

MAT 615: Complex Curves and Surfaces

Problem Set 5

Written Solutions (if any) due by Wednesday, 04/25, 9:45am

Please figure out all of the problems below and discuss them with others.

If you have not passed the orals yet, you are encouraged to write up concise solutions to problems adding up to 10 points in total.

Problem 1 (10 pts)

Let S be a (compact) complex surface, $\mathcal{C} \subset S$ be a (closed) connected curve, and $h : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ a normalization of \mathcal{C} . Let $S_0 = S, S_1, \dots$ be a sequence of surfaces and $\mathcal{C}_0 = \mathcal{C}, \mathcal{C}_1, \dots$ a sequence of curves, with $\mathcal{C}_k \subset S_k$, such that S_k is the blowup of S_{k-1} at a singular point $p_k \in \mathcal{C}_{k-1}$ and \mathcal{C}_k is the proper transform of \mathcal{C}_{k-1} under the blowup $S_k \rightarrow S_{k-1}$. It was shown in class that

$$a(\mathcal{C}_k) = a(\mathcal{C}) - \sum_{l=1}^k \binom{m_{p_l}(\mathcal{C}_{l-1})}{2}, \quad (1)$$

where $m_{p_l}(\mathcal{C}_{l-1})$ is the order of the point p_l in the curve \mathcal{C}_{l-1} . The above sequences terminate at the k -th stage if and only if $\mathcal{C}_k \subset S_k$ is a smooth curve. If this does happen, $\mathcal{C}_k \approx \tilde{\mathcal{C}}$ and equation (1) implies that

$$g(\tilde{\mathcal{C}}) \leq a(\mathcal{C}) - \sum_{p \in \mathcal{C}} \binom{m_p(\mathcal{C})}{2}, \quad a(\mathcal{C}) \in \mathbb{Z}^{\geq 0}. \quad (2)$$

In particular, $g(\tilde{\mathcal{C}}) \leq a(\mathcal{C})$ and the equality holds if and only if $\mathcal{C} \subset S$ is smooth. In class, the first statement in (2) was proved in a different way for S projective; it implies that $a(\mathcal{C}) \geq 1 - N(\mathcal{C})$ for any curve, where $N(\mathcal{C})$ is the number of connected components of \mathcal{C} . This was used along with (1) to show that the above sequence does terminate. The estimate $a(\mathcal{C}) \geq 1 - N(\mathcal{C})$ can also be obtained by showing either that $a(\mathcal{C}) + N(\mathcal{C}) - 1$ is the dimension of the space of holomorphic 1-forms on \mathcal{C} , properly defined (since \mathcal{C} is singular), or that $a(\mathcal{C})$ is the genus of a smooth curve in the same homology class as \mathcal{C} , which exists at least after deforming the complex structure. The purpose of this exercise is to show that the sequences terminate without an *a priori* lower bound on $a(\mathcal{C})$.

(a) Let $B_{i;0} \subset \mathcal{C}$ be a branch of \mathcal{C} at a singular point $p \in \mathcal{C}$ and $B_{i;k} \subset S_k$ be the proper transform of $B_{i;k-1}$ in S_k . Show that $\pi_{i;k} : B_{i;k} \rightarrow B_{i;0}$ is a bijection, $B_{i;k}$ contains at most 1 singular point, $p_{i;k} \equiv \pi_{i;k}^{-1}(p)$, and $p_k \neq p_{i;k}$ if k is sufficiently large (i.e. $B_{i;k}$ becomes smooth after enough blowups at $p_{i;k}$).

(b) If B_1 and B_2 are distinct smooth branches at $p_k \in \mathcal{C}_{k-1}$, the order of contact of the proper transforms of B_1 and B_2 in S_k is strictly less than $\text{mult}_{p_k}(B_1, B_2)$.

(c) Show that \mathcal{C}_k is a smooth curve if k is sufficiently large.

Problem 2 (10 pts)

(a) If S is a minimal surface and $D \subset S$ is an effective divisor such that $D \cdot K_S < 0$, then

$$P_n(S) \equiv h^0(K_S^n) = 0 \quad \forall n \in \mathbb{Z}^+.$$

(b) If S, S' are minimal (projective) surfaces and $f : S \rightarrow S'$ is a birational map, then either f is a biholomorphic map or $P_n(S) = P_n(S') = 0$ for all $n \in \mathbb{Z}^+$.

Note: This says that a surface S with Kodaira dimension $\kappa(S) \geq 0$ has a unique minimal model.

Problem 3 (10 pts)

Let S be a compact connected surface and D an effective divisor on S .

(a) If $D^2 \equiv D \cdot D < 0$, show that the base locus of the linear system of curves $|D|$ contains a curve. Is the converse true?

(b) If the base locus of $|D|$ does not contain a curve, show that it consists of at most D^2 points.

(c) If D is irreducible and $h^0(D) \geq 2$, then $D^2 \geq 0$. Show that the conclusion need not hold if D is not assumed to be irreducible.

(d) If S is a minimal rational surface and $D^2 \geq 0$, show that $h^0(D) \geq 2$. Show that the conclusion need not hold if S is not assumed to be minimal.

Hint: you can achieve arbitrarily high D^2 with $h^0(D) = 1$ by blowing up \mathbb{P}^2 at sufficiently many general points.

Problem 4 (5 pts)

Let $\mathbb{F}_n = S_n$ be the n -th Hirzebruch surface and $D \subset \mathbb{F}_n$ an irreducible curve. Compute $H^q(\mathbb{F}_n, \mathcal{O}(D))$ in terms of the decomposition of D relative to a standard basis for $\text{Pic}(\mathbb{F}_n)$.