MAT 615: Complex Curves and Surfaces
Spring 2009

Problem Set 6

Here is a final collection of exercises, that should not be too difficult; you do not need to hand in any written solutions.

Exercise 1

Let S be a complex surface such that $K_S \longrightarrow S$ is a negative line bundle.
(a) Using Noether's formula, show that $q(S) = 0$. Conclude that S is rational.
(b) If in addition S is minimal, show that S is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

Exercise 2

Let α be a primitive 5th root unity (so $\alpha^5 = 1$, but $\alpha \neq 1$) and

$$\tilde{S} = \big\{[X_0, X_1, X_2, X_3] \in \mathbb{P}^3 :\ X_0^5 + X_1^5 + X_2^5 + X_3^5 = 0\big\}.$$

Then $\mathbb{Z}_5$ acts on $\tilde{S}$ by

$$\alpha \cdot [X_0, X_1, X_2, X_3] = [X_0, \alpha X_1, \alpha^2 X_2, \alpha^3 X_3].$$

Show that
(a) $\tilde{S}$ and $S \equiv \tilde{S}/\mathbb{Z}_5$ are smooth projective surfaces;
(b) $q(S) = p_g(S) = 0$, but $K_S \longrightarrow S$ is a positive line bundle, and thus S is of general type (and in particular not rational).

Exercise 3

Let $\pi : S \longrightarrow \Sigma$ be an irrational ruled surface. Show that every irreducible rational curve is contained in a fiber of π and thus S is minimal.

Exercise 4

Let S be a projective surface containing infinitely many exceptional curves (such surfaces exist by PS4 #6). Show that S is rational.

Exercise 5

If S is a compact Kahler surface, each element $\gamma\in H_1(S;\mathbb{Z})$ defines a homomorphism

$$\int_\gamma \cdot : H^{1,0}(S) \longrightarrow \mathbb{C}, \qquad \omega \longrightarrow \int_\gamma \omega,$$

and thus an element of $H^{1,0}(S)^*$. Let

$$\Lambda_S = \left\{ \int_\gamma \cdot : \ \gamma\in H_1(S;\mathbb{Z}) \right\} \subset H^{1,0}(S)^*.$$

(a) Show that $\Lambda_S \subset H^{1,0}(S)^*$ is a lattice.
(b) If $\alpha : [0,1] \longrightarrow S$ is a path, show that the element

$$\int_\alpha \cdot \in \mathrm{Alb}(S) \equiv H^{1,0}(S)^*/\Lambda_S, \qquad \omega \longrightarrow \int_\alpha \omega \in \mathbb{C},$$

depends only on $\alpha(0)$ and $\alpha(1)$.
(c) Thus, for each $p\in S$, there is a well-defined map

$$\mu_p : S \longrightarrow \mathrm{Alb}(S), \qquad q \longrightarrow \int_p^q \cdot\,.$$

Show that the image of this map has dimension at least one, unless $q(S)=0$.
(d) If $q(S)\geq 1$ and $p_g(S)=0$, show that the image of μ_p is one-dimensional.

Exercise 6

Let S be a smooth compact complex surface, $L\longrightarrow S$ a holomorphic line bundle, and $C\subset S$ a curve, possibly singular. Let

$$\check{H}^i(C;L) = \check{H}^i(S;L|_C), \quad i\in\mathbb{Z}, \qquad K_C = K_S(C)|_C\,;$$

the latter is a sheaf on S, as is $L|_C$. If C is smooth, these definitions agree with the usual ones. Similarly, let

$$\chi(C,L) = \sum_{i\in\mathbb{Z}} (-1)^i \dim \check{H}^i(C;L)\,, \qquad \deg_C L = L\cdot C \equiv \langle c_1(L), [C]\rangle.$$

(a) Using Serre duality for line bundles on S and the Five Lemma, show that

$$\check{H}^i(C;L) \approx \check{H}^{1-i}(C;K_C(-L))^*\,.$$

Conclude that $\check{H}^i(C;L)=0$ unless $i=0,1$.
(b) Using Riemann-Roch for line bundles on S, show that

$$\chi(C,L) = 1 - a(C) + \deg_C L\,.$$

(c) Conclude that if C is connected, then

$$\dim \check{H}^0(C;K_C) = a(C).$$

Exercise 7

Let $\pi\colon S \longrightarrow \Sigma$ be a holomorphic map from a smooth compact connected complex surface to a smooth compact connected curve of genus g. Show that
(a) the fiber $F_\lambda = \pi^{-1}(\lambda)$ is smooth for all but finitely many $\lambda \in \Sigma$ (*hint:* get a pencil from Σ to use Bertini);
(b) the fibers F_λ and $F_{\lambda'}$ are not linearly equivalent if $\lambda \neq \lambda'$ and $g \geq 1$;
(c) if $g \geq 1$, every irreducible curve C such that the normalization $\tilde{C}$ of C is $\mathbb{P}^1$ is contained in a fiber;
(d) if in addition a generic fiber F of π is connected and of genus $a(F) \geq 1$, there are finitely many such curves.

Exercise 8

With $\pi\colon S \longrightarrow \Sigma$ and $F \subset S$ as in Exercise 7, assume also that $q(S)=1$, $g(\Sigma)=1$, $a(F)\geq 1$, and S contains a smooth irreducible curve C such that C is not contained in any fiber of π,

$$g(C) = 1, \qquad C^2 > 0, \qquad h^0(K_S + C) = 0.$$

Fixing $\lambda_0 \in \Sigma$, for any $\lambda \in \Sigma$ set

$$L_{C,\lambda} = C + F_\lambda - F_{\lambda_0}\,.$$

(a) Show that no element of the linear system $|L_{C,\lambda}|$ is contained in a finite union of fibers of π.
(b) Suppose mC', where C' is an irreducible curve and $m \in \mathbb{Z}^+$, is an element of $|L_{C,\lambda}|$. Show that $m=1$ and C' is smooth of genus 1.
(c) Suppose $\sum_{i=1}^{i=k} m_i C_i$, where $k \geq 2$, $C_i \subset S$ irreducible, $C_i \neq C_j$ for $i \neq j$, and $m_i \in \mathbb{Z}^+$, is an element of $|L_{C,\lambda}|$. Show that

$$C \cdot (K_S + C_i) < 0, \qquad h^0(K_S + C_i) = 0, \qquad a(C_i) \leq 1$$

for every i. If in addition C_i is not contained in any fiber, then C_i is smooth of genus 1. Furthermore, $C_i \cdot K_S < 0$ for some i; if S is minimal and F is connected, this C_i is not contained in any fiber.

Exercise 9

Let $\pi\colon S \longrightarrow \Sigma$ and C be as in Exercise 8, with S minimal and F connected.
(a) Show that there exists an irreducible curve D on S such that for every $\lambda \in \Sigma$ and every element D' of the linear system $|L_{D,\lambda}|$

$$a(D') = 1, \qquad D'^2 = D^2 \equiv d > 0, \qquad h^0(K_S + D') = 0;$$

furthermore, D' is smooth of genus 1 and is not contained in any fiber of π.
(b) Show that for all $\lambda, \lambda' \in \Sigma$ with $\lambda \neq \lambda'$ and $D' \in |L_{D,\lambda}|$, the restriction map

$$H^0(S; L_{D,\lambda'}) \longrightarrow H^0(D'; L_{D,\lambda'})$$

is injective. Conclude that $h^0(S, L_{D,\lambda}) = d$ for every $\lambda \in \Sigma$.
(c) Let $p_1, \ldots, p_{d-1} \in S$ be general points. Show that for each $\lambda \in \Sigma$, there is an element D_λ of $|L_{D,\lambda}|$ passing through the points $p_1, \ldots, p_{d-1}$; if λ is generic, this element is unique. If $\lambda, \lambda' \in \Sigma$ and $\lambda \neq \lambda'$, the set $D_\lambda \cap D_{\lambda'}$ consists of a single point which is independent of the choice of D_λ and $D_{\lambda'}$ (if there is a choice). Show that the map

$$\Sigma - \lambda \longrightarrow D_\lambda, \qquad \lambda' \longrightarrow D_\lambda \cap D_{\lambda'},$$

extends to a surjective map f_λ over Σ. Conclude that the choice of λ is in fact unique.
(d) Identify Σ with $\mathbb{C}/\Lambda$, where Λ is a lattice in $\mathbb{C}$. Show that the map

$$h \colon \Sigma \times \Sigma \longrightarrow S, \qquad (\mu, \lambda) \longrightarrow f_\lambda(\mu - \lambda),$$

is holomorphic surjective and induces a family of non-constant holomorphic maps

$$h_\mu \colon \mathbb{P}^1 \longrightarrow S, \qquad \lambda \longrightarrow h(\mu, \lambda),$$

which cover S. However, this contradicts Exercise 7d.

The purpose of Exercises 7-9 is to add details for G&H pp561-563, especially for p563. More precisely, suppose S is a minimal surface with $q(S) \geq 1$, which contains an irreducible curve C with $C \cdot K_S < 0$ and $h^0(K_S + C) = 0$, Σ is a smooth curve with $g(\Sigma) = q(S)$, and $\pi \colon S \longrightarrow \Sigma$ is a holomorphic map with generic fiber F irreducible and $\pi(C) = \Sigma$. By the top of p562, $g(C) = g(\Sigma)$ so that $\pi \colon C \longrightarrow \Sigma$ is either an isomorphism or $q(S) = 1$. By the middle of p562, $a(F) = 0$ if $q(S) \geq 2$. By a more involved argument on p563 and in Exercises 7-9, the same conclusion holds for $q(S) = 1$.

Exercise 10

Let $V \subset H^0(\mathbb{P}^2; \mathcal{O}(3))$ be the two-dimensional vector space spanned by two general cubic polynomials. Show that
(a) the base locus B of the linear system $\mathbb{P}V$ on $\mathbb{P}^2$ consists of 9 points;
(b) all of the cubics in $\mathbb{P}V$ are smooth, except for 12 that have exactly one node each;
(c) the proper transform $\tilde{V}$ of V in the blowup S of $\mathbb{P}^2$ at B induces a morphism

$$\pi \colon S \longrightarrow \mathbb{P}\tilde{V}^*,$$

expressing S as an elliptic surface over $\mathbb{P}^1$ with no multiple fibers and 12 nodal fibers. What do the 9 exceptional divisors have to do with π?