of orthonormal triples (w.r.t. the standard hermitian inner-product on $\mathbb{C}^n$). Since the maps

$$\mathcal{B}(3,n), \mathcal{B}^*(3,n) \longrightarrow \mathbb{G}(3,n), \qquad (v_1,v_2,v_3) \longrightarrow \mathbb{C}v_1 + \mathbb{C}v_2 + \mathbb{C}v_3,$$

are surjective, the topologies of $\mathcal{B}(3,n)$ and $\mathcal{B}^*(3,n)$ induce quotient topologies on $\mathbb{G}(3,n)$; see [15, Section 22]. The maps

$$\mathrm{GL}_n(\mathbb{C}), U(n) \longrightarrow \mathbb{G}(3,n)$$

sending each matrix to the span of the first three columns are also surjective. As with $\mathbb{G}(2,n)$ in Section 2, there are at least four ways to describe the topology on $\mathbb{G}(3,n)$.

Exercise 4.1. Let $n \in \mathbb{Z}^+$ be such that $n \geq 3$. Show that

(1) the four quotient topologies on $\mathbb{G}(3,n)$ are the quotient topologies

$$\mathcal{B}(3,n)/\mathrm{GL}_3(\mathbb{C}), \quad \mathcal{B}^*(3,n)/U(3), \quad \mathrm{GL}_n(\mathbb{C})/G_3, \quad U(n)/U(3)\times U(n-3)$$

for certain free group actions and for a certain subgroup $G_3 \subset \mathrm{GL}_n(\mathbb{C})$;

(2) the four quotient topologies on $\mathbb{G}(3,n)$ are in fact the same.

Exercise 4.2. Let $n \in \mathbb{Z}^+$ be such that $n \geq 3$. Show that

(1) $\mathbb{G}(3,n)$ is a compact complex manifold of dimension $3(n-3)$ and the projection maps

$$\mathcal{B}(3,n), \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathbb{G}(3,n)$$

defined above are holomorphic submersions;

(2) the bijections

$$\mathbb{G}(k,n) \longrightarrow \mathbb{G}(n-k,n), \qquad V \longrightarrow \mathrm{Ann}(V), \tag{4.1}$$

are holomorphic if $k, n-k \leq 3$ (this is true in general).

A stratification of $\mathbb{G}(3,n)$ and its cohomology are also described similarly to those of $\mathbb{G}(2,n)$. Given a flag $\mathbf{V}$ be in $\mathbb{C}^n$ as in Section 2 and nonnegative integers a,b,c, we define

$$\begin{aligned}\sigma^0_{abc}(\mathbf{V}) = \big\{V \in \mathbb{G}(3,n)\colon \dim(V \cap V_{n-2-a}) = 1,\, \dim(V \cap V_{n-3-a}) = 0;\\ \dim(V \cap V_{n-1-b}) = 2,\, \dim(V \cap V_{n-2-b}) = 1;\\ \dim(V \cap V_{n-c}) = 3,\, \dim(V \cap V_{n-1-c}) = 2\big\}.\end{aligned}$$

Since V is a linear subspace of $\mathbb{C}^n$ of dimension 3,

$$\begin{aligned}\sigma^0_{abc}(\mathbf{V}) = \big\{V \in \mathbb{G}(3,n)\colon V \cap V_{n-2-a} \neq \{0\},\, V \cap V_{n-3-a} = \{0\};\\ \dim(V \cap V_{n-1-b}) = 2,\, \dim(V \cap V_{n-2-b}) = 1;\\ V \subset V_{n-c},\, V \not\subset V_{n-1-c}\big\}.\end{aligned}$$

Since a generic element of $\mathbb{G}(3,n)$ is not contained in V_{n-1}, meets V_{n-2} in a one-dimensional subspace, and intersects V_{n-2} trivially, the numbers a, b, and c measure the extent of the deviation of the elements of σ^0_{abc} from a generic element of $\mathbb{G}(3,n)$. Note that

$$\sigma^0_{abc}(\mathbf{V}) \neq \emptyset \qquad \Longrightarrow \qquad n-3 \geq a \geq b \geq c\,.$$

Furthermore,

$$\mathbb{G}(3,n) = \bigsqcup_{n-3 \geq a \geq b \geq c \geq 0} \sigma^0_{abc}(\mathbf{V}). \tag{4.2}$$

The closure of $\sigma^0_{abc}(\mathbf{V})$ in $\mathbb{G}(3,n)$ is given by

$$\sigma_{abc}(\mathbf{V}) \equiv \bar{\sigma}^0_{abc}(\mathbf{V}) = \big\{V \in \mathbb{G}(3,n)\colon V \subset V_{n-c},\, \dim(V \cap V_{n-1-b}) \geq 2,\, V \cap V_{n-2-a} \neq \{0\}\big\}. \tag{4.3}$$

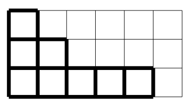
These subspaces of $\mathbb{G}(3,n)$ are called Schubert cells. We will write $\sigma_a(\mathbf{V})$ for $\sigma_{a00}(\mathbf{V})$ and $\sigma_{ab}(\mathbf{V})$ for $\sigma_{ab0}(\mathbf{V})$. If

$$\mathbf{V}^{n-c} \equiv (V_0, V_1, \ldots, V_{n-c})$$

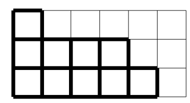
is a flag for $\mathbb{C}^{n-c}$, then

$$\sigma_{abc}(\mathbf{V}) = \sigma_{a-c,b-c}(\mathbf{V}^{n-c}) \subset \mathbb{G}(3, n-c). \tag{4.4}$$

These Schubert cycles can also be represented by Young diagrams, this time with 3 rows and $(n-3)$ columns:



$\mu = (5,2,1) \subset \mathbb{G}(3,9)$



$\mu^c = (5,4,1) \subset \mathbb{G}(3,9)$

For every Young diagram μ, we denote by μ^{tr} the transposed diagram, obtained by interchanging the rows and columns of μ. The first part of the next exercise is the $\mathbb{G}(3,n)$ analogue of Lemma 2.3 and Exercise 2.5.

Exercise 4.3. Let $n \in \mathbb{Z}^+$ be such that $n \geq 3$. Show

(1) the decomposition (4.2) is a stratification of $\mathbb{G}(3,n)$ with

$$\dim_{\mathbb{C}} \sigma^0_{abc}(\mathbf{V}) = 3(n-3) - (a+b+c) \qquad \text{if} \quad n-3 \geq a \geq b \geq c \geq 0\,; \tag{4.5}$$

(2) the subvariety $\sigma_\mu(\mathbf{V})$ of $\mathbb{G}(3,n)$ represented by a $3\times(n-3)$ Young diagram μ is smooth if and only if μ^c is a rectangle.

Similar to the $\mathbb{G}(2,n)$ case, the Schubert cycles $\sigma_{abc}(\mathbf{V})$ and $\sigma_{abc}(\mathbf{V}')$ corresponding to two different flags determine the same elements in the homology of $\mathbb{G}(3,n)$ and via the Poincare duality in the cohomology of $\mathbb{G}(3,n)$. Both of these elements will be denoted by σ_{abc}. By Exercise 4.3,

$$\sigma_{abc} \in H_{2(3(n-3)-(a+b+c))}\big(\mathbb{G}(3,n);\mathbb{Z}\big), H^{2(a+b+c)}\big(\mathbb{G}(3,n);\mathbb{Z}\big).$$

Furthermore, $H_*(\mathbb{G}(3,n);\mathbb{Z})$ and $H^*(\mathbb{G}(3,n);\mathbb{Z})$ are the free $\mathbb{Z}$-modules generated by σ_{abc} with $n-3 \geq a \geq b \geq c \geq 0$. The next lemma is the $\mathbb{G}(3,n)$ analogue of Lemma 2.16.

Exercise 4.4. Let $n \in \mathbb{Z}^+$ be such that $n \geq 3$. Show that

(1) the total chern class of the vector bundle $\gamma_3^* \longrightarrow \mathbb{G}(3,n)$ is given by

$$c(\gamma_3^*) = 1 + \sigma_1 + \sigma_{11} + \sigma_{111} \in H^*\big(\mathbb{G}(3,n);\mathbb{Z}\big);$$

(2) $\text{Ann}^* \sigma_\mu = \sigma_{\mu^{\text{tr}}}$, whenever $n \leq 6$ (this is true in general).

Similarly to Lemma 2.6, if $n-3 \geq a \geq b \geq c \geq 0$, then

$$\big\langle \sigma_{abc}\sigma_{a'b'c'}, \mathbb{G}(3,n)\big\rangle = \begin{cases} 1, & \text{if } a'=n-3-c,\ b'=n-3-b,\ c'=n-3-a; \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

Similarly to (2.9),

$$\big\langle \sigma_{a_1b_1c_1} \cdot \ldots \cdot \sigma_{a_kb_kc_k}, \mathbb{G}(3,n)\big\rangle = \big\langle \sigma_{a_1-c_1,b_1-c_1} \cdot \ldots \cdot \sigma_{a_k-c_k,b_k-c_k}, \mathbb{G}(3, n-c_1-\ldots-c_k)\big\rangle. \tag{4.7}$$

Exercise 4.5. Let $n\in\mathbb{Z}^+$ be such that $n\geq 3$.

(1) Verify (4.6) and (4.7).

(2) Suppose $0 \leq a_1, a_2, a_3, b_2, b_3 \leq n-3$. Show that

$$\big\langle \sigma_{a_1}\sigma_{a_2b_2}\sigma_{a_3b_3}, \mathbb{G}(3,n)\big\rangle = \begin{cases} 1, & \begin{array}{c}\text{if } b_2+b_3\leq n-3\leq a_2+b_3, a_3+b_2,\\ a_1+a_2+a_3 = 3(n-3);\end{array} \\ 0, & \text{otherwise.}\end{cases} \tag{4.8}$$

(3) Suppose $0 \leq a_1, a_2, b_2 \leq n-3$. Show that

$$\sigma_{a_1}\cdot\sigma_{a_2b_2} = \sum_{\substack{a'\geq a_2\geq b'\geq b_2\geq c'\geq 0\\ a'+b'+c'=a_1+a_2+b_2}} \sigma_{a'b'c'}. \tag{4.9}$$

Exercise 4.6. Let $n\in\mathbb{Z}^+$ be such that $n\geq 3$.

(1) Show that (4.6), (4.7), and (4.9) suffice to compute all intersection numbers

$$\langle \sigma_{a_1b_1c_1}\ldots\sigma_{a_kb_kc_k}, \mathbb{G}(3,n)\rangle \in \mathbb{Z}^{\geq 0}.$$

(2) Confirm the intersection numbers in Table 5.

To be added: constraints other than points and codimension 2, conics on complete intersection, genus 1 conics on complete intersections, twisted cubics

The number of conics on the quintic threefold, $X_{4;5}\subset\mathbb{P}^4$, is computed in [6, Section 3], by evaluating the euler class of a suitable bundle over the total space of the fibration

$$\mathbb{P}(\mathrm{Sym}^2\gamma_3^*) \longrightarrow \mathbb{G}(3,5).$$

$\ldots$	$\langle\ldots,\mathbb{G}(3,6)\rangle$	$\ldots$	$\langle\ldots,\mathbb{G}(3,6)\rangle$	$\ldots$	$\langle\ldots,\mathbb{G}(3,6)\rangle$
$\sigma_1\sigma_{11}\sigma_{33}$	0	$\sigma_1^3\sigma_2\sigma_{22}$	3	$\sigma_{11}\sigma_2^2\sigma_{21}$	2
$\sigma_1^3\sigma_{33}$	1	$\sigma_1^3\sigma_{11}\sigma_{22}$	3	$\sigma_{11}^2\sigma_2\sigma_{21}$	2
$\sigma_{11}^2\sigma_{32}$	0	$\sigma_1^5\sigma_{22}$	6	$\sigma_{11}^3\sigma_{21}$	2
$\sigma_1^2\sigma_2\sigma_{32}$	2	$\sigma_1\sigma_2\sigma_3^2$	1	$\sigma_1^2\sigma_2^2\sigma_{21}$	4
$\sigma_1^2\sigma_{11}\sigma_{32}$	1	$\sigma_1\sigma_{11}\sigma_3^2$	0	$\sigma_1^2\sigma_{11}\sigma_2\sigma_{21}$	4
$\sigma_1^4\sigma_{32}$	3	$\sigma_1^3\sigma_3^2$	1	$\sigma_1^2\sigma_{11}^2\sigma_{21}$	4
$\sigma_1^2\sigma_3\sigma_{31}$	1	$\sigma_1\sigma_2\sigma_{21}\sigma_3$	1	$\sigma_1\sigma_2^4$	3
$\sigma_{11}\sigma_{21}\sigma_{31}$	1	$\sigma_1\sigma_{11}\sigma_{21}\sigma_3$	1	$\sigma_1\sigma_{11}\sigma_2^3$	3
$\sigma_1^2\sigma_{21}\sigma_{31}$	2	$\sigma_1^3\sigma_{21}\sigma_3$	2	$\sigma_1\sigma_{11}^2\sigma_2^2$	2
$\sigma_1\sigma_2^2\sigma_{31}$	2	σ_{21}^3	2	$\sigma_1\sigma_{11}^3\sigma_2$	3
$\sigma_1\sigma_{11}\sigma_2\sigma_{31}$	1	$\sigma_1\sigma_2\sigma_{21}^2$	3	$\sigma_1\sigma_{11}^4$	3
$\sigma_1\sigma_{11}^2\sigma_{31}$	1	$\sigma_1\sigma_{11}\sigma_{21}^2$	3	$\sigma_1^3\sigma_2^3$	6
$\sigma_1^3\sigma_2\sigma_{31}$	3	$\sigma_1^3\sigma_{21}^2$	6	$\sigma_1^3\sigma_{11}\sigma_2^2$	5
$\sigma_1^3\sigma_{11}\sigma_{31}$	2	$\sigma_2^3\sigma_3$	1	$\sigma_1^3\sigma_{11}^2\sigma_2$	5
$\sigma_1^5\sigma_{31}$	5	$\sigma_{11}\sigma_2^2\sigma_3$	1	$\sigma_1^3\sigma_{11}^3$	6
$\sigma_1^2\sigma_3\sigma_{22}$	1	$\sigma_{11}^2\sigma_2\sigma_3$	0	$\sigma_1^5\sigma_2^2$	11
$\sigma_{11}\sigma_{21}\sigma_{22}$	1	$\sigma_{11}^3\sigma_3$	1	$\sigma_1^5\sigma_{11}\sigma_2$	10
$\sigma_1^2\sigma_{21}\sigma_{22}$	2	$\sigma_1^2\sigma_2^2\sigma_3$	2	$\sigma_1^5\sigma_{11}^2$	11
$\sigma_1\sigma_2^2\sigma_{22}$	1	$\sigma_1^2\sigma_{11}\sigma_2\sigma_3$	1	$\sigma_1^7\sigma_2$	21
$\sigma_1\sigma_{11}\sigma_2\sigma_{22}$	2	$\sigma_1^2\sigma_{11}^2\sigma_3$	1	$\sigma_1^7\sigma_{11}$	21
$\sigma_1\sigma_{11}^2\sigma_{22}$	1	$\sigma_2^3\sigma_{21}$	2	σ_1^9	42

Table 5: Top intersections of Schubert classes on $\mathbb{G}(3,6)$. The numbers directly obtainable from (4.6) and (4.9) are not shown (except for the first one).

2 Pseudocycles

5 Overview

A compact oriented k-manifold M carries a fundamental class $[M]\in H_k(M;\mathbb{Z})$. If X is a topological space and $M\subset X$, let

$$[M]_X \equiv \iota_{X,M*}\big([M]\big) \in H_k(X;\mathbb{Z})$$

denote the image of $[M]$ under the homomorphism induced by the inclusion $\iota_{X,M}: M \longrightarrow X$. More generally, if $f\colon M\longrightarrow X$ is a continuous map from a compact oriented k-manifold, let

$$[f] = f_*\big([M]\big) \in H_k(X;\mathbb{Z}).$$

If $-M$ is the same manifold with the reverse orientation and $-f$ denotes the same map as f, but with the domain $-M$,

$$[-f] = -[f] \in H_k(X;\mathbb{Z}).$$

Two maps $f_1\colon M_1\longrightarrow X$ and $f_2\colon M_2\longrightarrow X$ from compact oriented k-manifolds are cobordant if there exists a compact oriented $(k+1)$-manifold $\tilde{M}$ with boundary

$$\partial\tilde{M} = (-M_1)\sqcup M_2$$

and a continuous map $\tilde{f}\colon \tilde{M}\longrightarrow X$ such that $\tilde{f}|_{M_i}=f_i$ for $i=1,2$. For example, let $f\colon M\longrightarrow X$ be a continuous map from a compact oriented k-manifold M, $\mathbb{I}=[0,1]$, and $\pi_2\colon \mathbb{I}\times M\longrightarrow M$ be the projection onto the second component. The continuous map

$$f\circ\pi_2\colon \mathbb{I}\times M \longrightarrow X$$

is then a cobordism between f and itself, as well as between

$$f\sqcup(-f)\colon M\sqcup(-M) \longrightarrow X$$

as the empty set (viewed as a k manifold).

The set of equivalence classes of continuous maps $f\colon M\longrightarrow X$ from compact oriented k-manifolds forms an abelian group under the disjoint union with the inverse given by reversing the orientation of the domain. This group, denoted $\Omega_k(X)$, is called the k-th oriented cobordism group of X. Since two cobordant maps $f_i: M_i \longrightarrow X$ from compact oriented k-manifolds define the same homology class, the group homomorphism

$$\Omega_k(X) \longrightarrow H_k(X;\mathbb{Z}), \qquad [f\colon M\longrightarrow X] \longrightarrow f_*\big([M]\big), \tag{5.1}$$

is well-defined. If X is a smooth manifold, $\Omega_k(X)$ can be defined using smooth manifolds and smooth maps. This point of view provides a geometric way of representing homology cycles in a smooth manifold which is convenient especially when defining or computing specific geometrically meaningful counts. The homomorphism (5.1) tensored with $\mathbb{Q}$, i.e.

$$\Omega_k(X)\otimes_{\mathbb{Z}}\mathbb{Q} \longrightarrow H_k(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q} = H_k(X;\mathbb{Q}), \qquad [f\colon M \longrightarrow X]\otimes q \longrightarrow q\, f_*\big([M]\big), \tag{5.2}$$

is surjective, as a nonzero multiple of every homology class in an orientable manifold can be represented by an embedded submanifold; see [23, Théorème II.29]. By [12, Corollary 18.9],

$$\Omega_*(\mathrm{pt})\otimes_{\mathbb{Z}}\mathbb{Q} \approx \mathbb{Q}[\mathbb{P}^2,\mathbb{P}^4,\ldots];$$

thus (5.2) is never injective. The homomorphism (5.1) need not be even surjective in general. These deficiencies of the oriented cobordism ring can be resolved by relaxing the compactness assumption on the domain of the maps to an assumption regarding the image of the points away from compact subsets of the domain.

Definition 5.1. Let X be a topological space. The boundary of a continuous map $f\colon M \longrightarrow X$ from a topological space is the subspace

$$\mathrm{Bd}\, f = \bigcap_{K\subset M \text{ cmpt}} \overline{f(M-K)} \subset X.$$

Definition 5.2. Let X be a smooth manifold. A subset $Z\subset X$ is of dimension at most k if there exists a k-dimensional manifold Y and a smooth map $h\colon Y \longrightarrow X$ such that $Z\subset h(Y)$.

Definition 5.3. Let X be a smooth manifold.

(1) A smooth map $f: M \longrightarrow X$ from a smooth oriented k-manifold M is a k-pseudocycle if $\overline{f(M)}\subset X$ is compact and $\dim \mathrm{Bd}\, f \le k-2$.

(2) Two k-pseudocycles $f_1: M_1 \longrightarrow X$ and $f_2: M_2 \longrightarrow X$ are equivalent if there exists a smooth map $\tilde{f}\colon \tilde{M} \longrightarrow X$ from a smooth oriented $(k+1)$-manifold $\tilde{M}$ with boundary $\partial\tilde{M} = M_2 - M_1$ such that $\overline{\tilde{f}(M)}\subset X$ is compact, $\dim \mathrm{Bd}\,\tilde{f} \le k-1$, and $\tilde{f}|_{M_i} = f_i$ for $i=1,2$.

For example, the standard inclusion $\iota_{S^2,\mathbb{C}}\colon \mathbb{C} \longrightarrow S^2$ is a 2-pseudocycle, since the boundary of $\iota_{S^2,\mathbb{C}}$ consists of the single point ∞. On the other hand, the standard inclusion of the open unit disk into S^2 is not a 2-pseudocycle, since the boundary of this map is S^1, which is one-dimensional. If $f\colon M \longrightarrow X$ is a k-pseudocycle,

$$f\circ\pi_2\colon \mathbb{I}\times M \longrightarrow X$$

is an equivalence between f and itself, as well as between

$$f\sqcup(-f)\colon M\sqcup(-M) \longrightarrow X$$

as the empty set (viewed as a k-pseudocycle). The set of equivalence classes of k-pseudocycles, which we denote by $\mathcal{H}_k(X)$, forms an abelian group under the disjoint union with the inverse given by reversing the orientation of the domain.

The boundary of every smooth map $f\colon M \longrightarrow X$ from a compact manifold is empty. Thus, every smooth map $f\colon M \longrightarrow X$ from a compact oriented k-manifold is a k-pseudocycle and every two such maps equivalent in $\Omega_k(X)$ are also equivalent as elements of $\mathcal{H}_k(X)$. Thus, there is a natural group homomorphism

$$\Omega_k(X) \longrightarrow \mathcal{H}_k(X), \qquad [f\colon M \longrightarrow X] \longrightarrow [f\colon M \longrightarrow X]. \tag{5.3}$$

Since the homomorphism (5.1) is neither injective nor surjective in general, the first part of Theorem 5.4 below implies that neither is the homomorphism (5.3).

If X is oriented, $H_*(X;\mathbb{Z})$ carries a ring structure,

$$H_{k_1}(X;\mathbb{Z}) \otimes H_{k_2}(X;\mathbb{Z}) \longrightarrow H_{k_1+k_2-n}(X;\mathbb{Z}),$$

given by

$$A \otimes B \longrightarrow \mathrm{PD}\big(\mathrm{PD}_X(A) \cup \mathrm{PD}_X(B)\big)$$

where

$$\mathrm{PD}\colon H_*(X;\mathbb{Z}) \longrightarrow H_c^{n-*}(X;\mathbb{Z}) \qquad \text{and} \qquad H_c^*(X;\mathbb{Z}) \longrightarrow H_{n-*}(X;\mathbb{Z})$$

are the Poincare Duality isomorphisms and $n = \dim X$. In this case, $\mathcal{H}_*(X)$ also carries a ring structure,

$$\mathcal{H}_{k_1}(X) \otimes \mathcal{H}_{k_2}(X) \longrightarrow \mathcal{H}_{k_1+k_2-n}(X),$$

defined as follows. Suppose $i=1,2$, $f_i\colon M_i \longrightarrow X$ is a k_i-pseudocycle, and $h_i\colon Y_i \longrightarrow X$ is a smooth map from a (k_i-2)-manifold such that

(1) $\mathrm{Bd}\, f_1 \subset \mathrm{Im}\, h_1$ and $\mathrm{Bd}\, f_2 \subset \mathrm{Im}\, h_2$;

(2) $f_1 \pitchfork_X f_2$, $f_1 \pitchfork_X h_2$, $f_2 \pitchfork_X h_1$, and $h_1 \pitchfork_X h_2$.

Since X, M_1, and M_2 are oriented and $f_1 \pitchfork_X f_2$,

$$M_1\,{}_{f_1}\!\times_{f_2} M_2 \equiv \big\{(x_1,x_2)\colon\ f_1(x_1) = f_2(x_2)\big\}$$

is a smooth oriented manifold of dimension k_1+k_2-n and

$$f_1 \times_M f_2\colon M_1\,{}_{f_1}\!\times_{f_2} M_2 \longrightarrow X, \qquad (x_1,x_2) \longrightarrow f_1(x_1) = f_2(x_2).$$

The remaining assumptions insure that $f_1 \times_M f_2$ has sufficiently small boundary. By Proposition 11.2, every pair of equivalence classes in $\mathcal{H}_*(X)$ admits representatives $f_i : M_i \longrightarrow X$ satisfying (1) and (2).

Theorem 5.4. *Let X be a smooth manifold.*

(1) There exist natural homomorphisms of graded $\mathbb{Z}$-modules

$$\Phi\colon \mathcal{H}_*(X) \longrightarrow H_*(X;\mathbb{Z}) \qquad \textit{and} \qquad \Psi\colon H_*(X;\mathbb{Z}) \longrightarrow \mathcal{H}_*(X), \tag{5.4}$$

such that $\Phi \circ \Psi = \mathrm{Id}$, $\Psi \circ \Phi = \mathrm{Id}$, and the composition of Φ with the homomorphism (5.3) is the homomorphism (5.1).

(2) If in addition X is oriented, the isomorphisms (5.4) intertwine the ring structures on $\mathcal{H}_*(X)$ and $H_*(X;\mathbb{Z})$.

A smooth map $g\colon X \longrightarrow X'$ between smooth manifolds induces homomorphisms

$$g_*\colon H_*(X;\mathbb{Z}) \longrightarrow H_*(X';\mathbb{Z}) \qquad \text{and} \qquad g_*\colon \mathcal{H}_*(X) \longrightarrow \mathcal{H}_*(X')$$

by composition on the left. The naturality statement of Theorem 5.4 means that these maps commute with the isomorphisms Φ and Ψ corresponding to X and X'.

For the purposes of the first part of Theorem 5.4, it is sufficient to require that pseudocycle maps be continuous, as long as the same condition is imposed on pseudocycle equivalences. All arguments in this chapter concerning the first part of Theorem 5.4 go through for continuous pseudocycles; in fact, Lemma 7.2 would no longer be necessary. However, smooth pseudocycles are useful, including in algebraic geometry and symplectic topology, for describing intersections of cycles geometrically. In [16] and [18], pseudocycles are used to define Gromov-Witten invariants of compact semi-positive symplectic manifolds; they are then used to obtain a recursion for counts of rational curves in projective spaces.

Theorem 5.4 is the main subject of this chapter; its proof is outlined starting with the next paragraph. The proof of the first part of this theorem follows [25, 27]. Alternative treatments of this part of Theorem 5.4 appear in [7] and [20]. The latter is restricted to compact target manifolds X and considers only k-pseudocycles for which the boundary itself has vanishing homology in dimensions $k-1$ and k; see Remark 8.5. While non-compact manifolds are considered in [7], pseudocycles in [7] are not required to have compact closures. By [7, Proposition 1], there is then no surjective homomorphism from $H_*(X;\mathbb{Z})$ to $\mathcal{H}_*(X)$ for a non-compact manifold X, and so Theorem 5.4 fails for non-compact manifolds if pseudocycles are not required to have compact closures. The relevant target manifolds in [16, Section 7.1] and [18, Section 1] are compact, and a pseudocycle is not explicitly required to have a compact closure; the closure condition is made explicit in [17, Section 6.5].

Exercise 5.5. Let $f: M \longrightarrow X$ be a continuous map between topological spaces and U be an open neighborhood of $\operatorname{Bd} f$.

(1) Show that the subspace $f^{-1}(X-U)\subset M$ is compact if $\overline{f(M)}\subset X$ is.

(2) Give an example showing that the compactness requirement on $\overline{f(M)}$ cannot be dropped.

If $f: M \longrightarrow X$ is a k-pseudocycle, one can choose a compact k-submanifold with boundary, $\bar{V}\subset M$, so that $f(M-V)$ lies in an arbitrary small neighborhood U of $\operatorname{Bd} f$. In particular, $f|_{\bar{V}}$ determines the homology class

$$f_*[\bar{V},\partial\bar{V}] \in H_k(X,U;\mathbb{Z}).$$

By Corollary 8.4, U can be chosen so that $H_k(X,U;\mathbb{Z})$ is naturally isomorphic to $H_k(X;\mathbb{Z})$. In order to show that the resulting cycle in $H_k(X;\mathbb{Z})$ depends only on f (and not V or U), we use Proposition 6.4 to replace the singular chain complex $S_*(X)$ by a quotient complex $\bar{S}_*(X)$. The advantage of the latter complex is that cycles and boundaries between chains can be constructed more easily; see Remark 6.5.

In an analogous way, a pseudocycle equivalence $\tilde{f}\colon \tilde{M} \longrightarrow X$ between two pseudocycles

$$f_i\colon M_i \longrightarrow X, \qquad i=0,1,$$

gives rise to a chain equivalence between the corresponding cycles in $\bar{S}_*(X,W)$, for a small neighborhood W of $\mathrm{Bd}\,\tilde{f}$. In particular, the homology cycles determined by f_0 and f_1 are equal in $H_k(X,W;\mathbb{Z})$. On the other hand, by Corollary 8.4, W can be chosen so that $H_k(X;\mathbb{Z})$ naturally injects into $H_k(X,W;\mathbb{Z})$. Therefore, f_0 and f_1 determine the same elements of $H_k(X;\mathbb{Z})$ and the homomorphism Φ is well-defined. Its construction is described in detail in Section 8.

Remark 5.6. The homomorphism Φ of Section 8 induces the linear map

$$\mathcal{H}_*(X) \longrightarrow H_*(X;\mathbb{Z})/\mathrm{Tor}\big(H_*(X;\mathbb{Z})\big)$$

described in [16] and [18]. However, the construction of Φ in Section 8 differs from that of the induced map in [16] and [18]; the latter is in fact constructed via the homomorphism Ψ and the natural intersection pairing on $\mathcal{H}_*(X)$ defined whenever X is oriented. The construction of Φ in Section 8 is more direct.

Remark 5.7. The construction of Φ in Section 8 implies the following. Suppose M is an oriented k-manifold and $f\colon M \longrightarrow X$ is a continuous map with a pre-compact image. If R is a ring and $\mathrm{Bd}\, f$ has an arbitrary small neighborhood U so that $H_l(U;R)=0$ for all $l>k-2$, then f defines an element in $H_k(X;R)$. The analogous statement holds for equivalences between maps. It is not necessary for X to be a smooth manifold. These observations have a variety of applications. For example, the first statement implies that a compact complex algebraic variety carries a fundamental class. For essentially the same reason, (generalized) pseudocycles figure prominently in the approach in [24] to a large class of problems in enumerative geometry. Pseudocycles can also be used to give a more geometric interpretation of the virtual fundamental class construction of [4] and [10] and are used to define new symplectic invariants in [26]. This is a different type of generalization, as the ambient space X in these settings is a topological space stratified by infinite-dimensional orbifolds.

In order to construct the homomorphism Ψ, we show that a singular cycle gives rise to a pseudocycle and a chain equivalence between two cycles gives rise to a pseudocycle equivalence between the corresponding pseudocycles. The former works out precisely as outlined in [16, Section 7.1], with a reinterpretation for the complex $\bar{S}_*(X)$; this reinterpretation is not necessary to construct Ψ in Section 7, but is needed in Section 9 to show that the maps Φ and Ψ are isomorphisms. If s is a k-cycle, all codimension-one simplices of its k-simplices must cancel in pairs. By gluing the k-simplices along the codimension-one faces paired in this way, we obtain a continuous map from a compact topological space M' to X. The complement of the codimension-two simplices is a smooth manifold and the continuous map can be smoothed out in a fixed manner using Lemma 7.2. We thus obtain a pseudocycle from the cycle s.

On the other hand, turning a chain equivalence $\tilde{s}$ between two k-cycles, s_0 and s_1, into a pseudocycle equivalence between the corresponding pseudocycles,

$$f_0\colon M_0 \longrightarrow X \qquad \text{and} \qquad f_1\colon M_1 \longrightarrow X,$$

turns out to be less straightforward. Similarly to the previous paragraph, $\tilde{s}$ gives rise to a smooth map from a smooth $(k+1)$-manifold with boundary,

$$\tilde{f}\colon \tilde{M}^* \longrightarrow X.$$

However, if all codimension-two simplices (including those of dimension $k-1$) are dropped, the boundary of $\tilde{M}^*$ will be the complement in $M_0 \sqcup M_1$ of a subset of dimension $k-1$ (instead of being $M_0 \sqcup M_1$). One way to fix this is to keep the $(k-1)$-simplices that would lie on the boundary. In such a case, the entire space may no longer be a smooth manifold and its boundary may not be $M_0 \sqcup M_1$, because the $(k-1)$-simplices of the $(k+1)$-simplices of $\tilde{s}$ may be identified differently from the way the $(k-1)$-simplices of the k-simplices of s_0 and s_1 are identified. It is possible to modify $\tilde{s}$ so that all identifications are consistent. However, the required modification turns out to be quite laborious. We instead implement a less direct, but far simpler, construction suggested by D. McDuff. Instead of trying to reinsert $(k-1)$-simplices into the boundary of $\tilde{M}^*$, we attach to $\tilde{M}^*$ two collars,

$$\tilde{M}_0 \subset [0,1] \times M_0 \qquad \text{and} \qquad \tilde{M}_1 \subset [0,1] \times M_1.$$

The boundary of $\tilde{M}_i$ has two pieces, M_i and the complement in M_i of the $(k-1)$-simplices. We attach the latter to the piece of the boundary of $\tilde{M}^*$ corresponding to s_i. In this way, we obtain a smooth manifold $\tilde{M}$ with boundary $M_1 - M_0$ and a pseudocycle equivalence from f_0 to f_1; see Section 8 for details.

In Section 9, we verify that the homomorphisms Ψ and Φ are mutual inverses. It is fairly straightforward to see that the map $\Phi \circ \Psi$ is the identity on $\bar{H}_*(X;\mathbb{Z})$. However, showing the injectivity of Φ requires more care. The desired pseudocycle equivalence $\tilde{f}: \tilde{M} \longrightarrow X$, is constructed by taking a limit of the corresponding construction in Section 7. In particular, the smooth manifold $\tilde{M}$ is obtained as a subspace of a *non*-compact space.

6 Oriented homology groups

If X is a simplicial complex, the standard singular chain complex $S_*(X)$ most naturally corresponds to the *ordered* simplicial chain complex of X; see [14, Section 13]. In this section, we define a singular chain complex $\bar{S}_*(X)$ which corresponds to the standard, or *oriented*, simplicial chain complex. In particular, its homology is the same as the homology of the ordinary singular chain complex; see Proposition 6.4. On the other hand, it is much easier to construct cycles in $\bar{S}_*(X)$ than in $S_*(X)$; see Remark 6.5.

If A is a finite subset of $\mathbb{R}^k$, we denote by $\mathrm{CH}(A)$ and $\mathrm{CH}^0(A)$ the (closed) convex hull of A and the open convex hull of A, respectively, i.e.

$$CH(A) = \Big\{ \sum_{v\in A} t_v v : t_v \in [0,1],\ \sum_{v\in A} t_v = 1 \Big\} \qquad \text{and}$$
$$CH^0(A) = \Big\{ \sum_{v\in A} t_v v : t_v \in (0,1),\ \sum_{v\in A} t_v = 1 \Big\}.$$

For each $p=1,\ldots,k$, let e_p be the p-th coordinate vector in $\mathbb{R}^k$. Put $e_0=0\in\mathbb{R}^k$. Denote by

$$\Delta^k = \mathrm{CH}\big(e_0, e_1, \ldots, e_k\big) \qquad \text{and} \qquad \mathrm{Int}\,\Delta^k = \mathrm{CH}^0\big(e_0, e_1, \ldots, e_k\big)$$

the standard k-simplex and its interior. Let

$$b_k = \frac{1}{k+1}\Big(\sum_{q=0}^{q=k} e_q\Big) = \Big(\frac{1}{k+1}, \ldots, \frac{1}{k+1}\Big) \in \mathbb{R}^k$$

be the barycenter of Δ^k.

A map $f\colon \Delta^k \longrightarrow \mathbb{R}^m$ is linear if

$$f(t_0e_0+\ldots+t_ke_k) = t_0f(e_0) + \ldots + f_k(e_k) \quad \forall\ (t_0,\ldots,t_k) \in [0,1]^k \ \text{ s.t. } \sum_{q=0}^{k} t_qe_q = 1.$$

For each $k\in\mathbb{Z}^+$ and $p=0,\ldots,k-1$, define the linear map

$$\iota_{k,p}\colon \Delta^{k-1} \longrightarrow \Delta^k_p \subset \Delta^k \qquad \text{by} \qquad \iota_{k,p}(e_q) = \begin{cases} e_q, & \text{if } q<p; \\ e_{q+1}, & \text{if } q\geq p. \end{cases}$$

For any element τ in the group $\mathcal{S}_k$ of permutations of the set $\{0,\ldots,k\}$, we define the linear map

$$\tau\colon \Delta^k \longrightarrow \Delta^k \qquad \text{by} \quad \tau(e_q) = e_{\tau(q)} \quad \forall\ q = 0,\ldots,k.$$

We embed $\mathcal{S}_k$ into $\mathcal{S}_{k+1}$ by setting $\tau(k+1)=k+1$ for any $\tau\in\mathcal{S}_k$.

If X is a topological space, let $\big(S_*(X),\partial_X\big)$ denote its singular chain complex, i.e. the free abelian group on the set

$$\bigcup_{k=0}^{\infty} C(\Delta^k, X)$$

of all continuous maps from standard simplices to X, along with a map ∂_X of degree -1. Let $S'_k(X)$ denote the free subgroup of $S_*(X)$ spanned by the set

$$\big\{f-(\operatorname{sign}\tau)f\circ\tau\colon f\in C(\Delta^k,X),\ \tau\in\mathcal{S}_k,\ k=0,1,\ldots\big\}.$$

If $\tau\in\mathcal{S}_k$, put

$$\tilde{\tau} = \mathrm{Id}_{\Delta^k} - (\operatorname{sign}\tau)\tau \in S'_k(\Delta^k). \tag{6.1}$$

Thus, $S'_*(X)$ is the subgroup of $S_*(X)$ spanned by

$$\big\{f_\#\tilde{\tau}\colon f\in C(\Delta^k,X),\ \tau\in\mathcal{S}_k,\ k\in\mathbb{Z}^{\geq 0}\big\}.$$

If $h\colon X\longrightarrow Y$ is a continuous map, the homomorphism

$$h_\#\colon S_*(X) \longrightarrow S_*(Y)$$

maps $S'_*(X)$ into $S'_*(Y)$.

Lemma 6.1. *The free abelian group $S'_*(X)$ is a subcomplex of $\big(S_*(X),\partial_X\big)$, i.e. $\partial_X S'_*(X)\subset S'_*(X)$.*

Proof. Suppose $\tau\in\mathcal{S}_k$. For any $p=0,\ldots,k$, let $\tau_p\in\mathcal{S}_{k-1}$ be such that

$$\tau\circ\iota_{k,p} = \iota_{k,\tau(p)}\circ\tau_p\colon \Delta^{k-1} \longrightarrow \Delta^k_{\tau(p)}\subset\Delta^k. \tag{6.2}$$

Let $\tau_{k,p}\in\mathcal{S}_k$ be defined by

$$\tau_{k,p}(q) = \begin{cases} \iota_{k,p}(q), & \text{if } q<k; \\ p, & \text{if } q=k. \end{cases}$$

Since $\tau\circ\tau_{k,p}=\tau_{k,\tau(p)}\circ\tau_p\in\mathcal{S}_k$,

$$\operatorname{sign}\tau_p=(-1)^{(k-p)+(k-\tau(p))}\operatorname{sign}\ \tau=(-1)^{p+\tau(p)}\operatorname{sign}\tau\qquad\forall\ \tau\in\mathcal{S}_k. \tag{6.3}$$

By (6.2) and (6.3),

$$\begin{aligned}\partial_{\Delta^k}\tau=\sum_{p=0}^{k}(-1)^p\tau\circ\iota_{k,p}=\sum_{p=0}^{k}(-1)^p\iota_{k,\tau(p)}\circ\tau_p&=(\operatorname{sign}\tau)\sum_{p=0}^{k}(-1)^{\tau(p)}(\operatorname{sign}\tau_p)\iota_{k,\tau(p)}\circ\tau_p\\&=(\operatorname{sign}\tau)\sum_{p=0}^{k}(-1)^p(\operatorname{sign}\tau_{\tau^{-1}(p)})\iota_{k,p}\circ\tau_{\tau^{-1}(p)}.\end{aligned}$$

Thus,

$$\partial_{\Delta^k}\tilde{\tau}=\sum_{p=0}^{k}(-1)^p\big(\iota_{k,p}-(\operatorname{sign}\tau_{\tau^{-1}(p)})\iota_{k,p}\circ\tau_{\tau^{-1}(p)}\big)\in S'_{k-1}(\Delta^k).$$

It follows that for any $f\in S_k(X)$,

$$\partial_X(f_\#\tilde{\tau})=f_\#(\partial_{\Delta^k}\tilde{\tau})\in S'_{k-1}(X).$$

This establishes the claim. □

Lemma 6.2. *There exists a natural transformation of functors $D_X\colon S_*\longrightarrow S_{*+1}$ such that*

(1) if $f\colon\Delta^m\longrightarrow\Delta^k$ is a linear map, D_Xf is a linear combination of linear maps $\Delta^{m+1}\longrightarrow\Delta^k$ for all $k,m\in\mathbb{Z}^{\geq 0}$;

(2) $D_XS'_*(X)\subset S'_*(X)$ for all topological spaces X;

(3) $\partial_XD_X=(-1)^{k+1}\mathrm{Id}+D_X\partial_X$ on $S'_k(X)$.

Proof. (1) Suppose $k\in\mathbb{Z}^+$. If $f\colon\Delta^m\longrightarrow\Delta^k$ is a linear map, define a new linear map

$$P_kf\colon\Delta^{m+1}\longrightarrow\Delta^k\qquad\text{by}\qquad P_kf\,(e_q)=\begin{cases}f(e_q), & \text{if } q=0,\ldots,m;\\ b_k, & \text{if } q=m+1.\end{cases} \tag{6.4}$$

The transformation P_k induces a homomorphism on the sub-chain complex of $S_*(\Delta^k)$ spanned by the linear maps. If $\tau\in\mathcal{S}_m\subset\mathcal{S}_{m+1}$ and $f\in S_m(\Delta^k)$ is a linear map, then

$$P_k(f\circ\tau)=P_kf\circ\tau. \tag{6.5}$$

Thus, P_k maps the subgroup of $S'_*(\Delta^k)$ spanned by the linear maps into itself. Similarly,

$$\tau_\#(P_kf)\equiv\tau\circ P_kf=P_k(\tau\circ f)\equiv P_k(\tau_\#f) \tag{6.6}$$

if $\tau\in\mathcal{S}_k$ and f is a linear map as above. Furthermore,

$$\partial_{\Delta^k}P_kf=(-1)^{k+1}f+P_k\big(\partial_{\Delta^k}f\big) \tag{6.7}$$

for every linear map f and thus for linear combinations of linear maps.

(2) Let $D_X|_{S_k(X)}=0$ if $k<1$; then D_X satisfies (1)-(3). Suppose $k\ge 1$ and we have defined $D_X|_{S_{k-1}(X)}$ so that the three requirements are satisfied wherever D_X is defined. Put

$$D_{\Delta^k}(\mathrm{Id}_{\Delta^k}) = P_k\big(\mathrm{Id}_{\Delta^k} + (-1)^{k+1}D_{\Delta^k}\partial_{\Delta^k}\mathrm{Id}_{\Delta^k}\big) \in S_{k+1}(\Delta^k). \tag{6.8}$$

By the inductive assumption (1) and (6.4), $D_{\Delta^k}(\mathrm{Id}_{\Delta^k})$ is a well-defined linear combination of linear maps. For any $f\in C(\Delta^k,X)$, let

$$D_X f = f_\# D_{\Delta^k}\mathrm{Id}_{\Delta^k}. \tag{6.9}$$

This construction defines a natural transformation $S_k \longrightarrow S_{k+1}$. Since $D_{\Delta^k}(\mathrm{Id}_{\Delta^k})$ is a linear combination of linear maps, it is clear that the requirement (1) above is satisfied; it remains to check (2) and (3).

(3) Given $f\in C(\Delta^k,X)$ and $\tau\in\mathcal{S}_k$, let $s=f_\#\tilde{\tau}\in S'_k(X)$, with $\tilde{\tau}$ as in (6.1). By (6.9), (6.8), (6.6), and the naturality of $D_X|_{S_{k-1}}$,

$$\begin{aligned} D_X(f\circ\tau) &= f_\#\tau_\# D_{\Delta^k}\mathrm{Id}_{\Delta^k} = f_\#\tau_\# P_k\big(\mathrm{Id}_{\Delta^k} + (-1)^{k+1}D_{\Delta^k}\partial_{\Delta^k}\mathrm{Id}_{\Delta^k}\big)\\ &= f_\# P_k\big(\tau + (-1)^{k+1}\tau_\# D_{\Delta^k}\partial_{\Delta^k}\mathrm{Id}_{\Delta^k}\big) = f_\# P_k\big(\tau + (-1)^{k+1}D_{\Delta^k}\partial_{\Delta^k}\tau\big). \end{aligned} \tag{6.10}$$

Thus,

$$D_X s = f_\# P_k\big(\tilde{\tau} + (-1)^{k+1}D_{\Delta^k}\partial_{\Delta^k}\tilde{\tau}\big). \tag{6.11}$$

By Lemma 6.1, the induction assumption (2), and (6.6), $S'_k(\Delta^k)$ is mapped into $S'_*(\Delta^k)$ by $D_{\Delta^k}\partial_{\Delta^k}$ and by P_k. Thus, by (6.11), D_X maps $S'_k(X)$ into $S'_{k+1}(X)$. Finally, by (6.11), (6.7), and the inductive assumption (3),

$$\begin{aligned} \partial_X D_X s &= \partial_X f_\# P_k\big(\tilde{\tau}+(-1)^{k+1}D_{\Delta^k}\partial_{\Delta^k}\tilde{\tau}\big) = f_\#\partial_{\Delta^k}P_k\tilde{\tau} + (-1)^{k+1}f_\#\partial_{\Delta^k}P_k D_{\Delta^k}\partial_{\Delta^k}\tilde{\tau}\\ &= f_\#\big((-1)^{k+1}\tilde{\tau} + P_k\partial_{\Delta^k}\tilde{\tau}\big) + (-1)^{k+1}f_\#\big((-1)^{k+1}D_{\Delta^k}\partial_{\Delta^k}\tilde{\tau} + P_k\partial_{\Delta^k}D_{\Delta^k}\partial_{\Delta^k}\tilde{\tau}\big)\\ &= \big((-1)^{k+1}s+D_X\partial_X s\big) + f_\# P_k\partial_{\Delta^k}\tilde{\tau} + (-1)^{k+1}f_\# P_k\big((-1)^k\partial_{\Delta^k}\tilde{\tau}+D_{\Delta^k}\partial^2_{\Delta^k}\tilde{\tau}\big)\\ &= (-1)^{k+1}s + D_X\partial_X s. \end{aligned}$$

Thus, $D_X|_{S_k}$ satisfies the induction assumptions (2) and (3). □

Corollary 6.3. *All homology groups of the complex* $\big(S'_*(X),\partial_X|_{S'_*(X)}\big)$ *are zero.*

Let $\bar{S}_*(X)=S_*(X)/S'_*(X)$ and denote by

$$\pi\colon S_*(X)\longrightarrow \bar{S}_*(X) \tag{6.12}$$

the projection map. Let $\bar{\partial}_X$ be boundary map on $\bar{S}_*(X)$ induced by ∂_X. We denote by $\bar{H}_*(X;\mathbb{Z})$ the homology groups of $\big(\bar{S}_*(X),\bar{\partial}_X\big)$.

Proposition 6.4. *If X is a topological space, the projection map $\pi\colon S_*(X)\longrightarrow\bar{S}_*(X)$ induces a natural isomorphism $H_*(X;\mathbb{Z})\longrightarrow\bar{H}_*(X;\mathbb{Z})$. This isomorphism extends to relative homologies to give an isomorphism of homology theories.*

Proof. The first statement follows from the long exact sequence in homology for the short exact sequence of chain complexes

$$0 \longrightarrow S'_*(X) \longrightarrow S_*(X) \xrightarrow{\pi} \bar{S}_*(X) \longrightarrow 0$$

and Corollary 6.3. The second statement follows from the first and the Five Lemma; see [14, Lemma 24.3]. $\square$

For a simplicial complex K, let $|K| \subset \mathbb{R}^N$ denote its geometric realization; see [14, Section 3]. There is a natural chain map from the *ordered* simplicial complex $C'_*(K)$ to the singular chain complex $S_*(|K|)$, which induces an isomorphism in homology. If the vertices of K are ordered, there is also a chain map from $C'_*(K)$ to the *oriented* chain complex $C_*(K)$, which induces a natural isomorphism in homology. However, the chain map itself depends on the ordering of the vertices; see [14, Section 34]. The advantage of the complex $\bar{S}_*(K)$ is that there is a natural chain map from $C_*(K)$ to $\bar{S}_*(K)$, which induces an isomorphism in homology; this chain map is induced by the natural chain map from $C'_*(K)$ to $S_*(|K|)$ described in [14, Section 34].

If $|K| \subset \mathbb{R}^N$ is a geometric realization of a simplicial complex K, for each l-simplex σ of K there is an injective linear map $\iota_\sigma : \Delta^l \longrightarrow |K|$ taking Δ^l to $|\sigma|$. If M is an oriented n-manifold with boundary, an oriented triangulation of $(M, \partial M)$ is a triple $T = (K, K', \eta)$ consisting of a simplicial complex K, simplicial sub-complex K', and a homeomorphism

$$\eta\colon (|K|, |K'|) \longrightarrow (M, \partial M)$$

such that for every n-simplex Δ and every $x \in \mathrm{Int}\,\Delta^k$ the homeomorphism

$$\eta_\sigma \equiv \eta \circ \iota_\sigma : \Delta^n \longrightarrow M \tag{6.13}$$

takes the oriented generator of $H_n(\Delta^k, \Delta^k - x; \mathbb{Z})$ to the oriented generator of $H_n(M, M - x; \mathbb{Z})$.

Remark 6.5. Let M be a compact oriented n-manifold with boundary ∂M and $T = (K, K', \eta)$ be an oriented triangulation of $(M, \partial M)$. The fundamental homology class

$$[M, \partial M] \in H_n(M, \partial M; \mathbb{Z})$$

is represented in $\bar{S}_k(M, \partial M)$ by

$$\sum_{\sigma \in K, \dim \sigma = n} \{\eta_\sigma\} \equiv \sum_{\sigma \in K, \dim \sigma = n} \pi(\eta_\sigma),$$

where π is as before in (6.12). On the other hand,

$$\sum_{\sigma \in K, \dim \sigma = n} \eta_\sigma$$

may not even be a cycle in $S_n(M, \partial M)$. It is definitely *not* a cycle if $\partial M = \emptyset$ and n is an even positive integer, as the boundary of each term η_σ contains one more term with coefficient $+1$ than -1. Similarly, if $h\colon (M, \partial M) \longrightarrow (X, U)$ is a continuous map, $h_*([M, \partial M]) \in H_n(X, U; \mathbb{Z})$ is represented in $\bar{S}_k(X, U)$ by

$$\sum_{\sigma \in K, \dim \sigma = n} \{h \circ \eta_\sigma\}\,;$$

the obvious preimage under π of the above chain in $S_n(X, U; \mathbb{Z})$ may not be even a cycle.

We next characterize cycles and boundaries in $\bar{S}_*(X)$ in a manner suitable for converting them to pseudocycles and pseudocycle equivalences in Section 7. We use the statements of Exercises 6.6 and 6.7 below to glue maps from standard simplices together to construct smooth maps from smooth manifolds.

The homology groups of a smooth manifolds X can be defined with the space $C(\Delta^k, X)$ of continuous maps from Δ^k to X replaced by the space $C^\infty(\Delta^k, X)$ of smooth maps; this is a standard fact in differential topology. The operator D_X of Lemma 6.2 maps smooth maps into linear combinations of smooth maps. Thus, all constructions of this section go through for the chain complexes based on elements in $C^\infty(\Delta^k, X)$ instead of $C(\Delta^k, X)$. Below $\bar{S}_*(X)$ refers to the quotient complex based on such maps.

If $s = \sum_{j=1}^{j=N} f_j$, where $f_j\colon \Delta^k \longrightarrow X$ is a continuous map for each j, let

$$\mathcal{C}_s = \big\{(j,p)\colon j=1,\ldots,N,\ p=0,\ldots,k\big\}.$$

Exercise 6.6. Suppose $k\geq 1$ and $s \equiv \sum_{j=1}^{j=N} f_j$ is a cycle in $\bar{S}_k(X)$. Show that there exist a subset $\mathcal{D}_s \subset \mathcal{C}_s \times \mathcal{C}_s$ disjoint from the diagonal and a map

$$\tau\colon \mathcal{D}_s \longrightarrow \mathcal{S}_{k-1}, \qquad \big((j_1,p_1),(j_2,p_2)\big) \longrightarrow \tau_{(j_1,p_1),(j_2,p_2)},$$

such that

(1) if $\big((j_1,p_1),(j_2,p_2)\big) \in \mathcal{D}_s$, then $\big((j_2,p_2),(j_1,p_1)\big) \in \mathfrak{D}_s$;

(2) the projection $\mathcal{D}_s \longrightarrow \mathcal{C}_s$ on either coordinate is a bijection;

(3) for all $\big((j_1,p_1),(j_2,p_2)\big) \in \mathcal{D}_s$,

$$\tau_{(j_2,p_2),(j_1,p_1)} = \tau^{-1}_{(j_1,p_1),(j_2,p_2)}, \qquad f_{j_2} \circ \iota_{k,p_2} = f_{j_1} \circ \iota_{k,p_1} \circ \tau_{(j_1,p_1),(j_2,p_2)}, \tag{6.14}$$

$$\text{and} \qquad \operatorname{sign} \tau_{(j_1,p_1),(j_2,p_2)} = -(-1)^{p_1+p_2}. \tag{6.15}$$

(6.16)

The above claim follows from the assumption that $\bar{\partial}\{s\} = 0$ and from the definition of $\bar{S}_*(X)$ above. The terms appearing in the boundary of s are indexed by the set $\mathcal{C}_s$, and the coefficient of the (j,p)-th term is $(-1)^p$. Since s is a cycle in $\bar{S}_*(X)$, these terms cancel in pairs, possibly after composition with an element $\tau \in \mathcal{S}_{k-1}$ and multiplying by $\operatorname{sign}\tau$. This operation does not change the equivalence class of a $(k-1)$-simplex in $\bar{S}_{k-1}(X)$.

Exercise 6.7. Suppose $k\geq 1$,

$$s_0 \equiv \sum_{j=1}^{j=N_0} \{f_{0,j}\}, \qquad s_1 \equiv \sum_{j=1}^{j=N_1} \{f_{1,j}\}, \qquad \tilde{s} \equiv \sum_{j=1}^{j=\tilde{N}} \tilde{f}_j, \quad \text{and} \quad \bar{\partial}\{\tilde{s}\} = \{s_1\} - \{s_0\} \in \bar{S}_k(X).$$

Show that there exist a subset $\mathcal{D}_{\tilde{s}} \subset \mathcal{C}_{\tilde{s}} \times \mathcal{C}_{\tilde{s}}$ disjoint from the diagonal, subsets $\mathcal{C}_{\tilde{s}}^{(0)}, \mathcal{C}_{\tilde{s}}^{(1)} \subset \mathcal{C}_{\tilde{s}}$, and maps

$$\tilde{\tau}\colon \mathcal{D}_{\tilde{s}} \longrightarrow \mathcal{S}_k, \quad \big((j_1,p_1),(j_2,p_2)\big) \longrightarrow \tilde{\tau}_{(j_1,p_1),(j_2,p_2)},$$
$$(\tilde{j}_i,\tilde{p}_i)\colon \{1,\ldots,N_i\} \longrightarrow \mathcal{C}_{\tilde{s}}^{(i)}, \quad \text{and} \quad \tilde{\tau}_i\colon \{1,\ldots,N_i\} \longrightarrow \mathcal{S}_k, \quad j \longrightarrow \tilde{\tau}_{(i,j)}, \quad i=0,1,$$

such that

(1) if $\big((j_1,p_1),(j_2,p_2)\big) \in \mathcal{D}_{\tilde{s}}$, then $\big((j_2,p_2),(j_1,p_1)\big) \in \mathcal{D}_{\tilde{s}}$;

(2) the projection $\mathcal{D}_{\tilde{s}} \longrightarrow \mathcal{C}_{\tilde{s}}$ on either coordinate is a bijection onto the complement of $\mathcal{C}_{\tilde{s}}^{(0)} \cup \mathcal{C}_{\tilde{s}}^{(1)}$;

(3) for all $\big((j_1,p_1),(j_2,p_2)\big) \in \mathcal{D}_{\tilde{s}}$,

$$\tilde{\tau}_{(j_2,p_2),(j_1,p_1)} = \tilde{\tau}_{(j_1,p_1),(j_2,p_2)}^{-1}, \qquad \tilde{f}_{j_2} \circ \iota_{k+1,p_2} = \tilde{f}_{j_1} \circ \iota_{k+1,p_1} \circ \tilde{\tau}_{(j_1,p_1),(j_2,p_2)}, \tag{6.17}$$
$$\text{and} \qquad \operatorname{sign} \tilde{\tau}_{(j_1,p_1),(j_2,p_2)} = -(-1)^{p_1+p_2}; \tag{6.18}$$

(4) for all $i=0,1$ and $j=1,\ldots,N_i$,

$$\tilde{f}_{\tilde{j}_i(j)} \circ \iota_{k+1,\tilde{p}_i(j)} \circ \tilde{\tau}_{(i,j)} = f_{i,j} \qquad \text{and} \qquad \operatorname{sign} \tilde{\tau}_{(i,j)} = -(-1)^{i+\tilde{p}_i(j)}; \tag{6.19}$$

(5) $(\tilde{j}_i,\tilde{p}_i)$ is a bijection onto $\mathcal{C}_{\tilde{s}}^{(i)}$ for $i=0,1$.

This lemma follows from the assumption that

$$\bar{\partial}\{\tilde{s}\} = \{s_1\} - \{s_0\}.$$

The terms making up $\partial\tilde{s}$ are indexed by the set $\mathcal{C}_{\tilde{s}}$. By definition of $\bar{S}_*(X)$, there exist disjoint subsets $\mathcal{C}_{\tilde{s}}^{(0)}$ and $\mathcal{C}_{\tilde{s}}^{(1)}$ of $\mathcal{C}_{\tilde{s}}$ such that for each $(j,p) \in \mathcal{C}_{\tilde{s}}^{(1)}$ the (j,p)-th term of $\partial\tilde{s}$ equals one of the terms of s_i, after a composition with some $\tilde{\tau} \in \mathcal{S}_k$ and multiplying by $-(-1)^i \operatorname{sign} \tilde{\tau}$. The remaining terms of $\mathcal{C}_{\tilde{s}}$ must cancel in pairs, as in the case of Exercise 6.6.

7 From homology to pseudocycles

This section establishes Proposition 7.1 below. In the proof of Lemma 7.3, we construct a homomorphism from the subgroup of cycles in $\bar{S}_*(X)$ to $\mathcal{H}_*(X)$. We show that every $\mathbb{Z}$-homology cycle gives rise to a pseudocycle and every boundary between cycles gives rise to a pseudocycle equivalence. We use the conclusions of Exercises 6.6 and 6.7 to glue maps from standard simplicies into a continuous map from a manifold-like space. We then use Lemma 7.2 below to smooth out the map across the codimension 1 simplicies.

Proposition 7.1. *If X is a smooth manifold, there exists a homomorphism*

$$\Psi\colon H_*(X;\mathbb{Z}) \longrightarrow \mathcal{H}_*(X),$$

which is natural with respect to smooth maps.

Starting with a cycle $\{s\}$ as in Exercise 6.6, we glue the functions $f_j\circ\varphi_k$ together, where φ_k is the self-map of Δ^k provided by Lemma 7.2 below. These functions continue to satisfy the second equation in (6.14), i.e.

$$f_{j_2}\circ\varphi_k\circ\iota_{k,p_2}=f_{j_1}\circ\varphi_k\circ\iota_{k,p_1}\circ\tau_{(j_1,p_1),(j_2,p_2)} \qquad \forall\ \big((j_1,p_1),(j_2,p_2)\big)\in\mathcal{D}_s, \tag{7.1}$$

because $\varphi_k=\mathrm{id}$ on $\Delta^k-\mathrm{Int}\ \Delta^k$ by the first equation in (7.5). Using these modified functions insures that the glued map is smooth across the codimension 1 simplicies. The proof of Lemma 7.3 implements a construction suggested in [16, Section 7.1].

Starting with a chain $\{\tilde{s}\}$ as in Exercise 6.7, we glue the functions $\tilde{f}_j\circ\tilde{\varphi}_{k+1}\circ\varphi_{k+1}$ together, where $\tilde{\varphi}_{k+1}$ and φ_{k+1} are the self-maps of Δ^{k+1} provided by Lemma 7.2. If $i=0,1$ and $j=1,\ldots,N_i$, by the third equation in (7.6), the second equation in (7.5), and the first equation in (6.19)

$$\begin{aligned}\tilde{f}_{\tilde{j}_i(j)}\circ\tilde{\varphi}_{k+1}\circ\iota_{k+1,\tilde{p}_i(j)}\circ\tilde{\tau}_{(i,j)}=\tilde{f}_{\tilde{j}_i(j)}\circ\iota_{k+1,\tilde{p}_i(j)}\circ\varphi_k\circ\tilde{\tau}_{(i,j)}&=\tilde{f}_{\tilde{j}_i(j)}\circ\iota_{k+1,\tilde{p}_i(j)}\circ\tilde{\tau}_{(i,j)}\circ\varphi_k\\&=f_{i,j}\circ\varphi_k.\end{aligned}$$

Since $\varphi_{k+1}=\mathrm{id}$ on $\Delta^{k+1}-\mathrm{Int}\ \Delta^{k+1}$, it follows that

$$\tilde{f}_{\tilde{j}_i(j)}\circ\tilde{\varphi}_{k+1}\circ\varphi_{k+1}\circ\iota_{k+1,\tilde{p}_i(j)}\circ\tilde{\tau}_{(i,j)}=f_{i,j}\circ\varphi_k \qquad \forall\ j=1,\ldots,N_i,\ \ i=0,1. \tag{7.2}$$

Similarly, if $((j_1,p_1),(j_2,p_2))\in\mathcal{D}_{\tilde{s}}$, by the third equation in (7.6) used twice, the second equation in (6.17), and the second equation in (7.5),

$$\begin{aligned}\tilde{f}_{j_2}\circ\tilde{\varphi}_{k+1}\circ\iota_{k+1,p_2}&=\tilde{f}_{j_2}\circ\iota_{k+1,p_2}\circ\varphi_k=\tilde{f}_{j_1}\circ\iota_{k+1,p_1}\circ\tilde{\tau}_{(j_1,p_1),(j_2,p_2)}\circ\varphi_k\\&=\tilde{f}_{j_1}\circ\iota_{k+1,p_1}\circ\varphi_k\circ\tilde{\tau}_{(j_1,p_1),(j_2,p_2)}=\tilde{f}_{j_1}\circ\tilde{\varphi}_{k+1}\circ\iota_{k+1,p_1}\circ\tilde{\tau}_{(j_1,p_1),(j_2,p_2)}.\end{aligned}$$

Since $\varphi_{k+1}=\mathrm{id}$ on $\Delta^{k+1}-\mathrm{Int}\ \Delta^{k+1}$, it follows that

$$\tilde{f}_{j_2}\circ\tilde{\varphi}_{k+1}\circ\varphi_{k+1}\circ\iota_{k+1,p_2}=\tilde{f}_{j_1}\circ\tilde{\varphi}_{k+1}\circ\varphi_{k+1}\circ\iota_{k+1,p_1}\circ\tilde{\tau}_{(j_1,p_1),(j_2,p_2)} \quad \forall\ ((j_1,p_1),(j_2,p_2))\in\mathcal{D}_{\tilde{s}}. \tag{7.3}$$

Thus, the functions $\tilde{f}_j\circ\tilde{\varphi}_{k+1}\circ\varphi_{k+1}$ are the analogues (in the sense of Exercise 6.7) of the functions $\tilde{f}_j$ for the maps $f_{0,j}\circ\varphi_k$ and $f_{1,j}\circ\varphi_k$.

We continue with the notation set up at the beginning of Section 6. Define

$$\tilde{\pi}_p^k\colon\Delta^k-\{e_p\}\longrightarrow\Delta_p^k \qquad \text{by} \qquad \tilde{\pi}_p^k\Big(\sum_{q=0}^{q=k}t_qe_q\Big)=\frac{1}{1-t_p}\Big(\sum_{q\neq p}t_qe_q\Big).$$

Put

$$b_{k,p}=\iota_{k,p}(b_{k-1}), \qquad b'_{k,p}=\frac{1}{k+1}\bigg(b_k+\sum_{q\neq p}e_q\bigg).$$

The points $b_{k,p}$ and $b'_{k,p}$ are the barycenters of the $(k-1)$-simplex Δ_p^k and of the k-simplex spanned by b_k and the vertices of Δ_p^k; see Figure 2.1. Define a neighborhood of $\mathrm{Int}\,\Delta_p^k$ in Δ^k by

$$U_p^k=\big\{t_pb'_{k,p}+\sum_{\substack{0\le q\le k\\ q\neq p}}t_qe_q\colon t_p\ge0,\ \ t_q>0\ \forall q\neq p,\ \ \sum_{q=0}^{q=k}t_q=1\big\}.$$

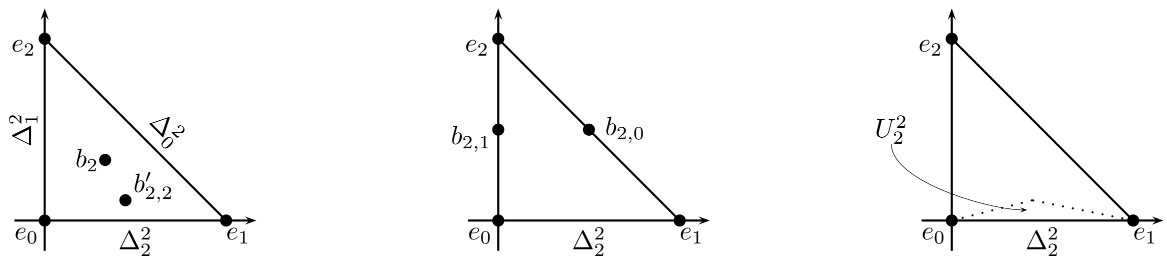


Figure 2.1: The standard 2-simplex and some of its distinguished subsets

If $p,q=0,1,\ldots,k$ and $p\neq q$, let

$$\Delta^k_{p,q}\equiv\Delta^k_p\cap\Delta^k_q$$

be the corresponding codimension 2 simplex. Define neighborhoods of $\operatorname{Int}\Delta^k_{p,q}$ in Δ^k by

$$\tilde{U}^k_{p,q}=\Big\{t_pb_{k,p}+t_qb_{k,q}+\sum_{\substack{0\le q\le k\\ r\neq p,q}}t_re_r:\,t_p,t_q\ge 0,\ t_r>0\ \forall\,r\neq p,q,\ \sum_{r=0}^{r=k}t_r=1\Big\},$$

$$U^k_{p,q}=\Big\{t_p\iota_{k,p}(b'_{k-1,\iota_{k,\iota^{-1}_{k,p}(q)}})+t_q\iota_{k,q}(b'_{k-1,\iota_{k,\iota^{-1}_{k,q}(p)}})+\sum_{\substack{0\le q\le k\\ q\neq p}}t_re_r:\,t_p,t_q\ge 0,\ t_r>0\,\forall r\neq p,q,\ \sum_{r=0}^{r=k}t_r=1\Big\};$$

see Figure 2.2. If

$$\sum_{r=0}^{k}t_re_r\in\tilde{U}^k_{p,q}\subset\Delta^k\qquad\text{with}\qquad t_r\ge 0,$$

then $t_p,t_q<t_r$ for all $r\neq p,q$. Thus,

$$\tilde{U}^k_{p_1,q_1}\cap\tilde{U}^k_{p_2,q_2}=\emptyset\qquad\text{if}\quad\{p_1,q_1\}\neq\{p_2,q_2\}.\tag{7.4}$$

Define a projection map

$$\tilde{\pi}^k_{p,q}\colon\Delta^k-\mathrm{CH}(e_p,e_q)\longrightarrow\Delta^k_{p,q}\qquad\text{by}\qquad\tilde{\pi}^k_{p,q}\Big(\sum_{r=0}^{r=k}t_re_r\Big)=\frac{1}{1-t_p-t_q}\Big(\sum_{r\neq p,q}t_re_r\Big).$$

Lemma 7.2. *If $k\ge 1$, Y is the $(k-2)$-skeleton of Δ^k, and $\tilde{Y}$ is the $(k-2)$-skeleton of Δ^{k+1}, there exist continuous functions*

$$\varphi_k\colon\Delta^k\longrightarrow\Delta^k\qquad\textit{and}\qquad\tilde{\varphi}_{k+1}\colon\Delta^{k+1}\longrightarrow\Delta^{k+1}$$

such that

(1) φ_k is smooth outside of Y and $\tilde{\varphi}_{k+1}$ is smooth outside of $\tilde{Y}$;

(2) for all $p=0,\ldots,k$ and $\tau\in\mathcal{S}_k$,

$$\varphi_k|_{U^k_p}=\tilde{\pi}^k_p|_{U^k_p}\qquad\textit{and}\qquad\varphi_k\circ\tau=\tau\circ\varphi_k;\tag{7.5}$$

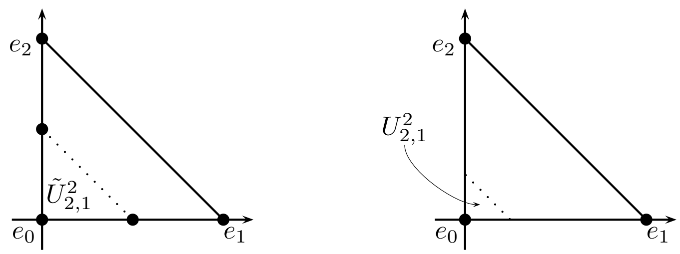


Figure 2.2: Open neighborhoods of codimension 2 simplicies

(3) for all $p,q=0,\ldots,k+1$ *with* $p\neq q$ *and* $\tilde\tau\in\mathcal{S}_{k+1}$,

$$\tilde\varphi_{k+1}|_{U^{k+1}_{p,q}}=\tilde\pi^{k+1}_{p,q}|_{U^{k+1}_{p,q}},\qquad \tilde\varphi_{k+1}\circ\tilde\tau=\tilde\tau\circ\tilde\varphi_{k+1},\quad \textit{and}\quad \tilde\varphi_{k+1}\circ\iota_{k+1,p}=\iota_{k+1,p}\circ\varphi_k. \tag{7.6}$$

Proof. (1) Choose a smooth function

$$\tilde\eta_{0,1}\colon \Delta^{k+1}-\Delta^{k+1}_{0,1}\cap\tilde Y\longrightarrow[0,1]$$

such that $\tilde\eta_{0,1}=1$ on $U^{k+1}_{0,1}$, $\tilde\eta_{0,1}=0$ outside of $\tilde U^{k+1}_{0,1}$, and $\tilde\eta_{0,1}$ is invariant under any permutation $\tilde\tau\in\mathcal{S}_{k+1}$ that preserves the set $\{0,1\}$. If $\tilde\tau\in\mathcal{S}_{k+1}$ is any permutation, let

$$\tilde\eta_{\tilde\tau(0),\tilde\tau(1)}=\tilde\eta_{0,1}\circ\tilde\tau^{-1}\colon \Delta^{k+1}-\Delta^{k+1}_{\tilde\tau(0),\tilde\tau(1)}\cap\tilde Y\longrightarrow[0,1].$$

By the assumptions on $\tilde\eta_{0,1}$, $\tilde\eta_{p,q}$ is a well-defined smooth function such that $\tilde\eta_{p,q}=1$ on $U^{k+1}_{p,q}$, $\tilde\eta_{p,q}=0$ outside of $\tilde U^{k+1}_{p,q}$, and

$$\tilde\eta_{\tilde\tau(p),\tilde\tau(q)}=\tilde\eta_{p,q}\circ\tilde\tau^{-1} \tag{7.7}$$

for all $\tilde\tau\in\mathcal{S}_{k+1}$ and distinct $p,q=0,\ldots,k+1$.

(2) Define

$$\tilde\varphi_{k+1}\colon \Delta^{k+1}\longrightarrow\Delta^{k+1}\qquad\text{by}\qquad \tilde\varphi_{k+1}(x)=x+\sum_{0\le p<q\le k+1}\tilde\eta_{p,q}(x)\cdot\big(\tilde\pi^{k+1}_{p,q}(x)-x\big).$$

Since $\tilde\pi^{k+1}_{p,q}$ restricts to the identity on $\Delta^{k+1}_{p,q}$ and $\tilde\eta_{p,q}$ vanishes on a neighborhood of $\mathrm{CH}^0(e_p,e_q)$, the function $\tilde\eta$ is well-defined, continuous everywhere, and smooth on $\Delta^{k+1}-\tilde Y$. By (7.4), $\tilde\varphi_{k+1}=\tilde\pi^{k+1}_{p,q}$ on $U^{k+1}_{p,q}$. By (7.7), for every $\tilde\tau\in\mathcal{S}_{k+1}$

$$\begin{aligned}\tilde\varphi_{k+1}\circ\tilde\tau&=\tilde\tau+\sum_{0\le p<q\le k+1}\tilde\eta_{p,q}\circ\tilde\tau\cdot\big(\tilde\pi^{k+1}_{p,q}\circ\tilde\tau-\tilde\tau\big)\\&=\tilde\tau+\sum_{0\le p<q\le k+1}\tilde\eta_{\tilde\tau^{-1}(p),\tilde\tau^{-1}(q)}\cdot\big(\tilde\tau\circ\tilde\pi^{k+1}_{\tilde\tau^{-1}(p),\tilde\tau^{-1}(q)}-\tilde\tau\big)\\&=\tilde\tau+\sum_{0\le p<q\le k+1}\tilde\eta_{p,q}\cdot\big(\tilde\tau\circ\tilde\pi^{k+1}_{p,q}-\tilde\tau\big)=\tilde\tau\circ\tilde\varphi_{k+1}.\end{aligned}$$

Thus, $\tilde\varphi_{k+1}$ satisfies the first two conditions in (7.6), as well as (1) above.

(3) We define φ_k by the third condition in (7.6). The function φ_k is independent of the choice of p and satisfies the second condition in (7.5) for the following reason. Suppose $p,q=0,\ldots,k+1$, $\tau\in\mathcal{S}_k$, and $\tilde\tau\in\mathcal{S}_{k+1}$ is defined by

$$\tilde\tau\circ\iota_{k+1,p}=\iota_{k+1,q}\circ\tau.$$

If $\varphi_{k,p}$ and $\varphi_{k,q}$ are the functions corresponding to p and q via the third equation in (7.6), then by the second equation in (7.6)

$$\begin{aligned}\iota_{k+1,q}\circ\tau\circ\varphi_{k,p}=\tilde\tau\circ\iota_{k+1,p}\circ\varphi_{k,p}&=\tilde\tau\circ\tilde\varphi_{k+1}\circ\iota_{k+1,p}=\tilde\varphi_{k+1}\circ\tilde\tau\circ\iota_{k+1,p}\\&=\tilde\varphi_{k+1}\circ\iota_{k+1,q}\circ\tau=\iota_{k+1,q}\circ\varphi_{k,q}\circ\tau.\end{aligned}$$

We conclude that

$$\tau\circ\varphi_{k,p}=\varphi_{k,q}\circ\tau\qquad\forall\ p,q=0,\ldots,k+1,\ \tau\in\mathcal{S}_k.$$

The function φ_k satisfies the first condition in (7.5) because

$$\begin{gathered}\iota_{k+1,p}\big(U_p^k\big)=U_{p,p+1}^{k+1}\cap\Delta_{p,p+1}^{k+1}\qquad\text{and}\\ \iota_{k+1,p}\circ\varphi_k=\tilde\varphi_{k+1}\circ\iota_{k+1,p}=\tilde\pi_{p,p+1}^{k+1}\circ\iota_{k+1,p}=\iota_{k+1,p}\circ\tilde\pi_p^k\quad\text{on}\quad U_p^k.\end{gathered}$$

Finally, φ_k satisfies (1) because $\tilde\varphi_{k+1}$ does. □

Lemma 7.3. *If X is a smooth manifold, every integral k-cycle in X, based on $C^\infty(\Delta^k;X)$, determines an element of $\mathcal{H}_k(X)$.*

Proof. If $k=0$, this is obvious. Suppose $k\ge1$ and

$$s\equiv\sum_{j=1}^{j=N}f_j$$

determines a cycle in $\bar S_k(X)$. Let $\mathcal{D}_s$ be the set provided by Exercise 6.6 and let $\tau\colon\mathcal{D}_s\longrightarrow\mathcal{S}_{k-1}$ be the corresponding map. Let

$$\begin{gathered}M'=\Big(\bigsqcup_{j=1}^{j=N}\{j\}\times\Delta^k\Big)/\sim,\qquad\text{where}\\ \big(j_1,\iota_{k,p_1}(\tau_{(j_1,p_1),(j_2,p_2)}(t))\big)\sim\big(j_2,\iota_{k,p_2}(t)\big)\quad\forall\ \big((j_1,p_1),(j_2,p_2)\big)\in\mathcal{D}_s,\ t\in\Delta^{k-1}.\end{gathered}$$

Let π be the quotient map. Define

$$F\colon M'\longrightarrow X\qquad\text{by}\quad F\big([j,t]\big)=f_j\big(\varphi_k(t)\big).\tag{7.8}$$

This map is well-defined by (7.1) and continuous by the universal property of the quotient topology; see [13, Theorem 22.2]. Let M be the complement in M' of the set

$$\pi\Big(\bigsqcup_{j=1}^{j=N}\{j\}\times Y\Big),$$

where Y is the $(k-2)$-skeleton of Δ^k. By continuity of F, compactness of M', and the first equation in (7.5),

$$\mathrm{Bd}\, F|_M = F(M'-M) = \bigcup_{j=1}^{j=N} f_j\big(\varphi_k(Y)\big) = \bigcup_{j=1}^{j=N} f_j(Y). \tag{7.9}$$

Since $f_j|_{\mathrm{Int}\,\sigma}$ is smooth for all $j=1,\ldots,N$ and all simplices $\sigma\subset\Delta^k$, $\mathrm{Bd}\,F|_M$ has dimension at most $k-2$ by (7.9). Thus, $F|_M$ is a k-pseudocycle, provided M is a smooth oriented manifold and $F|_M$ is a smooth map. □

Exercise 7.4. Complete the proof of Lemma 7.3.

The pseudocycle $F|_M$ constructed above depends on the choice of $\mathcal{D}_s$ and τ. However, as the next lemma shows, the image of $F|_M$ in $\mathcal{H}_k(X)$ depends only on $[\{s\}]$.

Lemma 7.5. *Under the construction of Lemma 7.3, homologous k-cycles determine the same equivalence class of pseudocycles in $\mathcal{H}_k(X)$.*

Proof. (1) If $k=0$, this is obvious. Suppose $k>0$ and

$$s_0 \equiv \sum_{j=1}^{j=N_0} f_{0,j} \qquad \text{and} \qquad s_1 \equiv \sum_{j=1}^{j=N_1} f_{1,j}$$

determine two homologous k-cycles in $\bar{S}_k(X)$. Let $\mathcal{D}_{s_0}$ and $\mathcal{D}_{s_1}$ be the sets provided by Exercise 6.6 and let τ_0 and τ_1 be the corresponding maps into $\mathcal{S}_{k-1}$. Denote by (M_0', M_0, F_0) and (M_1', M_1, F_1) the triples constructed in the proof of Lemma 7.3 corresponding to s_0 and s_1. Choose

$$\tilde{s} = \sum_{j=1}^{j=\tilde{N}} \tilde{f}_j \in S_{k+1}(X) \qquad \text{s.t.} \qquad \bar{\partial}\{\tilde{s}\} = \{s_1\} - \{s_0\} \in \bar{S}_k(X).$$

Denote by $\mathcal{C}_{\tilde{s}}^{(0)}$, $\mathcal{C}_{\tilde{s}}^{(1)}$, $\mathcal{D}_{\tilde{s}}$, $(\tilde{j}_i, \tilde{p}_i, \tilde{\tau}_i)$, and $\tilde{\tau}$ the corresponding objects of Exercise 6.7.

(2) Let $\mathbb{I}=[0,1]$ as before. Put

$$\tilde{M}' = \Big(\bigsqcup_{j=1}^{j=\tilde{N}} \{j\}\times\Delta^{k+1} \sqcup \bigsqcup_{i=0,1}\{i\}\times\mathbb{I}\times M_i'\Big)\Big/\sim, \qquad \text{where}$$

$$\big(j_1, \iota_{k+1,p_1}(\tilde{\tau}_{(j_1,p_1),(j_2,p_2)}(t))\big) \sim \big(j_2, \iota_{k+1,p_2}(t)\big) \quad \forall\ \big((j_1,p_1),(j_2,p_2)\big)\in\tilde{\mathcal{D}}_{\tilde{s}},\ t\in\Delta^k,$$
$$\big(i, 1-i, \pi(j,t)\big) \sim \big(\tilde{j}_i(j), \iota_{k+1,\tilde{p}_i(j)}(\tilde{\tau}_{i,j}(t))\big) \quad \forall\ t\in\Delta^k,\ j=1,\ldots,N_i,\ i=0,1.$$

Let

$$\tilde{\pi}\colon \bigsqcup_{j=1}^{j=\tilde{N}} \{j\}\times\Delta^{k+1} \sqcup \bigsqcup_{i=0,1}\{i\}\times\mathbb{I}\times M_i' \longrightarrow \tilde{M}'$$

be the quotient map. Define

$$\tilde{F}\colon \tilde{M}' \longrightarrow X \qquad \text{by} \qquad \begin{array}{ll} \tilde{F}\big([j,t]\big) = \tilde{f}_j\big(\tilde{\varphi}_{k+1}(\varphi_{k+1}(t))\big) & \forall\ t\in\Delta^{k+1},\ j=1,\ldots,\tilde{N}; \\ \tilde{F}\big([i,s,x]\big) = F_i(x) & \forall\ s\in\mathbb{I},\ x\in M_i',\ i=0,1. \end{array}$$

This map is well-defined by (7.2), (7.3), and (7.8) and is continuous by the universal property of the quotient topology. Let $\tilde{M}$ be the complement in $\tilde{M}'$ of the set

$$\tilde{\pi}\Big(\bigsqcup_{j=1}^{j=\tilde{N}}\{j\}\times\tilde{Y}\sqcup\bigsqcup_{i=0,1}\{i\}\times\mathbb{I}\times(M_i'-M_i)\Big),$$

where $\tilde{Y}$ is the $(k-1)$-skeleton of Δ^{k+1}. By continuity of $\tilde{F}$, compactness of $\tilde{M}'$, and the first equation in (7.6),

$$\mathrm{Bd}\,\tilde{F}|_{\tilde{M}}=\tilde{F}(\tilde{M}'-\tilde{M})=\bigcup_{j=1}^{j=\tilde{N}}\tilde{f}_j\big(\tilde{\varphi}_{k+1}(\varphi_{k+1}(\tilde{Y}))\big)\cup\bigcup_{i=0,1}f_{i,j}\big(\varphi_k(Y)\big)\;=\bigcup_{j=1}^{j=\tilde{N}}\tilde{f}_j\big(\tilde{Y}\big). \tag{7.10}$$

Since $\tilde{f}_j|_{\mathrm{Int}\,\sigma}$ is smooth for all $j=1,\ldots,\tilde{N}$ and all simplices $\sigma\subset\Delta^{k+1}$, $\mathrm{Bd}\,\tilde{F}|_{\tilde{M}}$ has dimension at most $k-1$ by (7.10). Thus, $\tilde{F}|_{\tilde{M}}$ is a pseudocycle equivalence between $F_0|_{M_0}$ and $F_1|_{M_1}$, provided $\tilde{M}$ is a smooth oriented manifold, $\tilde{F}|_{\tilde{M}}$ is a smooth map, and $\partial(\tilde{F}|_{\tilde{M}})=F_1|_{M_1}-F_0|_{M_0}$. □

Exercise 7.6. Complete the proof of Lemma 7.5.

Proceeding as in the proof of Lemma 7.3, we can turn $\tilde{s}$ into a pseudocycle equivalence $(\tilde{M}^*,\tilde{F})$ between some pseudocycles (M_0^*,F_0) and (M_1^*,F_1) by gluing across codimension 1 faces. Unfortunately, M_0^* and M_1^* are not the entire manifolds M_0 and M_1; they are missing the $(k-1)$-simplices of M_0 and M_1. This issue is resolved in (2) of the proof of Lemma 7.5 by adding collars to $\tilde{M}^*$: $(n+1)$-manifolds that begin with M_i^* and end with M_i^*.

8 From pseudocycles to homology

This section establishes Proposition 8.1 below. In the proofs of Lemmas 8.6 and 8.7, we construct a cycle and a boundary in the oriented singular complex $\bar{S}_*(X)$ out of a pseudocycle and a pseudocycle equivalence, respectively. In both cases, we use arbitrary small neighborhoods of the boundaries of these maps provided by Corollary 8.4 below; in a sense, these neighborhoods are analogous to tubular neighborhoods of embedded submanifolds.

Proposition 8.1. *If X is a smooth manifold, there exists a homomorphism*

$$\Phi\colon\mathcal{H}_*(X)\longrightarrow H_*(X;\mathbb{Z}),$$

which is natural with respect to smooth maps.

If K is a simplicial complex and σ is a simplex in K, the star of σ in K is the union of the subsets $\mathrm{Int}\,\sigma'$ taken over the simplices $\sigma'\in K$ such that $\sigma\subset\sigma'$; see [14, Section 62]. The barycentric subdivision of K is the simplicial complex $\mathrm{sd}\,K$ obtained from K by subdividing each simplex σ of K into simplicies with vertices at the barycenters $b_{\sigma'}$ of all simplicies $\sigma'\subset\sigma$; see [14, Section 17].

If X is a smooth manifold, a topological embedding $\mu\colon\Delta^l\longrightarrow X$ is a smooth embedding if there exist an open neighborhood Δ^l_μ of Δ^l in $\mathbb{R}^l$ and a smooth embedding $\tilde{\mu}:\Delta^l_\mu\longrightarrow X$ so that

$\tilde{\mu}|_{\Delta^l}=\mu$. A smooth triangulation of a smooth manifold X is a pair $T=(K,\eta)$ consisting of a simplicial complex and a homeomorphism $\eta\colon |K|\longrightarrow X$ such that

$$\eta_\sigma \equiv \eta\circ\iota_\sigma\colon \Delta^l \longrightarrow X$$

is a smooth embedding for every l-simplex σ in K and $l\in\mathbb{Z}^{\ge 0}$.

If $h\colon Y\longrightarrow X$ is a smooth map and k is a nonnegative integer, put

$$N_k(h)=\big\{y\in Y\colon \mathrm{rk}\,\mathrm{d}_y h\le k\big\}.$$

Lemma 8.2. *Let $h\colon Y\longrightarrow X$ be a smooth map. For every $k\in\mathbb{Z}^{\ge 0}$, there exists a neighborhood U of $h\big(N_k(h)\big)$ in X such that*

$$H_l(U;\mathbb{Z})=0 \qquad \forall\, l>k.$$

Proof. By Proposition 10.1, there exists a smooth triangulation $T=(K,\eta)$ of X such that the smooth map h is transverse to $\eta_\sigma|_{\mathrm{Int}\,\sigma}$ for all $\sigma\in K$. In particular,

$$h\big(N_k(h)\big)\subset \bigcup_{\sigma\in K,\dim\sigma\ge n-k}\eta(\mathrm{Int}\,\sigma)=\bigcup_{\sigma\in K,\dim\sigma\ge n-k}\eta\big(\mathrm{St}(b_\sigma,\mathrm{sd}\,K)\big), \tag{8.1}$$

where $n=\dim X$. Note that

$$\mathrm{St}(b_\sigma,\mathrm{sd}\,K)\cap\mathrm{St}(b_{\sigma'},\mathrm{sd}\,K)=\emptyset$$

unless $\sigma\subset\sigma'$ or $\sigma'\subset\sigma$. Furthermore, if $\sigma_1\subset\ldots\subset\sigma_m$,

$$\mathrm{St}(b_{\sigma_1},\mathrm{sd}\,K)\cap\ldots\cap\mathrm{St}(b_{\sigma_m},\mathrm{sd}\,K)=\mathrm{St}(b_{\sigma_1}\ldots b_{\sigma_m},\mathrm{sd}\,K);$$

the last set is contractible. Put

$$U'_m=\bigcup_{\sigma\in K,\dim\sigma=m}\mathrm{St}(b_\sigma,\mathrm{sd}\,K).$$

Thus, $U'_{l_m}\cap\ldots\cap U'_{m_j}$ is a disjoint union of contractible open sets in $|K|$. Let

$$U_m=\eta(U'_m),\quad m=n-k,\ldots,n;\qquad U=\bigcup_{m=n-k}^{n}U_m.$$

Since $\eta\colon |K|\longrightarrow X$ is a homeomorphism, $U_{m_1}\cap\ldots\cap U_{m_j}$ is a disjoint union of contractible open subsets of X. By (8.1), $h\big(N_k(h)\big)\subset U$. By Exercise 8.3 below, $H_l(U)=0$ for all $l>k$. □

Exercise 8.3. Let $\big\{U_m\big\}_{m=0}^{m=k}$ be an open cover of topological space X such that

$$H_l\big(U_{m_1}\cap\ldots\cap U_{m_j};\mathbb{Z}\big)=0 \qquad \forall\ l>0,\ m_1,\ldots,m_j=0,\ldots,k.$$

Use the Mayer-Vietoris Theorem [14, p186] to show that $H_l(U)=0$ for all $l>k$.

Corollary 8.4. *If $h\colon Y\longrightarrow X$ is a smooth map and W is an open neighborhood of a subset A of $\mathrm{Im}\,h$ in X, there exists a neighborhood U of A in W such that*

$$H_l(U;\mathbb{Z})=0 \qquad \forall\ l>\dim Y.$$

Proof. In Lemma 8.2, take $X=W$, $Y=h^{-1}(W)$, and $k=\dim Y$. □

Remark 8.5. It may not be true that $H_l(A;\mathbb{Z})=0$ if $l>\dim Y$. For example, let A be the subset of $X=\mathbb{R}^N$ consisting of countably many k-spheres of radii tending to 0 and having a single point in common. If $k\geq 2$, the set A has infinitely many nonzero homology groups; see [2].

Lemma 8.6. *Every k-pseudocycle determines a class in $H_k(X;\mathbb{Z})$.*

Proof. (1) Suppose $h\colon M\longrightarrow X$ is a k-pseudocycle and $f\colon N\longrightarrow X$ a smooth map such that

$$\dim N = k-2 \qquad \text{and} \qquad \mathrm{Bd}\, h \subset \mathrm{Im} f.$$

By Corollary 8.4, there exists an open neighborhood U of $\mathrm{Bd}\, h$ in X such that

$$H_l(U;\mathbb{Z}) = 0 \qquad \forall\, l > k-2.$$

Let $K=M-h^{-1}(U)$. Since the closure of $h(M)$ is compact in X, K is a compact subset of M by definition of $\mathrm{Bd}\, h$. Let V be an open neighborhood of K in M such that $\bar{V}$ is a compact manifold with boundary. It inherits an orientation from the orientation of M and thus defines a homology

$$[\bar{V},\partial\bar{V}] \in H_k(\bar{V},\partial\bar{V};\mathbb{Z}).$$

Put

$$[h] = h_*\big([\bar{V},\partial\bar{V}]\big) \in H_k(X,U;\mathbb{Z}) \approx H_k(X;\mathbb{Z}), \tag{8.2}$$

where

$$h_*\colon H_k(\bar{V},\partial\bar{V};\mathbb{Z})\longrightarrow H_k(X,U;\mathbb{Z}) \tag{8.3}$$

is the homology homomorphism induced by h. The isomorphism in (8.2) is induced by inclusion. It is an isomorphism by the assumption on the homology of U as follows from the long exact sequence in homology for the pair (X,U).

(2) The homology class $[h]$ is independent of the choice of V. Suppose V' is another choice such that $\bar{V}\subset V'$. Choose a triangulation of $\bar{V}'$ extending some triangulation of $\partial\bar{V}\cup\partial\bar{V}'$; such a triangulation exists by Section 16 in [14]. The cycles

$$h_*([\bar{V},\partial\bar{V}]), h_*([\bar{V}',\partial\bar{V}']) \in H_k(X,U;\mathbb{Z})$$

then differ by singular simplices lying in U and thus are the same; see Remark 6.5.

(3) The cycle $[h]$ is also independent of the choice of U. Suppose $U'\subset U$ is another choice. By (2), it can be assumed that V and V' chosen as in (1) are the same. Since the isomorphism in (8.2) is the composite of isomorphisms

$$H_k(X;\mathbb{Z}) \longrightarrow H_k(X,U';\mathbb{Z}) \longrightarrow H_k(X,U;\mathbb{Z})$$

induced by inclusions and the homomorphism (8.3) is the composition

$$H_k(\bar{V},\partial\bar{V};\mathbb{Z}) \longrightarrow H_k(X,U';\mathbb{Z}) \longrightarrow H_k(X,U;\mathbb{Z}),$$

the homology classes obtained in $H_k(X;\mathbb{Z})$ from U and U' are equal. Finally, if U and U' are two arbitrary choices of open sets in (1), by Corollary 8.4 there exists a third choice $U''\subset U\cap U'$. □

Lemma 8.7. *Equivalent k-pseudocycles determine the same class in* $H_k(X,\mathbb{Z})$.

Proof. Suppose $h_i\colon M_i \longrightarrow X$, $i=0,1$, are two equivalent k-pseudocycles and $\tilde{h}\colon \tilde{M} \longrightarrow X$ is an equivalence between them. In particular, $\tilde{M}$ is oriented,

$$\partial\tilde{M} = M_1 - M_0, \qquad \text{and} \qquad \tilde{h}|_{M_i} = h_i.$$

Let $\tilde{U}$ be an open neighborhood of $\mathrm{Bd}\,\tilde{h}$ in X such that

$$H_l(\tilde{U};\mathbb{Z}) = 0 \qquad \forall\ l > k-1.$$

Let U_i be an open neighborhood of $\mathrm{Bd}\,h_i \subset \mathrm{Bd}\,\tilde{h}$ in $\tilde{U}$ such that

$$H_l(U_i;\mathbb{Z}) = 0 \qquad \forall\ l > k-2,$$

as provided by Corollary 8.4. Let $V_i \subset M_i$ be a choice of an open set as in (1) of the proof of Lemma 8.6. For $i=0,1$, choose a triangulation of M_i that extends a triangulation of $\partial\bar{V}_i$. Extend these two triangulations to a triangulation $\tilde{T}=(\tilde{K},\tilde{\eta})$ of $\tilde{M}$. Let K be a finite sub-complex of $\tilde{K}$ such that

$$V_0, V_1 \subset \tilde{\eta}(|K|) \qquad \text{and} \qquad \tilde{M} - \tilde{h}^{-1}(\tilde{U}) \subset \tilde{\eta}(\mathrm{Int}\,|K|).$$

Such a subcomplex exists because $\tilde{h}(\tilde{M})$ is a pre-compact subset of X and thus $\tilde{M}-\tilde{h}^{-1}(\tilde{U})$ is a compact subset of $\tilde{M}$. Put

$$K_i = \big\{\sigma \in K : \eta(\sigma) \subset \bar{V}_i\big\} \qquad \text{for} \ \ i = 0,1.$$

By the proof of Lemma 8.6, $(K_i, \tilde{h}\circ\tilde{\eta}|_{|K_i|})$ determines the homology class $[h_i] \in H_k(X,U_i;\mathbb{Z})$. Let $[h_i']$ denote its image in $H_k(X,\tilde{U};\mathbb{Z})$ under the homomorphism induced by inclusion. The above assumptions on K imply that

$$\partial(K, \tilde{h}\circ\tilde{\eta}|_K) = (K_1, \tilde{h}\circ\tilde{\eta}|_{K_1}) - (K_0, \tilde{h}\circ\tilde{\eta}|_{K_0})$$

in $\bar{S}(M,\tilde{U})$. Thus,

$$[h_0'] = [h_1'] \in H_k(X,\tilde{U};\mathbb{Z}),$$

and this class lies in the image of the homomorphism

$$H_k(X;\mathbb{Z}) \longrightarrow H_k(X,\tilde{U};\mathbb{Z}) \tag{8.4}$$

induced by inclusion. This map is equal to the composites

$$H_k(X;\mathbb{Z}) \longrightarrow H_k(X,U_0;\mathbb{Z}) \longrightarrow H_k(X,\tilde{U};\mathbb{Z}),$$
$$H_k(X;\mathbb{Z}) \longrightarrow H_k(X,U_1;\mathbb{Z}) \longrightarrow H_k(X,\tilde{U};\mathbb{Z}).$$

Since $H_k(\tilde{U};\mathbb{Z})=0$, the homomorphism (8.4) is injective. Thus, $[h_0]$ and $[h_1]$ come from the same element of $H_k(X;\mathbb{Z})$. □

9 Isomorphism of homology theories

This section concludes the proof of the first part of Theorem 5.4. We show that $\Phi\circ\Psi$ is an isomorphism and Ψ is injective; see Lemmas 9.1 and 9.2, respectively.

Lemma 9.1. *If X is a smooth manifold, the composition*

$$\Phi\circ\Psi\colon H_*(X;\mathbb{Z})\longrightarrow\mathcal{H}_*(X)\longrightarrow H_*(X;\mathbb{Z})$$

is the identity map on $H_*(X;\mathbb{Z})$.

Proof. Suppose

$$\{s\}=\sum_{j=1}^{N}\{f_j\}\in\bar{S}_k(X)$$

is a cycle and $F\colon M\longrightarrow X$ is a pseudocycle corresponding to s via the construction of Lemma 7.3. Recall that M is the complement of the $(k-2)$-simplices in a compact space M' and F is the restriction of a continuous map $F'\colon M'\longrightarrow X$ induced by the maps

$$f_j\circ\varphi_k\colon\Delta^k\longrightarrow X,\qquad j=1,\ldots,N.$$

Since φ_k is homotopic to the identity on Δ^k, with boundary fixed,

$$f_j\circ\varphi_k-f_j\in\partial S_{k+1}(X)\qquad\forall\ j=1,\ldots,N. \tag{9.1}$$

Let U be a neighborhood of $\mathrm{Bd}\,F$ such that

$$H_l(U;\mathbb{Z})=0\qquad\forall\ l>k-2.$$

Put $K=M-f^{-1}(\varphi_k^{-1}(U))$. Let V be a pre-compact neighborhood of K such that $(\bar{V},\partial\bar{V})$ is a smooth manifold with boundary. Choose a triangulation $T=(K,\eta)$ of $(\bar{V},\partial\bar{V})$ such that every k-simplex of T is contained in a set of the form $\pi(\{j\}\times\Delta^k)$ for some $j=1,\ldots,N$, where π is as in the proof of Lemma 7.3. For each $j=1,\ldots,N$, put

$$K_j=\big\{\sigma\in K\colon\eta(\sigma)\subset\pi(\{j\}\times\Delta^k)\big\},\qquad K_j^{\mathrm{top}}=\big\{\sigma\in K_j\colon\dim\sigma=k\big\}.$$

Let $\tilde{T}_j=(\tilde{K}_j,\eta_j)$ be a triangulation of a subset of Δ^k that along with K_j gives a triangulation of Δ^k. Put

$$\tilde{K}_j^{\mathrm{top}}=\big\{\sigma\in\tilde{K}_j\colon\dim\sigma=k\big\}.$$

By definition of T,

$$f_j\circ\varphi_k\big(\eta_j(\sigma)\big)\subset U\qquad\forall\ \sigma\in\tilde{K}_j^{\mathrm{top}}. \tag{9.2}$$

Furthermore, by (9.1)

$$\begin{aligned}\{s\}&=\sum_{\sigma\in K^{\mathrm{top}}}\{f_j\circ\varphi_k\circ\eta\circ l_\sigma\}\\&=\sum_{j=1}^{N}\sum_{\sigma\in K_j^{\mathrm{top}}}\{f_j\circ\varphi_k\circ\eta\circ l_\sigma\}+\sum_{j=1}^{N}\sum_{\sigma\in\tilde{K}_j^{\mathrm{top}}}\{f_j\circ\varphi_k\circ\tilde{\eta}_j\circ l_\sigma\}\qquad\mathrm{mod}\ \bar{\partial}\bar{S}_{k+1}(X),\end{aligned} \tag{9.3}$$

since subdivisions of cycles do not change the homology class. By the proof of Lemma 8.6, the first sum on the right-hand side of (9.3) represents $[F]$ in $\bar{S}_k(X,U)$. By (9.2), the second sum lies in $\bar{S}_k(U)$. Since the sum of the two terms is a cycle in $\bar{S}_k(X)$, it must represent $[F]$ in $\bar{S}_k(X)$. Thus,

$$\{F\} = \{s\} \in H_k(X;\mathbb{Z}),$$

and the claim follows. □

Lemma 9.2. *If X is a smooth manifold, the homomorphism $\Phi\colon \mathcal{H}_*(X) \longrightarrow H_*(X;\mathbb{Z})$ is injective.*

Proof. Suppose a k-pseudocycle $h: M' \longrightarrow X$ determines the zero homology class via the construction of Lemma 8.6. We show that a modification of h is the boundary of a smooth map $\tilde{F}: \tilde{M} \longrightarrow X$ in the sense of pseudocycles. Since M' need not be compact, $\tilde{M}$ may need to be constructed from infinitely many $(k+1)$-simplices. This is achieved as the limit of finite stages $\tilde{M}_i$, so that as $i\in\mathbb{Z}^+$ increases M' is the pseudocycle boundary of $\tilde{M}_i$ "modulo" smaller and smaller neighborhoods U_i of $\mathrm{Bd}\, h$. As in the proof of Lemma 7.5, we also need to attach a collar to the $(k+1)$-manifold $\tilde{M}^*$ obtained directly from a bounding chain.

(1) It can be assumed that $k\ge 1$; otherwise, there is nothing to prove. Let $\{U_i\}_{i=1}^{\infty}$ be a sequence of open neighborhoods of $\mathrm{Bd}\, h$ in X such that $\bar{U}_i\subset X$ is compact,

$$U_{i+1} \subset U_i, \qquad \bigcap_{i=1}^{\infty} U_i = \mathrm{Bd}\, h, \quad \text{and} \quad H_l(U_i;\mathbb{Z}) = 0 \quad \forall\, l > k-2.$$

Existence of such a collection follows from Corollary 8.4 and metrizability of any manifold. Let $\{V_i\}_{i=1}^{\infty}$ be a corresponding collection of open sets in M' as in (1) of the proof of Lemma 8.6. It can be assumed that $\bar{V}_i \subset V_{i+1}$. Choose a triangulation $T=(K,\eta)$ of M' that extends a triangulation of $\bigcup_{i=1}^{\infty} \partial \bar{V}_i$. Let

$$K^{\mathrm{top}} = \{\sigma\in K : \dim\sigma = k\}, \qquad \mathcal{C}_\eta = \{(\sigma,p) : \sigma\in K^{\mathrm{top}},\ p=0,\ldots,k\}.$$

For each $\sigma\in K^{\mathrm{top}}$, let

$$\iota_\sigma\colon \Delta^k \longrightarrow \sigma \subset |K| \subset \mathbb{R}^\infty$$

be a linear map such that $\eta_\sigma \equiv \eta\circ\iota_\sigma$ is orientation-preserving. Put

$$f_\sigma = h\circ\eta_\sigma \qquad \forall\, \sigma\in K^{\mathrm{top}} \qquad \text{and}$$
$$\mathcal{D}_\eta = \{((\sigma_1,p_1),(\sigma_2,p_2))\in\mathcal{C}_\eta\times\mathcal{C}_\eta : (\sigma_1,p_1)\neq(\sigma_2,p_2),\ \iota_{\sigma_1}(\Delta^k_{p_1}) = \iota_{\sigma_2}(\Delta^k_{p_2})\}.$$

For each $((\sigma_1,p_1),(\sigma_2,p_2))\in\mathcal{D}_\eta$, define

$$\tau_{(\sigma_1,p_1),(\sigma_2,p_2)} \in \mathcal{S}_{k-1} \qquad \text{by} \qquad \iota_{\sigma_2}\circ\iota_{k,p_2} = \iota_{\sigma_1}\circ\iota_{k,p_1}\circ\tau_{(\sigma_1,p_1),(\sigma_2,p_2)}.$$

Since K is an oriented simplicial complex,

$$\mathcal{D}_\eta \subset \mathcal{C}_\eta\times\mathcal{C}_\eta \qquad \text{and} \qquad \tau\colon \mathcal{D}_\eta \longrightarrow \mathcal{S}_{k-1}$$

satisfy (1)-(3) of Exercise 6.6. Furthermore, M' is the topological space corresponding to $(\mathcal{C}_\eta,\mathcal{D}_\eta,\tau)$ via the construction of Lemma 7.3 and h is the continuous map described by

$$h|_{\pi(\sigma\times\Delta^k)} = f_\sigma.$$

As in the proof of Lemma 7.3, let M be the complement of the $(k-2)$-simplices in M'; the pseudocycles h and $h|_M$ are equivalent. Since φ_k is homotopic to the identity on Δ^k with boundary fixed, the pseudocycle $h|_M$ is in turn equivalent to the pseudocycle $F|_M$, where as in the proof of Lemma 7.3

$$F\colon M' \longrightarrow X, \qquad F\circ\eta_\sigma = f_\sigma\circ\varphi_k.$$

(2) For each $i\geq 1$, let

$$K_i^{\rm top} = \big\{\sigma\in K^{\rm top}\colon \eta(\sigma)\subset\bar{V}_i\big\}, \qquad \mathcal{C}_{\eta;i} = \big\{(\sigma,p)\in\mathcal{C}_\eta : \sigma\in K_i^{\rm top}\big\}, \qquad \mathcal{D}_{\eta;i} = \mathcal{D}_\eta\cap(\mathcal{C}_{\eta;i}\times\mathcal{C}_{\eta;i}).$$

By construction of $\Phi(h)$, for every $i\geq 1$ there exists a singular chain

$$s_i \equiv \sum_{j=1}^{N_i} f_{i,j} \in S_k(U_i) \qquad \text{s.t.} \qquad \sum_{\sigma\in K_i^{\rm top}}\{h\circ\eta_\sigma\} + \{s_i\}$$

is a cycle in $\bar{S}_k(X)$ representing $\Phi(h)$. Similarly to Exercise 6.6, there exist a symmetric subset

$$\mathcal{D}_i \subset (\mathcal{C}_{\eta;i}\sqcup\mathcal{C}_{s_i})\times(\mathcal{C}_{\eta;i}\sqcup\mathcal{C}_{s_i})$$

disjoint from the diagonal and a map $\tau_i\colon\mathcal{D}_i\longrightarrow\mathcal{S}_{k-1}$ such that

(1) $\mathcal{D}_{\eta;i}\subset\mathcal{D}_i$ and $\tau_i|_{\mathcal{D}_{\eta;i}}=\tau|_{\mathcal{D}_{\eta;i}}$;

(2) the projection map $\mathcal{D}_i \longrightarrow \mathcal{C}_{\eta;i}\sqcup\mathcal{C}_{s_i}$ on either coordinate is a bijection;

(3) for all $((j_1,p_1),(j_2,p_2))\in\mathcal{D}_i$,

$$\tau_{(j_2,p_2),(j_1,p_1)} = \tau^{-1}_{(j_1,p_1),(j_2,p_2)}, \qquad f_{i,j_2}\circ\iota_{k,p_2} = f_{i,j_1}\circ\iota_{k,p_1}\circ\tau_{(j_1,p_1),(j_2,p_2)},$$
$$\text{and} \qquad \text{sign}\,\tau_{(j_1,p_1),(j_2,p_2)} = -(-1)^{p_1+p_2},$$

where $f_{i,\sigma}\equiv f_\sigma$ for all $\sigma\in K_i^{\rm top}$.

(3) By (2), for each $i\geq 2$

$$\sum_{\sigma\in K_i^{\rm top}-K_{i-1}^{\rm top}}\{h\circ\eta_\sigma\} + \{s_i\} - \{s_{i-1}\} \in \bar{S}_k(U_{i-1})$$

is a cycle. Since $H_k(U_{i-1};\mathbb{Z})=0$, it is a boundary. If $i=1$, this conclusion is still true with $U_0=X$, $K_0^{\rm top}=\emptyset$, and $s_0=0$, since $\Phi(h)=0$ by assumption. Choose

$$\tilde{s}_i \equiv \sum_{j=1}^{\tilde{N}_i}\tilde{f}_{i,j} \in S_{k+1}(U_{i-1}) \qquad \text{s.t.} \qquad \sum_{\sigma\in K_i^{\rm top}-K_{i-1}^{\rm top}}\{h\circ\eta_\sigma\} + \{s_i\} - \{s_{i-1}\} = \bar{\partial}\big\{\tilde{s}_i\big\} \in \bar{S}_k(U_{i-1}).$$

Similarly to Exercise 6.7, there exist

$$\tilde{\mathcal{C}}_i^{(0)} \subset \tilde{\mathcal{C}}_i \equiv \bigsqcup_{i'=1}^{i'=i}\mathcal{C}_{\tilde{s}_{i'}},$$

a symmetric subset $\tilde{\mathcal{D}}_i\subset\tilde{\mathcal{C}}_i\times\tilde{\mathcal{C}}_i$ disjoint from the diagonal, and maps

$$\tilde{\tau}_i\colon \tilde{\mathcal{D}}_i \longrightarrow \mathcal{S}_k, \quad \big((j_1,p_1),(j_2,p_2)\big) \longrightarrow \tilde{\tau}_{i,((j_1,p_1),(j_2,p_2))},$$
$$(\tilde{j}_i,\tilde{p}_i)\colon K_i^{\mathrm{top}}\sqcup\{1,\ldots,N_i\} \longrightarrow \tilde{\mathcal{C}}_i^{(0)}, \quad \text{and} \quad \tilde{\tau}_i\colon K_i^{\mathrm{top}}\sqcup\{1,\ldots,N_i\} \longrightarrow \mathcal{S}_k, \quad j \longrightarrow \tilde{\tau}_{(i,j)},$$

such that

(1) $\tilde{\mathcal{D}}_i\subset\tilde{\mathcal{D}}_{i+1}$, $\tilde{\tau}_{i+1}|_{\tilde{\mathcal{D}}_i}=\tilde{\tau}_i$, and $(\tilde{j}_{i+1},\tilde{p}_{i+1},\tilde{\tau}_{i+1})|_{K_i^{\mathrm{top}}} = (\tilde{j}_i,\tilde{p}_i,\tilde{\tau}_i)|_{K_i^{\mathrm{top}}}$;

(2) the projection $\tilde{\mathcal{D}}_i\longrightarrow\tilde{\mathcal{C}}_i$ on either coordinate is a bijection onto the complement of $\tilde{\mathcal{C}}_i^{(0)}$;

(3) for all $((j_1,p_1),(j_2,p_2))\in\tilde{\mathcal{D}}_i\cap(\mathcal{C}_{\tilde{s}_{i_1}}\times\mathcal{C}_{\tilde{s}_{i_2}})$,

$$\tilde{\tau}_{i,((j_2,p_2),(j_1,p_1))} = \tilde{\tau}^{-1}_{i,((j_1,p_1),(j_2,p_2))}, \qquad \tilde{f}_{i_2,j_2}\circ\iota_{k+1,p_2} = \tilde{f}_{i_1,j_1}\circ\iota_{k+1,p_1}\circ\tilde{\tau}_{i,((j_1,p_1),(j_2,p_2))},$$
$$\text{and} \qquad \operatorname{sign}\tilde{\tau}_{i,((j_1,p_1),(j_2,p_2))} = -(-1)^{p_1+p_2};$$

(4) for all $\sigma\in K_i^{\mathrm{top}}-K_{i-1}^{\mathrm{top}}$,

$$\tilde{f}_{i,\tilde{j}_i(\sigma)}\circ\iota_{k+1,\tilde{p}_i(\sigma)}\circ\tilde{\tau}_{(i,\sigma)} = f_\sigma \qquad \text{and} \qquad \operatorname{sign}\tilde{\tau}_{(i,\sigma)} = -(-1)^{\tilde{p}_i(\sigma)};$$

(5) $(\tilde{j}_i,\tilde{p}_i)$ is a bijection onto $\tilde{\mathcal{C}}_i^{(0)}$.

(4) Put

$$\tilde{M}' = \Big(\bigsqcup_{i=1}^{\infty}\bigsqcup_{j=1}^{\tilde{N}_i}\{i\}\times\{j\}\times\Delta^{k+1}\sqcup\mathbb{I}\times M'\Big)\Big/\sim, \qquad \text{where}$$
$$\big(i_1,j_1,\iota_{k,p_1}(\tilde{\tau}_{i,((j_1,p_1),(j_2,p_2))}(t))\big)\sim\big(i_2,j_2,\iota_{k,p_2}(t)\big) \quad \forall\ ((j_1,p_1),(j_2,p_2))\in\tilde{\mathcal{D}}_i\cap(\mathcal{C}_{\tilde{s}_{i_1}}\times\mathcal{C}_{\tilde{s}_{i_2}}),\ t\in\Delta^k,$$
$$\big(1,\pi(\sigma,t)\big)\sim\big(i,\tilde{j}_i(\sigma),\iota_{k+1,\tilde{p}_i(\sigma)}(\tilde{\tau}_{(i,\sigma)}(t))\big) \quad \forall\ t\in\Delta^k,\ \sigma\in K_i^{\mathrm{top}}-K_{i-1}^{\mathrm{top}},\ i\in\mathbb{Z}^+.$$

Let

$$\tilde{\pi}\colon \bigsqcup_{i=1}^{\infty}\bigsqcup_{j=1}^{\tilde{N}_i}\{i\}\times\{j\}\times\Delta^{k+1}\sqcup\mathbb{I}\times M' \longrightarrow \tilde{M}'$$

be the quotient map. Define

$$\tilde{F}\colon \tilde{M}' \longrightarrow X \quad \text{by} \quad \begin{array}{ll} \tilde{F}\big([i,j,t]\big) = \tilde{f}_{i,j}\big(\tilde{\varphi}_{k+1}(\varphi_{k+1}(t))\big) & \forall\ t\in\Delta^{k+1},\ j=1,\ldots,\tilde{N}_i,\ i\in\mathbb{Z}^+; \\ \tilde{F}\big([s,x]\big) = F(x) & \forall\ s\in\mathbb{I},\ x\in M', \end{array}$$

where $\tilde{\varphi}_{k+1}$ and φ_{k+1} are the self-maps of Δ^{k+1} provided by Lemma 7.2. Similarly to Exercise 7.6, this map is well-defined and continuous. Since the image of

$$\bigsqcup_{i=2}^{\infty}\bigsqcup_{j=1}^{\tilde{N}_i}\{i\}\times\{j\}\times\Delta^{k+1}\sqcup\mathbb{I}\times\pi\Big(\bigsqcup_{\sigma\in K_2^{\mathrm{top}}}\{\sigma\}\times\Delta^{k-1}\Big)$$

under $\tilde{F}\circ\tilde{\pi}$ is contained U_1 and $\bar{U}_1\subset X$ is compact, $\overline{\tilde{F}(\tilde{M}')}\subset X$ is compact as well.

Let $\tilde{M}$ be the complement in $\tilde{M}'$ of the set

$$\tilde{\pi}\Big(\bigsqcup_{i=1}^{\infty}\bigsqcup_{j=1}^{j=\tilde{N}_i}\{i\}\times\{j\}\times\tilde{Y}\sqcup\mathbb{I}\times(M'-M)\Big),$$

where $\tilde{Y}\subset\Delta^{k+1}$ is the (k–1)-skeleton. Similarly to the proof of Lemma 7.5, $\mathrm{Bd}\,\tilde{F}|_{\tilde{M}}$ is of dimension at most $k-1$, $\tilde{M}$ is a smooth oriented manifold boundary $\partial\tilde{M}=-M$, $\tilde{F}|_{\tilde{M}}$ is a smooth map, and $\tilde{F}|_M=F|_M$. Since $\partial\big(\tilde{F}|_{\tilde{M}}\big)=-F|_M$, $F|_M$ and h represent the zero element in $\mathcal{H}_k(M)$. □

Exercise 9.3. Let $\tilde{F}\colon\tilde{M}'\longrightarrow X$ and $\tilde{M}\subset\tilde{M}'$ be as in the proof of Lemma 9.2. Show that

(1) the map $\tilde{F}$ is well-defined and continuous;

(2) $\dim\mathrm{Bd}\,\tilde{F}|_{\tilde{M}}\le k$–1, $\tilde{M}$ is a smooth oriented manifold boundary $\partial\tilde{M}=-M$, $\tilde{F}|_{\tilde{M}}$ is a smooth map, and $\tilde{F}|_M=F|_M$.

10 Existence of transverse triangulations

In this section, we show that every smooth manifold admits a smooth triangulation transverse to a given smooth map. Proposition 10.1 below is a key step in the proof of Theorem 5.4, as it is used in the proof of Lemma 8.6 via Corollary 8.4.

Proposition 10.1. *If X,Y are smooth manifolds and $h\colon Y\longrightarrow X$ is a smooth map, there exists a triangulation $T=(K,\eta)$ of X such that h is transverse to $\eta|_{\mathrm{Int}\,\sigma}$ for every simplex $\sigma\in K$.*

This claim appears clear and completely classical. It is established in [19] under the assumption that the smooth map h is proper (preimages of compact subsets are compact); the argument in [19] makes use of this assumption in an essential way. For our purposes, a transverse C^1-triangulation would suffice, and the existence of a such triangulation is fairly evident from the (infinite-dimensional) Sard-Smale Theorem [21, (1.3)]. On the other hand, according to M. Kreck, the existence of smooth transverse triangulations without the properness assumption is related to subtle issues arising in the topology of stratifolds [9]. A complete proof of Proposition 10.1, using only (the finite-dimensional) Sard's theorem [11, Section 2], is given in the rest of this section.

If $T=(K,\eta)$ is a smooth triangulation of X, as defined in Section 8, and $\psi\colon X\longrightarrow X$ is a diffeomorphism, then $(K,\psi\circ\eta)$ is also a smooth triangulation of X. By the proof of Proposition 10.1 below, $(K,\psi\circ\eta)$ is transverse to $h\colon Y\longrightarrow X$ for a generic diffeomorphism ψ of X.

For a simplicial complex K and $l\in\mathbb{Z}^{\ge 0}$, let K_l be the l-th skeleton of K, i.e. the subcomplex of K consisting of the simplices in K of dimension at most l. The main step in the proof of Proposition 10.1 is the following observation.

Lemma 10.2. *Let $h:Y\longrightarrow X$ be a smooth map between smooth manifolds. If (K,η) is a triangulation of X and σ is an l-simplex in K, there exists a diffeomorphism $\psi_\sigma:X\longrightarrow X$ restricting to the identity outside of $\eta(\mathrm{St}(b_\sigma,\mathrm{sd}\,K))$ so that $\psi_\sigma\circ\eta|_{\mathrm{Int}\,\sigma}$ is transverse to h.*

Corollary 10.3. *Let $h: Y \longrightarrow X$ be a smooth map between smooth manifolds. If (K,η) is a triangulation of X, for every $l=0,1,\ldots,\dim X$, there exists a diffeomorphism $\psi_l: X \longrightarrow X$ restricting to the identity on $\eta(|K_{l-1}|)$ so that $\psi_l \circ \eta|_{\mathrm{Int}\,\sigma}$ is transverse to h for every l-simplex σ in K.*

Proof. For each l-simplex $\sigma \in K$, let $\psi_\sigma: X \longrightarrow X$ be a diffeomorphism provided by Lemma 10.2. Since the collection

$$\{\mathrm{St}(b_\sigma, \mathrm{sd}\, K):\ \sigma \in K,\ \dim\sigma = l\}$$

is locally finite, ψ_σ is the identity outside of $\eta(\mathrm{St}(b_\sigma, \mathrm{sd}\, K))$, and

$$\mathrm{St}(b_\sigma, \mathrm{sd}\, K) \cap \mathrm{St}(b_{\sigma'}, \mathrm{sd}\, K) = \emptyset$$

for any two l-simplicies σ and σ' in K, the composition $\psi_l: X \longrightarrow X$ of the diffeomorphisms $\psi_\sigma: X \longrightarrow X$ taken over all l-simplices σ in K is a well-defined diffeomorphism of X. Since $\psi_l \circ \eta|_{|\sigma|} = \psi_\sigma \circ \eta|_{|\sigma|}$ for every l-simplex σ in K, ψ_l has the desired property. □

Proof of Proposition 10.1. By [13, Chapter II], X admits a triangulation (K, η_{-1}). By induction and Corollary 10.3, for each $l = 0, 1, \ldots, \dim X - 1$ there exists a triangulation

$$(K, \eta_l) \equiv (K, \psi_l \circ \eta_{l-1})$$

of X which is transverse to h on every open simplex in K of dimension at most l. □

In the remainder of this section, we establish Lemma 10.2.

Lemma 10.4. *For every $l \in \mathbb{Z}^+$, there exists a smooth function $\rho_l: \mathbb{R}^l \longrightarrow \bar{\mathbb{R}}^+$ such that*

$$\rho_l^{-1}(\mathbb{R}^+) = \mathrm{Int}\, \Delta^l.$$

Proof. Let $\rho: \mathbb{R} \longrightarrow \mathbb{R}$ be the smooth function given by

$$\rho(r) = \begin{cases} \mathrm{e}^{-1/r}, & \text{if } r > 0; \\ 0, & \text{if } r \le 0. \end{cases}$$

The smooth function $\rho_l: \mathbb{R}^l \longrightarrow \mathbb{R}$ given by

$$\rho_l(t_1, \ldots, t_n) = \rho\Bigg(1 - \sum_{i=1}^{i=l} t_i\Bigg) \cdot \prod_{i=1}^{i=l} \rho(t_i)$$

then has the desired property. □

Lemma 10.5. *Let (K,η) be a triangulation of a smooth n-manifold X and σ be an l-simplex in K. If $\Delta^l_\sigma \subset \mathbb{R}^l$ is an open neighborhood of Δ^l, $U_\sigma \subset X$ is an open neighborhood of $\eta(|\sigma|)$, and*

$$\tilde{\mu}_\sigma: \Delta^l_\sigma \times \mathbb{R}^{n-l} \longrightarrow U_\sigma \subset X$$

is a diffeomorphism such that $\tilde{\mu}_\sigma(t, 0) = \eta(\iota_\sigma(t))$ for all $t \in \Delta^l$, there exists $c_\sigma \in \mathbb{R}^+$ such that

$$\big\{(t,v) \in (\mathrm{Int}\, \Delta^l) \times \mathbb{R}^{n-l}:\ |v| \le c_\sigma \rho_l(t)\big\} \subset \tilde{\mu}_\sigma^{-1}\big(\eta(\mathrm{St}(b_\sigma, \mathrm{sd}\, K))\big).$$

Proof. If K' is the subdivision of K obtained by adding the vertices $b_{\sigma'}$ with $\sigma' \supsetneq \sigma$, then $\mathrm{St}(b_\sigma, \mathrm{sd}\, K) = \mathrm{St}(\sigma, K')$. Thus, it is sufficient to show that there exists $c_\sigma > 0$ such that

$$\big\{(t,v) \in (\mathrm{Int}\,\Delta^l) \times \mathbb{R}^{n-l} \colon |v| \le c_\sigma \rho_l(t)\big\} \subset \tilde{\mu}_\sigma^{-1}\big(\eta(\mathrm{St}(\sigma, K))\big).$$

We assume that $0 < l < n$. Suppose $(t_p, v_p) \in (\mathrm{Int}\,\Delta^l) \times (\mathbb{R}^{n-l} - 0)$ is a sequence such that

$$(t_p, v_p) \notin \tilde{\mu}_\sigma^{-1}\big(\eta(\mathrm{St}(\sigma, K))\big), \qquad |v_p| \le \frac{1}{p}\rho_l(t_p). \tag{10.1}$$

Since $\eta(\mathrm{St}(\sigma, K))$ is an open neighborhood of $\eta(\mathrm{Int}\,\sigma)$ in X, by shrinking v_p and passing to a subsequence we can assume that

$$(t_p, v_p) \in \tilde{\mu}_\sigma^{-1}\big(\eta(|\tau'|)\big) \subset \tilde{\mu}_\sigma^{-1}\big(\eta(|\tau|)\big) \tag{10.2}$$

for an n-simplex τ in K and a face τ' of τ so that $\sigma \not\subset \tau'$, $\tau' \not\subset \sigma$, and $\sigma \subset \tau$. Let $\iota_\tau \colon \Delta^n \longrightarrow |K|$ be an injective linear map taking Δ^n to $|\tau|$ so that

$$\iota_\tau^{-1}(|\sigma|) = \Delta^n \cap \mathbb{R}^l \times 0 \subset \mathbb{R}^l \times \mathbb{R}^{n-l}, \qquad \iota_\tau^{-1}(|\tau'|) = \Delta^n \cap 0 \times \mathbb{R}^{n-1} \subset \mathbb{R}^1 \times \mathbb{R}^{n-1}. \tag{10.3}$$

Choose a smooth embedding $\mu_\tau \colon \Delta^n_\tau \longrightarrow X$ from an open neighborhood of Δ^n in $\mathbb{R}^n$ such that $\mu_\tau|_{\Delta^n} = \eta \circ \iota_\tau$. Let ϕ be the first component of the diffeomorphism

$$\mu_\tau^{-1} \circ \tilde{\mu}_\sigma \colon \tilde{\mu}_\sigma^{-1}\big(\mu_\tau(\Delta^n_\tau)\big) \longrightarrow \mu_\tau^{-1}\big(\tilde{\mu}_\sigma(\Delta^l_\sigma \times \mathbb{R}^{n-l})\big) \subset \mathbb{R}^1 \times \mathbb{R}^{n-1}.$$

By (10.2), the second assumption in (10.3), the continuity of $\mathrm{d}\phi$, and the compactness of Δ^l,

$$\big|\phi(t_p, 0)\big| = \big|\phi(t_p, 0) - \phi(t_p, v_p)\big| \le C|v_p| \qquad \forall\, p, \tag{10.4}$$

for some $C > 0$. On the other hand, by the first assumption in (10.3), the vanishing of ρ_l on $\mathrm{Bd}\,\Delta^l$, the continuity of $\mathrm{d}\rho_l$, and the compactness of Δ^l,

$$\big|\rho_l(t_p)\big| \le C\big|\phi(t_p, 0)\big| \qquad \forall\, p, \tag{10.5}$$

for some $C > 0$. The second assumption in (10.1), (10.4), and (10.5) give a contradiction for $p > C^2$. This establishes the claim. □

Lemma 10.6. *Suppose $h \colon Y \longrightarrow X$ is a smooth map between smooth manifolds, (K, η) is a triangulation of X, and σ is an l-simplex in K. Let $\Delta^l_\sigma \subset \mathbb{R}^l$ be an open neighborhood of Δ^l, $U_\sigma \subset X$ be an open neighborhood of $\eta(|\sigma|)$, and*

$$\tilde{\mu}_\sigma \colon \Delta^l_\sigma \times \mathbb{R}^{n-l} \longrightarrow U_\sigma \subset X$$

be a diffeomorphism such that $\tilde{\mu}_\sigma(t, 0) = \eta(\iota_\sigma(t))$ for all $t \in \Delta^l$. For every $\epsilon > 0$, there exists $s_\sigma \in C^\infty(\mathrm{Int}\,\Delta^l; \mathbb{R}^{n-l})$ so that the map

$$\tilde{\mu}_\sigma \circ (\mathrm{id}, s_\sigma) \colon \mathrm{Int}\,\Delta^l \longrightarrow \mathrm{Int}\,\Delta^l \times \mathbb{R}^{n-l} \longrightarrow X \tag{10.6}$$

is transverse to h,

$$\big|s_\sigma(t)\big| < \epsilon^2 \rho_l(t) \quad \forall\, t \in \mathrm{Int}\,\Delta^l, \qquad \lim_{t \longrightarrow \mathrm{Bd}\,\Delta^l} \rho_l(t)^{-i}\big|\nabla^j s_\sigma(t)\big| = 0 \quad \forall i, j \in \mathbb{Z}^{\ge 0}, \tag{10.7}$$

where $\nabla^j s_\sigma$ is the multi-linear functional determined by the j-th partial derivatives of s_σ.

Proof. The smooth map

$$\phi\colon \mathrm{Int}\,\Delta^l\times\mathbb{R}^{n-l}\longrightarrow X, \qquad \phi(t,v)=\tilde\mu_\sigma\big(t,\mathrm{e}^{-1/\rho_l(t)}v\big),$$

is a diffeomorphism onto an open neighborhood U'_σ of $\eta(\mathrm{Int}\,\sigma)$ in X. The smooth map (10.6) with $s_\sigma=\mathrm{e}^{-1/\rho_l(t)}v$ is transverse to h if and only if $v\in\mathbb{R}^{n-l}$ is a regular value of the smooth map

$$\pi_2\circ\phi^{-1}\circ h\colon h^{-1}(U'_\sigma)\longrightarrow\mathbb{R}^{n-l},$$

where $\pi_2:\mathrm{Int}\,\Delta^l\times\mathbb{R}^{n-l}\longrightarrow\mathbb{R}^{n-l}$ is the projection onto the second component. By Sard's Theorem [11, Section 2], the set of such regular values is dense in $\mathbb{R}^{n-l}$. Thus, the map (10.6) with $s_\sigma=\mathrm{e}^{-1/\rho_l(t)}v$ is transverse to h for some $v\in\mathbb{R}^{n-l}$ with $|v|<\epsilon^2$. The second statement in (10.7) follows from $\rho_l|_{\mathrm{Bd}\,\Delta^l}=0$. □

Corollary 10.7. *Let X, Y, h, (K,η), l, and $\tilde\mu_\sigma$ be as in the statement of Lemma 10.6. For every $\epsilon>0$, there exists a diffeomorphism ψ'_σ of $\Delta^l_\sigma\times\mathbb{R}^{n-l}$ restricting to the identity outside of*

$$\big\{(t,v)\in(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{n-l}\colon\ |v|\le\epsilon\rho_l(t)\big\}$$

so that the map $\tilde\mu_\sigma\circ\psi'_\sigma|_{\mathrm{Int}\,\Delta^l\times 0}$ is transverse to h.

Proof. Choose $\beta\in C^\infty(\mathbb{R};[0,1])$ so that

$$\beta(r)=\begin{cases}1, & \text{if } r\le\frac12;\\ 0, & \text{if } r\ge 1.\end{cases}$$

Let $C_\beta=\sup_{r\in\mathbb{R}}|\beta'(r)|$ and s_σ be as provided by Lemma 10.6. Define

$$\psi'_\sigma\colon\Delta^l_\sigma\times\mathbb{R}^{n-l}\longrightarrow\Delta^l_\sigma\times\mathbb{R}^{n-l}\qquad\text{by}$$
$$\psi'_\sigma(t,v)=\begin{cases}\Big(t,v+\beta\Big(\frac{|v|}{\epsilon\rho_l(t)}\Big)s_\sigma(t)\Big), & \text{if } t\in\mathrm{Int}\,\Delta^l;\\ (t,v), & \text{if } t\notin\mathrm{Int}\,\Delta^l.\end{cases}$$

The restriction of this map to $(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{n-l}$ is smooth and its Jacobian is

$$\mathcal{J}\psi'_\sigma\big|_{(t,v)}=\begin{pmatrix}\mathbb{I}_l & 0\\ \beta\Big(\frac{|v|}{\epsilon\rho_l(t)}\Big)\nabla s_\sigma(t)-\beta'\Big(\frac{|v|}{\epsilon\rho_l(t)}\Big)\frac{|v|}{\epsilon\rho_l(t)}\frac{s_\sigma(t)}{\rho_l(t)}\nabla\rho_l & \mathbb{I}_{n-l}+\beta'\Big(\frac{|v|}{\epsilon\rho_l(t)}\Big)\frac{s_\sigma(t)}{\epsilon\rho_l(t)}\frac{v^{\mathrm{tr}}}{|v|}\end{pmatrix}. \tag{10.8}$$

By the first property in (10.7), this matrix is non-singular if $\epsilon<1/C_\beta$. If W is any linear subspace of $\mathbb{R}^{n-l}$ containing $s_\sigma(t)$,

$$\psi'_\sigma(t\times W)\subset t\times W, \qquad \psi'_\sigma(t,v)=(t,v)\quad\forall\,v\in W\ \text{s.t.}\ |v|\ge\epsilon\rho_l(t).$$

Thus, ψ'_σ is a bijection on $t\times W$, a diffeomorphism on $(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{n-l}$, and a bijection on $\Delta^l_\sigma\times\mathbb{R}^{n-l}$.

Since $\beta(r)=0$ for $r\ge1$, $\psi'_\sigma(t,v)=(t,v)$ unless $t\in\mathrm{Int}\,\Delta^l$ and $|v|<\epsilon\rho_l(t)$. It remains to show that ψ'_σ is smooth along

$$\overline{\big\{(t,v)\in(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{n-l}\colon\ |v|\le\epsilon\rho_l(t)\big\}}-(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{n-l}=(\mathrm{Bd}\,\Delta^l)\times 0.$$

Since $|s_\sigma(t)|\longrightarrow 0$ as $t\longrightarrow \mathrm{Bd}\,\Delta^l$ by the first property in (10.7), ψ'_σ is continuous along $(\mathrm{Bd}\,\Delta^l)\times 0$. By the first property in (10.7), ψ'_σ is also differentiable along $(\mathrm{Bd}\,\Delta^l)\times 0$, with the Jacobian equal to $\mathbb{I}_n$. By (10.8) and the compactness of Δ^l,

$$\big|\mathcal{J}\psi'_\sigma|_{(t,v)}-\mathbb{I}_n\big|\le C\big(|\nabla s_\sigma(t)|+\epsilon^{-1}\rho(t)^{-1}|s_\sigma(t)|\big)\quad \forall\,(t,v)\in(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{n-l}$$

for some $C>0$. So $\mathcal{J}\psi'_\sigma$ is continuous at $(t,0)$ by the second statement in (10.7), as well as differentiable, with the differential of $\mathcal{J}\psi'_\sigma$ at $(t,0)$ equal to 0. For $i\ge 1$, the i-th derivatives of the second component of ψ'_σ at $(t,v)\in(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{n-l}$ are linear combinations of the terms

$$\beta^{\langle i_1\rangle}\left(\frac{|v|}{\epsilon\rho_l(t)}\right)\cdot\left(\frac{|v|}{\epsilon\rho_l(t)}\right)^{i_1}\cdot\prod_{k=1}^{k=j_1}\left(\frac{\nabla^{p_k}\rho_l}{\rho_l(t)}\right)\cdot\frac{v_J}{|v|^{j_2}}\cdot\nabla^{i_2}s_\sigma(t)\,,$$

where $i_1,i_2,j_1,j_2\in\mathbb{Z}^{\ge 0}$ and $p_1,\ldots,p_{j_1}\in\mathbb{Z}^+$ are such that

$$p_1+p_2+\ldots+p_{j_1}+j_2\le i,\qquad i_1\le j_1+j_2\,,$$

and v_J is a product of $|J|\le j_2$ components of v. Such a term is nonzero only if $\epsilon\rho_l(t)/2<|v|<\epsilon\rho_l(t)$ or $i_1=0$ and $|v|<\epsilon\rho_l(t)$. Thus, the i-th derivatives of ψ'_σ at $(t,v)\in(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{n-l}$ are bounded by

$$C_i\sum_{i_1+i_2\le i}\rho_l(t)^{-i_1}\big|\nabla^{i_2}s_\sigma(t)\big|$$

for some constant $C_i>0$. By the second statement in (10.7), the last expression approaches 0 as $t\longrightarrow\mathrm{Bd}\,\Delta^l$ and does so faster than ρ_l. It follows that ψ'_σ is smooth at all $(t,0)\in(\mathrm{Bd}\,\Delta^l)\times 0$. □

Proof of Lemma 10.2. Let $\Delta^l_\sigma\subset\mathbb{R}^l$ be a contractible open neighborhood of Δ^l and $\mu_\sigma\colon\Delta^l_\sigma\longrightarrow X$ a smooth embedding so that $\mu_\sigma|_{\Delta^l}=\eta\circ\iota_\sigma$. By the Tubular Neighborhood Theorem [3, (12.11)], there exist an open neighborhood U_σ of $\mu_\sigma(\Delta^l_\sigma)$ in X and a diffeomorphism

$$\tilde\mu_\sigma\colon\Delta^l_\sigma\times\mathbb{R}^{n-l}\longrightarrow U_\sigma\qquad\text{s.t.}\qquad\tilde\mu_\sigma(t,0)=\mu_\sigma(t)\ \ \forall\,t\in\Delta^l_\sigma\,.$$

Let $c_\sigma>0$ be as in Lemma 10.5 and ψ'_σ be as in Corollary 10.7 with $\epsilon=c_\sigma$. The diffeomorphism

$$\psi_\sigma=\tilde\mu_\sigma\circ\psi'_\sigma\circ\tilde\mu_\sigma^{-1}\colon U_\sigma\longrightarrow U_\sigma$$

is then the identity on $U_\sigma-\mathrm{St}(b_\sigma,\mathrm{sd}\,K)$. Since ψ_σ is also the identity outside of a compact subset of U_σ, it extends by identity to a diffeomorphism on all of X. □

11 The ring isomorphism

remains to be written; proof of Proposition 11.1 is not quite complete

Proposition 11.1. *If X is a smooth oriented manifold, the homomorphism Ψ of Proposition 7.1 commutes with the ring structures.*

Proof. We need to show that

$$\Psi\big(\alpha\cap[X]\big)\cdot\Psi(B)=\Psi(\alpha\cap B)\qquad\forall\ \alpha\in H^l_c(X;\mathbb{Z}),\ B\in H_k(X;\mathbb{Z}),$$

with $l\le k\le n$, where $n=\dim X$. Let $\iota^f_{n;k}, \iota^b_{n;k}\colon \Delta^k \longrightarrow \Delta^n$ denote the natural inclusions as the front and back k-faces, i.e. the linear maps defined by

$$\iota^f_{n;k}(e_i)=e_i, \quad \iota^b_{n;k}(e_i)=e_{n-k+i}, \qquad i=0,1,\ldots,k.$$

Let (T,η) be a smooth oriented triangulation of X as in Sections 6 and 8. Similarly to Remark 6.5,

$$[X]=\sum_{\sigma\in K,\dim\sigma=n}\{\eta_\sigma\}\in \bar{S}_n(X),$$

where $n=\dim X$ and η_σ is as in (6.13). It can be assumed that B is a linear combination of the front l-simplicies of the singular simplicies η_σ, i.e.

$$B=\sum_{i=1}^N a_i\big\{\eta_{\sigma_i}\circ\iota^f_{n;k}\big\}\in \bar{S}_k(X),$$

for some $a_i\in\mathbb{Z}$ and $\sigma_i\in K$. Under these assumptions,

$$\begin{aligned}\alpha\cap[X]&=\sum_{\sigma\in K,\dim\sigma=n}\alpha\big(\eta_\sigma\circ\iota^f_{n;l}\big)\,\big\{\eta_\sigma\circ\iota^b_{n;n-l}\big\}\in \bar{S}_{n-l}(X)\,,\\ \alpha\cap B&=\sum_{i=1}^N a_i\,\alpha\big(\eta_{\sigma_i}\circ\iota^f_{n;k}\circ\iota^f_{k;l}\big)\,\big\{\eta_{\sigma_i}\circ\iota^f_{n;k}\circ\iota^b_{k;k-l}\big\}\\ &=\sum_{i=1}^N a_i\,\alpha\big(\eta_{\sigma_i}\circ\iota_{n;l}\big)\,\big\{\eta_{\sigma_i}\circ\iota^f_{n;k}\circ\iota^b_{k;k-l}\big\}\in \bar{S}_{k-l}(X)\,.\end{aligned}$$

Thus, $\alpha\cap B$ consists of the middle $k-l$ faces of the singular n-simplices η_{σ_i}; the same is the case for $(\alpha\cap[X])\cap B$. More formally, these intersections need to be made into transverse pseudo-cycles and the signs need to be checked. $\square$

An alternative argument should follow from a pseudo-cycle version of [12, Exercise 11C].

Proposition 11.2. *Let X be a smooth manifold and $A_i\in\mathcal{H}_{k_i}(X)$ for $i=1,2$. There exist representatives $f_i\colon M_i\longrightarrow X$ for A_i, with $i=1,2$, and smooth maps $h_i\colon Y_i\longrightarrow X$ from (k_i-2)-manifolds such that*

(1) $\mathrm{Bd}\, f_1\subset \mathrm{Im}\, h_1$ *and* $\mathrm{Bd}\, f_2\subset \mathrm{Im}\, h_2$*;*

(2) $f_1\overline{\pitchfork}_X f_2$*,* $f_1\overline{\pitchfork}_X h_2$*,* $f_2\overline{\pitchfork}_X h_1$*, and* $h_1\overline{\pitchfork}_X h_2$*.*

Lemma 11.3. *Let X be a smooth manifold and $f_i\colon Y_i\longrightarrow X$, with $i=1,2$, be smooth maps. For a generic diffeomorphism $\psi\colon X\longrightarrow X$*

The diffeomorphism ψ might perhaps be just C^k, for a fixed arbitrary large k, as happens in [17, Lemma 6.5.5].

A Review of Topology

A.1 Poincare Duality

A.2 Some topology

Proposition A.2.1. *Let* $\gamma_k \longrightarrow \mathbb{G}_k\mathbb{C}$ *denote the tautological* k*-plane bundle over the infinite Grassmannian of* k*-dimensional linear subspaces of* $\mathbb{C}^\infty$*. If* $f\colon (\mathbb{CP}^\infty)^k \longrightarrow \mathbb{G}_k\mathbb{C}$ *is a continuous map such that*

$$f^*\gamma_k = (\gamma_1)^k \equiv \bigoplus_{j=1}^{j=k} \pi_j^*\gamma_1 \longrightarrow (\mathbb{CP}^\infty)^k,$$

where $\pi_j\colon (\mathbb{CP}^\infty)^k \longrightarrow \mathbb{CP}^\infty$ *is the projection onto the* j*-th component, then*

$$f^*\colon H^*(\mathbb{G}_k\mathbb{C};\mathbb{Z}) \longrightarrow H^*\big((\mathbb{CP}^\infty)^k;\mathbb{Z}\big)$$

is an injective homomorphism.

Proof. Since the Schubert cells provide CW-decompositions of finite Grassmannians, they generate $H^*(\mathbb{G}_k\mathbb{C};\mathbb{Z})$. Based on intersection formulas for Schubert cycles, $H^*(\mathbb{G}_k\mathbb{C};\mathbb{Z})$ is in fact generated by

$$\sigma_1 = c_1(\gamma_k^*), \qquad \sigma_{11} = c_2(\gamma_k^*), \quad \dots \quad \sigma_{1\dots1} = c_k(\gamma_k^*)$$

as an algebra over $\mathbb{Z}$; see [12, Theorem 14.5]. By the product formula for chern classes,

$$f^*c_i(\gamma_k) = \mathfrak{s}_i \in H^*\big((\mathbb{CP}^\infty)^k\big) \approx R[\pi_1^*c_1(\gamma_1),\dots,\pi_k^*c_1(\gamma_1)]$$

is the i-th elementary symmetric polynomials in $\pi_1^*c_1(\gamma_1),\dots,\pi_1^*c_1(\gamma_k)$. Since the k elementary symmetric polynomials $\mathfrak{s}_1,\dots,\mathfrak{s}_k$ are algebraically independent [1, Corollary 14-(3.11)], it follows that f^* is injective. □

Exercise A.2.2. Let $\gamma_k \longrightarrow \mathbb{G}_k\mathbb{R}$ denote the tautological k-plane bundle over the infinite Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^\infty$ and $f\colon (\mathbb{RP}^\infty)^k \longrightarrow \mathbb{G}_k\mathbb{R}$ be a continuous map such that

$$f^*\gamma_k = (\gamma_1)^k \equiv \bigoplus_{j=1}^{j=k} \pi_j^*\gamma_1 \longrightarrow (\mathbb{RP}^\infty)^k,$$

where $\pi_j\colon (\mathbb{RP})^k \longrightarrow \mathbb{CP}^\infty$ is the projection onto the j-th component. Show that

$$f^*\colon H^*(\mathbb{G}_k\mathbb{C};\mathbb{Z}) \longrightarrow H^*\big((\mathbb{CP}^\infty)^k;\mathbb{Z}\big)$$

is an injective homomorphism.

Definition A.2.3. Let $p\colon E\longrightarrow B$ be an F-fiber bundle, $\iota_b: E_b\longrightarrow E$ be the inclusion of the fiber for each $b\in B$, and R be a ring. A cohomology extension of the fiber for p over R is a homomorphism

$$\theta\colon H^*(F;R)\longrightarrow H^*(E;R)$$

of R-modules such that $\iota_b^*\circ\theta\colon H^*(F;R)\longrightarrow H^*(E_b;R)$ is an isomorphism for every $b\in B$.

Example A.2.4. Let $V\longrightarrow B$ be a complex vector bundle of rank k. The projectivization of V,

$$p\colon \mathbb{P}V\longrightarrow B,$$

is the $\mathbb{CP}^{k-1}$-fiber bundle obtained by replacing each fiber of V by its projectivization over $\mathbb{C}$ and the transition maps between trivializations of V by the induced diffeomorphisms between their projectivizations. Let

$$\gamma_V=\big\{(\ell,v)\in\mathbb{P}V\times V:\ v\in\ell\subset V\big\}\longrightarrow\mathbb{P}V$$

denote the tautological line bundle and

$$\lambda_V=c_1(\gamma_V^*)\in H^2(\mathbb{P}V;\mathbb{Z}).$$

Since the restrictions of $\lambda_V^0=1,\lambda_V^1,\ldots,\lambda_V^{k-1}$ to each fiber $\mathbb{P}V_b$ form a basis for the R-module $H^*(\mathbb{P}V_b;\mathbb{Z})\approx H^*(\mathbb{CP}^{k-1};\mathbb{Z})$. Thus, the homomorphism

$$\theta\colon H^*(\mathbb{CP}^{k-1};\mathbb{Z})\longrightarrow H^*(\mathbb{P}V;\mathbb{Z}),\qquad \lambda^i\longrightarrow\lambda_V^i,\quad i=0,1,\ldots,k-1,$$

where $\lambda=c_1(\gamma^*)\in H^*(\mathbb{CP}^{k-1};\mathbb{Z})$, is a cohomology extension of the fiber for p over R.

Theorem A.2.5 ([22, Theorem 5.7.9])**.** *Let $p\colon E\longrightarrow B$ be an F-fiber bundle and R be a ring. If $\theta: H^*(F;R)\longrightarrow H^*(E;R)$ is a cohomology extension of the fiber for p over R, then the homomorphism*

$$H^*(B;R)\otimes_R H^*(F;R)\longrightarrow H^*(E;R),\qquad \alpha\otimes\beta\longrightarrow p^*\alpha\otimes\theta(\beta),$$

is an isomorphism of R-modules.

Exercise A.2.6. Let $V\longrightarrow B$ be a complex vector bundle of rank k and $p:\mathbb{P}V\longrightarrow B$ be its projectivization of V. Show that

(1) the vector bundle $\gamma^*\otimes p^*V\longrightarrow\mathbb{P}V$ admits a non-vanishing section;

(2) the homomorphism

$$H^*(B;\mathbb{Z})[\lambda_V]/\big(\lambda_V^k+c_1(V)\lambda_V^{k-1}+\ldots+c_k(V)\big)\longrightarrow H^*(\mathbb{P}V;\mathbb{Z}),\quad \alpha\lambda_V^i\longrightarrow p^*\alpha\cup\lambda_V^i\,,$$

is an isomorphism of $\mathbb{Z}$-algebras (preserves the product structure).

Corollary A.2.7. *Let B be a paracompact space. For every complex vector bundle $V\longrightarrow B$, there exists a topological space $\tilde{B}$ and a continuous map $\pi\colon\tilde{B}\longrightarrow B$ such that the homomorphism*

$$\pi^*\colon H^*(B;\mathbb{Z})\longrightarrow H^*(\tilde{B};\mathbb{Z})$$

is injective and the vector bundle $\pi^*V\longrightarrow\tilde{B}$ splits as a direct sum of line bundles.

Proof. Let $k = \mathrm{rk}_{\mathbb{C}} V \geq 2$ and assume that the statement holds for all vector bundles of rank less than k. Let $p\colon \mathbb{P}V \longrightarrow B$ be the projectivization of V. Since B is paracompact and $\gamma_V \subset p^*V$ is a vector subbundle,

$$p^*V \approx V' \oplus \gamma_V,$$

for some vector subbundle $V' \subset p^*V$ of rank $k-1$. By Theorem A.2.5, the homomorphism

$$p^*\colon H^*(B;\mathbb{Z}) \longrightarrow H^*(\mathbb{P}V;\mathbb{Z})$$

is injective. By the induction assumption, there exists a topological space $\tilde{B}$ and a continuous map $\pi'\colon \tilde{B} \longrightarrow \mathbb{P}V$ such that the homomorphism

$$\pi'^*\colon H^*(\mathbb{P}V;\mathbb{Z}) \longrightarrow H^*(\tilde{B};\mathbb{Z})$$

is injective and the vector bundle $\pi^*V' \longrightarrow \tilde{B}$ splits as a direct sum of line bundles. The projection

$$\pi = p\circ\pi'\colon \tilde{B} \longrightarrow B$$

then has the desired properties. □

Exercise A.2.8. Let $V \longrightarrow B$ be a real vector bundle of rank k. Show that

(1) there is a natural $\mathbb{RP}^{k-1}$-fiber bundle $p\colon \mathbb{P}V \longrightarrow B$ obtained by projectivizing each fiber of $V \longrightarrow B$ over $\mathbb{R}$.

(2) the fiber bundle $p\colon \mathbb{P}V \longrightarrow B$ admits a cohomology extension of the fiber over $\mathbb{Z}_2$;

(3) there is an isomorphism

$$H^*(B;\mathbb{Z}_2)[\lambda_V]\big/\big(\lambda_V^k + w_1(V)\lambda_V^{k-1} + \ldots + w_k(V)\big) \longrightarrow H^*(\mathbb{P}V;\mathbb{Z}_2), \quad \alpha\lambda_V^i \longrightarrow p^*\alpha \cup \lambda_V^i\,,$$

of $\mathbb{Z}_2$-algebras.

Exercise A.2.9. Let B be a paracompact space. Show that for every real vector bundle $E \longrightarrow V$ there exists a topological space $\tilde{B}$ and a continuous map $\pi\colon \tilde{B} \longrightarrow B$ such that the homomorphism

$$\pi^*\colon H^*(B;\mathbb{Z}_2) \longrightarrow H^*(\tilde{B};\mathbb{Z}_2)$$

is injective and the vector bundle $\pi^*V \longrightarrow \tilde{B}$ splits as a direct sum of real line bundles.

A.3 The splitting principle

Throughout this section, assume either

$\mathbb{C}$ case: all vector bundles are complex, all cohomology rings are with $\mathbb{Z}$-coefficients, $\mathbb{P}^1$ is the infinite complex projective space $\mathbb{CP}^\infty$, and $\mathbb{G}_n$ is the infinite complex Grassmannian $\mathrm{Gr}_n\mathbb{C}^\infty$, or

$\mathbb{R}$ case: all vector bundles are real, all cohomology rings are with $\mathbb{Z}_2$-coefficients, $\mathbb{P}^1$ is the infinite real projective space $\mathbb{RP}^\infty$, and $\mathbb{G}_n$ is the infinite real Grassmannian $\mathrm{Gr}_n\mathbb{R}^\infty$, unless explicitly stated otherwise. Base spaces B are assumed to be paracompact. Let $H^{\Pi}(B)$ be the product (rather than just sum) of all cohomology groups of B. So, an element of $H^{\Pi}(B)$ is a possibly infinite series

$$a_0 + a_1 + \ldots, \qquad \text{where} \qquad a_i \in H^i(B).$$

A vector bundle $V \longrightarrow B$ is split if it is isomorphic to a direct sum of line bundles $L_1, \ldots, L_k \longrightarrow B$.

Definition A.3.1. Let R ring. A rule assigning to every r-tuple of vector bundles $(V_1, \ldots, V_r)$ of ranks $(k_1, \ldots, k_r)$ over every base B a class $p(V_1, \ldots, V_r) \in H^{\Pi}(B; R)$ is natural if

$$p\big(f^*V_1, \ldots, f^*V_r\big) = f^*p(V_1, \ldots, V_r) \in H^{\Pi}(B'; R)$$

for every continuous map $f : B' \longrightarrow B$ and r-tuple of vector bundles $V_1, \ldots, V_r \longrightarrow B$ of ranks $k_1, \ldots, k_r$.

For example, in the complex case the rule assigning to each complex vector bundle $V \longrightarrow B$ the class

$$c_2(V \otimes V) \in H^4(B; \mathbb{Z})$$

is natural. So, is the rule assigning to each triple of complex vector bundles $V_1, V_2, V_3 \longrightarrow B$ the class

$$w_3(V_1 \otimes V_2 \otimes V_2 \otimes V_3) \in H^3(B; \mathbb{Z}_2).$$

Theorem A.3.2 (The Splitting Principle). *Let p, q be two natural rules assigning to every r-tuple of vector bundles $(V_1, \ldots, V_r)$ of ranks $(k_1, \ldots, k_r)$ over every base B classes*

$$p(V_1, \ldots, V_r),\; q(V_1, \ldots, V_r) \in H^{\Pi}(B).$$

If $p(E_1, \ldots, E_r) = q(E_1, \ldots, E_r)$ for every r-tuple of split vector bundles $E_1, \ldots, E_r$ over every base B, then

$$p(V_1, \ldots, V_r) = q(V_1, \ldots, V_r)$$

for every r-tuple of vector bundles $V_1, \ldots, V_r$ over every base B.

Proof 1. By Corollary A.2.7 in the $\mathbb{C}$ case and Exercise A.2.9 in the $\mathbb{R}$ case, there exists a topological space $\tilde{B}$ and a continuous map $\pi \colon \tilde{B} \longrightarrow B$ such that the homomorphism

$$\pi^* \colon H^*(B) \longrightarrow H^*(\tilde{B}) \tag{A.3.1}$$

is injective and the vector bundle $\pi^*V_i \longrightarrow \tilde{B}$ splits for every $i = 1, \ldots, r$. Since p and q are natural and agree on split vector bundles,

$$\pi^*p(V_1, \ldots, V_r) = p(\pi^*V_1, \ldots, \pi^*V_r) = q(\pi^*V_1, \ldots, \pi^*V_r) = \pi^*q(V_1, \ldots, V_r) \in H^{\Pi}(\tilde{B}; R).$$

Since the homomorphism (A.3.1) is injective, it follows that $p(V_1, \ldots, V_r) = q(V_1, \ldots, V_r)$. □

Proof 2. Let $\gamma_k \longrightarrow \mathbb{G}_k$ be the tautological k-plane bundle and

$$\pi_{\mathbb{G};i} \colon \mathbb{G}_{k_1} \times \ldots \times \mathbb{G}_{k_r} \longrightarrow \mathbb{G}_{k_i}$$

be the projection to the i-th factor. For each $i=1,\ldots,r$, choose a continuous map $f_i\colon (\mathbb{P}^1)^{k_i}\longrightarrow\mathbb{G}_{k_i}$ such that

$$f_i^*\gamma_{k_i} = (\gamma_1)^{k_i} \equiv \bigoplus_{j=1}^{j=k_i}\pi_j^*\gamma_1 \longrightarrow (\mathbb{P})^{k_i},$$

where $\pi_j : (\mathbb{P})^{k_i} \longrightarrow \mathbb{P}$ is the projection onto the j-th component; such a map exists by [12, Theorem 14.6] in the $\mathbb{C}$ case and by [12, Theorem 5.6] in the $\mathbb{R}$ case. Let

$$f=f_1\times\ldots\times f_r\colon (\mathbb{P})^{k_1+\ldots+k_r} \longrightarrow \mathbb{G}_{k_1}\times\ldots\times\mathbb{G}_{k_r}\,.$$

By Proposition A.2.1 in the $\mathbb{C}$ case and Exercise A.2.2 in the $\mathbb{R}$ case, along with the Kunneth formula [14, Theorem 60.3], the homomorphism

$$f^*\colon H^*\big(\mathbb{G}_{k_1}\times\ldots\times\mathbb{G}_{k_r}\big) \longrightarrow H^*\big((\mathbb{P}^1)^{k_1+\ldots+k_r}\big) \tag{A.3.2}$$

is injective. Since p and q are natural with respect to continuous maps and agree on split vector bundles,

$$\begin{aligned} f^*p\big(\pi_{\mathbb{G};1}^*\gamma_{k_1},\ldots,\pi_{\mathbb{G};r}^*\gamma_{k_r}\big) &= p\big(f^*\pi_{\mathbb{G};1}^*\gamma_{k_1},\ldots,f^*\pi_{\mathbb{G};r}^*\gamma_{k_r}\big)\\ &= q\big(f^*\pi_{\mathbb{G};1}^*\gamma_{k_1},\ldots,f^*\pi_{\mathbb{G};r}^*\gamma_{k_r}\big)\\ &= f^*q\big(\pi_{\mathbb{G};1}^*\gamma_{k_1},\ldots,\pi_{\mathbb{G};r}^*\gamma_{k_r}\big) \in H^{\Pi}\big((\mathbb{P}^1)^{k_1+\ldots+k_r}\big). \end{aligned}$$

Since the homomorphism (A.3.2) is injective, it follows that

$$p\big(\pi_{\mathbb{G};1}^*\gamma_{k_1},\ldots,\pi_{\mathbb{G};r}^*\gamma_{k_r}\big) = q\big(\pi_{\mathbb{G};1}^*\gamma_{k_1},\ldots,\pi_{\mathbb{G};r}^*\gamma_{k_r}\big) \in H^{\Pi}\big(\mathbb{G}_{k_1}\times\ldots\times\mathbb{G}_{k_r}\big).$$

If $V_1,\ldots,V_r\longrightarrow B$ are vector bundles of ranks $k_1,\ldots,k_r$, respectively, over a paracompact base, for each i there exists a continuous map $g_i\colon B\longrightarrow\mathbb{G}_{k_i}$ such that $V_i=g_i^*\gamma_{k_i}$. Let

$$g = g_1\times\ldots\times g_r\colon B \longrightarrow \mathbb{G}_{k_1}\times\ldots\times\mathbb{G}_{k_r}.$$

Since $g_i=\pi_{\mathbb{G};i}\circ g$, $V_i=g^*\pi_{\mathbb{G};i}^*\gamma_{k_i}$. Thus, by the naturality of p and q,

$$\begin{aligned} p\big(V_1,\ldots,V_r\big) &= p\big(g^*\pi_{\mathbb{G};1}^*\gamma_{k_1},\ldots,g^*\pi_{\mathbb{G};r}^*\gamma_{k_r}\big) = g^*p\big(\pi_{\mathbb{G};1}^*\gamma_{k_1},\ldots,\pi_{\mathbb{G};r}^*\gamma_{k_r}\big)\\ &= g^*q\big(\pi_{G;1}^*\gamma_{k_1},\ldots,\pi_{G;r}^*\gamma_{k_r}\big) = q\big(g^*\pi_{G;1}^*\gamma_{k_1},\ldots,g^*\pi_{\mathbb{G};r}^*\gamma_{k_r}\big)\\ &= q\big(V_1,\ldots,V_r\big) \in H^{\Pi}(B), \end{aligned}$$

as needed. □

This second proof of Theorem A.3.2 shows that it is sufficient to check only that

$$p\big(\pi_1^*\gamma_1^{k_1},\ldots,\pi_r^*\gamma_1^{k_r}\big) = q\big(\pi_1^*\gamma_1^{k_1},\ldots,\pi_r^*\gamma_1^{k_r}\big) \in H^{\Pi}\big((\mathbb{P}^1)^{k_1}\times\ldots\times(\mathbb{P}^1)^{k_r}\big).$$

Example A.3.3. Let B be a paracompact space and $V\longrightarrow B$ be a complex vector bundle of rank k. We use Theorem A.3.2 to show that

$$c_1\big(\Lambda_{\mathbb{C}}^{\mathrm{top}}V\big) \equiv c_1\big(\Lambda_{\mathbb{C}}^{k}V\big) = c_1(V).$$

For every complex vector bundle $V \longrightarrow B$ over every paracompact base B, let

$$p(V) = c_1\big(\Lambda_{\mathbb{C}}^{\mathrm{top}} V\big) \in H^2(B;\mathbb{Z}) \qquad \text{and} \qquad q(V) = c_1(V) \in H^2(B;\mathbb{Z}).$$

If $f\colon B' \longrightarrow B$ is any continuous map and $V \longrightarrow B$ is a complex vector bundle of rank k, then

$$\begin{aligned} p(f^*V) &\equiv c_1\big(\Lambda_{\mathbb{C}}^{\mathrm{top}}(f^*V)\big) = c_1\big(f^*(\Lambda_{\mathbb{C}}^{\mathrm{top}} V)\big) = f^*c_1\big(\Lambda_{\mathbb{C}}^{\mathrm{top}} V\big) \equiv f^*p(V) \in H^2(B';\mathbb{Z});\\ q(f^*V) &\equiv c_1\big(f^*V\big) = f^*c_1(V) \equiv f^*q(V) \in H^2(B';\mathbb{Z}). \end{aligned}$$

Thus, p and q are natural with respect to smooth maps. If $V = L_1 \oplus \ldots \oplus L_k$ is a sum of line bundles, then

$$\begin{aligned} \Lambda_{\mathbb{C}}^{\mathrm{top}} V = L_1 \otimes \ldots \otimes L_k \quad &\Longrightarrow \quad p(V) = c_1\big(L_1 \otimes \ldots \otimes L_k\big) = c_1(L_1) + \ldots + c_1(L_k);\\ c(V) = \big(1 + c_1(L_1)\big) \ldots \big(1 + c_1(L_k)\big) \quad &\Longrightarrow \quad q(V) \equiv c_1(V) = c_1(L_1) + \ldots + c_1(L_k). \end{aligned}$$

Thus, $p(V) = q(V)$ for every split vector bundle V of rank k. Since p and q are natural with respect to continuous maps, it follows that $p(V) = q(V)$ for every vector bundle V of rank k.

Exercise A.3.4. Let B be a paracompact space and $V \longrightarrow B$ be a complex vector bundle of rank 2. Show that

$$e\big(\mathrm{Sym}^3 V\big) = 9\, c_2(V)\, \big(c_1^2(V) + c_2(V)\big).$$

B Complex projective spaces

B.1 Definition and basic properties

The n-dimensional complex projective space, $\mathbb{P}^n$, is the quotient of $\mathbb{C}^{n+1}-0$ by the standard action of $\mathbb{C}^*\equiv\mathbb{C}-0$:

$$\mathbb{P}^n = \big(\mathbb{C}^{n+1}-0\big)/\sim, \qquad (X_0,\ldots,X_n)\sim(cX_0,\ldots,cX_n) \quad \forall\, c\in\mathbb{C}^*.$$

This space is a complex n-manifold; it can be covered by $n+1$ coordinate charts as follows. For $i=0,\ldots,n$, let

$$U_i = \big\{[X_0,\ldots,X_n]\in\mathbb{P}^1 : X_i\neq 0\big\},$$
$$\phi_i\colon \mathbb{C}^n \longrightarrow U_i, \quad \phi_i(w_1,\ldots,w_n) = \big[w_1,\ldots,w_i,1,w_{i+1},\ldots,w_n\big].$$

If $i<j$, then

$$\phi_i^{-1}(U_j) = \big\{(w_1,\ldots,w_n)\in\mathbb{C}^n : w_j\neq 0\big\},$$
$$\phi_j^{-1}(U_i) = \big\{(w_1,\ldots,w_n)\in\mathbb{C}^n : w_{i+1}\neq 0\big\}.$$

The corresponding overlap map

$$\phi_{ij}\equiv\phi_i^{-1}\circ\phi_j\colon \phi_j^{-1}(U_i)\longrightarrow\phi_i^{-1}(U_j)$$

is given by

$$(w_1,\ldots,w_n)\longrightarrow\Big(\frac{w_1}{w_{i+1}},\ldots,\frac{w_i}{w_{i+1}},\frac{w_{i+2}}{w_{i+1}},\ldots,\frac{w_j}{w_{i+1}},w_{i+1}^{-1},\frac{w_{j+1}}{w_{i+1}},\ldots,\frac{w_n}{w_{i+1}}\Big). \qquad \text{(B.1.1)}$$

Since each of the maps ϕ_{ij} is bi-holomorphic, $\big\{(U_i,\phi_i,\mathbb{C}^n)\big\}$ is an atlas of holomorphic charts on $\mathbb{P}^n$. We will call the chart $(U_i,\phi_i,\mathbb{C}^n)$ the i-th standard coordinate chart on $\mathbb{P}^n$. By the next exercise, $\mathbb{P}^n$ is compact.

Exercise B.1.1. Show that the inclusions of the unit sphere S^{2n+1} into $\mathbb{C}^{n+1}$ and of S^1 into $\mathbb{C}^*$ induce a homeomorphism

$$S^{2n+1}/S^1\longrightarrow\big(\mathbb{C}^{n+1}-0\big)/\mathbb{C}^*$$

with respect to the quotient topologies.

If V is any vector space over $\mathbb{C}$, the projectivization of V, $\mathbb{P}V$, is the quotient of $V-0$ by the standard $\mathbb{C}^*$-action. An invertible linear transformation A of V gives rise to a bijection on $\mathbb{P}V$:

$$\bar{A}\colon\mathbb{P}^n\longrightarrow\mathbb{P}^n, \qquad [v]\longrightarrow[Av] \quad \forall\, v\in V-0.$$

If $V = \mathbb{C}^n$, this bijection is a biholomorphism. Thus, if V is any $(n+1)$-dimensional vector space over $\mathbb{C}$, $\mathbb{P}V$ is a complex manifold bi-holomorphic to $\mathbb{P}^n$.

If V is a linear subspace of $\mathbb{C}^{n+1}$ of dimension $k+1$, $\mathbb{P}V \approx \mathbb{P}^k$ is a complex submanifold of $\mathbb{P}^n$. We will call such a submanifold of $\mathbb{P}^n$ a linear k-dimensional subspace. If $k = n-1$ ($k=1$), $\mathbb{P}V$ is called a hyperplane in $\mathbb{P}^n$ (line in $\mathbb{P}^n$).

B.2 CW-structure

The n-dimensional complex projective space is a CW-complex with one cell in dimensions $0, 2, \ldots, 2n$, described as follows. For each $k = 0, \ldots, n$, let

$$\sigma_k^0(\mathbf{V}_{\rm std}) = \big\{[X_0, \ldots, X_{n-k}, 0, \ldots, 0] \in \mathbb{P}^n \colon X_{n-k} \neq 0\big\}. \tag{B.2.1}$$

This is a smooth submanifold of $\mathbb{P}^n$ diffeomorphic to the open unit ball $B^{2(n-k)}$ around 0 in $\mathbb{C}^{n-k}$; in particular, the map

$$\begin{aligned} \iota_k \colon B^{2(n-k)} &\longrightarrow \sigma_k^0(\mathbf{V}_{\rm std}) \subset \mathbb{P}^n, \\ \mathbf{w} \equiv (w_1, \ldots, w_{n-k}) &\longrightarrow \Big[w_1, \ldots, w_{n-k}, 1-|\mathbf{w}|^2, \ldots, 0\Big], \end{aligned} \tag{B.2.2}$$

is a diffeomorphism. It extends continuously (in fact, smoothly) over the closed ball $\bar{B}^{2(n-k)}$; the image of the boundary $S^{2(n-k)-1}$ of $\bar{B}^{2(n-k)}$ is contained in

$$\big\{[X_0, \ldots, X_n] \in \mathbb{P}^n \colon X_i = 0 \ \forall\ i \geq n-k\big\} = \bigsqcup_{l>k} \sigma_l^0(\mathbf{V}_{\rm std}).$$

We conclude that

$$\mathbb{P}^n = \bigsqcup_{k=0}^{k=n} \sigma_k^0(\mathbf{V}_{\rm std})$$

is a CW-decomposition, $\sigma_k^0(\mathbf{V}_{\rm std})$ is an open cell of dimension $2(n-k)$, and ι_k is an attaching map for $\sigma_k^0(\mathbf{V}_{\rm std})$. The closure of $\sigma_k^0(\mathbf{V}_{\rm std})$ in $\mathbb{P}^n$ is

$$\sigma_k(\mathbf{V}_{\rm std}) = \big\{[X_0, \ldots, X_n] \in \mathbb{P}^n \colon X_i = 0 \ \forall\ i > n-k\big\} = \mathbb{P}^{n-k}. \tag{B.2.3}$$

Since all cells are of even dimension,

$$\mathrm{H}_k(\mathbb{P}^n; \mathbb{Z}), \mathrm{H}^k(\mathbb{P}^n; \mathbb{Z}) \approx \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \ldots, 2n; \\ 0, & \text{otherwise.} \end{cases} \tag{B.2.4}$$

We denote by

$$[\sigma_k(\mathbf{V}_{\rm std})] \in \mathrm{H}_{2(n-k)}(\mathbb{P}^n; \mathbb{Z}) \qquad \text{and} \qquad [\sigma_k(\mathbf{V}_{\rm std})]^* \in \mathrm{H}^{2(n-k)}(\mathbb{P}^n; \mathbb{Z})$$

the generators corresponding to the attaching map ι_k. Since the diffeomorphism

$$\iota_k \colon B^{2(n-k)} \longrightarrow \mathbb{P}^n$$

is orientation-preserving with respect to the standard orientations on $B^{2(n-k)}$ and $\mathbb{P}^n$, $[\sigma_k(\mathbf{V}_{\text{std}})]$ is the image of $[\mathbb{P}^{n-k}]$ under the homology homomorphism induced by the standard embedding

$$\mathbb{P}^{n-k} \longrightarrow \mathbb{P}^n, \qquad [X_0, \ldots, X_{n-k}] \longrightarrow [X_0, \ldots, X_{n-k}, 0, \ldots, 0].$$

In particular, $[\sigma_0]$ and $[\sigma_0]^*$ are the fundamental and orientation classes of $\mathbb{P}^n$, $[\sigma_n]$ is the homology class of a point in $\mathbb{P}^n$, and $[\sigma_n]^* = 1$.

By (B.2.3), the standard embedding $\mathbb{P}^{n-k} \longrightarrow \mathbb{P}^n$ induces isomorphisms

$$\mathrm{H}_l(\mathbb{P}^{n-k}; \mathbb{Z}) \longrightarrow \mathrm{H}_l(\mathbb{P}^n; \mathbb{Z}) \qquad \text{and} \qquad \mathrm{H}^l(\mathbb{P}^n; \mathbb{Z}) \longrightarrow \mathrm{H}^l(\mathbb{P}^{n-k}; \mathbb{Z})$$

for $l \le 2(n-k)$. Since $\mathrm{GL}_{n+1}\mathbb{C}$ is connected, it follows that the same is the case for any *linear embedding* $\mathbb{P}^{n-k} \longrightarrow \mathbb{P}^n$, i.e. an embedding given by

$$\mathbb{P}^{n-k} \longrightarrow \mathbb{P}^n, \qquad [v] \longrightarrow [Av] \quad \forall\, v \in \mathbb{C}^{n-k+1} - 0,$$

for some injective homomorphism $A\colon \mathbb{C}^{n-k+1} \longrightarrow \mathbb{C}^{n+1}$.

A (complete) flag on $\mathbb{C}^{n+1}$ is a sequence of $n+2$ linear subspaces of $\mathbb{C}^{n+1}$,

$$\mathbf{V} \equiv \big(V_0 = \{0\} \subsetneq V_1 \subsetneq \ldots \subsetneq V_n \subsetneq V_{n+1} = \mathbb{C}^{n+1}\big). \tag{B.2.5}$$

The standard flag $\mathbf{V}_{\text{std}}$ on $\mathbb{C}^{n+1}$ is given by

$$V_k = \mathbb{C}^k \times \{0\}^{n+1-k} \subset \mathbb{C}^{n+1}.$$

For any flag $\mathbf{V}$ on $\mathbb{C}^{n+1}$ as in (B.2.5), let

$$\begin{aligned} \sigma_k^0(\mathbf{V}) &= \big\{[v] \in \mathbb{P}^n \colon v \in V_{n+1-k} - V_{n-k}\big\}; \\ \sigma_k(\mathbf{V}) &= \big\{[v] \in \mathbb{P}^n \colon v \in V_{n+1-k}\big\} = \mathbb{P}V_{n+1-k}. \end{aligned} \tag{B.2.6}$$

If $\mathbf{V} = \mathbf{V}_{\text{std}}$, these definitions agree with (B.2.1) and (B.2.3). As in the $\mathbf{V} = \mathbf{V}_{\text{std}}$ case, $\sigma_k(\mathbf{V})$ is the closure of $\sigma_k^0(\mathbf{V})$ in $\mathbb{P}^n$ and

$$\mathbb{P}^n = \bigsqcup_{k=0}^{k=n} \sigma_k^0(\mathbf{V})$$

is a CW-decomposition. The attaching map ι_k for $\sigma_k^0(\mathbf{V})$ can be defined as before, but with $B^{2(n-k)}$ replaced by the unit ball in V_{n-k}. Since $\mathrm{GL}_n\mathbb{C}$ is connected, the generators

$$[\sigma_k(\mathbf{V})] \in \mathrm{H}_{2(n-k)}(\mathbb{P}^n; \mathbb{Z}) \qquad \text{and} \qquad [\sigma_k(\mathbf{V})]^* \in \mathrm{H}^{2(n-k)}(\mathbb{P}^n; \mathbb{Z})$$

are then independent of the choice of $\mathbf{V}$. We denote them by $[\sigma_k]$ and $[\sigma_k]^*$, respectively. Let

$$\sigma_k \in \mathrm{H}^{2k}(\mathbb{P}^n; \mathbb{Z})$$

be the Poincare dual of $[\sigma_k]$.

Exercise B.2.1. Using Poincare duality, show by induction on n that

$$\mathrm{H}^{2*}(\mathbb{P}^n; \mathbb{Z}) \approx \mathbb{Z}[\sigma_1]/\sigma_1^{n+1}.$$

We next show that $\sigma_k \cup \sigma_l = \sigma_{k+l}$ if $k,l \geq 0$ and $k+l \leq n$. Choose flags $\mathbf{V}$ and $\mathbf{V}'$ on $\mathbb{C}^{n+1}$ such that V_{n+1-k} and V'_{n+1-l} intersect in a subspace $V''_{n+1-k-l}$ of $\mathbb{C}^{n+1}$ of codimension $k+l$. Then, $\sigma_k(\mathbf{V}) \equiv \mathbb{P}V_{n+1-k}$ and $\sigma_l(\mathbf{V}') \equiv \mathbb{P}V'_{n+1-l}$ intersect transversally in $\mathbb{P}^n$, and their intersection is the complex manifold $\mathbb{P}V''_{n+1-k-l}$ with its standard orientation. Since $\mathbb{P}V''_{n+1-k-l} = \sigma_{k+l}(\mathbf{V}'')$ for some flag $\mathbf{V}''$ on $\mathbb{C}^{n+1}$, we conclude from Proposition 0.9 that

$$\begin{aligned} \sigma_k \cup \sigma_l &\equiv \mathrm{PD}_{\mathbb{P}^n}\big([\mathbb{P}V_{n+1-k}]_{\mathbb{P}^n}\big) \cup \mathrm{PD}_{\mathbb{P}^n}\big([\mathbb{P}V'_{n+1-l}]_{\mathbb{P}^n}\big) \\ &= \mathrm{PD}_{\mathbb{P}^n}\big([\mathbb{P}V''_{n+1-k-l}]_{\mathbb{P}^n}\big) \equiv \sigma_{k+l}. \end{aligned}$$

Since σ_n is the orientation class of $\mathbb{P}^n$, we find that

$$1 = \big\langle \sigma_k \cup \sigma_{n-k}, [\mathbb{P}^n] \big\rangle = \big\langle \sigma_k, \sigma_{n-k} \cap [\mathbb{P}^n] \big\rangle = \big\langle \sigma_k, [\sigma_{n-k}] \big\rangle.$$

Thus, by (B.2.4),

$$\sigma_k = [\sigma_{n-k}]^* \in \mathrm{H}^{2k}(\mathbb{P}^n; \mathbb{Z})\,. \tag{B.2.7}$$

Since $\sigma_1^k = \sigma_k$ by the above, it follows that

$$\big\langle \iota^* \sigma_1^k, [\mathbb{P}^k] \big\rangle = 1$$

for any linear embedding $\iota\colon \mathbb{P}^k \longrightarrow \mathbb{P}^n$.

Exercise B.2.2. Using local coordinates on $\mathbb{P}^n$, show that $\mathbb{P}V_{n+1-k}$ and $\mathbb{P}V'_{n+1-l}$ intersect transversally in $\mathbb{P}^n$ as claimed above.

B.3 Tautological line bundle

We will usually view $\mathbb{P}^n$ as the space of one-dimensional linear subspaces of $\mathbb{C}^{n+1}$. With this understanding, let

$$\gamma = \big\{ (\ell, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : v \in \ell \subset \mathbb{C}^{n+1} \big\}. \tag{B.3.1}$$

Exercise B.3.1. Show that γ is a complex submanifold of $\mathbb{P}^n \times \mathbb{C}^{n+1}$

In fact, the projection $\pi : \gamma \longrightarrow \mathbb{P}^n$ defines a holomorphic line bundle, which we will call the tautological line bundle over $\mathbb{P}^n$. A trivialization f_i of γ over the open subset U_i of $\mathbb{P}^n$ is given by

$$f_i\colon \gamma|_{U_i} \longrightarrow U_i \times \mathbb{C}, \qquad f_i\big([X_0, \ldots, X_n], (Z_0, \ldots, Z_n)\big) = \big([X_0, \ldots, X_n], Z_i\big). \tag{B.3.2}$$

The corresponding overlap data is given by

$$g_{ij}\colon U_i \cap U_j \longrightarrow \mathbb{C}^*, \qquad g_{ij}\big([X_0, \ldots, X_n]\big) = \frac{X_i}{X_j}, \tag{B.3.3}$$

$$\text{i.e.} \qquad \pi_2 f_i(\ell, v) = g_{ij}(\ell) \pi_2 f_j(\ell, v) \quad \forall\, (\ell, v) \in \gamma|_{U_i \cap U_j},$$

where $\pi_2\colon U_i \times \mathbb{C}, U_j \times \mathbb{C} \longrightarrow \mathbb{C}$ are the projections onto the second component. We will call f_i the standard trivialization of γ over the i-th coordinate chart on $\mathbb{P}^n$.

If $a \geq 0$, a holomorphic section s of $\gamma^{*\otimes a}$ corresponds to a holomorphic map

$$s\colon \gamma \longrightarrow \mathbb{C} \qquad \text{s.t.} \qquad s(\ell, cv) = c^a s(\ell, v) \quad \forall\, (\ell, v) \in \gamma,\ c \in \mathbb{C}\,.$$

Thus, any degree a homogeneous polynomial Q in $X_0, \ldots, X_n$ induces a section s_Q of $\gamma^{*\otimes a}$ by

$$s_Q\colon \gamma \longrightarrow \mathbb{C}, \qquad s_Q(\ell, v) = Q(v) \quad \forall\, (\ell, v) \in \gamma.$$

By Lemma B.3.3 below, all holomorphic sections of $\gamma^{*\otimes a}$, for any $a \in \mathbb{Z}$, are of this form.

Exercise B.3.2. If Q is linear function of $X_0, \ldots, X_n$, show that the induced section s_Q of $\gamma^* \longrightarrow \mathbb{P}^n$ is transverse to the zero set.

By this exercise and Theorem 0.4, the first chern class of the line bundle γ^* over $\mathbb{P}^n$ is the Poincare dual of a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$:

$$c_1(\gamma^*) = \sigma_1 \in \mathrm{H}^2(\mathbb{P}^n; \mathbb{Z}) . \tag{B.3.4}$$

Lemma B.3.3. *If $a \in \mathbb{Z}^+$, the line bundle $\gamma^a \longrightarrow \mathbb{P}^n$ admits no nonzero holomorphic section. If $a \in \mathbb{Z}^{\geq 0}$, every holomorphic section of $\gamma^{*\otimes a}$ is of the form s_Q for some degree a homogeneous polynomial Q in $X_0, \ldots, X_n$.*

Proof. (1) It is sufficient to prove the first claim for $n=1$. If s is a nonzero holomorphic section of the line bundle $\gamma^a \longrightarrow \mathbb{P}^1$, $s^{-1}(0)$ is a finite set. By Corollary 0.7 and (B.3.4), its cardinality counted with some positive multiplicities is

$$\big\langle e(\gamma^a), [\mathbb{P}^1] \big\rangle = -a \big\langle e(\gamma^*), [\mathbb{P}^1] \big\rangle = -a \big\langle \sigma_1, [\mathbb{P}^1] \big\rangle = -a < 0.$$

However, this is impossible if $a > 0$.

(2) Suppose $a \geq 0$ and s is a holomorphic sections of the line bundle $\gamma^{*\otimes a}$ over $\mathbb{P}^n$. Since the projection map

$$\pi_2 \colon \gamma - \mathbb{P}^n \longrightarrow \mathbb{C}^{n+1} - 0$$

is a biholomorphism, s induces a holomorphic function

$$\tilde{s} \colon \mathbb{C}^{n+1} - 0 \longrightarrow \mathbb{C} \qquad \text{s.t.} \qquad \tilde{s}(cv) = c^a \tilde{s}(v) \quad \forall\, v \in \mathbb{C}^{n+1} - 0,\ c \in \mathbb{C}^*.$$

By Hartog's Theorem [5, p7], $\tilde{s}$ extends to a holomorphic function

$$Q \colon \mathbb{C}^{n+1} \longrightarrow \mathbb{C} \qquad \text{s.t.} \qquad Q(cv) = c^a Q(v) \quad \forall\, v \in \mathbb{C}^{n+1},\ c \in \mathbb{C}.$$

Thus, Q is a degree a homogeneous polynomial as claimed. □

to be added: Castelnuovo bound, every degree d curves lies in a $\mathbb{P}^d$, in a $\mathbb{P}^{d-1}$ if at least of genus 1

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