MAT 615: Complex Curves and Surfaces

Problem Set 4

Written Solutions (if any) due by Wednesday, 04/05, 9:45am

Please figure out all of the problems below and discuss them with others.

If you have not passed the orals yet, *you are encouraged to write up concise solutions to problems adding up to 10 points in total.*

Problem 1 (5 pts)

Let $\mathcal{U} \longrightarrow \overline{\mathcal{M}}_{1,1}$ be the universal family of stable genus 1 1-marked curves with section s; see Hain's Section 5.2. Let

$$\mathbb{L}_1 \equiv s^*(T\mathcal{U}^{\mathrm{vert}})^*, \; \mathcal{L}_1 \longrightarrow \overline{\mathcal{M}}_{1,1}$$

be the universal cotangent line bundle at the (first) marked point and the line bundle defined in Hain's Section 4, respectively.
(a) Show that $\mathbb{L}_1$ is indeed a line bundle in the orbifold category (i.e. describe equivariant trivializations over the charts and the isomorphism on the overlap).
(b) Show that $\mathbb{L}_1 \approx \mathcal{L}_1$ in the orbifold category.

Problem 2 (10 pts)

Fix any 8 general points, $p_1, \ldots, p_8$, in $\mathbb{P}^2$.
(a) Show that the space of cubics passing through the 8 points is a linearly embedded $\mathbb{P}^1$ in $\mathbb{P}H^0(\mathbb{P}^2; \mathcal{O}(3)) \approx \mathbb{P}^9$.
(b) Show that

$$X \equiv \big\{([f], q) \in \mathbb{P}^1 \times \mathbb{P}^2 : \; f(q) = 0\big\}$$

is a smooth submanifold of $\mathbb{P}^1 \times \mathbb{P}^2$. What is its Hodge diamond?
(c) Show that $\pi\colon X \longrightarrow \mathbb{P}^1$ with the holomorphic section

$$s\colon \mathbb{P}^1 \longrightarrow X, \qquad [f] \longrightarrow \big([f], p_1\big),$$

is a family of stable genus 1 1-marked curves (i.e. $(\pi^{-1}(b), s(b))$ is a stable genus 1 1-marked curve for every $b \in \mathbb{P}^1$) and the generic fiber is smooth. Show that the number of singular fibers is 12.
Hint: the set of nodes of the fibers is a subset of X which can be written as the zero set of a holomorphic section of a rank-two vector bundle over X.
(d) Let $L_1 = s^*(TX^{\mathrm{vert}})^* \longrightarrow \mathbb{P}^1$. Show that $L_1 \approx \mathcal{O}(1) \longrightarrow \mathbb{P}^1$ and $L_1 \approx \Phi^*\mathbb{L}_1$, where $\Phi\colon \mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}_{1,1}$ is the morphism corresponding to the family in (c). Conclude that

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24} \, ,$$

where $\psi_1 = c_1(\mathbb{L}_1) \in H^2(\overline{\mathcal{M}}_{1,1})$.

Problem 3 (15 pts)

Let S be a compact complex surface ($\dim_{\mathbb{C}} S=2$). Show that
(a) if $f\colon S\longrightarrow\mathbb{C}$ is a smooth function s.t. $\bar\partial\partial f=0$, then f is constant;
(b) there are natural injections $\overline{H^{1,0}(S)}\longrightarrow H^{0,1}(S)$ and $H^{1,0}(S)\oplus\overline{H^{1,0}(S)}\longrightarrow H^1(S;\mathbb{C})$;
(c) $b_1(S)\le h^{1,0}(S)+h^{0,1}(S)$;
(d) there are natural injections $\overline{H^{2,0}(S)}\longrightarrow H^{0,2}(S)$ and $H^{2,0}(S)\oplus\overline{H^{2,0}(S)}\longrightarrow H^2(S;\mathbb{C})$;
(e) $\big(b_2^+(S)-2h^{0,2}(S)\big)+\big(2h^{0,1}(S)-b_1(S)\big)=1$;
(f) $b_1(S)=h^{1,0}(S)+h^{0,1}(S)$ and either $h^{1,0}(S)=h^{0,1}(S)$ and $b_2^+(S)=2h^{0,2}(S)+1$
or $h^{1,0}(S)=h^{0,1}(S)-1$ and $b_2^+(S)=2h^{0,2}(S)$;
(g) $h^{1,0}(S), h^{0,1}(S), h^{2,1}(S), h^{1,2}(S)$ (resp. $h^{2,0}(S), h^{0,2}(S)$) are topological invariants of the unoriented (resp. oriented) manifold S.
Hints: (a) show that f is harmonic in each variable in each coordinate chart;
(b) for $\alpha\in H^{1,0}(S)$, show that $\int_S(\mathrm{d}\alpha)\wedge(\mathrm{d}\bar\alpha)=0$;
(c) let $\mathcal{Z}$ denote the sheaf of closed holomorphic 1-forms on S; use the short exact sequence

$$0\longrightarrow\underline{\mathbb{C}}\longrightarrow\mathcal{O}_S\xrightarrow{\ \mathrm{d}\ }\mathcal{Z}\longrightarrow 0$$

of sheaves on S;
(d) for $\beta\in H^{2,0}(S)-\{0\}$, show that $\int_S\beta\wedge\bar\beta\neq 0$;
(e) use Noether's and Hirzerbruch's formulas:

$$\chi(\mathcal{O}_S)=\frac{1}{12}\big(K_S^2+\chi(S)\big)\qquad\text{and}\qquad b_2^+(S)-b_2^-(S)=\frac{1}{3}\big(K_S^2-2\chi(S)\big).$$

Note: From Serre Duality and Riemann-Roch for $T^*S\longrightarrow S$,

$$2h^{1,0}(S)-h^{1,1}(S)=\chi(T^*S)=\frac{1}{6}\big(K_S^2-5\chi(S)\big),$$

we find that $h^{1,1}(S)$ is also a topological invariant of the oriented manifold S and that

$$b_2(S)=h^{2,0}(S)+h^{1,1}(S)+h^{0,2}(S).$$

Problem 4 (5 pts)

Let $f: C\longrightarrow\mathbb{P}^2$ be an immersion with only simple normal crossing singularities (thus, $|f^{-1}(p)|\le 2$ for all $p\in\mathbb{P}^2$; if $f^{-1}(p)=\{z_1,z_2\}$ with $z_1\neq z_2$, $d_{z_1}f(T_{z_1}C)\neq d_{z_2}f(T_{z_2}C)$). Let S be the blowup of $\mathbb{P}^2$ at the double points of $f(C)$ (the singular values of f).
(a) Show that f lifts to an embedding $\tilde f\colon C\longrightarrow S$.
(b) Use Adjunction Formula in S to show that if $C\subset\mathbb{P}^2$ is of degree d and has δ double points, then

$$g(C)=\binom{d-1}{2}-\delta\,.$$

Problem 5 (5 pts)

Let X be the blowup of $\mathbb{P}^2$ at one point and $E \subset X$ the exceptional divisor. Show that
(a) X is not biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$;
(b) the blowup of X at a point of $X - E$ is biholomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point;
(c) $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are minimal surfaces (contain no exceptional curves).

Problem 6 (5 pts)

Let $\pi \colon S \longrightarrow \mathbb{P}^2$ be the blowup at k general points, $p_1, \ldots, p_k \in \mathbb{P}^2$, with the corresponding exceptional divisors $E_1, \ldots, E_k$. Let

$$D = dH - \sum_{i=1}^{i=k} m_i E_i, \qquad m_i \in \mathbb{Z},$$

and L_{ij} be the proper transform of the line $\overline{p_i p_j}$, with $i \neq j$. Suppose $|D| \neq \emptyset$ (i.e. there is an effective divisor linearly equivalent to D). Show that
(a) E_i is a fixed component of every curve in $|D|$ iff $m_i < 0$;
(b) $L_{i,j}$ is a fixed component of every curve in $|D|$ iff $d < m_i + m_j$.

Problem 7 (10 pts)

Let $\pi \colon S \longrightarrow \mathbb{P}^2$ be the blowup at k general points, $p_1, \ldots, p_k \in \mathbb{P}^2$.
(a) Determine the number of exceptional curves in S for $k \leq 8$.
(b) Show that there are infinitely many exceptional curves in S for $k \geq 9$.
Hint. The structure of $\mathrm{Pic}(S)$ and the intersection theory on S give necessary numerical conditions for which classes in $\mathrm{Pic}(S)$ *may* be representable by exceptional curves. These conditions readily imply that the number of exceptional curves in S is finite if $k \leq 8$. It then remains to show that all the necessary conditions are sufficient if $k \leq 8$ and that there are infinitely many exceptional curves if $k = 9$. For both purposes, the Cremona transform on pp496,7 can be useful.