MAT 615: Complex Curves and Surfaces

Partial Problem Set 5 Solutions

Problem 2 (10 pts)

(a) *If S is a minimal surface and $D\subset S$ is an effective divisor such that $D\cdot K_S<0$, then*

$$P_n(S)\equiv h^0(K_S^n)=0 \qquad \forall n\in\mathbb{Z}^+.$$

(b) *If S,S' are minimal (projective) surfaces and $f:S\longrightarrow S'$ is a birational map, then either f is a biholomorphic map or $P_n(S)=P_n(S')=0$ for all $n\in\mathbb{Z}^+$.*
Note: This says that a surface S with Kodaira dimension $\kappa(S)\geq 0$ has a unique minimal model.

(a) We can assume that D is irreducible. If $C\subset S$ is a divisor in the linear system nK_S for some $n\in\mathbb{Z}^+$, then

$$D\cdot C=D\cdot nK_S<0.$$

Thus, D is one of the irreducible components of C and $D^2<0$. Since $D\cdot K_S<0$ as well, D is an exceptional divisor for a blowup by the Castelnuovo-Enriques Criterion, contrary to the assumption that S is minimal.

(b) We first note the following. If $\pi:\widetilde{S}\longrightarrow S$ is the blowup at a point $p'\in S'$ with the exceptional divisor E and $\overline{C}\subset\widetilde{S}$ is the proper transform of a curve $C\subset S$, then

$$K_{\widetilde{S}}=\pi^*K_S+E,\quad \overline{C}=\pi^*C-mE \qquad\Longrightarrow\qquad \overline{C}\cdot K_{\widetilde{S}}=C\cdot K_S+m$$

for some $m\geq 0$. Thus, $C\cdot K_S$ does not decrease under blowdowns.

Suppose now that $f:S\longrightarrow S'$ is a birational map between minimal surfaces which is not biholomorphic. Thus, there exist blowups $\pi:\widetilde{S}\longrightarrow S$ and $\pi':\widetilde{S}\longrightarrow S'$ so that $\pi'=f\circ\pi$. Take minimal such blowup, i.e. $\widetilde{S}$ contains an exceptional curve $\widetilde{C}\subset\widetilde{E}$ which is contracted by π', but not by π (and the other way around). Let $C=\pi(\widetilde{C})\subset S$. Since $\widetilde{C}\cdot K_{\widetilde{S}}=-1$, $C\cdot K_S<0$ by the previous paragraph and thus $P_n(S)=0$ by part (a). By symmetry, $P_n(S')=0$ as well.