MAT 615: Complex Curves and Surfaces

Problem Set 2 Solutions

Problem 1 (5 pts)

Describe all special divisors on a smooth compact Riemann surface of genus 0,1 and 2.

If D is an effective divisor on a smooth compact Riemann surface S of genus g,

$$0 \le \deg D \le \deg K_S = 2g-2.$$

Thus, there are no effective divisors if $g=0$. If $g=1$, any effective divisor D on S is of degree 0 and thus must be the zero divisor. If $g=2$, any effective divisor D on S is of degree 0, 1, or 2. In this case, the canonical map

$$\iota_{K_S}: S \longrightarrow \mathbb{P}\big(H^0(S;K_S)^*\big)$$

is a degree 2 branched cover, which determines a holomorphic involution $\tau: S \longrightarrow S$. Since $K_S = \iota_{K_S}^*\mathcal{O}(1)$,

$$K_S = \big[p+\tau(p)\big] \qquad \forall\, p\in S.$$

Thus, the effective divisors on S are $0, p, p+\tau(p)$ (all of the degree 1 divisors p are not linearly equivalent, while all of the degree 2 divisors $p+\tau(p)$ are linearly equivalent).

Problem 2 (5 pts)

Let $C, D_1, D_2 \subset \mathbb{P}^2$ be smooth cubics so that

$$C \cdot D_1 = \sum_{i=1}^{i=9} p_i$$

as divisors on C and D_2 passes through $p_1, \ldots, p_8$. Show that $p_9 \in D_2$.

Let $p_9' \in C \cap D_2$ be the 9-th point of the intersection so that

$$C \cdot D_2 = \sum_{i=1}^{i=8} p_i + p_9'$$

as divisors on C. Since D_1, D_2 are linearly equivalent divisors on $\mathbb{P}^2$, the points p_9, p_9' are linearly equivalent on C. Since C is of genus 1, this implies that $p_9 = p_9'$ (because C does not admit a meromorphic function with a single simple pole).

Alternatively, suppose $D_1 = (f_1)$ and $D = (f_2)$ for some degree 3 homogeneous polynomials in 3 variables. The restriction of f_1/f_2 to C is then a meromorphic function on S with a single simple pole at p_9' if $p_9 \ne p_9'$. Such a function does not exist because C is of genus 1.

Problem 3 (5 pts)

Let $C \subset \mathbb{P}^n$ with $n \geq 3$ be a smooth (connected) curve of genus 1 and degree 4. Show that C is contained in some linearly embedded $\mathbb{P}^3 \subset \mathbb{P}^n$ and is the intersection of two quadric (degree 2) surfaces in that $\mathbb{P}^3$.

Since $L \equiv \mathcal{O}_{\mathbb{P}^n}(1)|_C \longrightarrow C$ is a holomorphic line bundle of degree 4, Riemann-Roch gives

$$h^0(L) = 1-1+4+h^0\big(K_C \otimes L^*\big) = 4.$$

Thus, C is a contained in a $\mathbb{P}^3$.

Since the dimensions of $H^0(\mathbb{P}^3; \mathcal{O}_{\mathbb{P}^3}(2))$ and $H^0(C; \mathcal{O}_{\mathbb{P}^3}(2)|_C)$ are

$$\binom{2+3}{3} = 10 \qquad \text{and} \qquad 1-1+2\cdot4 = 8\,,$$

respectively, there exist two distinct quadric hypersurfaces $H_1, H_2 \subset \mathbb{P}^3$ so that $C \subset H_1 \cap H_2$. Since $4 = 2\cdot2$, this inclusion must be an equality.