

# LECTURES ON MODULI SPACES OF ELLIPTIC CURVES

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ABSTRACT. The goal of these notes is to introduce and motivate basic concepts and constructions (such as orbifolds and stacks) important in the study of moduli spaces of curves and abelian varieties through the example of elliptic curves. Moduli spaces of elliptic curves are rich enough so that one encounters most of the important issues associated with moduli spaces, yet simple enough that most of the constructions are elementary and explicit. These notes touch on all four aspects of the study of moduli spaces of curves – complex analytic, topological, algebro-geometric, and number theoretic.

## CONTENTS

1. Introduction to Elliptic Curves and the Moduli Problem	3
2. Families of Elliptic Curves and the Universal Curve	11
3. The Orbifold $\mathcal{M}_{1,1}$	17
4. The Orbifold $\overline{\mathcal{M}}_{1,1}$ and Modular Forms	29
5. Cubic Curves and the Universal Curve $\overline{\mathcal{E}} \rightarrow \overline{\mathcal{M}}_{1,1}$	38
6. The Picard Groups of $\mathcal{M}_{1,1}$ and $\overline{\mathcal{M}}_{1,1}$	53
7. The Algebraic Topology of $\overline{\mathcal{M}}_{1,1}$	60
8. Concluding Remarks	64
Appendix A. Background on Riemann Surfaces	69
Appendix B. A Very Brief Introduction to Stacks	77
References	80

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2008.<sup>1</sup> Their goal is to introduce and motivate basic concepts and constructions important in the study of moduli spaces of curves and abelian varieties through the example of elliptic curves. The advantage of working with elliptic curves is that most constructions are elementary and explicit. All four approaches to moduli spaces of curves — complex analytic, topological, algebro-geometric, and number theoretic — are considered. Topics covered reflect my own biases. Very little, if anything, in these notes is original, except perhaps the selection of topics and the point of view.

Many moduli spaces are usefully regarded as orbifolds or stacks. The notes include a detailed exposition of orbifolds, which is motivated by a discussion of how the quotient of the upper half plane by the modular group  $SL_2(\mathbb{Z})$  is related to families of elliptic curves. The moduli space of elliptic curves  $\mathcal{M}_{1,1}$  and its Deligne-Mumford compactification  $\overline{\mathcal{M}}_{1,1}$  are constructed as orbifolds. Modular forms arise naturally as holomorphic sections of powers of the Hodge bundle over the orbifold  $\overline{\mathcal{M}}_{1,1}$ . They, in turn, are used to construct the extension of the universal elliptic curve  $\mathcal{E} \rightarrow \mathcal{M}_{1,1}$  to the universal stable elliptic curve  $\overline{\mathcal{E}} \rightarrow \overline{\mathcal{M}}_{1,1}$ . The homotopy types and Picard groups of the orbifolds  $\mathcal{M}_{1,1}$  and  $\overline{\mathcal{M}}_{1,1}$  are computed explicitly. The discussion of orbifolds is used to motivate the definition of stacks, which is discussed very briefly in Appendix B.

**Note to the reader:** These notes are intended for students. The exposition is generally elementary, but some sections, especially those later in the notes, are more demanding.

- (i) The best way to learn about moduli spaces and orbifolds (and stacks) is to work with them. For this reason, these notes contain over 100 exercises. The reader is encouraged to work as many of them as possible.
- (ii) Sections not central to the exposition are marked with an asterisk \*. These can be skipped.
- (iii) Some basic background material on Riemann surfaces is reviewed in Appendix A.

**Background:** These notes assume familiarity with the definition of and basic facts about Riemann surfaces, including the definition of holomorphic and meromorphic functions and 1-forms, and of holomorphic line bundles. They also assume familiarity with the basic concepts

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<sup>1</sup>*Geometry of Teichmüller spaces and moduli spaces of curves*, Zhejiang University, July 14–20, 2008.

of algebraic topology, including homology, fundamental groups and covering spaces. Some familiarity with sheaves is desirable, but not essential. Good basic references for Riemann surfaces include Forster's book [4] and Griffiths' China notes [5]; Clemens' book [3] is an excellent supplement.

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## 1. INTRODUCTION TO ELLIPTIC CURVES AND THE MODULI PROBLEM

A Moduli space of Riemann surfaces is a space whose points correspond to all isomorphism classes of Riemann surface structures on a fixed compact oriented surface. They themselves are algebraic varieties (or *orbifolds*). In this section, we will construct the moduli space of elliptic curves, which is itself a Riemann surface.

Before attempting to understand the moduli space of a structure such as a Riemann surface, it is desirable to first understand the basic properties of the structure itself. As we shall see in the case of elliptic curves, properties of the object are reflected in properties of the moduli space. We therefore begin with some basic facts from the theory of elliptic curves.

An elliptic curve is a “1-pointed” genus 1 curve:

**Definition 1.1.** An *elliptic curve* is a compact Riemann surface  $X$  of genus 1 together with the choice of a point  $P \in X$ .

Since the genus of a compact Riemann surface is, by definition, the dimension of its space of holomorphic 1-forms, the space of holomorphic 1-forms of a genus 1 Riemann surface has dimension 1.

*Exercise 1.* Use the Riemann-Roch formula (Appendix A.4) to prove that if  $w$  is a non-zero holomorphic 1-form on an elliptic curve  $X$ , then  $w$  has no zeros. Deduce that the canonical divisor  $K_X$  of every elliptic curve is zero.

**Definition 1.2.** A subgroup  $\Lambda$  of a finite dimensional real vector space  $V$  is a *lattice* if it is discrete and if  $V/\Lambda$  is compact.

*Exercise 2.* Show that a subgroup  $\Lambda$  of the finite dimensional real vector space  $V$  is a lattice if and only if  $\Lambda$  is a finitely generated abelian group such that every  $\mathbb{Z}$ -basis  $\lambda_1, \dots, \lambda_n$  of  $\Lambda$  is an  $\mathbb{R}$ -basis of  $V$ . Deduce that if  $\Lambda$  is a lattice in  $V$ , then  $V/\Lambda$  is diffeomorphic to the real  $n$ -torus  $\mathbb{R}^n/\mathbb{Z}^n$ .

The simplest examples of elliptic curves are 1-dimensional complex tori

$$(X, P) = (\mathbb{C}/\Lambda, 0)$$

which are quotients of  $\mathbb{C}$  by a lattice  $\Lambda$ . It is easy to write down the holomorphic differentials on a complex torus:

$$H^0(X, \Omega_X^1) = \mathbb{C} dz$$

where  $z$  is the coordinate in  $\mathbb{C}$ .

We shall show shortly that every elliptic curve is isomorphic to a 1-dimensional complex torus. Before we do this, we need to introduce *periods*.

Suppose that  $(X, P)$  is an elliptic curve. Fix a holomorphic 1-form  $\omega$  on  $X$ . Define

$$\Lambda = \left\{ \int_{\gamma} \omega : \gamma \in H_1(X; \mathbb{Z}) \right\}$$

This is a group, elements of which are called the periods of  $\omega$ .

**Lemma 1.3.** *The group  $\Lambda$  is a lattice in  $\mathbb{C}$ .*

*Proof.* Choose a basis  $\sigma_1, \sigma_2$  of  $H_1(X; \mathbb{Z})$ . Set

$$\lambda_j = \int_{\sigma_j} \omega, \quad j = 1, 2.$$

To prove that  $\Lambda$  is a lattice, we have to show that  $\lambda_1$  and  $\lambda_2$  are linearly independent over  $\mathbb{R}$ . If  $\lambda_1 = a\lambda_2$  for some  $a \in \mathbb{R}$ , then

$$\int_{a\sigma_1 - \sigma_2} \bar{\omega} = \int_{a\sigma_1 - \sigma_2} \omega = 0,$$

which implies that  $\int \omega$  and  $\int \bar{\omega}$  are linearly dependent over  $\mathbb{C}$  as functions  $H_1(X; \mathbb{Z}) \rightarrow \mathbb{C}$ . This implies that they represent proportional elements of  $H^1(C; \mathbb{C})$  and therefore that

$$\int_C \omega \wedge \bar{\omega} = 0.$$

On the other hand, for each local holomorphic coordinate  $w = u + iv$  on  $C$ , we can write (locally)  $\omega = h(w)dw$ . Consequently

$$i \omega \wedge \bar{\omega} = 2|h(w)|^2 du \wedge dv > 0$$

from which it follows that

$$i \int_C \omega \wedge \bar{\omega} > 0.$$

It follows that  $\lambda_1$  and  $\lambda_2$  are linearly independent over  $\mathbb{R}$  and that  $\Lambda$  is indeed a lattice in  $\mathbb{C}$ .  $\square$

**Proposition 1.4.** *Every elliptic curve is isomorphic to a 1-dimensional complex torus.*

*Proof.* Let  $(X, P)$  be an elliptic curve. Choose a non-zero holomorphic differential  $\omega$  on  $X$ . We will show that  $(X, P)$  is isomorphic to  $(\mathbb{C}/\Lambda, 0)$  where  $\Lambda$  is the period lattice of  $\omega$ . Define a holomorphic mapping

$$F : X \rightarrow \mathbb{C}/\Lambda$$

by

$$F(x) = \int_P^x \omega \pmod{\Lambda}$$

Here the integral is over any path in  $X$  from  $P$  to  $x$ . Since any two such paths differ by an element of  $H_1(X; \mathbb{Z})$ , the function  $F$  is well defined.

By elementary calculus, the derivative of  $F$  is  $\omega$ . Since this is holomorphic, this implies that  $F$  is holomorphic. Further, since  $\omega$  has no zeros,  $F$  is a local biholomorphism at each point of  $X$ . By Exercise 77, this implies that  $F$  is a covering map. To complete the proof, we show that  $F$  has degree 1. To do this, it suffices to show that the induced mapping

$$F_* : H_1(X; \mathbb{Z}) \rightarrow H_1(\mathbb{C}/\Lambda; \mathbb{Z}),$$

which is injective by covering space theory, is surjective. But this follows as there is a natural isomorphism  $H_1(\mathbb{C}/\Lambda; \mathbb{Z}) \cong \Lambda$  and as, under this identification,

$$F_*(\gamma) = \int_\gamma \omega.$$

□

*Remark 1.5.* This also follows directly from the Uniformization Theorem, which implies that the universal covering of  $X$  is biholomorphic to  $\mathbb{C}$ .

Since every elliptic curve is isomorphic to a complex torus, and since every complex torus is a group, we obtain:

**Corollary 1.6.** *Every elliptic curve  $(X, P)$  has the structure of a group with identity  $P$  and where the multiplication  $X \times X \rightarrow X$  and inverse  $X \rightarrow X$  are holomorphic.*

Shortly we will show that this group structure is unique.

**Corollary 1.7.** *If  $X$  is a compact Riemann surface of genus 1 and if  $P, Q \in X$ , then the elliptic curves  $(X, P)$  and  $(X, Q)$  are isomorphic.*

*Proof.* It suffices to prove this when  $X$  is a complex torus  $\mathbb{C}/\Lambda$ . In this case the isomorphism is given by translation by  $Q - P$ :

$$(\mathbb{C}/\Lambda, P) \rightarrow (\mathbb{C}/\Lambda, Q), \quad x \mapsto x - P + Q.$$

□

*Remark 1.8.* It is easier to construct moduli spaces of structures that have at most a finite number of automorphisms. Since every genus 1 Riemann surface  $X$  is isomorphic to  $\mathbb{C}/\Lambda$ , its automorphism group  $\text{Aut } X$  contains  $X$  as a group of translations. For this reason, moduli problem for genus 1 curves is not well behaved. We will see shortly that the automorphism group  $\text{Aut}(X, P)$  of an elliptic curve is finite, which is one reason why we study the moduli problem for elliptic curves rather than for genus 1 curves. In general, the automorphism group of an  $n$ -pointed compact Riemann surface  $(X, \{x_1, \dots, x_n\})$  is finite if and only if  $2g - 2 + n > 0$ . This condition may seem mysterious, but it is equivalent to the condition that the Euler characteristic of the punctured surface  $X' := X - \{x_1, \dots, x_n\}$  be negative. This, in turn (by the Uniformization Theorem) is equivalent to the condition that  $X'$  has a complete hyperbolic metric.

**Lemma 1.9.** *Suppose that  $\Lambda_1$  and  $\Lambda_2$  are two lattices in  $\mathbb{C}$ . If  $f : (\mathbb{C}/\Lambda_1, 0) \rightarrow (\mathbb{C}/\Lambda_2, 0)$  is a holomorphic mapping, then there exists  $c \in \mathbb{C}$  such that  $c\Lambda_1 \subseteq \Lambda_2$  and*

$$f(z + \Lambda_1) = cz + \Lambda_2.$$

*In particular,  $f$  is a group homomorphism.*

*Proof.* Note that  $(\mathbb{C}, 0) \rightarrow (\mathbb{C}/\Lambda, 0)$  is a pointed universal covering of  $\mathbb{C}/\Lambda$ . Covering space theory implies that there is a holomorphic map  $F : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$  such that

$$f(z + \Lambda_1) = F(z) + \Lambda_2.$$

The result will follow if we show that  $F$  is linear. For  $j = 1, 2$  set

$$\omega_j = dz \in H^0(\mathbb{C}/\Lambda_j, \Omega_{\mathbb{C}/\Lambda_j}^1).$$

Then there is a constant  $c \in \mathbb{C}$  such that  $f^*\omega_2 = c\omega_1$ . Consequently  $dF = cdz$ . Since  $F(0) = 0$ , this implies that  $F(z) = cz$ . □

This yields the following fact, which will give us the leverage we need to construct and understand the moduli space of elliptic curves.

**Corollary 1.10.** *Two complex tori  $(\mathbb{C}/\Lambda_1, 0)$  and  $(\mathbb{C}/\Lambda_2, 0)$  are isomorphic if and only if there is  $c \in \mathbb{C}^*$  such that  $\Lambda_2 = c\Lambda_1$ .*

*Exercise 3.* Show that

$$\text{Aut}(\mathbb{C}/\Lambda, 0) = \{u \in \mathbb{C}^* : u\Lambda = \Lambda\}.$$

Note that  $\pm 1$  are automorphisms. Show that every  $u \in \text{Aut}(\mathbb{C}/\Lambda, 0)$  has modulus 1. Deduce that  $\text{Aut}(\mathbb{C}/\Lambda, 0)$  is isomorphic to the group  $\mu_{2n}$  of  $2n$ th roots of unity for some  $n \geq 1$ . Deduce that the automorphism group of every elliptic curve  $(X, P)$  is a finite cyclic group of even order.

The proposition also yields the following useful fact, which follows as every elliptic curve is isomorphic to a 1-dimensional torus.

**Corollary 1.11.** *Every holomorphic mapping  $f : (X, P) \rightarrow (Y, Q)$  between elliptic curves is a group homomorphism.*

*Remark 1.12.* Another consequence of Proposition 1.4 is the well known statement that every compact Riemann surface of genus 1 has a flat riemannian metric whose conformal class is determined by the complex structure. The metric is unique up to multiplication by a constant. Lemma 1.9 implies that holomorphic maps between genus 1 Riemann surfaces are orientation preserving homotheties with respect to their flat metrics. In higher genus, a similar statement holds with flat replaced by hyperbolic. The main difference with the genus 1 case is that there is a unique hyperbolic metric in each conformal class.

**1.1. Moduli of elliptic curves.** To determine the moduli space of elliptic curves, we need only determine the moduli space of lattices in  $\mathbb{C}$ .

As is typical in constructing the moduli space of curves in higher genus via Teichmüller theory, and when constructing the moduli of principally polarized abelian varieties, we begin by “framing” the object of interest.

**Definition 1.13.** A *framed elliptic curve* is an elliptic curve  $(X, P)$  together with an ordered basis  $\mathbf{a}, \mathbf{b}$  of  $H_1(X, \mathbb{Z})$  with the property that the intersection number  $\mathbf{a} \cdot \mathbf{b}$  is 1.

If  $\Lambda$  is a lattice in  $\mathbb{C}$  then  $\lambda, \lambda' \in \Lambda$  are linearly independent over  $\mathbb{R}$  if and only if  $\text{Im}(\lambda'/\lambda) \neq 0$ . The condition that the corresponding elements of  $H_1(\mathbb{C}/\Lambda)$  intersect positively is that  $\text{Im}(\lambda'/\lambda) > 0$ .

**Definition 1.14.** A framing of a lattice  $\Lambda$  in  $\mathbb{C}$  is an ordered basis  $\lambda_1, \lambda_2$  such that  $\lambda_2/\lambda_1$  has positive imaginary part.

Since  $H_1(\mathbb{C}/\Lambda; \mathbb{Z})$  is naturally isomorphic to  $\Lambda$ , a framing of  $(\mathbb{C}/\Lambda, 0)$  corresponds to a framing of  $\Lambda$ .

Isomorphism of framed elliptic curves is defined in the obvious way. Two framed lattices  $(\Lambda; \lambda_1, \lambda_2)$  and  $(\Lambda'; \lambda'_1, \lambda'_2)$  are isomorphic if there is a non-zero complex number  $u$  such that  $\lambda'_j = u\lambda_j$ .

Clearly a framed lattice  $(\Lambda; \lambda_1, \lambda_2)$  is determined by its framing  $\lambda_1, \lambda_2$  as

$$\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2.$$

*Exercise 4.* Show that the framed lattice with basis  $\lambda_1, \lambda_2$  is isomorphic to the framed lattice with basis  $\omega_1, \omega_2$  if and only if  $\lambda_2/\lambda_1 = \omega_2/\omega_1$ .

An immediate consequence is that every framed lattice is isomorphic to a unique framed lattice of the form

$$(\mathbb{Z} \oplus \mathbb{Z}\tau; 1, \tau)$$

where  $\tau$  lies in the upper half plane  $\mathfrak{h}$ .

In summary:

**Proposition 1.15.** *There are natural bijections*

$$\mathfrak{h} \leftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of framed lattices} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{framed elliptic curves} \end{array} \right\}.$$

Under this correspondence,  $\tau \in \mathfrak{h}$  corresponds to the framed elliptic curve  $(\mathbb{C}/\Lambda_\tau; 1, \tau)$  and the framed elliptic curve  $(X, P; \mathbf{a}, \mathbf{b})$  corresponds to

$$\int_{\mathbf{b}} \omega / \int_{\mathbf{a}} \omega \in \mathfrak{h},$$

where  $\omega$  is a non-zero holomorphic differential on  $X$  and

$$\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau.$$

At this stage, these correspondences are simply bijections of sets. In the next lecture, we will show that the right-hand set has a natural Riemann surface structure and that the bijection with  $\mathfrak{h}$  is a biholomorphism.

As a set, the moduli space of elliptic curves is the set of isomorphism classes of elliptic curves. It is the quotient of the set of isomorphism classes of framed elliptic curves that is obtained by forgetting the framing.

Two framings  $(\mathbf{a}, \mathbf{b})$  and  $(\mathbf{a}', \mathbf{b}')$  of an elliptic curve are related by

$$(1) \quad \begin{pmatrix} \mathbf{b}' \\ \mathbf{a}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix}.$$

where

$$(2) \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$



is a  $2 \times 2$  integral matrix. Since  $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}' \cdot \mathbf{b}' = 1$ ,  $\gamma$  has determinant 1 and is thus an element of  $\mathrm{SL}_2(\mathbb{Z})$ .

Denote the isomorphism class of the framed elliptic curve  $(X, P; \mathbf{a}, \mathbf{b})$  by  $[X, P; \mathbf{a}, \mathbf{b}]$ . Define a left action of  $\mathrm{SL}_2(\mathbb{Z})$  on

$$\left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{framed elliptic curves} \end{array} \right\}$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : [X, P; \mathbf{a}, \mathbf{b}] \mapsto [X, P; \mathbf{a}', \mathbf{b}']$$

where  $\mathbf{a}', \mathbf{b}'$  are given by (1). The quotient is the set of isomorphism classes of elliptic curves.

**Proposition 1.16.** *The set of isomorphism classes of elliptic curves is the quotient*

$$\mathrm{SL}_2(\mathbb{Z}) \backslash \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{framed elliptic curves} \end{array} \right\}.$$

Let's compute the corresponding action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathfrak{h}$ :

$$\begin{array}{ccc} \tau & \longleftrightarrow & (\mathbb{C}/\Lambda_\tau; 1, \tau) \\ \gamma \downarrow \text{dotted} & & \downarrow \gamma \\ \frac{a\tau + b}{c\tau + d} & \longleftrightarrow & (\mathbb{C}/\Lambda_\tau; c\tau + d, a\tau + b) \end{array}$$

To summarize:

**Proposition 1.17.** *The set of isomorphism classes of elliptic curves is isomorphic to the quotient  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$  of the upper half plane by the  $\mathrm{SL}_2(\mathbb{Z})$ -action*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

*Exercise 5.* Show that the isotropy group

$$\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma\tau = \tau\}$$

of  $\tau \in \mathfrak{h}$  is isomorphic to  $\mathrm{Aut}(\mathbb{C}/\Lambda_\tau, 0)$ .

The problem of finding a fundamental domain for the  $\mathrm{SL}_2(\mathbb{Z})$ -action on  $\mathfrak{h}$  can be solved easily by thinking of the upper half plane as the moduli space of framed lattices in  $\mathbb{C}$ . We seek a natural basis (possibly up to finite ambiguity) of every lattice  $\Lambda$  in  $\mathbb{C}$ . A natural choice for the first basis vector is a shortest vector  $u \in \Lambda$ . Since  $\Lambda$  is a discrete subset of  $\mathbb{C}$ , there is a finite number (generically 1) of these.

*Exercise 6.* Show that if  $v \in \Lambda$  is a shortest vector that is not a real multiple of  $u$ , then  $u, v$  is a basis of  $\Lambda$ .

By replacing  $v$  by  $-v$  if necessary, we may assume that  $\text{Im } v/u > 0$ . The framed lattice  $(\Lambda; u, v)$  is isomorphic to  $(u^{-1}\Lambda, 1, \tau)$ , where  $\tau = v/u$ , which we assume to be in  $\mathfrak{h}$ .

*Exercise 7.* Show that, with these choices,  $|\tau| \geq 1$  and that  $|\text{Re } \tau| \leq 1/2$ .

With a little more work (cf. [10, p. 78]), we have:

**Proposition 1.18.** *Every framed lattice in  $\mathbb{C}$  is isomorphic to one with basis  $1, \tau$ , where  $\tau \in \mathfrak{h}$  lies in the region*

$$F := \{\tau \in \mathfrak{h} : |\text{Re}(\tau)| \leq 1/2 \text{ and } |\tau| \geq 1\}.$$

If  $\tau, \tau' \in F$  lie in the same  $\text{SL}_2(\mathbb{Z})$  orbit, then either

$$|\text{Re } \tau| = |\text{Re } \tau'| = 1/2 \text{ and } \tau' = \tau \pm 1,$$

or

$$|\tau| = 1 \text{ and } \tau' = -1/\tau.$$

If  $\gamma\tau = \tau$ , then either  $\gamma = \pm \text{id}$  or

$$\tau = i \text{ and } \gamma \text{ is a power of } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\tau = \rho := e^{2\pi i/3} \text{ and } \gamma \text{ is a power of } \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

or

$$\tau = -1/\rho \text{ and } \gamma \text{ is a power of } \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is convenient (and standard) to set

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and } U = ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

Then  $S$  has order 4,  $U$  has order 6 and  $S^2 = U^3 = -I$ . The stabilizer of  $i$  is generated by  $S$ , the stabilizer of  $\rho$  is generated by  $U$ . Serre [10, p. 78] proves that  $\text{SL}_2(\mathbb{Z})$  is generated by  $S$  and  $T$  and has presentation:<sup>2</sup>

$$(3) \quad \text{SL}_2(\mathbb{Z}) = \langle S, T : S^2 = (ST)^3, S^4 \rangle = \langle S, U : S^2 = U^3, S^4 \rangle.$$

<sup>2</sup>This is easily proved. Let  $\Gamma$  be the subgroup of  $\text{SL}_2(\mathbb{Z})$  generated by  $S$  and  $T$ . Show that  $F$  is a fundamental domain for the action of  $\Gamma$  on  $\mathfrak{h}$ . This is essentially the LLL algorithm.

*Exercise 8.* Show that if  $\tau \in \mathfrak{h}$ , then

$$\text{Aut}(\mathbb{C}/\Lambda_\tau, 0) \cong \{\gamma \in \text{SL}_2(\mathbb{Z}) : \gamma(\tau) = \tau\}.$$

Deduce that

$$\text{Aut}(\mathbb{C}/\Lambda_\tau) \cong \{\pm 1\}$$

unless  $\gamma$  lies in the  $\text{SL}_2(\mathbb{Z})$ -orbit of  $i$  or  $\rho$ . Show that

$$\text{Aut}(\mathbb{C}/\Lambda_i, 0) \cong \boldsymbol{\mu}_4 \text{ and } \text{Aut}(\mathbb{C}/\Lambda_\rho, 0) \cong \boldsymbol{\mu}_6,$$

where  $\boldsymbol{\mu}_n$  denotes the group of  $n$ th roots of unity.

**Corollary 1.19.**  $\text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$  is homeomorphic to the disk.

By Exercise 96, we have:

**Theorem 1.20.** *The quotient  $\text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$  has a unique Riemann surface structure such that the quotient mapping  $\mathfrak{h} \rightarrow \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$  is holomorphic.*

This is our first, but not final, version of the moduli space of elliptic curves.

## 2. FAMILIES OF ELLIPTIC CURVES AND THE UNIVERSAL CURVE

In the first lecture, we showed that the quotient  $\text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$  of the upper half plane  $\mathfrak{h}$  by the standard action of  $\text{SL}_2(\mathbb{Z})$  is a Riemann surface whose points correspond to the isomorphism classes of elliptic curves. Denote this quotient by  $M_{1,1}$ .<sup>3</sup>

A (holomorphic) family of elliptic curves over a complex manifold  $T$  is a complex manifold  $X$  together with a holomorphic mapping  $\pi : X \rightarrow T$  of maximal rank and a section  $s : T \rightarrow X$  of  $\pi$  such that for each  $t \in T$ , each fiber  $(\pi^{-1}(t), s(t))$  is an elliptic curve.

$$X \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{\pi} \end{array} T$$

For convenience, we denote the fiber  $\pi^{-1}(t)$  of  $\pi$  over  $t \in T$  by  $X_t$ .

To such a family we can associate the function  $\Phi : T \rightarrow M_{1,1}$  defined by

$$\Phi : t \mapsto [X_t, s(t)].$$

We would like such a family of elliptic curves to be “classified” by mappings  $T \rightarrow M_{1,1}$ . More precisely, we would like  $M_{1,1}$  to satisfy:

- (i) the function  $\Phi : T \rightarrow M_{1,1}$  is holomorphic;
- (ii) every holomorphic mapping from a complex manifold  $T$  to  $M_{1,1}$  corresponds to a family of elliptic curves over  $T$ ;

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<sup>3</sup>In general, when  $2g - 2 + n > 0$ ,  $M_{g,n}$  will denote the moduli space of compact Riemann surfaces of genus  $g$  with  $n$  marked points.

- (iii) there should be a holomorphic family of elliptic curves  $E \rightarrow M_{1,1}$  that is universal in the sense that the family  $\pi : X \rightarrow T$  should be isomorphic to the pullback family<sup>4</sup>

$$\begin{array}{ccccc} X & \xrightarrow{\cong} & \Phi^*E & \longrightarrow & E \\ \pi \downarrow & & \downarrow & & \downarrow \\ T & \xlongequal{\quad} & T & \xrightarrow{\Phi} & M_{1,1}. \end{array}$$

The isomorphism  $X \rightarrow \Phi^*E$  is unique up to an automorphism of the family  $X \rightarrow T$  that is the identity on the zero section.

We will see shortly that the Riemann surface  $M_{1,1}$  possesses the first property, but not the second or third. Later in this lecture, we will define the *orbifold*  $\mathcal{M}_{1,1}$ , which is endowed with a universal elliptic curve  $\mathcal{E} \rightarrow \mathcal{M}_{1,1}$  and has all three properties. In preparation for this, we first consider families of *framed* elliptic curves.<sup>5</sup>

**2.1. The universal elliptic curve  $\mathcal{E}_{\mathfrak{h}}$  over  $\mathfrak{h}$ .** Recall that  $\Lambda_{\tau}$  denotes the lattice  $\mathbb{Z} \oplus \mathbb{Z}\tau$ . It is easy to construct a family of elliptic curves over  $\mathfrak{h}$  whose fiber over  $\tau$  is  $\mathbb{C}/\Lambda_{\tau}$ .

The group  $\mathbb{Z}^2$  acts on  $\mathbb{C} \times \mathfrak{h}$  on the left:

$$(4) \quad (m, n) : (z, \tau) \mapsto \left( z + (m \ n) \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \tau \right)$$

This action is properly discontinuous and fixed point free. Therefore the quotient

$$\mathbb{Z}^2 \backslash (\mathbb{C} \times \mathfrak{h})$$

is a 2-dimensional complex manifold. Denote it by  $\mathcal{E}_{\mathfrak{h}}$ . The projection  $\mathbb{C} \times \mathfrak{h} \rightarrow \mathfrak{h}$  induces a projection  $\mathcal{E}_{\mathfrak{h}} \rightarrow \mathfrak{h}$  whose fiber over  $\tau$  is  $\mathbb{C}/\Lambda_{\tau}$ .

**2.2. Families of framed elliptic curves.** Every family of elliptic curves

$$X \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{\pi} \end{array} T$$

is locally trivial as a  $C^{\infty}$  fiber bundle. For an open subset  $U$  of  $T$  set  $X_U = \pi^{-1}(U)$ . Suppose that  $U$  is an open ball in  $T$  over which  $X$  is topologically trivial and  $o \in U$ :

$$X_U \cong U \times X_o \text{ as a smooth manifolds, where } o \in U.$$

<sup>4</sup>A slightly stronger version of (ii) implies (iii). Namely, if in (ii) we also insist that the period mapping of the pullback family  $f^*X \rightarrow S$  of elliptic curves associated to  $f : S \rightarrow T$  be  $\Phi \circ f : S \rightarrow M_{1,1}$ , then the universal family in (iii) is the family corresponding to the identity  $M_{1,1} \rightarrow M_{1,1}$ .

<sup>5</sup>The fancy terminology is that  $M_{1,1}$  is a *coarse* moduli space. The orbifold  $\mathcal{M}_{1,1}$  is a *fine* moduli space.

Since  $U$  is contractible, the inclusion  $j_t : X_t \hookrightarrow X_U$  is a homotopy equivalence for each  $t \in U$ . So if  $s, t \in U$ , then there are natural isomorphisms

$$(5) \quad H_1(X_t; \mathbb{Z}) \xrightarrow{j_{t*}} H_1(X_U; \mathbb{Z}) \xleftarrow{j_{s*}} H_1(X_s; \mathbb{Z}).$$

A family of elements

$$\{c(t) \in H_1(X_t; \mathbb{Z}) : t \in T\}$$

is locally constant if for each open ball  $U$  in  $T$  over which  $X_U$  is topologically trivial and each pair  $s, t$  of elements of  $U$ ,  $c(s)$  and  $c(t)$  correspond under the isomorphism (5).

A family of framings

$$\{\mathbf{a}(t), \mathbf{b}(t) \in H_1(X_t; \mathbb{Z}) : \mathbf{a}(t) \cdot \mathbf{b}(t) = 1, t \in T\}$$

is locally constant if  $\mathbf{a}(t)$  and  $\mathbf{b}(t)$  are locally constant.

**Definition 2.1.** A family of elliptic curves is *framed* if it has a locally constant framing.

**Example 2.2.** The family  $\mathcal{E}_{\mathfrak{h}} \rightarrow \mathfrak{h}$  is framed. The basis  $\mathbf{a}(\tau), \mathbf{b}(\tau)$  of  $H_1(\mathbb{C}/\Lambda_{\tau}; \mathbb{Z}) \cong \Lambda_{\tau}$  is  $1, \tau$ .

*Exercise 9.* Show that every family of elliptic curves  $X \rightarrow T$  over a simply connected base  $T$  has a framing.

Since  $\mathfrak{h}$  is the set of isomorphism classes of framed elliptic curves, a framed family of elliptic curves  $\pi : X \rightarrow T$  determines a function

$$\Phi : T \rightarrow \mathfrak{h}.$$

It is defined by

$$t \mapsto \int_{\mathbf{b}(t)} \omega_t / \int_{\mathbf{a}(t)} \omega_t$$

where  $\omega_t$  is any non-zero holomorphic differential on  $X_t$ . The mapping  $\Phi$  is called the *period mapping* of the family.

**Proposition 2.3.** *If  $\pi : X \rightarrow T$  is a family of framed elliptic curves, then the period mapping  $\Phi$  is holomorphic.*

*Sketch of Proof.* The main task is to show that we can choose the holomorphic differential  $\omega_t$  to vary holomorphically with  $t$ . In other words, we need to show that each  $o \in T$  has an open neighbourhood  $U$  such that there is a holomorphic 1-form  $\omega$  on  $X_U$ , defined modulo 1-forms that vanish on the fibers, whose restriction to  $X_o$  is non-zero. More precisely, we need to construct an element  $\omega$  of  $H^0(U, \pi_* \Omega_{X/T}^1)$  whose restriction to  $X_o$  is non-zero. By shrinking  $U$  if necessary, the restriction  $\omega_t$  of  $\omega$  to  $X_t$  will be non-zero for all  $t \in U$ .

Once we have done this, after further shrinking  $U$  if necessary, we can construct continuous mappings  $\alpha$  and  $\beta$  from  $S^1 \times U \rightarrow X_U$  such that

$$\begin{array}{ccc} S^1 \times U & \xrightarrow{\alpha, \beta} & X_U \\ & \searrow \text{pr}_2 & \swarrow \pi \\ & & U \end{array}$$

commutes and

- (i) for each  $t \in U$ ,  $\alpha_t : \theta \mapsto \alpha(\theta, t)$  and  $\beta_t : \theta \mapsto \beta(\theta, t)$  are piecewise smooth representatives of  $\mathbf{a}(t)$  and  $\mathbf{b}(t)$ ;
- (ii) for each  $\theta$ ,  $t \mapsto \alpha(\theta, t)$  and  $t \mapsto \beta(\theta, t)$  are holomorphic. (That this is possible follows from the holomorphic implicit function theorem.)

Basic calculus now implies that

$$\int_{\alpha_t} \omega_t \text{ and } \int_{\beta_t} \omega_t$$

vary holomorphically with  $t \in U$  from which it follows that  $\Phi(t)$  varies holomorphically with  $t \in T$ .

We now establish the existence of  $\omega$ . Let  $N$  be the holomorphic normal bundle in  $X$  of the zero section  $\text{im } s$  of  $X \rightarrow T$ . This is a holomorphic line bundle on the zero section of  $X \rightarrow T$ . Denote by  $L$  the pullback to  $T$  of the dual of  $N$  along the zero section  $s : T \rightarrow X$ . This has a local holomorphic section  $\sigma$  defined in a neighbourhood  $U$  of  $o \in T$  that does not vanish at  $o$ . Since the holomorphic cotangent bundle of each  $X_t$  is trivial, there is a unique holomorphic differential  $\omega_t$  on  $X_t$  whose value at the identity  $s(t)$  is  $\sigma(t)$ . The form defined is a holomorphic section  $\omega$  of the sheaf  $\pi_* \Omega_{X/T}^1$ , as required.  $\square$

Each framed family  $X \rightarrow T$  of elliptic curves determines a family of elliptic curves by pulling back the family  $\mathcal{E}_{\mathfrak{h}} \rightarrow \mathfrak{h}$  along the period mapping:

$$\begin{array}{ccc} \Phi^* \mathcal{E}_{\mathfrak{h}} & \longrightarrow & \mathcal{E}_{\mathfrak{h}} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\Phi} & \mathfrak{h} \end{array}$$

*Exercise 10.* Show that the framed families  $X \rightarrow T$  and  $\Phi^* \mathcal{E}_{\mathfrak{h}} \rightarrow T$  are naturally isomorphic. That is, there is a biholomorphism  $F : X \rightarrow$

$\Phi^*\mathcal{E}_{\mathfrak{h}}$  that commutes with the projections to  $T$ :

$$\begin{array}{ccc} X & \xrightarrow{F} & \Phi^*\mathcal{E}_{\mathfrak{h}} \\ \downarrow & & \downarrow \\ T & \xlongequal{\quad} & T \end{array}$$

takes the zero section of  $X$  to the zero section of  $\Phi^*\mathcal{E}_{\mathfrak{h}}$ , and preserves the framings. Hint: First observe that  $(X_t, s(t); \mathbf{a}(t), \mathbf{b}(t))$  is canonically isomorphic to  $(\mathbb{C}/\Lambda_{\Phi(t)}, 0; 1, \Phi(t))$ . Show that these isomorphisms can be assembled (locally) into a holomorphic mapping  $X \rightarrow \mathcal{E}_{\mathfrak{h}}$  by taking  $x \in X_t$  to

$$\int_c \omega_t / \int_{\alpha(t)} \omega_t \pmod{\Lambda_{\Phi(t)}}$$

where  $c$  is a smooth path in  $X_t$  from  $s(t)$  to  $x$ .

This proves that  $\mathfrak{h}$  is a fine moduli space for framed families of elliptic curves.

**Proposition 2.4.** *There is a 1-1 correspondence between framed families of elliptic curves  $X \rightarrow T$  and holomorphic mappings  $\Phi : T \rightarrow \mathfrak{h}$ . Moreover, if  $X \rightarrow T$  corresponds to  $\Phi : T \rightarrow \mathfrak{h}$ , then the framed family  $X \rightarrow T$  is isomorphic to the framed family  $\Phi^*\mathcal{E}_{\mathfrak{h}} \rightarrow T$ .*

*Remark 2.5.* Every family  $X \rightarrow T$  of elliptic curves can be framed locally — that is, each  $t \in T$  has a neighbourhood  $U$  such that the restricted family  $X_U \rightarrow U$  has a framing and therefore admits a period mapping  $\Phi_U : U \rightarrow \mathfrak{h}$  so that  $X_U \rightarrow U$  is the pullback of the universal framed family  $\mathcal{E}_{\mathfrak{h}} \rightarrow \mathfrak{h}$  along  $\Phi_U$ . The period mapping  $T \rightarrow M_{1,1}$  associated to a family of elliptic curves is thus “locally liftable” to a mapping  $T \rightarrow \mathfrak{h}$ . Since the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathfrak{h}$  has fixed points, the identity  $M_{1,1} \rightarrow M_{1,1}$  is not locally liftable, and therefore not the period mapping of a family of elliptic curves. It is this phenomenon which will lead us naturally to orbifolds and stacks.

The group  $\mathrm{SL}_2(\mathbb{Z})$  acts on the set of framings of a framed family  $X \rightarrow T$  of elliptic curves via the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix}.$$

*Exercise 11.* Show that if  $\Phi : T \rightarrow \mathfrak{h}$  is the period map of a family of elliptic curves with framing  $\mathbf{a}, \mathbf{b}$ , then the period mapping with respect to the framing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix}$$

is  $(a\Phi + b)/(c\Phi + d) : T \rightarrow \mathfrak{h}$ .

### 2.3. The universal elliptic curve. If

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

then the isomorphism

$$(\mathbb{C}/\Lambda_\tau, 0) \rightarrow (\mathbb{C}/\Lambda_{\gamma\tau}, 0)$$

is induced by the mapping  $z \mapsto (c\tau + d)^{-1}z$ . This suggests that we consider the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathbb{C} \times \mathfrak{h}$  defined by

$$\gamma : (z, \tau) \mapsto (z/(c\tau + d), (a\tau + b)/(c\tau + d)).$$

*Exercise 12.* Prove that this is indeed an action.

We would like to combine this with the action of  $\mathbb{Z}^2$  that we used to define the universal curve over  $\mathfrak{h}$ .

The group  $\mathrm{SL}_2(\mathbb{Z})$  acts on  $\mathbb{Z}^2$  by *right* multiplication:

$$(6) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (m \ n) \mapsto (m \ n) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Denote the corresponding semi-direct product  $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$  by  $\Gamma$ . This is the set  $\mathrm{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2$  with multiplication:

$$(\gamma_1, v_1)(\gamma_2, v_2) = (\gamma_1\gamma_2, v_1\gamma_2 + v_2)$$

where  $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$  and  $v_1, v_2 \in \mathbb{Z}^2$ .

*Exercise 13.* Show that  $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$  is isomorphic to the group

$$\left\{ \begin{pmatrix} \gamma & 0 \\ v & 1 \end{pmatrix} : \gamma \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } v \in \mathbb{Z}^2 \right\}$$

*Exercise 14.* Show that (4) and (6) determine a well defined left action of  $\Gamma$  on  $\mathbb{C} \times \mathfrak{h}$ . Show that if  $(\gamma, v) : (z, \tau) \rightarrow (z', \tau')$ , where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } v = (m, n),$$

then

$$\begin{pmatrix} \tau' \\ 1 \\ z' \end{pmatrix} = (c\tau + d)^{-1} \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{pmatrix} \begin{pmatrix} \tau \\ 1 \\ z \end{pmatrix}$$

Set

$$E = (\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \backslash (\mathbb{C} \times \mathfrak{h}).$$

There is a projection  $E \rightarrow M_{1,1}$ .



*Exercise 15.* Show that the fiber of  $E$  over the point  $[X, P]$  of  $M_{1,1}$  corresponding to the elliptic curve  $(X, P)$  is  $X/\text{Aut}(X, P)$ . (Cf. Exercise 8.) Show that if  $\text{Aut}(X, P)$  is cyclic of order 2, then  $X/\text{Aut}(X, P)$  is isomorphic to the Riemann sphere  $\mathbb{P}^1$ . In particular, no fiber of  $E$  is an elliptic curve.

This problem can be rectified by pulling back the family  $X \rightarrow T$  of elliptic curves to the universal covering  $p: \tilde{T} \rightarrow T$  of  $T$ :

$$\begin{array}{ccc} p^*X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \tilde{T} & \xrightarrow{p} & T \end{array}$$

Since  $\tilde{T}$  is simply connected, the family  $p^*X \rightarrow \tilde{T}$  admits a framing. It is therefore obtained by pulling back the universal framed family  $\mathcal{E}_{\mathfrak{h}} \rightarrow \mathfrak{h}$  along the period mapping  $\Phi: \tilde{T} \rightarrow \mathfrak{h}$ .

Note that the fiber of  $p^*X$  over  $t$  is canonically isomorphic to the fiber of  $X \rightarrow T$  over  $p(t)$ . This means that if  $\gamma \in \text{Aut}(\tilde{T}/T)$  and  $t \in \tilde{T}$ , then the fibers of  $p^*X$  over  $t$  and  $\gamma t$  are canonically isomorphic. So if  $\mathbf{a}(t), \mathbf{b}(t)$  is a framing of  $p^*X \rightarrow \tilde{T}$ , then  $\mathbf{a}(t), \mathbf{b}(t)$  and  $\mathbf{a}(\gamma t), \mathbf{b}(\gamma t)$  are both framings of  $H_1(X_{p(t)}; \mathbb{Z})$ , and therefore differ by an element of  $\text{SL}_2(\mathbb{Z})$ .

Define a homomorphism  $\phi: \text{Aut}(\tilde{T}/T) \rightarrow \text{SL}_2(\mathbb{Z})$  from the group of deck transformations to  $\text{SL}_2(\mathbb{Z})$  by

$$\begin{pmatrix} \mathbf{b}(\gamma t) \\ \mathbf{a}(\gamma t) \end{pmatrix} = \phi(\gamma) \begin{pmatrix} \mathbf{b}(t) \\ \mathbf{a}(t) \end{pmatrix}.$$

*Exercise 16.* Show that the period mapping  $\Phi$  is equivariant with respect to  $\phi$  in the sense that

$$\Phi(\gamma t) = \phi(\gamma)\Phi(t)$$

for all  $t \in \tilde{T}$  and  $\gamma \in \text{Aut}(\tilde{T}/T)$ .

*Exercise 17.* Show that the action of  $\text{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2$  on  $\mathbb{C} \times \mathfrak{h}$  induces an action of  $\text{SL}_2(\mathbb{Z})$  on  $E_{\mathfrak{h}}$ .

### 3. THE ORBIFOLD $\mathcal{M}_{1,1}$

**3.1. Local theory: basic orbifolds.** The discussion in the previous section suggests a generalization of topological spaces which includes quotients  $\Gamma \backslash X$  and in which morphisms  $\Gamma \backslash X \rightarrow \Gamma' \backslash X'$  are  $\Gamma$ -equivariant mappings  $X \rightarrow X'$  with respect to a group homomorphism  $\phi: \Gamma \rightarrow \Gamma'$ .

**Definition 3.1.** A basic *pointed orbifold* is a triple  $(X, \Gamma, \rho)$  where  $X$  is a connected, simply connected topological space  $X$  (typically a smooth manifold) and  $\Gamma$  is a discrete group that acts on  $X$  via the homomorphism  $\rho : \Gamma \rightarrow \text{Aut } X$ . A *pointed morphism*

$$(f, \phi) : (X, \Gamma, \rho) \rightarrow (X', \Gamma', \rho')$$

of orbifolds consists of a continuous mapping  $f : X \rightarrow X'$  and a group homomorphism  $\phi : \Gamma \rightarrow \Gamma'$  such that for all  $\gamma \in \Gamma$ , the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \rho(\gamma) \downarrow & & \downarrow \rho'(\phi(\gamma)) \\ X & \xrightarrow{f} & X' \end{array}$$

commutes. A morphism  $(X, \Gamma, \rho) \rightarrow (X', \Gamma', \rho')$  of orbifolds is a  $\Gamma$ -orbit of pointed morphisms where  $\Gamma$  acts on the pointed morphism  $(f, \phi) : (X, \Gamma, \rho) \rightarrow (X', \Gamma', \rho')$  by conjugation:

$$g : (f, \phi) \rightarrow (gfg^{-1}, g\phi g^{-1}), \quad g \in \Gamma.$$

Define the pointed orbifolds  $(X, \Gamma, \rho)$  and  $(X, \Gamma, \rho')$  to be *equivalent* if there exists  $g \in \Gamma$  such that  $\rho' = g\rho g^{-1}$ . In this case, there is an isomorphism

$$(g, \text{conjugation by } g) : (X, \Gamma, \rho) \rightarrow (X, \Gamma, \rho').$$

A *basic orbifold* is an equivalence class of pointed orbifolds.

We will usually omit  $\rho$  from the notation. We shall write  $\Gamma \backslash\backslash X$  for  $(X, \Gamma)$ , which we will regard as the orbifold quotient of  $X$  by  $\Gamma$ . When  $\Gamma$  is trivial we shall denote the orbifold  $(X, \mathbf{1})$  by  $X$ . The identity  $X \rightarrow X$  induces a natural quotient morphism  $p : X \rightarrow \Gamma \backslash\backslash X$ , which we shall regard as a universal covering of  $\Gamma \backslash\backslash X$ .

The quotient mapping  $p : X \rightarrow \Gamma \backslash\backslash X$  should be thought of as a base point of  $\Gamma \backslash\backslash X$ . Pointed morphisms preserve these base points.

A path connected topological space  $X$  with a universal covering  $p : \tilde{X} \rightarrow X$  will be regarded as the orbifold  $(\tilde{X}, \text{Aut}(\tilde{X}/X))$ .

*Exercise 18.* Show that if  $\Gamma$  acts on  $X$ , then there is an orbifold mapping from the orbifold  $\Gamma \backslash\backslash X$  to the topological space  $\Gamma \backslash X$ .

*Exercise 19.* Suppose that the discrete group  $\Gamma$  acts (trivially) on the one point space  $*$ . Show that there is a 1-1 correspondence between orbifold mappings  $T \rightarrow \Gamma \backslash\backslash *$  from a topological space  $T$  to  $\Gamma \backslash\backslash *$  and isomorphism classes of principal  $\Gamma$ -bundles over  $T$ .

We shall regard any  $\Gamma$ -invariant structure on  $X$  as a structure on the orbifold quotient  $\Gamma \backslash X$ . For example, if  $X$  is a Riemann surface with a  $\Gamma$ -invariant complex structure, then we will regard  $\Gamma \backslash X$  as a Riemann surface in the category of orbifolds. The holomorphic functions on  $\Gamma \backslash X$  are, by definition, the  $\Gamma$ -invariant holomorphic functions on  $X$ . Properties of holomorphic functions between Riemann surfaces extend to holomorphic mappings  $f : \Gamma \backslash X \rightarrow \Gamma' \backslash X'$  between orbifold Riemann surfaces. For example, we say that  $f$  is unramified if the mapping  $X \rightarrow X'$  of “universal coverings” is unramified.

The quotient of a non-simply connected space  $X$  by a discrete group  $\Gamma$  has a natural orbifold structure. Suppose, for simplicity, that  $X$  is path connected. Suppose that  $p : \tilde{X} \rightarrow X$  is a universal covering of  $X$ . Set  $G = \text{Aut}(\tilde{X}/X) = \pi_1(X, p)$ . Define  $\tilde{\Gamma}$  to be the group consisting of the pairs  $(\gamma, g) \in \Gamma \times \text{Aut } \tilde{X}$  such that the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & \tilde{X} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\gamma} & X \end{array}$$

commutes. This is an extension

$$1 \rightarrow G \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 1$$

Define  $\Gamma \backslash X$  to be the orbifold  $\tilde{\Gamma} \backslash \tilde{X}$ .

*Remark 3.2.* Orbifolds are examples of stacks. Stacks can be defined as groupoids in an appropriate category, such as the category of complex analytic manifolds. (See Appendix B for the definition.) A basic orbifold  $(X, \Gamma)$  may be viewed as a groupoid in the category of topological spaces. The set of objects of the groupoid is  $X$ , and the set of morphisms is  $\Gamma \times X$ . The morphism  $(\gamma, x)$  has source  $x$  and target  $\gamma x$ . Two morphisms  $(\gamma, x)$  and  $(\mu, y)$  are composable when  $y = \gamma x$ . Their composition is given by

$$(\mu, \gamma x) \circ (\gamma, x) = (\mu\gamma, x).$$

The inverse of  $(\gamma, x)$  is  $(\gamma^{-1}, \gamma x)$ . For more details, see Appendix B.

**3.2. Points\***. One has to distinguish various kinds of “points” of orbifolds. Suppose that  $\Gamma \backslash X$  is an orbifold. A point  $x$  of  $X$  corresponds to an inclusion  $i_x : * \rightarrow X$ . The composite

$$* \xrightarrow{i_x} X \longrightarrow \Gamma \backslash X$$

will be denoted  $x : * \rightarrow \Gamma \backslash X$  and regarded as a *closed point* of  $\Gamma \backslash X$ . The closed point  $x$  induces an orbifold mapping

$$\bar{\iota}_x : \Gamma_x \backslash * \rightarrow \Gamma \backslash X$$

where  $\Gamma_x$  denotes the isotropy group of  $x$  in  $\Gamma$ . When  $\Gamma_x$  is finite and non-trivial we will call  $\bar{\iota}_x$  an *orbifold point* of  $\Gamma \backslash X$ . In this case, we define the *degree*  $\deg(x)$  of  $x$  to be the order of its isotropy group  $\Gamma_x$ .

Two closed points  $x, x' \in X$  are said to be *conjugate* if they lie in the same  $\Gamma$  orbit. Denote the conjugacy class of  $x$  by  $(x)$ . Conjugacy classes of points of  $X$  are in 1-1 correspondence with the points of the orbit space  $\Gamma \backslash X$ . If  $\Gamma$  acts virtually freely<sup>6</sup> on  $X$ , then conjugate closed points have the same degree.

**3.3. Homotopy theory of basic orbifolds.** Suppose that  $(X, \Gamma, \rho)$  is a pointed orbifold. Denote the unit interval  $[0, 1]$  by  $I$ . Define  $I \times (X, \Gamma, \rho)$  to be the pointed orbifold  $(I \times X, \Gamma, \text{id}_I \times \rho)$ , where  $\text{id}_I \times \rho : \Gamma \rightarrow \text{Aut}(I \times X)$  is given by

$$\text{id}_I \times \rho(\gamma) : (t, x) \mapsto (t, \rho(\gamma)(x)).$$

**Definition 3.3.** A homotopy between two morphisms  $(f, \phi), (g, \psi) : (X, \Gamma) \rightarrow (X', \Gamma')$  of pointed orbifolds is a morphism

$$(F, \phi) : I \times (X, \Gamma) \rightarrow (X', \Gamma')$$

of pointed orbifolds that satisfies

- (i)  $\phi = \psi$ ,
- (ii)  $\phi : \Gamma \rightarrow \Gamma'$  is a homomorphism,
- (iii)  $f(x) = F(0, x)$  and  $g(x) = F(1, x)$  for all  $x \in X$ .

Homotopy of orbifold morphisms is an equivalence relation. Two pointed orbifolds  $(X, \Gamma)$  and  $(X', \Gamma')$  are defined to be *homotopy equivalent* if there are morphisms  $(f, \phi) : (X, \Gamma) \rightarrow (X', \Gamma')$  and  $(g, \psi) : (X', \Gamma') \rightarrow (X, \Gamma)$  such that  $(g, \psi) \circ (f, \phi)$  is homotopic to  $\text{id}_{(X, \Gamma)}$  and  $(f, \phi) \circ (g, \psi)$  is homotopic to  $\text{id}_{(X', \Gamma')}$ .

Denote by  $S^1$  the orbifold  $(\mathbb{R}, \mathbb{Z})$ , where  $\mathbb{Z}$  acts on  $\mathbb{R}$  by translation. The fundamental group  $\pi_1(\Gamma \backslash X, p)$  of the pointed orbifold  $(X, \Gamma)$  with respect to  $p : X \rightarrow \Gamma \backslash X$  is defined by

$$\begin{aligned} & \pi_1(\Gamma \backslash X, p) \\ &= \{ \text{homotopy classes of pointed morphisms } (f, \phi) : S^1 \rightarrow (X, \Gamma) \}. \end{aligned}$$

Denote the homotopy class of  $(f, \phi) : S^1 \rightarrow (X, \Gamma)$  by  $[f, \phi]$ .

---

<sup>6</sup>That is,  $\Gamma$  has a finite index subgroup that acts fixed point freely on  $X$ .

*Exercise 20.* Show that the fundamental group of a basic orbifold is a group. Show that the function  $\pi_1(\Gamma \backslash X, p) \rightarrow \Gamma$  that takes  $[f, \phi]$  to  $\phi(1)$  is a group isomorphism.

The orbifold and usual fundamental groups of  $\Gamma \backslash X$  agree when  $\Gamma$  acts freely and discontinuously on  $X$ . In particular, if  $X$  is a topological space with universal covering  $p : \tilde{X} \rightarrow X$ , then  $\pi_1(X, p) = \text{Aut}(\tilde{X}/X)$ , which agrees with the standard definition of  $\pi_1(X, p)$ .

Define a morphism  $(f, \phi) : (X, \Gamma) \rightarrow (X', \Gamma')$  between two pointed orbifolds to be a weak homotopy equivalence if  $\phi$  is an isomorphism and  $f : X \rightarrow X'$  induces an isomorphism  $H_\bullet(X) \rightarrow H_\bullet(X')$ .<sup>7 8</sup>

Every basic orbifold  $(X, \Gamma)$  is weak homotopy equivalent to a topological space. Indeed, if  $(X, \Gamma)$  is a basic orbifold, then we can consider the orbifold

$$(E\Gamma \times X, \Gamma)$$

where  $E\Gamma$  is any contractible space on which  $\Gamma$  acts properly discontinuously and fixed point freely and where  $\Gamma$  acts diagonally on  $E\Gamma \times X$ .<sup>9</sup> The projection

$$(E\Gamma \times X, \Gamma) \rightarrow (X, \Gamma)$$

is a weak homotopy equivalence.<sup>10</sup>

In other words, the weak homotopy type of a basic orbifold  $(X, \Gamma)$  is the homotopy type of the homotopy quotient  $E\Gamma \times_\Gamma X := \Gamma \backslash (E\Gamma \times X)$  of  $X$  by  $\Gamma$ . This motivates the following definition of the homology, cohomology and higher homotopy groups of a basic orbifold, which agree with the standard definitions on topological spaces.

A *local system*  $\mathbb{V}$  on the pointed orbifold  $(X, \Gamma)$  with fiber  $V$  over the base point  $p$  is simply a representation  $\Gamma \rightarrow \text{Aut } V$ . We will denote the corresponding local system on  $E\Gamma \times_\Gamma X$  by  $\mathbb{V}$ .

---

<sup>7</sup>Since  $X$  and  $X'$  are simply connected, this is equivalent, by a classical theorem of Hurewicz, to the statement that  $X \rightarrow X'$  induces an isomorphism on homotopy groups. A proof may be found in a standard text such as [12].

<sup>8</sup>A classical theorem of J. H. C. Whitehead states that a weak homotopy equivalence of CW-complexes is a homotopy equivalence. This is proved in many standard texts, such as [12].

<sup>9</sup>One can take  $E\Gamma$  to be the simplicial complex whose set of  $n$ -simplices is  $\Gamma^{n+1}$ . The group  $\Gamma$  acts diagonally on this complex. This complex is contractible as it is a cone with vertex the identity.

<sup>10</sup>If  $X$  has the homotopy type of a CW-complex and  $\Gamma$  acts properly discontinuously and fixed point freely on  $X$ , then this morphism is a homotopy equivalence. The homotopy inverse is given by any  $\Gamma$ -invariant mapping  $X \rightarrow E\Gamma \times X$  that is the identity in the second factor.

**Definition 3.4.** Suppose that  $(X, \Gamma)$  is a basic orbifold and that  $V$  is a  $\Gamma$ -module. Define the higher homotopy, homology, and cohomology groups of the orbifold  $\Gamma \backslash X$  by

- (i)  $\pi_n(\Gamma \backslash X, p) := \pi_n(X)$ , when  $n \geq 2$ ,
- (ii)  $H_\bullet(\Gamma \backslash X; \mathbb{V}) = H_\bullet(E\Gamma \times_\Gamma X; \mathbb{V})$ ,
- (iii)  $H^\bullet(\Gamma \backslash X; \mathbb{V}) = H^\bullet(E\Gamma \times_\Gamma X; \mathbb{V})$ .

Note that  $H^\bullet(E\Gamma \times_\Gamma X; \mathbb{V})$  is the  $\Gamma$ -equivariant cohomology  $H_\Gamma^\bullet(X; \mathbb{V})$  of  $X$ .

*Exercise 21.* Show that there is a natural isomorphism

$$\pi_1(\Gamma \backslash X, p) \cong \pi_1(E\Gamma \times_\Gamma X, p')$$

where  $p'$  is the quotient mapping  $E\Gamma \times X \rightarrow E\Gamma \times_\Gamma X$ .

**Example 3.5.** The homotopy type of the orbifold quotient  $\Gamma \backslash *$  of a one-point space  $*$  by a discrete group  $\Gamma$  is that of the classifying space  $B\Gamma := \Gamma \backslash E\Gamma$  of  $\Gamma$ . The cohomology groups of  $\Gamma \backslash *$  with coefficients in the local system that corresponds to the  $\Gamma$ -module  $V$  are those of the  $B\Gamma$ , which, by definition, are the cohomology groups of  $\Gamma$ :

$$H^\bullet(\Gamma \backslash *; \mathbb{V}) = H^\bullet(\Gamma, V) := H^\bullet(B\Gamma; \mathbb{V}).$$

The higher homotopy groups of  $\Gamma \backslash *$  vanish.

**Example 3.6.** The orbifold quotient  $\mu_d \backslash \mathbb{D}$  of the unit disk  $\mathbb{D}$  by the natural action of the group  $\mu_d$  of  $d$ th roots of unity on  $\mathbb{D}$ . The quotient of  $\mathbb{D}$  by  $\mu_d$  in the category of topological spaces is the disk, which is simply connected. Thus the topological and orbifold quotients can be different, even when the group action is effective. The projection  $\mathbb{D} \rightarrow *$  is a  $\mu_d$ -equivariant homotopy equivalence which induces a homotopy equivalence

$$\mu_d \backslash \mathbb{D} \simeq \mu_d \backslash * = B\mu_d.$$

The cohomology groups of  $\mu_d \backslash \mathbb{D}$  are therefore those of the group  $\mu_d$ . In particular,  $\mu \backslash \mathbb{D}$  does not have finite cohomological dimension.

A vector bundle over the orbifold  $\Gamma \backslash X$  is a vector bundle  $V \rightarrow X$  together with a lift of the  $\Gamma$ -action on  $X$  to  $V$ . The bundle over  $\Gamma \backslash X$  is denoted by  $\Gamma \backslash V \rightarrow \Gamma \backslash X$ . Sections of this bundle correspond precisely to  $\Gamma$  invariant sections of  $V \rightarrow X$ .

**Example 3.7.** A vector bundle over  $\Gamma \backslash *$  is simply a  $\Gamma$ -module  $V$ . Its space of sections consists of all  $\Gamma$ -equivariant functions  $f : * \rightarrow V$  and is therefore the subspace  $V^\Gamma$  of  $\Gamma$ -invariant vectors in  $V$ :

$$H^0(\Gamma \backslash *, V) = V^\Gamma.$$

**Example 3.8.** If  $X$  is a manifold, then a smooth  $\Gamma$ -action on  $X$  lifts naturally to a smooth  $\Gamma$ -action on the tangent bundle  $TX$  of  $X$ . In this case, the quotient  $\Gamma \backslash X$  can be regarded as a manifold in the category of orbifolds with tangent bundle  $\Gamma \backslash (TX) \rightarrow \Gamma \backslash X$ . Denote this by  $T(\Gamma \backslash X)$ . Sections of  $T(\Gamma \backslash X)$  are  $\Gamma$ -invariant vector fields on  $X$ . This example extends to all natural bundles on  $X$ , such as the cotangent bundle and its exterior powers. In particular, differential  $k$ -forms on  $\Gamma \backslash X$  are  $\Gamma$ -invariant sections of  $\Lambda^k T^*X \rightarrow X$ , which are just  $\Gamma$ -invariant differential forms on  $X$ .

Note that the de Rham theorem does not hold for all orbifolds. For example, it does not hold for  $\mathbb{Z} \backslash *$ .

**Definition 3.9.** An action of a group  $\Gamma$  on a space  $X$  is *virtually free* if  $\Gamma$  has a finite index subgroup  $\Gamma'$  that acts freely.

The action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathfrak{h}$  is virtually free by Exercise 96.

*Exercise 22.* Show that if  $\Gamma$  is a finitely generated discrete group that acts properly discontinuously and virtually freely on the simply connected manifold  $X$ , then there is a natural ring isomorphism

$$H^\bullet(\Gamma \backslash X; \mathbb{R}) \cong H^\bullet(\Gamma\text{-invariant, smooth real-valued forms on } X).$$

**3.4. Orbifold Euler characteristic.** The notion of Euler characteristic of a finite complex extends to orbifolds that satisfy some finiteness restrictions.

Suppose that the discrete group  $\Gamma$  acts virtually freely and properly discontinuously on  $X$ . If  $\Gamma'$  is a finite index normal subgroup  $\Gamma'$  that acts freely and discontinuously on  $X$ , then the quotient mapping  $X \rightarrow \Gamma' \backslash X$  is an unramified covering. The map  $\Gamma' \backslash X \rightarrow \Gamma \backslash X$  is an orbifold morphism that can be thought of as an unramified covering of degree  $[\Gamma : \Gamma']$ .

When  $\Gamma' \backslash X$  is a finite complex, we can define the *orbifold Euler characteristic* of  $\Gamma \backslash X$  by

$$\chi(\Gamma \backslash X) = \frac{1}{[\Gamma : \Gamma']} \chi(\Gamma' \backslash X)$$

where  $\chi(\Gamma' \backslash X)$  is the usual Euler characteristic of  $\Gamma' \backslash X$ .

*Exercise 23.* Show that the orbifold Euler characteristic of  $\Gamma \backslash X$  is well defined — that is, it is independent of the choice of the finite index subgroup  $\Gamma'$ .

**Example 3.10.** The orbifold Euler characteristic of  $\mu_d \backslash \mathbb{D}$  is  $1/d$ .

*Exercise 24.* Suppose that  $X$  is a finite simplicial complex and that  $\Gamma$  acts simplicially on  $X$ . Show that if  $\Gamma$  acts virtually freely on  $X$ , then the orbifold Euler characteristic of the semi-simplicial complex  $\Gamma \backslash X$  is given by

$$(7) \quad \chi(\Gamma \backslash X) = \sum_{k \geq 0} (-1)^k \sum_{\sigma \in (\Gamma \backslash X)_k} \frac{1}{|\Gamma_\sigma|}$$

where  $(\Gamma \backslash X)_k$  denotes the set of  $k$ -simplices of  $\Gamma \backslash X$  and  $\Gamma_\sigma$  denotes the isotropy group of  $\sigma$ .

**Example 3.11.** Suppose that  $K$  is the hexagonal decomposition of the disk. Let  $\Gamma$  be the symmetric group  $S_3$ , which is generated by the reflections in the 3 diagonals. This action is simplicial. The orders

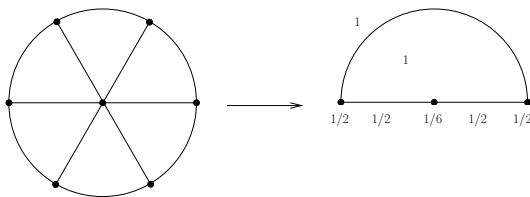


FIGURE 1. The quotient map

of the stabilizers of orbit representatives are shown in Figure 1. For example, the origin is “1/6 of a point” and the orbit of a diagonal consists of two copies of a “1/2” edge. Formula (7) thus gives

$\chi(S_3 \backslash K) = (1/6 + 1/2 + 1/2) - (1/2 + 1/2 + 1) + 1 = 1/6 = \chi(K)/|S_3|$   
as it should.

**3.5. The orbifold  $\mathcal{M}_{1,1}$ .** Our primary example of an orbifold is the moduli space of elliptic curves.

**Definition 3.12.** Define  $\mathcal{M}_{1,1}$  to be the orbifold  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$ . It is a Riemann surface in the category of orbifolds. There is a natural isomorphism

$$\pi_1(\mathcal{M}_{1,1}, p) \cong \mathrm{SL}_2(\mathbb{Z})$$

where the base point is the covering projection  $p : \mathfrak{h} \rightarrow \mathcal{M}_{1,1}$ .

Recall that  $M_{1,1}$  is the quotient  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$  in the category of Riemann surfaces. It is called the *coarse moduli space associated to  $\mathcal{M}_{1,1}$* . There is a natural morphism  $\mathcal{M}_{1,1} \rightarrow M_{1,1}$ . Each elliptic curve  $(E, P)$  determines a point  $[E, P]$  of  $\mathcal{M}_{1,1}$ , which is called the *moduli point of  $(E, P)$* .

Families of elliptic curves  $X \rightarrow T$  whose coarse period mapping  $T \rightarrow M_{1,1}$  is constant are said to be *isotrivial*.



*Exercise 25* (isotrivial families). Show that if  $T$  is a compact Riemann surface and  $X \rightarrow T$  is a family of elliptic curves over  $T$ , then the coarse period mapping  $T \rightarrow M_{1,1}$  is constant. (Hint:  $M_{1,1}$  is a non-compact Riemann surface.) Examples of such families over a (not necessarily compact) base  $T$  can be constructed as follows: fix an elliptic curve  $(E, 0)$  and an automorphism  $\sigma \in \text{Aut}(E, 0)$  of order  $d$ . Choose a cyclic unramified covering  $S \rightarrow T$  of degree  $d$ . Choose a generator  $\phi$  of  $\text{Aut}(S/T)$ . Then  $\mathbb{Z}/d\mathbb{Z}$  acts diagonally on  $E \times S$  by  $k : (x, s) \mapsto (\sigma^k(x), \phi^k(s))$ . Define  $X \rightarrow T$  to be the quotient

$$(\mathbb{Z}/d\mathbb{Z}) \backslash (E \times S) \rightarrow (\mathbb{Z}/d\mathbb{Z}) \backslash S.$$

Show that  $X \rightarrow T$  is a family of elliptic curves and that the coarse period mapping  $T \rightarrow M_{1,1}$  takes the constant value  $[E, 0]$ . Show that the period mapping  $T \rightarrow \mathcal{M}_{1,1}$  factors through  $\text{Aut}(E, 0) \backslash *$ . Finally, show that a family of elliptic curves  $X \rightarrow T$  is isotrivial if and only if its coarse period mapping is constant.

**Proposition 3.13.** *The low-dimensional homology and cohomology groups of  $\mathcal{M}_{1,1}$  are*

$$H_1(\mathcal{M}_{1,1}; \mathbb{Z}) = \mathbb{Z}/12\mathbb{Z}, \quad H^1(\mathcal{M}_{1,1}; \mathbb{Z}) = 0, \quad H^2(\mathcal{M}_{1,1}; \mathbb{Z}) = \mathbb{Z}/12\mathbb{Z}.$$

*The morphism  $\mathcal{M}_{1,1} \rightarrow M_{1,1}$  induces an isomorphism on rational homology and rational cohomology, so that  $\mathcal{M}_{1,1}$  has the rational homology and cohomology of a point.*

*Proof.* Since  $\mathfrak{h}$  is contractible,  $\mathcal{M}_{1,1}$  has the homotopy type of the classifying space  $B\text{SL}_2(\mathbb{Z})$  of  $\text{SL}_2(\mathbb{Z})$ . Therefore

$$H_*(\mathcal{M}_{1,1}; \mathbb{Z}) \cong H_*(\text{SL}_2(\mathbb{Z}); \mathbb{Z}) \text{ and } H^*(\mathcal{M}_{1,1}; \mathbb{Z}) \cong H^*(\text{SL}_2(\mathbb{Z}); \mathbb{Z}).$$

Since  $\text{SL}_2(\mathbb{Z})$  is finitely presented, this implies that its homology and cohomology are finitely generated in degrees 1 and 2.<sup>11</sup> In particular,  $H_1(\mathcal{M}_{1,1}; \mathbb{Z})$  is the maximal abelian quotient of  $\text{SL}_2(\mathbb{Z})$ . Using the presentation (3) of  $\text{SL}_2(\mathbb{Z})$ , we have

$$H_1(\mathcal{M}_{1,1}; \mathbb{Z}) = (\mathbb{Z}s \oplus \mathbb{Z}u) / \langle 2s - 3u, 4s \rangle \cong \mathbb{Z}/12\mathbb{Z}$$

from which it follows that

$$H^1(\mathcal{M}_{1,1}; \mathbb{Z}) = \text{Hom}(H_1(\mathcal{M}_{1,1}), \mathbb{Z}) = 0.$$

<sup>11</sup>It is not difficult to see that they are finitely generated in all degrees as  $B\text{SL}_2(\mathbb{Z})$  can be realized as a CW-complex with a finite number of cells in each degree.

By Exercise 89, the finite index, normal subgroup  $\mathrm{SL}_2(\mathbb{Z})[m]$  of  $\mathrm{SL}_2(\mathbb{Z})$  is free when  $m \geq 3$ .<sup>12</sup> Standard arguments imply that

$$H^k(\mathrm{SL}_2(\mathbb{Z}); V) = H^k(\mathrm{SL}_2(\mathbb{Z})[m]; V)^{\mathrm{SL}_2(\mathbb{Z}/m)}$$

whenever  $V$  is a  $\mathbb{Q}$ -module. Since  $\mathrm{SL}_2(\mathbb{Z})[m]$  is free, these vanish when  $k \geq 2$ . It follows that  $H^2(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Z})$  is a finitely generated torsion group. The universal coefficient theorem implies that

$$H^2(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Z}) \cong \mathrm{Hom}(H_1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q}/\mathbb{Z})) \cong \mathbb{Z}/12\mathbb{Z}.$$

□

The purpose of the following examples and exercises is to give some feel for the orbifold structure of  $\mathcal{M}_{1,1}$ .

*Exercise 26.* Consider the orbifold morphism  $\mathcal{M}_{1,1} \rightarrow M_{1,1}$ . Show that if  $j : \mathbb{D} \hookrightarrow M_{1,1}$  is the inclusion of a coordinate disk centered at the orbit  $[i]$  of  $i$  (or  $[\rho]$  of  $\rho$ ), then there is no orbifold lift  $\tilde{j} : \mathbb{D} \rightarrow \mathcal{M}_{1,1}$  of the restriction of  $j$  to  $\mathbb{D}$ . Deduce that there is no universal curve over the coarse moduli space  $M_{1,1}$ .

**Example 3.14.** Denote the subgroup  $\{\pm I\}$  of  $\mathrm{SL}_2(\mathbb{Z})$  by  $C_2$ . It acts trivially on  $\mathfrak{h}$ . The fundamental group of the quotient  $\{\pm I\} \backslash \mathfrak{h}$  is cyclic of order 2. The projection  $\mathfrak{h} \rightarrow C_2 \backslash \mathfrak{h}$  is viewed as a 2:1 cover of orbifolds, even though it is a homeomorphism in the category of topological spaces. The group  $\mathrm{PSL}_2(\mathbb{Z})$ , which is the quotient  $\mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ , acts faithfully on  $\mathfrak{h}$ . The orbifold quotient  $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathfrak{h}$  has fundamental group  $\mathrm{PSL}_2(\mathbb{Z})$  and is not isomorphic to  $\mathcal{M}_{1,1}$ .

Given two sets  $\{a_1, a_2, a_3\}$  and  $\{b_1, b_2, b_3\}$  of 3 distinct points of the Riemann sphere  $\mathbb{P}^1$ , there is a unique  $\phi \in \mathrm{Aut} \mathbb{P}^1$  such that  $\phi(a_j) = b_j$  for all  $j$ . The permutations of  $\{0, 1, \infty\}$  therefore define an action of the symmetric group  $S_3$  on  $\mathbb{P}^1$  which restricts to a homomorphism  $S_3 \hookrightarrow \mathrm{Aut} \mathbb{P}^1 - \{0, 1, \infty\}$ . The group  $C_2 \times S_3$  acts on  $\mathbb{P}^1 - \{0, 1, \infty\}$  via the projection onto  $S_3$ :

$$C_2 \times S_3 \rightarrow S_3 \hookrightarrow \mathrm{Aut}(\mathbb{C} - \{0, 1\}).$$

**Proposition 3.15.** *The orbifold Riemann surface  $\mathcal{M}_{1,1}$  is isomorphic to the orbifold  $(C_2 \times S_3) \backslash (\mathbb{C} - \{0, 1\})$ .*

*Proof.* The level 2 subgroup  $\mathrm{SL}_2(\mathbb{Z})[2]$  of  $\mathrm{SL}_2(\mathbb{Z})$  is the subgroup of  $\mathrm{SL}_2(\mathbb{Z})$  consisting of matrices congruent to the identity mod 2. The quotient  $\mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})[2]$  is isomorphic to  $\mathrm{SL}_2(\mathbb{F}_2)$ , which is isomorphic to the symmetric group  $S_3$ . It follows from Exercise 89 that

<sup>12</sup>A group  $\Gamma$  is *virtually free* if it has a free subgroup of finite index. Thus  $\mathrm{SL}_2(\mathbb{Z})$  is virtually free.

the image  $\mathrm{PSL}_2(\mathbb{Z})[2]$  of  $\mathrm{SL}_2(\mathbb{Z})$  in  $\mathrm{PSL}_2(\mathbb{Z})$  is torsion free. The quotient  $\mathrm{PSL}_2(\mathbb{Z})[2]\backslash\mathfrak{h}$  is thus a Riemann surface. It is biholomorphic to  $\mathbb{P}^1 - \{0, 1, \infty\}$  as can be seen, for example, by considering the fundamental domain of the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathfrak{h}$ . (See [3].) It follows that  $\mathrm{PSL}_2(\mathbb{Z})[2]$  is a free group  $F_2$  of rank 2. Choose a splitting of  $\mathrm{SL}_2(\mathbb{Z})[2] \rightarrow F_2$  (not unique!) and use it to identify  $\mathrm{SL}_2(\mathbb{Z})[2]$  with  $F_2 \times C_2$ . Then

$$\mathrm{SL}_2(\mathbb{Z})[2]\backslash\mathfrak{h} \cong C_2 \backslash (F_2 \backslash \mathfrak{h}) \cong C_2 \backslash (\mathbb{C} - \{0, 1\}).$$

Since the actions of  $C_2$  and  $S_3$  on  $\mathbb{C} - \{0, 1\}$  commute, we have

$$\mathcal{M}_{1,1} \cong S_3 \backslash (C_2 \backslash (\mathbb{C} - \{0, 1\})) \cong (C_2 \times S_3) \backslash (\mathbb{C} - \{0, 1\}).$$

□

Since  $\chi(\mathbb{C} - \{0, 1\}) = -1$  and  $C_2 \times S_3$  has order 12, we have:

**Corollary 3.16.** *The orbifold Euler characteristic of  $\mathcal{M}_{1,1}$  is  $-1/12$ .*

*Exercise 27.* Show that the function  $q : \mathfrak{h} \rightarrow \mathbb{D}^*$  defined by  $q(\tau) = \exp(2\pi i\tau)$  induces an orbifold isomorphism

$$\{\pm 1\} \left( \begin{array}{c|c} 1 & \mathbb{Z} \\ \hline 0 & 1 \end{array} \right) \backslash \mathfrak{h} \rightarrow C_2 \backslash \mathbb{D}^*.$$

Deduce that there is an orbifold morphism  $\mathbb{D}^* \rightarrow \mathcal{M}_{1,1}$  which factors through the quotient of  $\mathbb{D}^*$  by the *trivial*  $C_2$ -action:

$$\mathbb{D}^* \longrightarrow C_2 \backslash \mathbb{D}^* \longrightarrow \mathcal{M}_{1,1}.$$

**Example 3.17.** The cyclic group  $C_2 = \{\pm I\}$  acts on the line bundle  $\mathbb{C} \times \mathfrak{h} \rightarrow \mathfrak{h}$  by  $-I : (z, \tau) \mapsto (-z, \tau)$ . Sections of the orbifold line bundle

$$C_2 \backslash (\mathbb{C} \times \mathfrak{h}) \rightarrow C_2 \backslash \mathfrak{h}$$

correspond to  $C_2$  invariant functions  $f : \mathfrak{h} \rightarrow \mathbb{C}$ , and are thus zero.

*Exercise 28.* Show that the function

$$\mathrm{SL}_2(\mathbb{Z}) \times \mathbb{C} \times \mathfrak{h} \rightarrow \mathbb{C} \times \mathfrak{h}$$

defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z, \tau) \mapsto ((c\tau + d)^k z, (a\tau + b)/(c\tau + d))$$

is an action that lifts the standard action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathfrak{h}$ . Set

$$\mathcal{L}_k = \mathrm{SL}_2(\mathbb{Z}) \backslash (\mathbb{C} \times \mathfrak{h})$$

This is an orbifold line bundle over  $\mathcal{M}_{1,1}$ . Show that  $\mathcal{L}_k = \mathcal{L}_1^{\otimes k}$ . Show that the holomorphic sections of  $\mathcal{L}_k$  correspond to holomorphic functions  $f : \mathfrak{h} \rightarrow \mathbb{C}$  that satisfy

$$f((a\tau + b)/(c\tau + d)) = (c\tau + d)^k f(\tau).$$

Show that  $\mathcal{L}_k$  has no non-zero sections when  $k$  is odd.

**3.6. The universal elliptic curve  $\mathcal{E} \rightarrow \mathcal{M}_{1,1}$ .** Define  $\mathcal{E}$  to be the orbifold quotient

$$(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \backslash (\mathbb{C} \times \mathfrak{h})$$

where the action is defined in Exercise 14. The projection  $\mathbb{C} \times \mathfrak{h} \rightarrow \mathfrak{h}$  induces an orbifold morphism

$$\mathcal{E} \rightarrow \mathcal{M}_{1,1}.$$

It is a family of elliptic curves over  $\mathcal{M}_{1,1}$  in the category of orbifolds.

*Exercise 29.* Show that every orbifold morphism  $T \rightarrow \mathcal{M}_{1,1}$  is a locally liftable mapping  $T \rightarrow M_{1,1}$ . Show that the universal elliptic curve  $\mathcal{E} \rightarrow \mathcal{M}_{1,1}$  pulls back along an orbifold morphism  $\Phi : T \rightarrow \mathcal{M}_{1,1}$  to a family of elliptic curves  $X \rightarrow T$ . Use this to prove the following theorem.

**Theorem 3.18.** *There is a 1-1 correspondence between isomorphism classes of families of elliptic curves  $X \rightarrow T$  over a complex manifold and holomorphic orbifold morphisms  $T \rightarrow \mathcal{M}_{1,1}$ . This orbifold morphism is induced by the period mapping. The morphism  $\Phi : T \rightarrow \mathcal{M}_{1,1}$  corresponds to the isomorphism class of the pullback family  $\Phi^*\mathcal{E} \rightarrow T$ .*

*Exercise 30.* Denote the  $\mathrm{SL}_2(\mathbb{Z})$ -orbit of  $\tau \in \mathfrak{h}$  by  $[\tau]$ . Show that the inclusion  $j : M_{1,1} - \{[i], [\rho]\} \hookrightarrow M_{1,1}$  is locally liftable to a map to  $\mathfrak{h}$ , but that there is no orbifold mapping  $\tilde{j}$  such that the diagram

$$\begin{array}{ccc} & & \mathcal{M}_{1,1} \\ & \nearrow \tilde{j} & \downarrow \\ M_{1,1} - \{[i], [\rho]\} & \longrightarrow & M_{1,1} \end{array}$$

commutes. Deduce that there is no universal elliptic curve over either  $M_{1,1}$  or  $M_{1,1} - \{[i], [\rho]\}$ .

Theorem 3.18 is more subtle than it may at first appear due to the subtleties of orbifold mappings. This is illustrated by *isotrivial families* — non-trivial families with constant period mappings.

**Example 3.19.** This example is a continuation of Exercise 25. Let  $X \rightarrow T$  be the isotrivial family of elliptic curves associated to  $\sigma \in \text{Aut}(E, 0)$  and an unramified covering  $S \rightarrow T$ . The coarse period mapping  $T \rightarrow M_{1,1}$  takes the constant value  $[E, 0]$ . Even though the period mapping  $T \rightarrow M_{1,1}$  to the coarse moduli space is constant, the orbifold period mapping  $T \rightarrow \mathcal{M}_{1,1}$  is non-trivial when  $\sigma$  is non-trivial. To compute the period mapping, fix a framing  $\mathbf{a}, \mathbf{b}$  of  $H_1(E; \mathbb{Z})$ . Let

$$\tau = \int_{\mathbf{b}} \omega / \int_{\mathbf{a}} \omega.$$

This is the point of  $\mathfrak{h}$  that corresponds to the framed elliptic curve  $(E, 0; \mathbf{a}, \mathbf{b})$ . Define  $A \in \text{SL}_2(\mathbb{Z})$  by

$$\sigma_* \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} = A \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix}$$

Then  $A\tau = \tau$  and the induced mapping  $\rho : \pi_1(T) \rightarrow \pi_1(\mathcal{M}_{1,1}) = \text{SL}_2(\mathbb{Z})$  is the composite  $\pi_1(T) \rightarrow \text{Aut}(S/T) \rightarrow \text{SL}_2(\mathbb{Z})$  where  $\phi \in \text{Aut}(S/T)$  is mapped to  $A$ . The period mapping  $T \rightarrow \mathcal{M}_{1,1}$  is represented by the mapping  $(\tilde{T}, \pi_1(T)) \rightarrow (\mathfrak{h}, \text{SL}_2(\mathbb{Z}))$  that takes  $(t, \gamma)$  to  $(\tau, \rho(\gamma))$ .

#### 4. THE ORBIFOLD $\overline{\mathcal{M}}_{1,1}$ AND MODULAR FORMS

In this section we explain how to construct the orbifold compactification  $\overline{\mathcal{M}}_{1,1}$  of  $\mathcal{M}_{1,1}$ ; it is the prototypical example of an orbifold obtained by patching. We are able to do this as the orbifold  $\overline{\mathcal{M}}_{1,1}$  is easy to define because it is obtained by gluing two basic orbifolds along another basic orbifold. In general, to construct an orbifold, it is necessary to glue more than two basic orbifolds. In this case, one has a compatibility condition for the gluing maps that is analogous to the familiar cocycle condition  $g_{\alpha\gamma} = g_{\alpha\beta}g_{\beta\gamma}$ . For completeness, we discuss stacks briefly in Appendix B. Analytical (resp. topological) orbifolds are stacks in the category of analytic varieties (resp. topological spaces).

To construct the orbifold  $\overline{\mathcal{M}}_{1,1}$ , we begin with the diagram

$$\begin{array}{ccc}
 & C_2 \times \mathbb{Z} & \\
 & \curvearrowright & \\
 & \mathfrak{h} & \\
 p \swarrow & & \searrow q \\
 \text{SL}_2(\mathbb{Z}) \curvearrowright \mathfrak{h} & & \mathbb{D} \curvearrowright C_2
 \end{array}$$

of spaces with compatible group actions. Here  $C_2 \times \mathbb{Z}$  acts on  $\mathfrak{h}$  by  $(\pm 1, n) : \tau \mapsto \tau + n$ ,  $C_2$  acts trivially on  $\mathbb{D}$ ,  $q(\tau) = \exp(2\pi i\tau)$ ,  $p$  is the

identity, and  $(\pm 1, n) \in C_2 \times \mathbb{Z}$  is mapped to  $\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  and to  $\pm 1$  in  $C_2$ . These induce orbifold mappings

$$\begin{array}{ccc} & C_2 \backslash \mathbb{D}^* & \\ & \swarrow \quad \searrow & \\ \mathcal{M}_{1,1} & & C_2 \backslash \mathbb{D} \end{array}$$

where we identify  $\mathbb{Z} \backslash \mathfrak{h}$  with the punctured  $q$ -disk  $\mathbb{D}^*$ , via the mapping  $\tau \mapsto \exp(2\pi i\tau)$ , and the right hand arrow is the quotient of the inclusion  $\mathbb{D}^* \hookrightarrow \mathbb{D}$  by the trivial  $C_2$  action.

The compactification  $\overline{\mathcal{M}}_{1,1}$  of  $\mathcal{M}_{1,1}$  is essentially obtained by adding one point with automorphism group  $C_2$  to  $\mathcal{M}_{1,1}$ . Formally,  $\overline{\mathcal{M}}_{1,1}$  is the orbifold obtained by gluing  $\mathcal{M}_{1,1}$  and  $C_2 \backslash \mathbb{D}$  together via  $C_2 \backslash \mathbb{D}^*$ . It is a Riemann surface in the category of orbifolds. The parameter  $q$  is a local holomorphic coordinate about the new closed point  $\infty$ . One works with  $\overline{\mathcal{M}}_{1,1}$  in the obvious way. For example, a line bundle on  $\overline{\mathcal{M}}_{1,1}$  consists of a line bundle on each of the orbifolds  $\mathcal{M}_{1,1}$  and  $C_2 \backslash \mathbb{D}$ , together with an isomorphism of their pullbacks to  $C_2 \backslash \mathbb{D}^*$ . Sections of a line bundle over  $\overline{\mathcal{M}}_{1,1}$  consist of sections of the line bundle over the charts  $\mathcal{M}_{1,1}$  and  $C_2 \backslash \mathbb{D}$  that agree on their pullbacks to  $C_2 \backslash \mathbb{D}^*$ .

The coarse moduli space associated to  $\overline{\mathcal{M}}_{1,1}$  is

$$\overline{M}_{1,1} := M_{1,1} \cup_{\mathbb{D}^*} \mathbb{D}$$

where  $\mathbb{D}$  is the  $q$ -disk.

The orbifold  $\overline{\mathcal{M}}_{1,1}$  can also be expressed as a finite quotient of the Riemann sphere. This gives an algebraic description of  $\overline{\mathcal{M}}_{1,1}$ .

*Exercise 31.* Show that the orbifold isomorphism of Proposition 3.15 extends to an orbifold isomorphism

$$\overline{\mathcal{M}}_{1,1} \cong (C_2 \times S_3) \backslash \mathbb{P}^1$$

where  $C_2$  acts trivially on  $\mathbb{P}^1$  and  $S_3$  acts on  $\mathbb{P}^1$  by permuting  $\{0, 1, \infty\}$ . Show that the quotient mapping  $\mathbb{P}^1 \rightarrow \overline{\mathcal{M}}_{1,1}$  is ramified at  $\{0, 1, \infty\}$  and that it is locally 2:1 about each of these points. Deduce that the orbifold Euler characteristic of  $\overline{\mathcal{M}}_{1,1}$  is  $5/12$ .

The line bundles  $\mathcal{L}_k$  extend naturally to  $\overline{\mathcal{M}}_{1,1}$ .

**Proposition 4.1.** *The orbifold line bundle  $\mathcal{L}_k \rightarrow \mathcal{M}_{1,1}$  extend naturally to a holomorphic line bundles  $\overline{\mathcal{L}}_k \rightarrow \overline{\mathcal{M}}_{1,1}$ .*

*Proof.* Define  $\overline{\mathcal{L}}_k$  to be the line bundle over  $\overline{\mathcal{M}}_{1,1}$  whose restriction to  $\mathcal{M}_{1,1}$  is  $\mathcal{L}_k$ , whose restriction to  $C_2 \backslash \mathbb{D}$  is the quotient of the trivial bundle  $\mathbb{C} \times \mathbb{D} \rightarrow \mathbb{D}$  by the action  $\pm 1 : (z, q) \mapsto ((\pm 1)^k z, q)$ . The isomorphism

$$C_2 \backslash (\mathbb{C} \times \mathbb{D}^*) \cong p^* \mathcal{L}_k \rightarrow q^* \overline{\mathcal{L}}_k \cong C_2 \backslash (\mathbb{C} \times \mathbb{D}^*)$$

is the identity.  $\square$

*Exercise 32.* Prove that  $\overline{\mathcal{L}}_k$  is isomorphic to  $\overline{\mathcal{L}}_1^{\otimes k}$ .

**4.1. Modular forms.** A holomorphic (resp. meromorphic) modular function of weight  $k \in \mathbb{N}$  is a holomorphic (resp. meromorphic) function  $f : \mathfrak{h} \rightarrow \mathbb{C}$  that satisfies

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$

for all

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Since  $-I \in \mathrm{SL}_2(\mathbb{Z})$ , each modular function of weight  $k$  satisfies  $f(\tau) = (-1)^k f(\tau)$ , from which it follows that all modular functions of odd weight vanish. As we have seen in Exercise 28, holomorphic (resp. meromorphic) modular functions of weight  $k$  are precisely the holomorphic (resp. meromorphic) sections of the orbifold line bundle  $\mathcal{L}_k$  over  $\mathcal{M}_{1,1}$ .

**Example 4.2** (Eisenstein Series). Fix an integer  $k > 2$ . For a lattice  $\Lambda$  in  $\mathbb{C}$  define  $S_k$  by the absolutely convergent series

$$S_k(\Lambda) = \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{1}{\lambda^k}.$$

Observe that when  $u \in \mathbb{C}^*$

$$(8) \quad S_k(u\Lambda) = u^{-k} S_k(\Lambda)$$

Since  $\Lambda = -\Lambda$ , this implies that  $S_k$  is identically zero when  $k$  is odd.

Recall that for  $\tau \in \mathfrak{h}$ ,  $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$ . For  $\tau \in \mathfrak{h}$ , define

$$G_k(\tau) := S_k(\Lambda_\tau).$$

This is holomorphic on  $\mathfrak{h}$ . Since  $\Lambda_{\gamma\tau} = (c\tau + d)^{-1} \Lambda_\tau$  for all

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

the identity (8) becomes  $G_k(\gamma\tau) = (c\tau + d)^k G_k(\tau)$ . That is,  $G_k$  is a holomorphic modular function of weight  $k$ . For more details, see any standard book on modular forms, such as [10, pp. 81–84].

Since  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  each modular function satisfies  $f(\tau + 1) = f(\tau)$  and thus has a Fourier expansion (its “ $q$ -expansion”):

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau}.$$

The  $q$ -expansion of  $G_{2k}$  is

$$(9) \quad G_{2k}(\tau) = 2\zeta(2k) + 2 \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where  $\zeta$  denotes the Riemann zeta function and  $\sigma_k(n) := \sum_{d|n} d^k$ . Details can be found in [10, p. 92].

**Definition 4.3.** Suppose that  $k \in \mathbb{N}$ . A *modular form of weight  $k$*  is a holomorphic modular function that is holomorphic at  $q = 0$ . That is, the coefficients  $a_n$  of its  $q$ -expansion vanish when  $n < 0$ . A *cuspidal form* is a modular form whose  $q$ -expansion vanishes at  $q = 0$ . A *meromorphic modular form of weight  $k$*  is a meromorphic modular function of weight  $k$  whose  $q$ -expansion is meromorphic on the  $q$ -disk.

*Exercise 33.* Show that the holomorphic modular forms of weight  $k$  correspond to holomorphic sections of  $\overline{\mathcal{L}}_k \rightarrow \overline{\mathcal{M}}_{1,1}$ . Show that cuspidal forms correspond to those sections that vanish at the point  $\infty$ , the origin of the  $q$ -disk.

Equation (9) implies that each  $G_{2k}$  is a modular form of weight  $2k$  when  $k \geq 2$ . Since

$$(60 \cdot 2 \cdot \zeta(4))^3 = \frac{64}{27} \pi^{12} = 27(140 \cdot 2 \cdot \zeta(6))^2,$$

the *Ramanujan tau function*

$$\Delta(\tau) := g_2(\tau)^3 - 27g_3(\tau)^2$$

is a cuspidal form of weight 12, where  $g_2(\tau) := 60G_4(\tau)$  and  $g_3(\tau) := 140G_6(\tau)$ . It has  $q$ -expansion [10, p. 95]

$$\Delta = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

The function  $\Delta$  has no zeros in  $\mathfrak{h}$  [10, p. 88] and a simple zero at  $q = 0$ .<sup>13</sup>

<sup>13</sup>A geometric proof is given in Section 5. Cf. Corollary 5.5.



For a line bundle  $L$  over  $\overline{\mathcal{M}}_{1,1}$  and  $d \in \mathbb{Z}$ , define  $L(d\infty) = L \otimes \mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(d\infty)$ .<sup>14</sup>

*Exercise 34.* Show that  $\overline{\mathcal{L}}_{12} \cong \mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(\infty)$ .

**Proposition 4.4.** *The log canonical bundle  $\Omega_{\overline{\mathcal{M}}_{1,1}}^1(\infty)$  of  $\overline{\mathcal{M}}_{1,1}$  is isomorphic to  $\overline{\mathcal{L}}_2$ .*

*Proof.* Since  $q = \exp(2\pi i\tau)$ , we have that  $2\pi i d\tau = dq/q$ . That is,  $d\tau$  is a trivialization of the pullback of the log canonical bundle of  $\overline{\mathcal{M}}_{1,1}$  to the  $q$ -disk  $\mathbb{D}$ . On the other hand, since

$$d\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{d\tau}{(c\tau + d)^2},$$

a meromorphic form  $f(\tau)d\tau$  on  $\mathfrak{h}$  descends to a meromorphic section of  $\Omega_{\overline{\mathcal{M}}_{1,1}}^1$  if and only if  $f(\tau)$  is a meromorphic modular function of weight 2. In particular,

$$\omega = \frac{G_4(\tau)}{G_2(\tau)} d\tau$$

is such a meromorphic form. Since  $G_4$  and  $G_2$  are both non-zero at infinity (Cf. equation (9)), the restriction of  $\omega$  to a neighbourhood of  $\infty$  is a nowhere vanishing holomorphic multiple of  $dq/q$ . From this it follows that  $f(\tau)$  is a meromorphic section of  $\overline{\mathcal{L}}_2$  if and only if  $f(\tau)d\tau$  is a meromorphic section of  $\Omega_{\overline{\mathcal{M}}_{1,1}}^1(\infty)$ . The result follows.  $\square$

Since  $g_2$  does not vanish at  $q = 0$ , the modular function

$$j(\tau) := 1728 g_2(\tau)^3 / \Delta(\tau)$$

of weight 0 is holomorphic on  $\mathfrak{h}$  and has a simple pole at  $q = 0$ . It has  $q$ -expansion

$$j(\tau) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c_n q^n,$$

where each  $c_n \in \mathbb{Z}$ .

---

<sup>14</sup>Sections of the orbifold sheaf  $\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(d\infty)$  are meromorphic functions on  $\overline{\mathcal{M}}_{1,1}$  that are holomorphic on  $\mathcal{M}_{1,1}$  and whose Fourier expansion in the  $q$ -disk has a pole of order  $\leq d$ . Cf. Section 6.

*Exercise 35.* Show that  $j$  may be viewed as a holomorphic function  $j : \overline{\mathcal{M}}_{1,1} \rightarrow \mathbb{P}^1$ . Show that its restriction to  $\mathcal{M}_{1,1}$  induces a biholomorphism  $\mathcal{M}_{1,1} \rightarrow \mathbb{C}$ :

$$\begin{array}{ccccc}
 & & \mathcal{M}_{1,1} & \longrightarrow & \overline{\mathcal{M}}_{1,1} \\
 & \swarrow & \downarrow j & & \downarrow j \\
 M_{1,1} & & \mathbb{C} & \longrightarrow & \mathbb{P}^1 \\
 & \searrow \approx & & & \\
 & & & & 
 \end{array}$$

Note that  $j$  can be used to define a local parameter about each point of  $M_{1,1}$  except the points  $[i]$  and  $[\rho]$  as the map  $\mathfrak{h} \rightarrow M_{1,1}$  is ramified above  $[i]$  and  $[\rho]$ .

*Exercise 36.* Show that if  $X \rightarrow T$  is a family of smooth elliptic curves over a compact Riemann surface  $T$ , then the period mapping  $T \rightarrow M_{1,1}$  is constant. Deduce that every such family is isotrivial.

Denote the space of holomorphic modular forms of weight  $k$  by  $M_k$ . These form an evenly graded ring  $M_\bullet$  with respect to multiplication of functions. For each even  $k$ , evaluation at  $q = 0$  defines a linear surjection  $M_k \rightarrow \mathbb{C}$  whose kernel is  $M_k^o$ , the space of cusp forms of weight  $k$ . Since  $G_{2k}$  does not vanish at  $q = 0$

$$M_{2k} = M_{2k}^o \oplus \mathbb{C}G_{2k}$$

for each  $k > 1$ .

The following basic result is proved in [10, p. 89]. It can be deduced from the results of Exercise 62.

**Proposition 4.5.** *The graded ring  $M_\bullet$  is generated freely by  $G_4$  and  $G_6$ . The subspace of cusp forms  $M_\bullet^o$  is the ideal of  $M_\bullet$  generated by  $\Delta$ . In addition*

$$\dim M_{2k} = 1 + \dim M_{2k}^o = \begin{cases} \lfloor k/6 \rfloor & k \equiv 1 \pmod{6}, k \geq 0; \\ 1 + \lfloor k/6 \rfloor & k \not\equiv 1 \pmod{6}, k \geq 0. \end{cases}$$

□

**4.2. Level structures\*.** In the first Chapter, we used a framing of  $H_1(E; \mathbb{Z})$  to rigidify and then solve the moduli problem for elliptic curves. Since  $H_1(E; \mathbb{Z})$  has an infinite number of framings, the moduli space of framed elliptic curves is an infinite, and therefore transcendental, covering of  $\mathcal{M}_{1,1}$ . To construct finite coverings of  $\mathcal{M}_{1,1}$ , we consider the moduli space of elliptic curves plus a framings of  $H_1(E; \mathbb{Z}/m\mathbb{Z})$ , where  $m \geq 1$ .

**Definition 4.6.** Suppose that  $m$  is a positive integer. A *level  $m$  structure* on an elliptic curve is a basis  $\mathbf{a}, \mathbf{b}$  of  $H_1(E; \mathbb{Z}/m\mathbb{Z})$ , where the mod  $m$  intersection number  $\mathbf{a} \cdot \mathbf{b}$  is 1.

*Exercise 37.* Show that the homomorphism  $\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$  that takes a matrix to its reduction mod  $m$  is surjective. Its kernel is called the *level  $m$  subgroup of  $\mathrm{SL}_2(\mathbb{Z})$*  and will be denoted by  $\mathrm{SL}_2(\mathbb{Z})[m]$ . Show that the set of isomorphism classes of elliptic curves with a level  $m$  structure is the quotient  $\mathrm{SL}_2(\mathbb{Z})[m] \backslash \mathfrak{h}$ . Show that this is a Riemann surface with fundamental group  $\mathrm{SL}_2(\mathbb{Z})[m]$  for all  $m \geq 3$ . (Cf. Exercise 89.)

Denote the orbifold Riemann surface  $\mathrm{SL}_2(\mathbb{Z})[m] \backslash \mathfrak{h}$  by  $\mathcal{M}_{1,1}[m]$ . It is called the moduli space of elliptic curves with a level  $m$  structure. Points of the corresponding coarse moduli space  $M_{1,1}[m] := \mathrm{SL}_2(\mathbb{Z})[m] \backslash \mathfrak{h}$  are isomorphism classes of elliptic curves with a level  $m$  structure. Since  $\mathrm{SL}_2(\mathbb{Z})[m]$  is torsion free for all  $m \geq 3$ ,  $\mathcal{M}_{1,1}[m] = M_{1,1}[m]$  for all  $m \geq 3$ . The group  $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$  acts on  $\mathcal{M}_{1,1}[m]$  and  $\mathcal{M}_{1,1}$  is the orbifold quotient  $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) \backslash \mathcal{M}_{1,1}[m]$ . The projection  $\mathcal{M}_{1,1}[m] \rightarrow \mathcal{M}_{1,1}$  takes the isomorphism class of  $(X, P; \mathbf{a}, \mathbf{b})$  to the isomorphism class of  $(X, P)$ . It has orbifold degree equal to the order of  $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$ .

*Exercise 38.* Use the Chinese Remainder Theorem to show that

$$\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) \cong \prod_p \mathrm{SL}_2(\mathbb{Z}/p^{\nu_p}),$$

where  $p$  ranges over all prime numbers and where  $\nu_p := \mathrm{ord}_p(m)$ . Show that  $I + pA \in \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$  for all  $A \in \mathfrak{gl}_2(\mathbb{Z}/p^{n-1}\mathbb{Z})$ .<sup>15</sup> Deduce that there is an exact sequence

$$1 \rightarrow I + p\mathfrak{gl}_2(\mathbb{Z}/p^{n-1}\mathbb{Z}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z}) \rightarrow \mathrm{GL}_2(\mathbb{F}_p) \rightarrow 1.$$

Use this to show that, for all  $n \geq 1$ ,

$$|\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})| = p^{4n-3}(p^2 - 1)(p - 1)$$

and that

$$|\mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z})| = |\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})| / |(\mathbb{Z}/p^n\mathbb{Z})^\times| = p^{3n}(1 - p^{-2}).$$

Deduce that

$$|\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})| = m^3 \prod_{p|m} \left(1 - \frac{1}{p^2}\right).$$

<sup>15</sup>Here  $\mathfrak{gl}_n(R)$  denotes the set of  $n \times n$  matrices over  $R$ .

The moduli space  $\mathcal{M}_{1,1}[m]$  can be compactified by adding a finite number of points, called *cusps*, as we now explain. (Cf. Exercise 93.)

The boundary of the upper half plane is  $\mathbb{R} \cup \{\infty\}$ , which is usefully regarded as the real projective line  $\mathbb{P}^1(\mathbb{R})$ . The  $\mathrm{SL}_2(\mathbb{R})$ -action on the upper half plane extends to its boundary  $\mathbb{P}^1(\mathbb{R})$ ; it acts by fractional linear transformations.

*Exercise 39.* Show that the  $\mathrm{SL}_2(\mathbb{Z})$ -orbit of  $\infty \in \mathbb{P}^1(\mathbb{R})$  is the subset  $\mathbb{P}^1(\mathbb{Q})$  of  $\mathbb{P}^1(\mathbb{R})$ . Deduce that  $\mathrm{SL}_2(\mathbb{Z})[m] \backslash \mathbb{P}^1(\mathbb{Q})$  is finite.

Let  $U_\infty$  denote the subset  $\mathrm{Im}(\tau) > 1$  of  $\mathfrak{h}$ . The stabilizer of  $U_\infty$  in  $\mathrm{SL}_2(\mathbb{Z})$  is the isotropy group of  $\infty$ , which is

$$\Gamma_\infty := \{\pm 1\} \times \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}.$$

The stabilizer of  $U_\infty$  in  $\mathrm{SL}_2(\mathbb{Z})[m]$  is  $\Gamma_\infty[m] := \Gamma_\infty \cap \mathrm{SL}_2(\mathbb{Z})[m]$ . For each  $x \in \mathbb{P}^1(\mathbb{Q})$ , choose  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  such that  $x = \gamma\infty$ . Set

$$U_x = \gamma U_\infty.$$

Its stabilizer in  $\mathrm{SL}_2(\mathbb{Z})[m]$  is  $\Gamma_x[m] := \gamma \Gamma_\infty[m] \gamma^{-1}$ . Both  $U_x$  and  $\Gamma_x[m]$  depend only on  $x$  and not on the choice of  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ . Further  $\gamma$  induces a biholomorphism

$$\Gamma_\infty[m] \backslash U_\infty \xrightarrow{\cong} \Gamma_x[m] \backslash U_x.$$

For each  $x \in \mathbb{P}^1(\mathbb{Q})$ , the quotient mapping (in the category of Riemann surfaces)

$$\Gamma_x[m] \backslash U_x \cong \Gamma_\infty[m] \backslash U_\infty \rightarrow \Gamma_\infty \backslash U_\infty \cong \mathbb{D}_R^*$$

has degree  $m$  where  $R = \exp(-2\pi)$ . It follows that  $\Gamma_x[m] \backslash U_x$  is a punctured disk with coordinate the  $m$ th root  $\exp(2\pi i\tau/m)$  of  $q$ .

Fix  $m \geq 3$ . The inclusion  $U_x \rightarrow \mathfrak{h}$  induces an inclusion

$$\Gamma_x[m] \backslash U_x \rightarrow \mathcal{M}_{1,1}[m]$$

of a punctured disk. With the identifications above, this map depends only on the  $\mathrm{SL}_2(\mathbb{Z})[m]$ -orbit  $c$  of  $x$ . We shall therefore denote this punctured disk by  $V_c$ .

**Definition 4.7.** For each  $m \geq 3$ , define  $\overline{\mathcal{M}}_{1,1}[m]$  to be the Riemann surface whose underlying set is

$$\mathrm{SL}_2(\mathbb{Z})[m] \backslash (\mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})).$$

As a Riemann surface, it is obtained from  $\mathcal{M}_{1,1}[m]$  by attaching one disk  $\mathbb{D}_S$  of radius  $S = \sqrt[m]{R}$  for each  $c \in D_m := \mathrm{SL}_2(\mathbb{Z})[m] \backslash \mathbb{P}^1(\mathbb{Q})$

by identifying the punctured disk  $\mathbb{D}_S^*$  with the punctured disk  $V_c$  in  $\mathcal{M}_{1,1}[m]$ . Elements of  $C_m := \mathrm{SL}_2(\mathbb{Z})[m]$  are called *cusps*.<sup>16</sup>

*Exercise 40.* Suppose that  $m \geq 3$ . Show that the Riemann surface  $\overline{\mathcal{M}}_{1,1}[m]$  is compact, that the action of  $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$  on  $\mathcal{M}_{1,1}[m]$  extends to  $\overline{\mathcal{M}}_{1,1}[m]$ , and that the isomorphism  $\mathcal{M}_{1,1} \cong \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) \backslash \mathcal{M}_{1,1}[m]$  extends to an orbifold isomorphism  $\overline{\mathcal{M}}_{1,1} \cong \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) \backslash \overline{\mathcal{M}}_{1,1}[m]$ .

When  $m = 2$ , one attaches one copy of  $D_2 \backslash \mathbb{D}_{\sqrt{R}}$  to  $\mathcal{M}_{1,1}[2]$  for each cusp  $c \in \{0, 1, \infty\} = \mathrm{SL}_2(\mathbb{Z})[2] \backslash \mathbb{P}^1(\mathbb{Q})$  to obtain  $\overline{\mathcal{M}}_{1,1}[2]$ .

*Exercise 41.* Show that  $\overline{\mathcal{M}}_{1,1}[2]$  is isomorphic to the quotient of  $\mathbb{P}^1$  by the trivial  $C_2$ -action. (Cf. the proof of Prop. 3.15.)

*Exercise 42.* Suppose that  $m \geq 3$ . Set  $d_m = |\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})|$  and  $c_m = \#D_m$ . Show that  $\chi(\mathcal{M}_{1,1}[m]) = d_m \chi(\mathcal{M}_{1,1}) = -d_m/12$  and that

$$c_m = \frac{d_m}{2m} = \frac{m^2}{2} \prod_{p|m} \left(1 - \frac{1}{p^2}\right).$$

Use this to show that

$$\chi(\overline{\mathcal{M}}_{1,1}[m]) = c_m + \chi(\mathcal{M}_{1,1}[m]) = \frac{m^2}{2} \left(1 - \frac{m}{6}\right) \prod_{p|m} \left(1 - \frac{1}{p^2}\right).$$

Deduce that the genus  $g_m$  of  $\overline{\mathcal{M}}_{1,1}[m]$  is given by

$$g_m = 1 - \frac{m^2}{4} \left(1 - \frac{m}{6}\right) \prod_{p|m} \left(1 - \frac{1}{p^2}\right).$$

Except when  $m$  is small,  $\overline{\mathcal{M}}_{1,1}[m]$  is not rational. For example,  $g_3 = g_4 = g_5 = 0$ ,  $g_7 = 3$ ,  $g_8 = 5$ ,  $g_{41} = 2451$ ,  $g_{5^3} = 74376$ .

Modular forms of weight  $k$  for  $\mathrm{SL}_2(\mathbb{Z})[m]$  are simply holomorphic sections of  $\overline{\mathcal{L}}_k$  over  $\overline{\mathcal{M}}_{1,1}[m]$ ; cusp forms are holomorphic sections of  $\overline{\mathcal{L}}_k$  that vanish at the cusps.

*Remark 4.8.* Moduli spaces of elliptic curves with a level are frequently used by number theorists. They typically work with more refined level structures, such as the moduli space of elliptic curves  $E$  together with an element order  $m$  of  $H_1(E; \mathbb{Z}/m\mathbb{Z})$ , or of elliptic curves plus a cyclic subgroup of  $H_1(E; \mathbb{Z}/m\mathbb{Z})$  of order  $m$ .

<sup>16</sup>This terminology is confusing as each  $c \in D_m$  is a smooth point of  $\overline{\mathcal{M}}_{1,1}[m]$ . Elements of  $D_m$  are not cusps in the sense of singularity theory, but they are related to cusp forms, which are sections of powers of  $\overline{\mathcal{L}}$  that vanish on  $D_m$ .

5. CUBIC CURVES AND THE UNIVERSAL CURVE  $\overline{\mathcal{E}} \rightarrow \overline{\mathcal{M}}_{1,1}$ 

In this section we construct the extension of the universal curve  $\mathcal{E} \rightarrow \mathcal{M}_{1,1}$  to the universal stable elliptic curve  $\overline{\mathcal{E}} \rightarrow \overline{\mathcal{M}}_{1,1}$ .

**5.1. Plane cubics.** The description of an elliptic curve as the quotient of  $\mathbb{C}$  by a lattice is very transcendental. In algebraic geometry it is more natural to consider an elliptic curve as a smooth plane cubic curve rather than as the quotient of  $\mathbb{C}$  by a lattice.

*Exercise 43.* Suppose that  $f(x) \in \mathbb{C}[x]$  is a cubic polynomial. Show that the curve  $C$  in  $\mathbb{P}^2$  defined by  $y^2 = f(x)$  is smooth if and only if  $f(x)$  has 3 distinct roots. (To study this curve at infinity, use the homogenized version  $y^2z = z^3f(x/z)$ .) Show that the algebraic differential  $dx/y$  is a non-zero holomorphic differential on  $C$ .

**Proposition 5.1.** *Every smooth plane cubic curve has genus 1.*

*Proof.* This is an immediate consequence of the genus formula.<sup>17</sup> It can also be proved directly as follows. Suppose that  $C$  is a smooth plane cubic curve. Consider the exact sequence

$$0 \rightarrow TC \rightarrow T\mathbb{P}^2|_C \rightarrow N \rightarrow 0$$

of vector bundles over  $C$ . Since  $C$  is a cubic, its normal bundle  $N$  is the restriction of  $\mathcal{O}_{\mathbb{P}^2}(3)$  to  $C$ . This has degree 9. The first Chern class of the  $T\mathbb{P}^2$  is the negative of the first Chern class of the canonical bundle  $K_{\mathbb{P}^2}$  of  $\mathbb{P}^2$ . Since the canonical bundle of  $\mathbb{P}^2$  is  $\mathcal{O}_{\mathbb{P}^2}(-3)$ , its restriction to  $C$  has degree  $-9$ . By the standard formula,

$$c_1(T\mathbb{P}^2|_C) = -c_1(K_{\mathbb{P}^2}|_C) = c_1(TC) + c_1(N) \in H^2(C; \mathbb{Z}).$$

Since the degree of a line bundle  $L$  on  $C$  is  $\int_C c_1(L)$ , we have

$$2 - 2g(C) = \deg TC = -\deg(K_{\mathbb{P}^2}|_C) - \deg N = 9 - 9 = 0.$$

Thus  $g(C) = 1$ . □

*Exercise 44.* Show that  $y^2 = x^3 - x$  and  $y^2 = x^3 - 1$  are smooth elliptic curves (with distinguished point  $[0, 1, 0]$ ). Show that the first has an automorphism of order 4 and the second an automorphism of order 6. Deduce that they are isomorphic to  $\mathbb{C}/\mathbb{Z}[i]$  and  $\mathbb{C}/\mathbb{Z}[\rho]$ , respectively. Cf. Exercise 8.

The discriminant of the polynomial

$$f(x) = 4x^3 - ax - b$$

---

<sup>17</sup>The genus formula states that the genus of a smooth curve in  $\mathbb{P}^2$  of degree  $d$  is  $(d-1)(d-2)/2$ .

is  $16(a^3 - 27b^2)$ . For convenience, we will divide it by 16 and instead call  $D(a, b) := a^3 - 27b^2$  the discriminant of  $f(x)$ . Every curve of the form

$$y^2 = 4x^3 - ax - b$$

is an elliptic curve, with distinguished point  $[0, 1, 0]$ . The converse is also true.

**Proposition 5.2.** *Every elliptic curve  $(X, P)$  is isomorphic to a smooth plane cubic of the form*

$$y^2 = 4x^3 - ax - b,$$

where  $P \in X$  corresponds to  $[0, 1, 0] \in \mathbb{P}^2$  and  $D(a, b) \neq 0$ . Moreover, the elliptic curve  $(y^2 = 4x^3 - ax - b, [0, 1, 0])$  is isomorphic to

$$(y^2 = 4x^3 - Ax - B, [0, 1, 0])$$

if and only if there exists  $u \in \mathbb{C}^*$  such that  $A = u^2a$  and  $B = u^3b$ .

*Proof.* This is an exercise in the use of the Riemann-Roch formula. There is an inclusion of vector spaces

$$L(P) \subseteq L(2P) \subseteq L(3P) \subseteq L(4P) \subseteq L(5P) \subseteq L(6P).$$

The Riemann-Roch formula implies that, when  $n \geq 1$ ,

$$\ell(nP) := \dim L(nP) = n.$$

Since  $\mathbb{C} \subseteq L(P)$ ,  $L(P)$  is spanned by the constant function 1. Since  $\ell(2P) = 2$ , there exists a non-constant function  $x : X \rightarrow \mathbb{P}^1$  that is holomorphic away from  $P$  and where  $P$  is at worst a double pole. If the pole had degree 1, then  $x : X \rightarrow \mathbb{P}^1$  would have degree 1, and would therefore be a biholomorphism. This is impossible as  $g(X) = 1$ . Thus  $x$  has a double pole at  $P$  and  $x : X \rightarrow \mathbb{P}^1$  has degree 2. The Riemann-Hurwitz formula (Exercise 78) implies that  $x$  has 4 critical values, one of which is infinity. Let  $c_1, c_2, c_3$  be the 3 critical values in  $\mathbb{C}$ . By adding a constant to  $x$  if necessary, we may assume that  $c_1 + c_2 + c_3 = 0$ . This condition determines  $x$  up to a constant multiple.

Since  $\ell(3P) = 3$ , there is a function  $y : X \rightarrow \mathbb{P}^1$  whose only pole is  $P$  and which does not lie in  $L(2P)$ . The pole is therefore a triple pole. Denote the deck transformation of the covering  $x : X \rightarrow \mathbb{P}^1$  by  $\sigma$ . Observe that  $\sigma$  acts trivially on  $L(2P)$ , but that  $\sigma^*y \neq y$  as  $y$  has an odd order pole at  $P$ . By replacing  $y$  by  $y - \sigma^*y$  if necessary, we may assume that  $\sigma^*y = -y$ . This condition determines  $y$  up to a constant. Thus  $1, x, y$  is a basis of  $L(3P)$ .

Since  $\ell(4P) = 4$ , and since  $1, x, y, x^2 \in L(4P)$  are linearly independent, they comprise a basis. Likewise,  $1, x, y, x^2, xy$  is a basis of  $L(5P)$ .

Since  $L(6P)$  contains the 7 functions  $1, x, y, x^2, xy, x^3, y^2$ , and since  $\ell(6P) = 6$ , it follows that they are linearly dependent. This linear dependence can be written as the sum of a  $\sigma$ -invariant term and a  $\sigma$ -anti-invariant term. Since the  $\sigma$ -anti-invariant functions  $y, xy$  are in  $L(5P)$ , they are linearly independent. Consequently, the  $\sigma$ -invariant basis elements must be linearly dependent. The coefficients of  $x^3$  and  $y^2$  in this dependence are both non-zero, otherwise  $x^3$  or  $y^2 \in L(5P)$ , which is a contradiction. We therefore have a relation of the form

$$ey^2 = 4x^3 - ux^2 - ax - b$$

where  $e \neq 0$ . Since the critical values  $c_1, c_2, c_3$  are the roots of the right-hand side, and since we chose  $x$  so that their sum is 0,  $u = 0$ . By rescaling  $y$ , we may assume that  $e = 1$ . That is,  $X$  has an equation of the form

$$y^2 = 4x^3 - ax - b$$

in which  $x(P) = \infty$ . As remarked,  $x$  is unique up to multiplication by a constant. If we multiply it by  $u^2$ , we have to multiply  $y$  by  $u^3$  so that  $y^2 - 4x^3$  remains in  $L(2P)$ .

The uniqueness statement is easily proved and is left as an exercise for the reader.  $\square$

It is useful to give a second proof that every elliptic curve is isomorphic to a curve of the form  $y^2 = 4x^3 - ax - b$ . Recall from Proposition 1.4 that every elliptic curve is isomorphic to a 1-dimensional complex torus  $(\mathbb{C}/\Lambda, 0)$ . So it suffices to show that every 1-dimensional complex torus is isomorphic to a plane cubic.

Suppose that  $\Lambda$  is a lattice in  $\mathbb{C}$ . The Weierstrass  $\wp_\Lambda$ -function is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right).$$

This converges to a meromorphic function on  $\mathbb{C}$  that is periodic with respect to  $\Lambda$  and has a double pole at each lattice point and is holomorphic elsewhere. Consequently, the induced holomorphic function  $\wp_\Lambda : \mathbb{C}/\Lambda \rightarrow \mathbb{P}^1$  has a unique double pole at  $0 \in \mathbb{C}/\Lambda$  and is 2:1. Since  $\wp_\Lambda(z) = \wp_\Lambda(-z)$ , the automorphism of the map  $x : \mathbb{C}/\Lambda \rightarrow \mathbb{P}^1$  is  $z \mapsto -z$ .

For  $\tau \in \mathfrak{h}$ , denote the Weierstrass  $\wp$ -function of the lattice  $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$  by  $\wp_\tau$ :

$$\wp_\tau(z) := \wp_{\Lambda_\tau}(z).$$



*Exercise 45.* Show that if  $u \in \mathbb{C}^*$ , then

$$\wp_{u\Lambda}(uz) = u^{-2}\wp_{\Lambda}(z).$$

Deduce that if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

then

$$\wp_{\tau}(\gamma\tau)(z/(c\tau + d)) = (c\tau + d)^2\wp_{\tau}(z).$$

**Proposition 5.3.** For all  $\tau \in \mathfrak{h}$

$$(\wp'_{\tau})^2 = 4\wp_{\tau}^3 - g_2(\tau)\wp_{\tau} - g_3(\tau)$$

where  $\wp'_{\tau} := \partial\wp_{\tau}/\partial z$  and  $g_2 = 60G_4$  and  $g_3 = 140G_6$  are Eisenstein series of weights 4 and 6, respectively.

*Proof.* The identity

$$\frac{1}{(z - \lambda)^2} = \frac{d}{dz} \left( \frac{1}{\lambda - z} \right) = \frac{1}{\lambda^2} \sum_{m=0}^{\infty} (m+1) \frac{z^m}{\lambda^m}$$

implies that

$$(10) \quad \wp_{\tau}(z) = \frac{1}{z^2} + \sum_{m=1}^{\infty} (2m+1)G_{2m+2}(\tau)z^{2m}.$$

Differentiating, we obtain the expansion

$$\wp'_{\tau}(z) = -\frac{2}{z^3} + \sum_{m=1}^{\infty} 2m(2m+1)G_{2m+2}(\tau)z^{2m-1}.$$

Since

$$\wp_{\tau}(z)^3 \equiv \frac{1}{z^6} + 9G_4(\tau)\frac{1}{z^2} + 15G_6(\tau) \pmod{z}$$

and

$$\wp'_{\tau}(z)^2 \equiv 4\frac{1}{z^6} - 24G_4(\tau)\frac{1}{z^2} - 80G_6(\tau) \pmod{z}$$

it follows that

$$4\wp_{\tau}(z)^3 - g_2(\tau)\wp_{\tau}(z) - g_3(\tau) - \wp'_{\tau}(z)^2 \equiv 0 \pmod{z}.$$

Since  $\wp_{\tau}$  and  $\wp'_{\tau}$  are holomorphic away from  $\Lambda_{\tau}$ , the left hand side of the previous expression is a holomorphic function on  $\mathbb{C}/\Lambda_{\tau}$  that vanishes at 0. It is therefore zero.  $\square$

**Proposition 5.4.** *For all  $\tau \in \mathfrak{h}$ , the polynomial  $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$  has non-vanishing discriminant and the holomorphic map*

$$[\wp_\tau, \wp'_\tau, 1] : \mathbb{C}/\Lambda_\tau \rightarrow \mathbb{P}^2$$

*imbeds  $E_\tau$  in  $\mathbb{P}^2$  as the smooth cubic  $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$ . Moreover, the rational differential  $dx/y$  on  $\mathbb{P}^2$  pulls back to the holomorphic differential  $dz$  on  $\mathbb{C}/\Lambda_\tau$ .*

*Proof.* Set  $E_\tau = \mathbb{C}/\Lambda_\tau$ . Since  $E_\tau$  has genus 1 and  $\wp_\tau : (E_\tau, 0) \rightarrow (\mathbb{P}^1, 0)$  has degree two, the Riemann-Hurwitz formula (Exercise 78) implies that  $\wp_\tau$  has 4 distinct critical points.<sup>18</sup> Consequently,  $\wp_\tau$  has three distinct critical values in  $\mathbb{C}$ . Since

$$(\wp'_\tau)^2 = 4\wp_\tau^3 - g_2(\tau)\wp_\tau - g_3(\tau),$$

these are the three roots of the cubic  $4x^3 - g_2(\tau)x - g_3(\tau)$ . Since they are distinct, its discriminant

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$$

is non-zero. By Exercise 43, this implies that  $[\wp_\tau, \wp'_\tau, 1]$  imbeds  $E_\tau$  as a smooth cubic.

The last statement holds because  $x = \wp_\tau$  and  $y = \wp'_\tau$ , so that

$$\frac{dx}{y} = \frac{\wp'_\tau dz}{\wp'_\tau} = dz.$$

□

An immediate consequence of the proof is a topological/geometric proof that the Ramanujan tau function has no zeros in  $\mathfrak{h}$ . (Cf. [10, p. 84].)

**Corollary 5.5.** *The Ramanujan tau function  $\Delta := g_2^3 - 27g_3^2$  has no zeros in  $\mathfrak{h}$ .*

**5.2. Extending the universal curve.** The description of an elliptic curve as a plane cubic curve allows us to extend explicitly the universal elliptic curve over  $\mathcal{M}_{1,1}$  to  $\overline{\mathcal{M}}_{1,1}$ .

Consider the family

$$E = \{([x, y, z], q) \in \mathbb{P}^2 \times \mathbb{D} : zy^2 = 4x^3 - g_2(\tau)xz^2 - g_3(\tau)z^3, q = e^{2\pi i\tau}\}$$

of cubic curves over the disk. This family has a natural  $C_2$ -action in which the generator acts by taking  $([x, y, z], q)$  to  $([x, -y, z], q)$ .

*Exercise 46.* Show that  $E$  is a smooth surface in  $\mathbb{P}^2 \times \mathbb{D}$ . Show that the projection  $E \rightarrow \mathbb{D}$  is proper.

<sup>18</sup>Since  $\wp_\tau(z) = \wp_\tau(-z)$ , these are  $\infty$  and the 3 non-zero points of order 2 of  $E_\tau$ .

**Lemma 5.6.** *The restriction of  $E \rightarrow \mathbb{D}$  to the punctured disk  $\mathbb{D}^*$  is the pullback of the universal elliptic  $\mathcal{E} \rightarrow \mathcal{M}_{1,1}$  along the natural mapping*

$$\mathbb{D}^* = \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \backslash \mathfrak{h} \rightarrow \mathcal{M}_{1,1}.$$

which is equivariant with respect to the natural  $C_2$ -actions on  $\mathcal{E}_\mathfrak{h}$  and  $E$ .

*Proof.* Proposition 5.4 implies that the mapping  $\mathbb{C} \times \mathfrak{h} \rightarrow E$  defined by

$$(11) \quad (z, \tau) \mapsto [\wp_\tau(z), \wp'_\tau(z), e^{2\pi i\tau}]$$

induces a holomorphic mapping  $q : \mathcal{E}_\mathfrak{h} \rightarrow E$  such that the diagram

$$\begin{array}{ccc} \mathcal{E}_\mathfrak{h} & \xrightarrow{q} & E \\ \downarrow & & \downarrow \\ \mathfrak{h} & \xrightarrow{\tau \mapsto e^{2\pi i\tau}} & \mathbb{D} \end{array}$$

commutes and that it is an isomorphism on each fiber. The first assertion follows.

The generator of  $C_2$  acts on  $\mathcal{E}_\mathfrak{h}$  by  $(z, \tau) \mapsto (-z, \tau)$ . It acts on  $E$  by  $([x, y, z], q) \mapsto ([x, -y, z], q)$ . The  $C_2$  equivariance  $q$  follows as  $\wp_\tau(-z) = \wp_\tau(z)$  and  $\wp'_\tau(-z) = -\wp'_\tau(z)$ .  $\square$

We can thus extend the universal curve  $\mathcal{E}_\mathfrak{h} \rightarrow \mathcal{M}_{1,1}$  to  $\overline{\mathcal{M}}_{1,1}$  by gluing it to a copy of  $E$ :

$$\begin{array}{ccc} & \begin{array}{c} \pm \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \curvearrowright \\ \mathcal{E}_\mathfrak{h} \\ \downarrow \\ \mathfrak{h} \end{array} & \\ & \swarrow p \quad \searrow q & \\ \begin{array}{c} \text{SL}_2(\mathbb{Z}) \curvearrowright \\ \mathcal{E}_\mathfrak{h} \\ \downarrow \\ \mathfrak{h} \end{array} & & \begin{array}{c} E_{\mathbb{D}} \\ \downarrow \\ \mathbb{D} \\ C_2 \curvearrowright \end{array} \end{array}$$

The extended family  $\mathcal{E} \rightarrow \overline{\mathcal{M}}_{1,1}$  has smooth total space. Its fiber over  $\infty$  is the nodal cubic:

**Proposition 5.7.** *The fiber  $E_0$  of  $\overline{\mathcal{E}} \rightarrow \overline{\mathcal{M}}_{1,1}$  over  $q = 0$  is isomorphic to the nodal cubic*

$$Y^2 = \frac{4}{27}(3X + 2)(3X - 1)^2.$$

*Proof.* Set  $x = (\pi i)^2 X$  and  $y = (\pi i)^3 Y$ . In these coordinates the equation of  $E$  is

$$Y^2 = 4X^3 - \frac{g_2}{(\pi i)^4} X - \frac{g_3}{(\pi i)^6}.$$

When  $q = 0$ ,

$$g_2 = 60 G_4|_{q=0} = \frac{2^2}{3} \pi^4 \text{ and } g_3 = 140 G_6|_{q=0} = \frac{2^3}{27} \pi^6$$

so that the equation of  $E_0$  is

$$(12) \quad Y^2 = 4X^3 - \frac{4}{3} X + \frac{8}{27} = \frac{4}{27} (3X + 2)(3X - 1)^2.$$

□

*Exercise 47.* Use the identity (10) to show that when  $q = 0$

$$(2\pi i)^{-2} \wp_0(z) = \frac{1}{(2\pi iz)^2} - \sum_{m=1}^{\infty} \frac{B_{2m+2}}{(2m+2)(2m)!} (2\pi iz)^{2m}$$

where  $B_n$  denotes the  $n$ th Bernoulli number.<sup>19</sup> Deduce that when  $q = 0$ , the mapping (11) factors through the quotient mapping  $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$  defined by  $w = \exp(2\pi iz)$ . Show that the rational differential  $2\pi i dx/y$  on  $\mathbb{P}^2$  pulls back to  $dw/w \in H^0(\mathbb{P}^1, \Omega^1([0] + [\infty]))$ .

Differentiate the identity

$$\frac{1}{2} \coth(u/2) = \sum_{m=0}^{\infty} \frac{B_{2m}}{(2m)!} u^{2m-1}$$

(that is obtained by manipulating the defining series for Bernoulli numbers) to show that

$$\frac{1/4}{\sinh^2(u/2)} = \frac{1}{u^2} - \sum_{m=0}^{\infty} \frac{B_{2m+2}}{(2m+2)(2m)!} u^{2m+1}.$$

Deduce that

$$(2\pi i)^{-2} \wp_0(z) = \frac{1}{12} + \frac{1/4}{\sinh^2(\pi iz)} = \frac{1}{12} + \frac{w}{(w-1)^2}.$$

Now use the fact that  $X/4 = x/(2\pi i)^2 = (2\pi i)^{-2} \wp_0$  and the equation (12) to show that

$$X = \frac{1}{3} + \frac{4w}{(w-1)^2}, \text{ and } Y = \frac{8w(w+1)}{(w-1)^3}.$$

---

<sup>19</sup>Bernoulli numbers are defined by the power series  $x/(e^x - 1) = \sum_{n=0}^{\infty} B_n x^n/n!$ . The first few Bernoulli numbers are  $B_0 = 1$ ,  $B_1 = -1/2$ ,  $B_2 = 1/6$ . When  $k \geq 1$ ,  $B_{2k+1} = 0$ . Bernoulli numbers are related to values of the Riemann zeta function at positive even integers by  $\zeta(2k) = -(2\pi i)^{2k} B_{2k}/4(2k)!$

Finally show that the map  $\mathbb{P}^1 \rightarrow \mathbb{P}^2$  defined by  $w \mapsto [X(w), Y(w), 1]$  maps  $\mathbb{P}^1$  onto  $E_0$ . Show that it takes the identity  $1 \in \mathbb{C}^*$  to the identity  $[0, 1, 0]$  of  $E_0$  and that  $0$  and  $\infty$  are both mapped to the double point  $[1/3, 0, 1]$  of  $E_0$ ; show that the map is otherwise injective.

*Remark 5.8.* The nodal cubic  $(E_0, [0, 1, 0])$  is an example of a *stable* pointed curve. In general, a stable pointed curve is a pointed compact, connected, complex analytic (or algebraic) curve all of whose singularities are nodes (i.e., analytically isomorphic to  $xy = 0$ ) and whose automorphism group (as a pointed curve) is finite. The marked point is required to be distinct from the nodes.

*Exercise 48.* Prove that all singular stable 1-pointed genus 1 curves are isomorphic to  $E_0$ .

**5.3. Families of stable elliptic curves.** The extension of the universal curve to  $\overline{\mathcal{M}}_{1,1}$  allows us to study the period mappings of algebraic families of smooth elliptic curves.

*Exercise 49.* Suppose that  $F$  is a non-empty finite subset of a compact Riemann surface  $T$ . Let  $D$  be the divisor  $\sum_{P \in F} [P]$ . Show that there exists a positive integer  $n$  such that the linear system  $H^0(T, \mathcal{O}(nD))$  embeds  $T$  into projective space. Deduce that  $T - F$  is an affine complex algebraic curve.

Suppose that  $T$  is a compact Riemann surface and that  $F$  is a (possibly empty) finite subset of  $T$ . Set  $T' = T - F$ . The following result establishes the Stable Reduction Theorem (cf. [8, p. 118]) in the special case of elliptic curves.

**Theorem 5.9** (Stable reduction for families of elliptic curves). *If  $X \rightarrow T'$  is a family of smooth elliptic curves over  $T'$ , then*

- (i) *the coarse period mapping  $T' \rightarrow \mathcal{M}_{1,1}$  extends to a holomorphic mapping  $T \rightarrow \overline{\mathcal{M}}_{1,1}$ ;*
- (ii) *after passing to a finite covering  $S \rightarrow T$  unramified over  $T'$ , the period mapping  $\Phi : T' \rightarrow \mathcal{M}_{1,1}$  extends to a morphism  $\tilde{\Phi} : S \rightarrow \overline{\mathcal{M}}_{1,1}$ :*

$$\begin{array}{ccccc}
 & & S & \xrightarrow{\tilde{\Phi}} & \overline{\mathcal{M}}_{1,1} \\
 & \nearrow & \searrow & & \uparrow \\
 S' & & T & & \\
 \text{unramified} & \searrow & \nearrow & & \\
 & & T' & \xrightarrow{\Phi} & \mathcal{M}_{1,1}
 \end{array}$$

In other words, after passing to a finite unramified covering of  $T'$ , the family  $X$  extends to a family of stable elliptic curves.

This says that étale locally on  $T'$  every family of elliptic curves can be extended to a family of elliptic curves over a compact curve  $S$ , each of whose fibers is stable. Note, however, that the total space of the extended family  $\tilde{\Phi}^*\tilde{\mathcal{E}}$  over  $S$  is typically singular over  $S - S'$ .

To prove Theorem 5.9, we first study the local version of stable reduction.

**Proposition 5.10.** *If  $f : \mathbb{D}^* \rightarrow \mathcal{M}_{1,1}$  is a holomorphic mapping, then either*

- (i) *the image of  $f_* : \pi_1(\mathbb{D}^*) \rightarrow \mathrm{SL}_2(\mathbb{Z})$  is finite. In this case, there is a finite covering  $p : \mathbb{D}^* \rightarrow \mathbb{D}^*$  and a holomorphic mapping  $\tilde{f} : \mathbb{D} \rightarrow \mathcal{M}_{1,1}$  whose restriction to  $\mathbb{D}^*$  is  $f \circ p$ ; or*
- (ii) *the image of  $f_* : \pi_1(\mathbb{D}^*) \rightarrow \mathrm{SL}_2(\mathbb{Z})$  is infinite. In this case, there is a double covering  $p : \mathbb{D}^* \rightarrow \mathbb{D}^*$  and a holomorphic mapping  $\tilde{f} : \mathbb{D} \rightarrow \overline{\mathcal{M}}_{1,1}$  whose restriction to  $\mathbb{D}^*$  is  $f \circ p$ .*

If the image of  $f_* : \pi_1(\mathbb{D}^*) \rightarrow \mathrm{SL}_2(\mathbb{Z})$  lies  $\mathrm{SL}_2(\mathbb{Z})[m]$  for some  $m \geq 3$ , then  $f$  extends to a holomorphic mapping  $\tilde{f} : \mathbb{D} \rightarrow \overline{\mathcal{M}}_{1,1}$  without passing to a finite covering  $p$ .

*Proof.* Denote the image of the positive generator of  $\pi_1(\mathbb{D}^*)$  under  $f_* : \pi_1(\mathbb{D}^*) \rightarrow \mathrm{SL}_2(\mathbb{Z})$  by  $A$ . Identify  $(\overline{\mathcal{M}}_{1,1}, \infty)$  with  $(\mathbb{P}^1, \infty)$  via the modular function  $j$ . Let  $q : \mathfrak{h} \rightarrow \mathbb{D}^*$  be the universal covering  $z \mapsto \exp(2\pi iz)$ . Let  $F : \mathfrak{h} \rightarrow \mathfrak{h}$  be a  $\pi_1$ -equivariant lift of  $f$ :

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{F} & \mathfrak{h} \\ q \downarrow & & \downarrow \\ \mathbb{D}^* & \xrightarrow{f} & \mathcal{M}_{1,1} \end{array}$$

The Schwartz Lemma [1] implies that  $F$  is distance decreasing in the Poincaré metric. This implies that the composite

$$\mathbb{D}^* \xrightarrow{f} \mathcal{M}_{1,1} \xrightarrow{j} \mathbb{P}^1$$

cannot have an essential singularity at the origin as we now explain. If it did, the image of each angular sector of each subdisk about the origin of  $\mathbb{D}_\epsilon^*$  would be dense in  $\mathbb{P}^1$ . But this implies that the image of every strip  $\mathrm{Im}(z) \geq c$ ,  $|\mathrm{Re}(z - z_o)| \leq \epsilon$  has dense image in  $\mathcal{M}_{1,1}$ , which contradicts Schwartz's Lemma. It follows that  $j \circ f$  has a removable singularity.<sup>20</sup> Denote its extension to  $\mathbb{D}$  by  $G : \mathbb{D} \rightarrow \mathbb{P}^1$ . If  $G(0) \in \mathcal{M}_{1,1}$ ,

<sup>20</sup>A pole is a removable singularity of a map to  $\mathbb{P}^1$ .

then there is a finite covering  $p : \mathbb{D} \rightarrow \mathbb{D}$  and a holomorphic mapping  $\tilde{f} : \mathbb{D} \rightarrow \mathfrak{h}$  that lifts  $G$ . In this case  $A$  fixes  $\tilde{f}(0)$ , and thus has finite order.

Suppose now that  $G(0) = \infty$ . By standard complex variables, one can choose a holomorphic coordinate  $w$  on  $\mathbb{D}$  centered at the origin such that  $G$  is given by  $q = w^n$  in a neighbourhood of the origin for some positive integer  $n$ . By choosing  $r > 0$  to be small enough, we may assume that  $\mathbb{D}$  is the disk  $|w| < r$ . But this implies that

$$A = \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

If the diagonal entries of  $A$  are 1, then  $f : \mathbb{D}^* \rightarrow \mathcal{M}_{1,1}$  extends to a holomorphic mapping  $\mathbb{D} \rightarrow \overline{\mathcal{M}}_{1,1}$ . If the diagonal entries of  $A$  are  $-1$ , the composition of  $f$  with a double covering  $p : \mathbb{D}^* \rightarrow \mathbb{D}^*$  extends to a holomorphic function  $\tilde{f} : \mathbb{D} \rightarrow \overline{\mathcal{M}}_{1,1}$ .

Finally, if  $A \in \mathrm{SL}_2(\mathbb{Z})[m]$  where  $m \geq 3$ , then  $A$  cannot have finite order (as  $\mathrm{SL}_2(\mathbb{Z})[m]$  is torsion free), and  $A$  cannot be the negative of a unipotent matrix:

$$A \notin \begin{pmatrix} -1 & \mathbb{Z} \\ 0 & -1 \end{pmatrix}$$

It follows that one can take  $p$  to be the identity when  $A$  lies in a subgroup of level  $m \geq 3$ .  $\square$

*Proof of Theorem 5.9.* Suppose that  $T' = T - F$ , where  $T$  is a compact Riemann surface and  $F$  is a finite subset. Suppose that  $\Phi : T' \rightarrow \mathcal{M}_{1,1}$  is the period mapping of a family  $X \rightarrow T'$  of smooth elliptic curves. Fix an integer  $m \geq 3$ . The kernel of the homomorphism

$$\Phi_* : \pi_1(T') \rightarrow \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$$

is a finite index subgroup of  $\pi_1(T')$ . It determines a finite, unramified covering  $p : S' \rightarrow T'$ . By standard arguments (cf. Exercise 93), there is a compact Riemann surface  $S$  and with a finite subset  $F_S$  such that  $S' = S - F_S$  and a holomorphic mapping  $S \rightarrow T$  whose restriction to  $S'$  is  $p$ . The composite

$$S' \xrightarrow{p} T' \xrightarrow{\Phi} \mathcal{M}_{1,1}$$

is the period mapping of the family  $p^*X \rightarrow S'$ . The associated monodromy representation  $(\Phi \circ p)_*$  is the composite

$$\pi_1(S') \rightarrow \pi_1(T') \rightarrow \mathrm{SL}_2(\mathbb{Z}).$$

The image of  $(\Phi \circ p)_*$  lies in  $\mathrm{SL}_2(\mathbb{Z})[m]$ .

For each  $P \in F_S$ , choose a coordinate disk  $U_P \cong \mathbb{D}$  centered at  $P$  such that  $U_P \cap F = \{P\}$ . Since the image of a generator of  $\pi_1(U_P^*)$  under  $(\Phi \circ p)_*$  lies in  $\mathrm{SL}_2(\mathbb{Z})[m]$  ( $m \geq 3$ ), it follows from Proposition 5.10 that the period mapping  $S' \rightarrow \mathcal{M}_{1,1}$  extends across  $P$  and that the period mapping extends to a holomorphic mapping

$$\tilde{\Phi} : S \rightarrow \overline{\mathcal{M}}_{1,1}.$$

The family  $p^*X \rightarrow S'$  of smooth elliptic curves extends to the family  $\tilde{\Phi}^*\overline{\mathcal{E}} \rightarrow S$  of stable curves.  $\square$

**Example 5.11.** Suppose that  $\sigma$  is a non-trivial automorphism of the elliptic curve  $(E, 0)$ . Let  $d$  be the order of  $\sigma$ . Let  $X \rightarrow \mathbb{D}^*$  be the isotrivial family associated to  $\sigma$  and the  $d$ -fold covering  $p : \mathbb{D}^* \rightarrow \mathbb{D}^*$ . (See Exercise 25 for the construction.) It follows from Example 3.19 that the period mapping  $\mathbb{D}^* \rightarrow \mathcal{M}_{1,1}$  does not extend to a mapping  $\mathbb{D} \rightarrow \overline{\mathcal{M}}_{1,1}$ , for if it did extend, the induced mapping  $\pi_1(\mathbb{D}^*) \rightarrow \mathrm{SL}_2(\mathbb{Z})$  would be trivial. Since the pullback of  $X \rightarrow \mathbb{D}^*$  along the  $d$ -fold covering  $p : \mathbb{D}^* \rightarrow \mathbb{D}^*$  is the trivial family  $E \times \mathbb{D}^* \rightarrow \mathbb{D}^*$ , the period mapping of  $p^*X$  is the constant map with value  $[E]$ , which trivially extends to a mapping  $\mathbb{D} \rightarrow \mathcal{M}_{1,1}$ .

*Exercise 50.* For  $e \in \mathbb{Z}$  set

$$X_e = \{([x, y, z], t) \in \mathbb{P}^2 \times \mathbb{D}^* : t^e zy^2 = (x^2 - tz)(x - z)\}.$$

This is a family of elliptic curves over  $\mathbb{D}^*$  with zero section  $t \mapsto [0, 1, 0]$ . Show that the fiber of  $X_e$  over  $t \in \mathbb{D}^*$  is isomorphic to the fiber of  $X_0$  over  $t$ . Show that the monodromy representation  $\pi_1(\mathbb{D}^*) \rightarrow \mathrm{SL}_2(\mathbb{Z})$  takes the positive generator of  $\pi_1(\mathbb{D}^*)$  to a conjugate of

$$(-1)^e \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(Hint: set  $Y = (\sqrt{t})^e y$ .) Deduce that the families  $X_0$  and  $X_1$  are not isomorphic.<sup>21</sup> Show that the period mapping  $\mathbb{D}^* \rightarrow \mathcal{M}_{1,1}$  extends to  $\mathbb{D}$  if and only if  $e$  is even.

One consequence of the stable reduction theorem is that families of elliptic curves over affine algebraic curves are either isotrivial (cf. Exercise 25 and Example 3.19) or have “large monodromy”.

**Corollary 5.12.** *Suppose that  $X \rightarrow T'$  is a family of elliptic curves over a Riemann surface  $T'$ . If the coarse period mapping  $T' \rightarrow \mathcal{M}_{1,1}$  is constant, then the monodromy representation*

$$\phi : \pi_1(T') \rightarrow \mathrm{SL}_2(\mathbb{Z})$$

<sup>21</sup>The families  $X_0$  and  $X_1$  are said to differ by a *quadratic twist*.



has finite image and the family is isotrivial. If  $T' = T - F$  where  $F$  is a finite subset of a compact Riemann surface  $T$ , and if the coarse period mapping is non-constant, then the image of the monodromy representation  $\phi$  has finite index in  $\mathrm{SL}_2(\mathbb{Z})$ .

*Sketch of Proof.* Denote the period mapping of the family  $X \rightarrow T'$  by  $\Phi : T' \rightarrow \mathcal{M}_{1,1}$ . Fix a universal covering  $Y \rightarrow T'$ . If the coarse period mapping  $T' \rightarrow \mathcal{M}_{1,1}$  is constant, then the framed period mapping  $\tilde{\Phi} : Y \rightarrow \mathfrak{h}$  is constant. Let  $\{\tau\}$  be the image of  $\tilde{\Phi}$ . Since  $\tilde{\Phi}$  is equivariant with respect to  $\pi_1(T') \rightarrow \mathrm{SL}_2(\mathbb{Z})$ , this implies that the image of  $\phi : \pi_1(T') \rightarrow \mathrm{SL}_2(\mathbb{Z})$  fixes  $\tau$  and is therefore finite. The pullback of the family  $X \rightarrow T'$  to the covering  $S' \rightarrow T'$  determined by  $\ker \phi$  is trivial. The family  $X \rightarrow T'$  is a quotient of the trivial family  $E_\tau \times S' \rightarrow S'$  by  $\pi_1(T')/\ker \phi$  and is thus isotrivial.

Now suppose that  $T' = T - F$  where  $F$  is a finite subset of a compact Riemann surface  $T$ . Suppose also that the period mapping  $T' \rightarrow \mathcal{M}_{1,1}$  is non-constant. Fix an integer  $m \geq 3$ . The inverse image of  $\mathrm{SL}_2(\mathbb{Z})[m]$  in  $\pi_1(T)$  is a finite index normal subgroup of  $\pi_1(T)$ . It determines a finite covering  $S' \rightarrow T'$ . This extends to a finite holomorphic mapping  $S \rightarrow T$ , where  $S$  is a compact Riemann surface that contains  $S'$  as the complement of a finite subset. The period mapping  $T' \rightarrow \mathcal{M}_{1,1}$  lifts to a holomorphic mapping  $S' \rightarrow \mathcal{M}_{1,1}[m]$  to the level- $m$  moduli space.<sup>22</sup> It extends to a holomorphic mapping  $S \rightarrow \overline{\mathcal{M}}_{1,1}[m]$ . Since  $S$  is compact and the period mapping is non-constant,  $S \rightarrow \overline{\mathcal{M}}_{1,1}[m]$  is surjective. Exercise 51 (below) implies that the image of  $\pi_1(S') \rightarrow \pi_1(\overline{\mathcal{M}}_{1,1}[m]) = \mathrm{SL}_2(\mathbb{Z})[m]$  has finite index. Since the diagram

$$\begin{array}{ccc} \pi_1(S') & \longrightarrow & \pi_1(T') \\ \downarrow & & \downarrow \\ \mathrm{SL}_2(\mathbb{Z})[m] & \longrightarrow & \mathrm{SL}_2(\mathbb{Z}) \end{array}$$

commutes, the image of  $\pi_1(T') \rightarrow \mathrm{SL}_2(\mathbb{Z})$  has finite index in  $\mathrm{SL}_2(\mathbb{Z})$ .  $\square$

*Exercise 51.* Suppose that  $f : X \rightarrow Y$  is a non-constant mapping of compact Riemann surfaces. Show that if  $F_X$  and  $F_Y$  are finite subsets of  $X$  and  $Y$ , respectively such that  $f(F_X) \supseteq F_Y$ , then the image of

$$f_* : \pi_1(X - F_X, x) \rightarrow \pi_1(Y - F_Y, f(x))$$

has finite index in  $\pi_1(Y - F_Y, f(x))$ . Hints: (1) first show that if  $F_Z$  is a discrete subset of a Riemann surface  $Z$  then  $Z - F_Z \hookrightarrow Z$  induces a

<sup>22</sup>See Section 4.2.

surjection on fundamental groups; (2) reduce to the case where  $f$  is an unramified covering by enlarging  $F_X$  and  $F_Y$ .

**5.4. The Hodge bundle\*.** The *Hodge bundle* is defined to be the line bundle

$$\pi_*\Omega_{\overline{\mathcal{E}}/\overline{\mathcal{M}}_{1,1}}^1(\log E_0)$$

over  $\overline{\mathcal{M}}_{1,1}$ , where  $\pi : \overline{\mathcal{E}} \rightarrow \overline{\mathcal{M}}_{1,1}$  is the universal curve and  $E_0$  is the fiber of  $\overline{\mathcal{E}}$  over  $q = 0$ .<sup>23</sup> It (and its generalizations in higher genus) play an important role in the enumerative geometry of algebraic curves and their moduli. In Section 6 we show that the Picard groups of  $\mathcal{M}_{1,1}$  and  $\overline{\mathcal{M}}_{1,1}$  are both generated by the Hodge bundle. In this section, we show that the Hodge bundle is isomorphic to  $\overline{\mathcal{L}}$ .

The first step in proving that the Hodge bundle is  $\overline{\mathcal{L}}$  is to show that its restriction to  $\mathcal{M}_{1,1}$  is  $\mathcal{L}$ . The restriction of the Hodge bundle to  $\mathcal{M}_{1,1}$  is the line bundle  $\pi_*\Omega_{\mathcal{E}/\mathcal{M}_{1,1}}^1$ , whose fiber over  $[E] \in \mathcal{M}_{1,1}$  is the space of holomorphic differentials  $H^0(E, \Omega_E^1)$  of  $E$ . That this is isomorphic to  $\mathcal{L}$  follows from the next result:

**Lemma 5.13.** *The set of isomorphism classes of triples  $(X, P, \omega)$ , where  $(X, P)$  is an elliptic curve and  $\omega$  is a holomorphic differential on  $X$ , is isomorphic in bijective correspondence with  $\mathcal{L}$ .*

*Proof.* We begin by considering isomorphism classes of framed triples  $(X, P, \omega)$ . That is, isomorphism classes of 5-tuples  $(X, P, \omega; \mathbf{a}, \mathbf{b})$ , where  $\mathbf{a}, \mathbf{b}$  is a basis of  $H_1(X; \mathbb{Z})$  with  $\mathbf{a} \cdot \mathbf{b} = 1$ .

Since every framed elliptic curve is isomorphic to one of the form

$$(\mathbb{C}/\Lambda_\tau, 0; 1, \tau),$$

we need only consider isomorphism classes of 5-tuples

$$(\mathbb{C}/\Lambda_\tau, 0, \omega; 1, \tau).$$

The differential  $\omega_\tau := dz$  is the unique holomorphic differential on  $\mathbb{C}/\Lambda_\tau$  such that  $\int_{\mathbf{a}} \omega_\tau = 1$ . There is therefore a bijection

$$(13) \quad \mathbb{C} \times \mathfrak{h} \rightarrow \{\text{isomorphism classes of 5-tuples } (X, P, \omega; \mathbf{a}, \mathbf{b})\}$$

<sup>23</sup>If  $X$  is a smooth variety and  $D$  is a normal crossings divisor in  $X$ , then  $\Omega_X^1(\log D)$  is the  $\mathcal{O}_X$ -module that is generated locally by  $du_1/u_1, \dots, du_r/u_r$  and  $du_{r+1}, \dots, du_n$ , where  $D$  is defined locally by  $u_1 u_2 \dots u_r = 0$  with respect to local holomorphic coordinates  $(u_1, \dots, u_n)$ . It is a locally free  $\mathcal{O}_X$ -module of rank equal to  $\dim X$ . If  $f : X \rightarrow T$  is a holomorphic family over a smooth curve  $T$  whose fiber  $X_t$  over  $t \in T$  is smooth when  $t$  is not in the finite subset  $F$  of  $T$  and where the fiber  $X_t$  over each  $t \in F$  is reduced and has normal crossings, then  $\Omega_{X/T}^1(\log D)$  is defined to be the sheaf  $\Omega_X^1(\log D)/f^*\Omega_T^1(F)$ , where  $D = f^{-1}F$ . It is a locally free  $\mathcal{O}_X$ -module of rank  $\dim X - 1$ .

defined by

$$(u, \tau) \mapsto (\mathbb{C}/\Lambda_\tau, 0, u\omega_\tau; 1, \tau).$$

To complete the proof, we will show that the correspondence is  $\mathrm{SL}_2(\mathbb{Z})$ -equivariant.

The element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of  $\mathrm{SL}_2(\mathbb{Z})$  takes  $(u, \tau)$  to  $((c\tau + d)u, \gamma\tau)$  and takes the framing  $(1, \tau)$  of  $\mathbb{C}/\Lambda_\tau$  to the framing  $(c\tau + d, a\tau + b)$ . The isomorphism

$$(\mathbb{C}/\Lambda_\tau, 0; c\tau + d, a\tau + b) \rightarrow (\mathbb{C}/\Lambda_{\gamma\tau}, 0; 1, \gamma\tau)$$

is obtained by dividing by  $c\tau + d$ , which implies that this isomorphism takes  $\omega_\tau$  to  $(c\tau + d)^{-1}\omega_{\gamma\tau}$ . Since  $u\omega_\tau = (c\tau + d)u\omega_{\gamma\tau}$  the mapping (13) is  $\mathrm{SL}_2(\mathbb{Z})$ -equivariant.  $\square$

**Corollary 5.14.** *The restriction of the Hodge bundle to  $\mathcal{M}_{1,1}$  is isomorphic to  $\mathcal{L}$ .*

*Exercise 52.* Show that the fiber of the Hodge bundle over  $\overline{\mathcal{M}}_{1,1}$  over the moduli points  $[E_0]$  of the nodal cubic is

$$H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}([0] + [\infty])) = \mathbb{C} \frac{dw}{w}.$$

Here we are identifying  $E_0$  with  $\mathbb{P}^1$  (coordinate  $w$ ) with 0 and  $\infty$  identified.

*Exercise 53.* Show that the rational differential  $dx/y$  on  $\mathbb{P}^2$  pulls back to a section of the restriction of the Hodge bundle  $\pi_*\Omega_{E/\mathbb{D}}^1(\log E_0)$  to the  $q$ -disk. Deduce that it trivializes the Hodge bundle over the  $q$ -disk.

Proposition 5.4 and Exercise 47 imply that the local framing  $dx/y$  of the Hodge bundle takes the value  $\omega_\tau \in H^0(E_\tau, \Omega_{E_\tau}^1)$  if  $q = \exp(2\pi i\tau)$  and  $(2\pi i)^{-1}dw/w$  when  $q = 0$ . In other words, the local framing of the Hodge bundle about  $q = 0$  agrees with the local framing of  $\overline{\mathcal{L}}$  about  $q = 0$  when restricted to the punctured  $q$ -disk when the Hodge bundle over  $\mathcal{M}_{1,1}$  is identified with  $\mathcal{L}$ . This implies that the Hodge bundle over  $\overline{\mathcal{M}}_{1,1}$  is isomorphic to  $\overline{\mathcal{L}}$ .

**Theorem 5.15.** *The Hodge bundle over  $\overline{\mathcal{M}}_{1,1}$  is isomorphic to  $\overline{\mathcal{L}}$ .*

In the elliptic curve case, the Hodge bundle is also isomorphic to the conormal bundle of the identity section of  $\overline{\mathcal{E}}$ . Denote the identity section of  $\pi : \overline{\mathcal{E}} \rightarrow \overline{\mathcal{M}}_{1,1}$  and its image by  $Z$ . The *relative cotangent bundle* of  $\pi$  is defined to be the dual  $\check{N}$  of the normal bundle  $N$  of  $Z$  in  $\overline{\mathcal{E}}$ .

**Proposition 5.16.** *The Hodge bundle is isomorphic to the relative cotangent bundle  $s^*\check{N}$  of the zero section.*

*Proof.* Since the holomorphic tangent bundle of every smooth elliptic curve is trivial, there is a natural isomorphism

$$T_0E \cong H^0(E, \Omega_E^1)^*$$

for all smooth elliptic curves  $E$ . There is also a natural isomorphism

$$T_1E_0 \cong T_1\mathbb{C}^* \cong H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1([0] + [\infty]))$$

for the nodal cubic. The result follows.  $\square$

*Remark 5.17.* In Hao Xu's talk,  $\psi_1$  denotes the first Chern class of the relative cotangent bundle of the universal elliptic curve  $\bar{\mathcal{E}} \rightarrow \bar{\mathcal{M}}_{1,1}$  and  $\lambda_1$  denotes the first Chern class of the Hodge bundle. Theorem 5.15 implies that  $\lambda_1$  is the class of  $\bar{\mathcal{L}}$  and Proposition 5.16 implies (in the case of elliptic curves) that  $\lambda_1 = \psi_1$ . Xu denotes the class of the nodal cubic by  $-\bullet\bigcirc$  which, by Proposition 5.7, is the class of the boundary point  $\infty$ . Exercise 34 states that  $[\infty] = 12\lambda_1$ . In Xu's notation, this reads:

$$\psi_1 = \lambda_1 = \frac{1}{12} \left( -\bullet\bigcirc \right)$$

**5.5. Natural metrics\*.** The restriction of the Hodge bundle to  $\mathcal{M}_{1,1}$  has a natural metric. This is because there is a natural metric on the space of holomorphic 1-forms on an elliptic curve  $X$ . Namely:

$$\|\omega\|^2 = \frac{i}{2} \int_X \omega \wedge \bar{\omega}, \quad \omega \in H^0(X, \Omega_X^1).$$

In particular,

$$\|w_\tau\|^2 = \int_{X_\tau} dz \wedge d\bar{z} = \text{Im}(\tau).$$

Since this metric is intrinsically defined, it follows that the metric

$$\|(u, \tau)\|^2 = |u|^2 \text{Im}(\tau)$$

on the line bundle  $\mathbb{C} \times \mathfrak{h} \rightarrow \mathfrak{h}$  is invariant under the action (20) of  $\text{SL}_2(\mathbb{Z})$  and thus descends to a metric on  $\mathcal{L} \rightarrow \mathcal{M}_{1,1}$ . (This is easy to check directly.) The  $k$ th power of this metric

$$\|(u, \tau)\|^2 = |u|^2 \text{Im}(\tau)^k$$

defines a metric on  $\mathcal{L}_k \rightarrow \mathcal{M}_{1,1}$ .

These metrics do not extend to metrics on  $\bar{\mathcal{L}}_k$ . To see this, write  $q = re^{i\theta}$ , so that

$$\text{Im}(\tau) = \text{Im}((\log q)/2\pi i) = -(\log r)/2\pi = -(\log |q|)/2\pi,$$

which blows up as  $|q| \rightarrow 0$ . Nonetheless, this metric is still useful as it is  $L^1$  on the  $q$ -disk as  $|\log r|$  is  $L^1$  on the unit disk.

*Exercise 54.* Show that the metric on the tangent bundle  $T\mathcal{M}_{1,1}$  induced by the isomorphism  $T\mathcal{M}_{1,1} \cong \mathcal{L}_{-2}$  equals the hyperbolic metric

$$ds^2 = \operatorname{Im}(\tau)^{-2} d\tau d\bar{\tau}.$$

Show that the punctured  $q$ -disk  $\mathbb{D}_R^*$  has finite volume in this metric, where  $R = \exp(-2\pi)$ . Deduce that  $\mathcal{M}_{1,1}[m]$  has finite volume for all  $m \geq 1$ .

The metric on  $\mathcal{L}_k$  can be used to define an inner product of two modular forms  $f$  and  $g$  of weight  $k$  of  $\operatorname{SL}_2(\mathbb{Z})[m]$  by integrating the  $\operatorname{SL}_2(\mathbb{Z})[m]$ -invariant function

$$f(\tau)\overline{g(\tau)} \operatorname{Im}(\tau)^k$$

over a fundamental domain of the action of  $\operatorname{SL}_2(\mathbb{Z})[m]$  on  $\mathfrak{h}$  with respect to the invariant volume form. This defines a positive definite hermitian form on the space of modular forms  $H^0(\overline{\mathcal{M}}_{1,1}[m], \overline{\mathcal{L}}_k)$  of weight  $k$  of  $\operatorname{SL}_2(\mathbb{Z})[m]$ :

$$(f, g) := \frac{i}{2} \int_{\mathcal{M}_{1,1}[m]} f(\tau)\overline{g(\tau)} \operatorname{Im}(\tau)^{k-2} d\tau \wedge d\bar{\tau}.$$

It is called the *Petersson inner product*.

## 6. THE PICARD GROUPS OF $\mathcal{M}_{1,1}$ AND $\overline{\mathcal{M}}_{1,1}$

In this section we compute the Picard groups of  $\mathcal{M}_{1,1}$  and  $\overline{\mathcal{M}}_{1,1}$ . This requires a detailed discussion of divisors and line bundles on orbifold Riemann surfaces.

**6.1. Assumptions.** We consider only orbifolds that are locally of the form  $\Gamma \backslash X$  where  $\Gamma$  acts virtually freely on  $X$ . In particular, the isotropy group of  $X$

$$\Gamma_X := \{g \in \Gamma : gx = x \text{ for all } x \in X\}$$

is finite. In addition, we will always assume that  $\Gamma_X$  is cyclic and central in  $\Gamma$ . These conditions are satisfied by  $\mathcal{M}_{1,1}$ ,  $\overline{\mathcal{M}}_{1,1}$  and the universal curves over them.

The group  $\Gamma/\Gamma_X$  acts effectively on  $X$ . The *reduced orbifold* associated to  $\Gamma \backslash X$  is defined by

$$(\Gamma \backslash X)^{\text{red}} := (\Gamma/\Gamma_X) \backslash X.$$

We say that  $\Gamma \backslash X$  is *reduced* if  $\Gamma$  acts effectively on  $X$ . That is, when  $\Gamma_X$  is trivial.

**Example 6.1.** The moduli space  $\mathcal{M}_{1,1}$  is not reduced. The corresponding reduced orbifold  $\mathcal{M}_{1,1}^{\text{red}}$  is  $\text{PSL}_2(\mathbb{Z}) \backslash \mathfrak{h}$ .

There are natural morphisms

$$\Gamma \backslash X \rightarrow (\Gamma \backslash X)^{\text{red}} \rightarrow \Gamma \backslash X$$

which are induced by the obvious morphisms

$$(X, \Gamma) \rightarrow (X, \Gamma/\Gamma_X) \rightarrow (\Gamma \backslash X, \mathbf{1}).$$

If  $\Gamma \backslash X$  is an orbifold in the category of Riemann surfaces, then so is  $(\Gamma \backslash X)^{\text{red}}$  and the natural morphisms above are both holomorphic.

We define the *degree* of each of the morphisms

$$\Gamma \backslash X \rightarrow (\Gamma \backslash X)^{\text{red}} \text{ and } \Gamma \backslash X \rightarrow \Gamma \backslash X$$

to be  $|\Gamma_X|$ . Note that  $(\Gamma \backslash X)^{\text{red}} \rightarrow \Gamma \backslash X$  has degree 1.

**6.2. Local theory.** Here we develop the theory for basic orbifolds. For simplicity, we consider only the 1-dimensional case.

Suppose that  $\Gamma \backslash X$  is a Riemann surface in the category of orbifolds, where  $\Gamma$  acts virtually freely on  $X$ . Denote by  $[x]$  the  $\Gamma$ -orbit of  $x \in X$ . To this we can associate the order  $|\Gamma_x|$  of the isotropy group of  $x$ . This depends only on the orbit  $[x]$  and not on the choice of the representative  $x$ .

Define a *divisor* on  $\Gamma \backslash X$  to be a locally finite, formal linear combination

$$\sum_{[x] \in \Gamma \backslash X} \frac{n_x}{|\Gamma_x|} [x]$$

of points of  $\Gamma \backslash X$ , where each  $n_x \in \mathbb{Z}$ . Denote the group of divisors on  $\Gamma \backslash X$  by  $\text{Div}(\Gamma \backslash X)$ .

*Remark 6.2.* Motivation for the definition of a divisor as an integral linear combinations of the  $[x]/|\Gamma_x|$  comes from the discussion of orbifold Euler characteristic in Paragraph 3.4.

To each section of a holomorphic line bundle over  $\Gamma \backslash X$ , we can associate a divisor. A section  $s$  of a holomorphic line bundle  $\Gamma \backslash L \rightarrow \Gamma \backslash X$  is a  $\Gamma$ -equivariant holomorphic section  $\tilde{s}$  of  $L \rightarrow X$ . Define the *order*  $\nu_{[x]}(s)$  of  $s$  at  $[x]$  to be the order of  $\tilde{s}$  at  $x \in X$ . This is well defined as  $s$  is  $\Gamma$ -equivariant. Define the *divisor* of a non-zero section  $s$  by

$$\text{div}(s) = |\Gamma_X| \sum_{[x] \in \Gamma \backslash X} \frac{\nu_{[x]}(s)}{|\Gamma_x|} [x] \in \text{Div}(\Gamma \backslash X).$$

The factor  $|\Gamma_X|$  is present as all non-zero sections  $s$  are  $\Gamma_X$ -invariant, which means that such  $s$  are pulled back from  $(\Gamma \backslash X)^{\text{red}}$ .

*Exercise 55.* Suppose that  $f : \Gamma' \backslash X' \rightarrow \Gamma \backslash X$  is a holomorphic mapping between orbifolds. Show that a holomorphic line bundle  $L \rightarrow \Gamma' \backslash X'$  pulls back to a holomorphic line bundle  $f^*L \rightarrow \Gamma \backslash X$  and that a section  $s$  of  $L$  pulls back to a section  $f^*s$  of  $f^*L$ . Show that if  $\Gamma$  and  $\Gamma'$  act virtually freely on  $X$  and  $X'$ , respectively, then there is a homomorphism

$$f^* : \text{Div}(\Gamma' \backslash X') \rightarrow \text{Div}(\Gamma \backslash X)$$

such that  $f^* \text{div}(s) = \text{div}(f^*s)$ .

Suppose that  $D$  is divisor on  $\Gamma \backslash X$ . Let  $\pi : X \rightarrow \Gamma \backslash X$  be the natural projection. For each open subset  $U$  of  $\Gamma \backslash X$ , define  $\mathcal{O}_{\Gamma \backslash X}(D)$  to consist of the  $\Gamma$ -invariant sections  $\mathcal{O}_X(\pi^*D)(\pi^{-1}(U))$  of  $\mathcal{O}_X(\pi^*D)$  over  $\pi^{-1}(U)$ . Then  $\mathcal{O}_{\Gamma \backslash X}(D)$  is an example of a sheaf on  $\Gamma \backslash X$ .

The group of divisors of the Riemann surface  $\Gamma \backslash X$  consists of all formal linear combinations

$$\sum_{[x] \in \Gamma \backslash X} n_x [x]$$

where each  $n_x \in \mathbb{Z}$ .

*Exercise 56.* Show that the mapping

$$\text{Div}(\Gamma \backslash X) \rightarrow \text{Div}((\Gamma \backslash X)^{\text{red}}) \rightarrow \text{Div}(\Gamma \backslash X)$$

induced by the canonical quotient mappings  $\Gamma \backslash X \rightarrow (\Gamma \backslash X)^{\text{red}} \rightarrow \Gamma \backslash X$  satisfy  $[x] \mapsto [x]$ . In particular, these mappings are injective.

*Exercise 57.* Show that if  $Y$  is a Riemann surface, then every holomorphic mapping  $\Gamma \backslash X \rightarrow Y$  factors through the quotient mapping  $\pi : \Gamma \backslash X \rightarrow \Gamma \backslash X$ . In particular, every meromorphic function  $\Gamma \backslash X \rightarrow \mathbb{P}^1$  is pulled back from a meromorphic function  $\Gamma \backslash X \rightarrow \mathbb{P}^1$ .

The definitions of divisor class groups and Picard groups can be extended to basic orbifolds.

**Definition 6.3.** A *principal divisor* on  $\Gamma \backslash X$  is the divisor of a non-zero meromorphic function  $f : \Gamma \backslash X \rightarrow \mathbb{P}^1$ . The *divisor class group* of  $\Gamma \backslash X$  is the group

$$\mathcal{C}l(\Gamma \backslash X) := \text{Div}(\Gamma \backslash X) / \{\text{principal divisors}\}.$$

The *Picard group* of  $\Gamma \backslash X$  is defined to be the group of isomorphism classes of holomorphic line bundles over  $\Gamma \backslash X$ , where the group operation is tensor product of line bundles. Denote it by  $\text{Pic}(\Gamma \backslash X)$ .

*Exercise 58.* Show that if  $\Gamma \backslash X$  is reduced (i.e.,  $\Gamma_X$  is trivial), there is a well defined group homomorphism

$$\text{Pic}(\Gamma \backslash X) \rightarrow \mathcal{C}\ell(\Gamma \backslash X)$$

that takes the isomorphism class of a holomorphic line bundle to the divisor class of a non-zero meromorphic section.<sup>24</sup> Show that it is an isomorphism.

When  $\Gamma \backslash X$  is not reduced, there are line bundles that have no meromorphic sections.

**Example 6.4.** The line bundle  $\mathcal{L}_k \rightarrow \mathcal{M}_{1,1}$  has no non-zero meromorphic sections when  $k$  is odd.

Because of this, we compute  $\text{Pic } \mathcal{M}_{1,1}^{\text{red}}$  before computing  $\text{Pic } \mathcal{M}_{1,1}$ .

**Proposition 6.5.** *There are natural isomorphisms*

$$\text{Pic } \mathcal{M}_{1,1}^{\text{red}} \cong \mathcal{C}\ell(\mathcal{M}_{1,1}^{\text{red}}) \cong \mathbb{Z}/6\mathbb{Z}.$$

*The Picard group is generated by the class of  $\mathcal{L}_2$ .*

*Proof.* To compute  $\text{Pic } \mathcal{M}_{1,1}^{\text{red}}$ , it suffices, by Exercise 56, to compute  $\mathcal{C}\ell(\mathcal{M}_{1,1}^{\text{red}})$ . Since  $M_{1,1} \cong \mathbb{C}$ , it follows that  $[\tau]$  is trivial in  $\mathcal{C}\ell(\mathcal{M}_{1,1})$  for all  $\tau \in \mathfrak{h}$ . Consequently,

$$\mathcal{C}\ell(\mathcal{M}_{1,1}^{\text{red}}) \cong \left\{ \frac{n_i}{2}[i] + \frac{n_\rho}{3}[\rho] : n_i, n_\rho \in \mathbb{Z} \right\} / \mathbb{Z}[i] \oplus \mathbb{Z}[\rho],$$

which is isomorphic to  $\mathbb{Z}/6\mathbb{Z}$ .

To see that  $\mathcal{L}_2$  generates  $\text{Pic } \mathcal{M}_{1,1}^{\text{red}}$  we use the facts [10, p. 80]<sup>25</sup>

$$(14) \quad \nu_i(G_4) = 0, \quad \nu_\rho(G_4) = 1, \quad \nu_i(G_6) = 1, \quad \nu_\rho(G_6) = 0.$$

where  $G_k$  denotes the Eisenstein series of weight  $k$ . Since  $G_6/G_4$  is a meromorphic section of  $\mathcal{L}_2$ , its divisor  $[\rho]/3 - [i]/2$  generates  $\mathcal{C}\ell(\mathcal{M}_{1,1}^{\text{red}})$ . This implies that  $\mathcal{L}_2$  generates  $\text{Pic } \mathcal{M}_{1,1}^{\text{red}}$ .  $\square$

To compute  $\text{Pic } \mathcal{M}_{1,1}$ , we need to relate it to  $\text{Pic } \mathcal{M}_{1,1}^{\text{red}}$ .

<sup>24</sup>For this you will need to show that every orbifold line bundle  $L \rightarrow \Gamma \backslash X$  has a non-zero meromorphic section. This can be proved by first noting that, since the action of  $\Gamma$  on  $X$  is virtually free and effective,  $L \rightarrow \Gamma \backslash X$  is the quotient of a line bundle  $M \rightarrow Y$  over a Riemann surface by a finite group  $G$ . One can then use standard results about Riemann surfaces to show that such a line bundle has a non-zero  $G$ -invariant meromorphic section. When  $Y$  is compact, you can do this using Riemann-Roch. When  $Y$  is non-compact, you can use the fact that  $Y$  is Stein, so that  $M$  is trivial.

<sup>25</sup>A direct proof of these facts using results developed in these notes can be given. See Exercise 62.



*Exercise 59.* Show that if  $X$  is a simply connected Riemann surface, then there is an exact sequence

$$0 \rightarrow \text{Pic}((\Gamma \backslash X)^{\text{red}}) \rightarrow \text{Pic}(\Gamma_X \backslash X) \rightarrow \text{Char}(\Gamma_X) \rightarrow 1$$

where  $\text{Char}(\Gamma_X)$  denotes the group of characters  $\chi : \Gamma_X \rightarrow \mathbb{C}^*$ .

**Theorem 6.6.** *The group  $\text{Pic } \mathcal{M}_{1,1}$  is cyclic of order 12. It is generated by the class of  $\mathcal{L}_1$ .*

*Proof.* Since the square of  $[\mathcal{L}] \in \text{Pic } \mathcal{M}_{1,1}$  generates  $\text{Pic } \mathcal{M}_{1,1}^{\text{red}}$ , which has order 6, it follows that  $[\mathcal{L}]$  has order 12 in  $\text{Pic } \mathcal{M}_{1,1}$ . By Exercise 59 the sequence

$$0 \rightarrow \text{Pic } \mathcal{M}_{1,1}^{\text{red}} \rightarrow \text{Pic } \mathcal{M}_{1,1} \rightarrow \text{Char}(C_2) \rightarrow 0$$

is exact. Since  $[\mathcal{L}]$  maps to the non-trivial character  $C_2 \rightarrow \mathbb{C}^*$ , it follows that  $\text{Pic } \mathcal{M}_{1,1}$  is generated by  $[\mathcal{L}]$  and has order 12.  $\square$

*Remark 6.7.* These definitions in this section generalize easily to complex analytic orbifolds of higher dimension.

**6.3. The Picard group of  $\overline{\mathcal{M}}_{1,1}$ .** The constructions of the previous section generalize to all orbifold Riemann surfaces. In this section we explain how to do this for  $\overline{\mathcal{M}}_{1,1}$  and the corresponding reduced orbifold  $\overline{\mathcal{M}}_{1,1}^{\text{red}}$ , which we define below.

We shall view  $\overline{\mathcal{M}}_{1,1}$  as the union of the basic orbifolds  $\mathcal{M}_{1,1}$  and  $C_2 \backslash \mathbb{D}_R$ , where  $R = e^{-2\pi}$ , which “intersect” in the basic orbifold  $C_2 \backslash \mathbb{D}_R^*$ . In both cases, the  $C_2$ -action is trivial. Thus, to each  $P \in \overline{\mathcal{M}}_{1,1}$ , we can associate an “automorphism group”  $\text{Aut}(P)$ , which is well defined up to isomorphism. If  $P = [x] \in \Gamma \backslash X$ , where  $(X, \Gamma) = (\mathfrak{h}, \text{SL}_2(\mathbb{Z}))$  or  $(\mathbb{D}_R, C_2)$ , and  $x$  is a lift of  $P$  to  $X$ , then  $\text{Aut}(P)$  is the isomorphism class of  $\Gamma_x$ . For all but  $[i], [\rho] \in \mathcal{M}_{1,1}$ , this isotropy group is isomorphic to  $C_2$ ;  $\text{Aut}([i]) = \mu_4$  and  $\text{Aut}([\rho]) = \mu_6$ .

The orbifold  $\overline{\mathcal{M}}_{1,1}^{\text{red}}$  is obtained by gluing  $\mathcal{M}_{1,1}^{\text{red}} := \text{PSL}_2(\mathbb{Z}) \backslash \mathfrak{h}$  to  $\mathbb{D}$  along the orbifold  $\mathbb{D}_R^*$ . The only points in  $\overline{\mathcal{M}}_{1,1}^{\text{red}}$  with non-trivial automorphism groups are  $[i]$  and  $[\rho]$ , whose automorphism groups are cyclic of orders 2 and 3, respectively.

A divisor on  $\overline{\mathcal{M}}_{1,1}$  is a finite sum

$$\sum_{P \in \overline{\mathcal{M}}_{1,1}} \frac{n_P}{|\text{Aut}(P)|} P$$

where each  $n_P \in \mathbb{Z}$  for all  $P \in \overline{\mathcal{M}}_{1,1}$ . These form a group  $\text{Div}(\overline{\mathcal{M}}_{1,1})$ .

Since the size of the stabilizer of  $P$  in  $\mathrm{PSL}_2(\mathbb{Z})$  is  $|\mathrm{Aut}(P)|/2$ , divisors on  $\overline{\mathcal{M}}_{1,1}^{\mathrm{red}}$  are finite sums

$$\sum_{P \in \overline{\mathcal{M}}_{1,1}} \frac{2n_P}{|\mathrm{Aut}(P)|} P$$

where  $n_P \in \mathbb{Z}$  for all  $P \in \overline{\mathcal{M}}_{1,1}$ . These form a subgroup  $\mathrm{Div}(\overline{\mathcal{M}}_{1,1}^{\mathrm{red}})$  of  $\mathrm{Div}(\overline{\mathcal{M}}_{1,1})$ .

The group of divisors on the Riemann surface  $\overline{\mathcal{M}}_{1,1}$  is the free abelian group generated by the  $[x] \in \overline{\mathcal{M}}_{1,1}$ . The quotient mappings  $\overline{\mathcal{M}}_{1,1} \rightarrow \overline{\mathcal{M}}_{1,1}^{\mathrm{red}} \rightarrow \overline{\mathcal{M}}_{1,1}$  induce the inclusions

$$\mathrm{Div}(\overline{\mathcal{M}}_{1,1}) \hookrightarrow \mathrm{Div}(\overline{\mathcal{M}}_{1,1}^{\mathrm{red}}) \hookrightarrow \mathrm{Div}(\overline{\mathcal{M}}_{1,1})$$

which take  $P$  to  $P$ .

The divisor

$$\mathrm{div}(s) \in \mathrm{Div}(\overline{\mathcal{M}}_{1,1})$$

of a section  $s$  of a holomorphic line bundle  $L \rightarrow \overline{\mathcal{M}}_{1,1}$  is computed locally on the two basic orbifold patches as in the previous section. A principal divisor on  $\overline{\mathcal{M}}_{1,1}$  is the divisor of a non-zero rational function  $f : \overline{\mathcal{M}}_{1,1} \rightarrow \mathbb{P}^1$ . All such functions are pulled back from rational functions  $\overline{\mathcal{M}}_{1,1} \rightarrow \mathbb{P}^1$ .

At this point, one can define the sheaves  $\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(D)$  locally in the two patches  $\mathcal{M}_{1,1}$  and  $C_2 \setminus \mathbb{D}$ .

*Exercise 60.* Show that if  $D \in \mathrm{Div}(\overline{\mathcal{M}}_{1,1})$  and  $L$  is a holomorphic line bundle over  $\overline{\mathcal{M}}_{1,1}$ , then a meromorphic section  $s$  of  $L$  is a holomorphic section of  $L(D)$  if and only if  $\mathrm{div}(s) + D \geq 0$ . In particular, if  $P \in \overline{\mathcal{M}}_{1,1}$ , then  $\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(P)$  is the pullback of  $\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(P)$  to  $\overline{\mathcal{M}}_{1,1}$ . This proves that the ad hoc definition of the twist  $L(d\infty)$  of  $L$  given in the discussion preceding Exercise 34 is consistent with the definitions given in this section.

As in the local case, not every line bundle over  $\overline{\mathcal{M}}_{1,1}$  has a meromorphic section. Because of this, we define the divisor class group only for  $\overline{\mathcal{M}}_{1,1}^{\mathrm{red}}$ .

Define the divisor class group of  $\overline{\mathcal{M}}_{1,1}^{\mathrm{red}}$  to be

$$\mathcal{C}\ell(\overline{\mathcal{M}}_{1,1}^{\mathrm{red}}) := \mathrm{Div}(\overline{\mathcal{M}}_{1,1}^{\mathrm{red}}) / \{\text{principal divisors}\}$$

*Exercise 61.* Show that there is an exact sequence

$$0 \rightarrow \mathcal{C}\ell(\overline{\mathcal{M}}_{1,1}) \rightarrow \mathcal{C}\ell(\overline{\mathcal{M}}_{1,1}^{\mathrm{red}}) \rightarrow \mathcal{C}\ell(\mathcal{M}_{1,1}^{\mathrm{red}}) \rightarrow 0.$$

Show that  $\mathcal{C}\ell(\overline{\mathcal{M}}_{1,1})$  is infinite cyclic and is generated by the class of any point of  $\overline{\mathcal{M}}_{1,1}$ . Show directly that  $\mathcal{C}\ell(\mathcal{M}_{1,1}^{\text{red}})$  is cyclic of order 6. Deduce that  $\mathcal{C}\ell(\overline{\mathcal{M}}_{1,1}^{\text{red}})$  is infinite cyclic.<sup>26</sup>

Define the Picard group of an orbifold Riemann surface  $\mathcal{X}$  to be the group of isomorphism classes of holomorphic line bundles over  $\mathcal{X}$  with operation tensor product. Holomorphic mappings between orbifolds induce mappings on their Picard groups. In particular, we have natural pullback homomorphisms

$$\text{Pic } \overline{\mathcal{M}}_{1,1} \rightarrow \text{Pic } \overline{\mathcal{M}}_{1,1}^{\text{red}} \rightarrow \text{Pic } \overline{\mathcal{M}}_{1,1}.$$

**Proposition 6.8.** *There is a natural isomorphism*

$$\text{Pic } \overline{\mathcal{M}}_{1,1}^{\text{red}} \xrightarrow{\cong} \mathcal{C}\ell(\overline{\mathcal{M}}_{1,1}^{\text{red}}).$$

*Both are isomorphic to  $\mathbb{Z}$  and generated by the class of  $\overline{\mathcal{L}}_2$ . For all  $P \in \overline{\mathcal{M}}_{1,1}$ ,*

$$[\mathcal{O}_{\overline{\mathcal{M}}_{1,1}^{\text{red}}}(P)] = 6[\overline{\mathcal{L}}_2] \in \text{Pic } \overline{\mathcal{M}}_{1,1}^{\text{red}}.$$

*Sketch of Proof.* We first construct a homomorphism

$$(15) \quad \text{Pic } \overline{\mathcal{M}}_{1,1}^{\text{red}} \rightarrow \mathcal{C}\ell(\overline{\mathcal{M}}_{1,1}^{\text{red}}).$$

To do this, we have to show that every holomorphic line bundle over  $\overline{\mathcal{M}}_{1,1}^{\text{red}}$  has a non-zero meromorphic section. The homomorphism is defined by taking the isomorphism class of a line bundle to the divisor class of any non-zero meromorphic section.

To see that every holomorphic line bundle over  $\overline{\mathcal{M}}_{1,1}^{\text{red}}$  has a non-zero meromorphic section, we use the fact that every line bundle  $L \rightarrow \overline{\mathcal{M}}_{1,1}^{\text{red}}$  is the quotient of a holomorphic line bundle  $N \rightarrow Y$  over a Riemann surface  $Y$  by the action of a finite group  $G$ , where  $G$  acts effectively on  $Y$ .<sup>27</sup> The Riemann-Roch Theorem implies that  $N \rightarrow Y$  has a non-zero meromorphic section  $s$  such that

$$\text{Tr}(s) := \sum_{g \in G} g \cdot s$$

is a non-zero,  $G$ -invariant, meromorphic section of  $N$  over  $Y$ . It is an easy exercise to show that the homomorphism (15) is injective. This implies that  $\text{Pic } \overline{\mathcal{M}}_{1,1}^{\text{red}}$  is infinite cyclic.

<sup>26</sup>It is not necessary to use Proposition 6.5. Bypassing Prop. 6.5 is desirable as one can then deduce (14).

<sup>27</sup>For example, one can take  $Y$  to be the level  $m$  moduli space  $\overline{\mathcal{M}}_{1,1}[m]$  for any  $m \geq 3$ . It is constructed in Section 4.2.

Since the Ramanujan tau function  $\Delta$  is a section of  $\overline{\mathcal{L}}_{12} = \overline{\mathcal{L}}_2^{\otimes 6}$  and since  $\text{div}(\Delta) = [\infty]$ , it follows that  $6[\overline{\mathcal{L}}_2] = [\mathcal{O}_{\overline{\mathcal{M}}_{1,1}^{\text{red}}}(P)]$ . This establishes the surjectivity (15) and that  $\text{Pic}(\overline{\mathcal{M}}_{1,1}^{\text{red}})$  is generated by  $[\overline{\mathcal{L}}_2]$ .  $\square$

*Exercise 62* (cf. [10, Thm. 3, p. 80]). Use the preceding result to show that the meromorphic modular form  $f : \mathfrak{h} \rightarrow \mathbb{C}$  is a section of  $\overline{\mathcal{L}}_{2k}$  over  $\overline{\mathcal{M}}_{1,1}$  (and  $\overline{\mathcal{M}}_{1,1}^{\text{red}}$ ) if and only if

$$\nu_i(f)/2 + \nu_\rho(f)/3 + \sum_{\substack{P \in \overline{\mathcal{M}}_{1,1} \\ P \neq [i], [\rho]}} \nu_P(f) = \frac{k}{6}.$$

Use this to prove the statements (14).<sup>28</sup>

As in the local case, there is a short exact sequence

$$0 \rightarrow \text{Pic } \overline{\mathcal{M}}_{1,1}^{\text{red}} \rightarrow \text{Pic } \overline{\mathcal{M}}_{1,1} \rightarrow \text{Char}(C_2) \rightarrow 1.$$

Since the class of  $\overline{\mathcal{L}}$  in  $\text{Pic } \overline{\mathcal{M}}_{1,1}$  maps to the generator of  $\text{Char}(C_2)$ , the previous result implies:

**Theorem 6.9.** *The Picard group of  $\overline{\mathcal{M}}_{1,1}$  is infinite cyclic and is generated by the class of the Hodge bundle  $\overline{\mathcal{L}}$ . For all  $P \in \overline{\mathcal{M}}_{1,1}$ ,*

$$[\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(P)] = 12[\overline{\mathcal{L}}] \in \text{Pic } \overline{\mathcal{M}}_{1,1}.$$

Consequently, the sequence

$$0 \rightarrow \mathbb{Z}[\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(\infty)] \rightarrow \text{Pic } \overline{\mathcal{M}}_{1,1} \rightarrow \text{Pic } \mathcal{M}_{1,1} \rightarrow 0$$

is exact.

## 7. THE ALGEBRAIC TOPOLOGY OF $\overline{\mathcal{M}}_{1,1}$

The homotopy type of a basic orbifold has already been discussed in Section 3.3. Global orbifolds, such as  $\overline{\mathcal{M}}_{1,1}$ , also have a well defined homotopy type. In this section we discuss the homotopy type of  $\overline{\mathcal{M}}_{1,1}$  and use it to compute its low dimensional (co)homology groups.

**7.1. The homotopy type of  $\overline{\mathcal{M}}_{1,1}$ .** Let  $U$  be a contractible topological space on which  $\text{SL}_2(\mathbb{Z})$  acts properly discontinuously and fixed point freely, such as the standard model of  $ESL_2(\mathbb{Z})$ . The groups

$$C_2 = \{\pm \text{id}\} \text{ and } \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$$

<sup>28</sup>These results imply Proposition 4.5 as in [10, p. 88].

are subgroups of  $\mathrm{SL}_2(\mathbb{Z})$ , and thus act freely and discontinuously on  $U$  as well. We can therefore consider the diagram

$$(16) \quad \begin{array}{ccc} & (C_2 \times \mathbb{Z}) \backslash (U \times \mathfrak{h}) & \\ \mathrm{id}_U \times p \swarrow & & \searrow \mathrm{id}_U \times q \\ \mathrm{SL}_2(\mathbb{Z}) \backslash (U \times \mathfrak{h}) & & C_2 \backslash (U \times \mathbb{D}) \end{array}$$

of topological spaces, where  $\mathrm{SL}_2(\mathbb{Z})$  acts diagonally on  $U \times \mathfrak{h}$ , etc. Because  $U \times \mathfrak{h}$  and  $U \times \mathbb{D}$  are contractible and each of the groups acts freely and discontinuously, the homotopy type of the 3 pieces are:<sup>29</sup>

$$BSL_2(\mathbb{Z}), \quad BC_2, \quad B(C_2 \times \mathbb{Z}).$$

So we can represent the diagram above as

$$\begin{array}{ccc} & B(C_2 \times \mathbb{Z}) & \\ P \swarrow & & \searrow Q \\ BSL_2(\mathbb{Z}) & & BC_2 \end{array}$$

One can form the space

$$\mathcal{M}_{1,1} \cup_{BC_2 \times \mathbb{D}^*} (BC_2 \times \mathbb{D}^*) := BSL_2(\mathbb{Z}) \cup_{B(C_2 \times \mathbb{Z})} BC_2$$

by taking the pushout of the diagram (16) in the homotopy category. Explicitly, it is the homotopy type of the space

$$[\mathrm{SL}_2(\mathbb{Z}) \backslash (U \times \mathfrak{h}) \dot{\cup} [0, 1] \times (C_2 \times \mathbb{Z}) \backslash (U \times \mathfrak{h}) \dot{\cup} C_2 \backslash (U \times \mathbb{D})] / \sim$$

obtained by identifying the  $C_2 \times \mathbb{Z}$  orbit of  $(0, u, \tau) \in [0, 1] \times (C_2 \times \mathbb{Z}) \backslash (U \times \mathfrak{h})$  with the  $\mathrm{SL}_2(\mathbb{Z})$  orbit of  $(u, \tau) \in U \times \mathfrak{h}$ , and the orbit of  $(1, u, \tau) \in [0, 1] \times (C_2 \times \mathbb{Z}) \backslash (U \times \mathfrak{h})$  with the  $C_2$  orbit of  $(u, q(\tau))$ . Its homotopy type is well defined. There is a well defined morphism

$$(17) \quad \mathcal{M}_{1,1} \cup_{BC_2 \times \mathbb{D}^*} (BC_2 \times \mathbb{D}^*) \rightarrow \overline{\mathcal{M}}_{1,1}$$

of topological orbifolds obtained by projecting the diagram (16) to the atlas of  $\overline{\mathcal{M}}_{1,1}$  along  $U$ .

**Definition 7.1.** The homotopy type of  $\overline{\mathcal{M}}_{1,1}$  is defined to be the homotopy type of the space  $BSL_2(\mathbb{Z}) \cup_{B(C_2 \times \mathbb{Z})} BC_2$  defined above.

*Exercise 63.* Use the presentation (3) to prove that  $\mathrm{PSL}_2(\mathbb{Z})$  is isomorphic to the free product of  $C_2 * C_3$  and that  $\mathrm{SL}_2(\mathbb{Z})$  is an extension

$$1 \rightarrow C_2 \rightarrow \mathrm{SL}_2(\mathbb{Z}) \rightarrow C_2 * C_3 \rightarrow 1.$$

Deduce that  $BSL_2(\mathbb{Z})$  is a  $BC_2$  bundle over  $BC_2 \vee BC_3$ .

<sup>29</sup>These spaces are not that exotic:  $BC_2 \simeq \mathbb{R}\mathbb{P}^\infty$ ,  $B\mathbb{Z} \simeq S^1$  and  $B(\mathbb{Z} \times C_2) \simeq B\mathbb{Z} \times BC_2 \simeq S^1 \times \mathbb{R}\mathbb{P}^\infty$ .

*Exercise 64.* Show that  $\mathcal{M}_{1,1}$  and  $\overline{\mathcal{M}}_{1,1}$  are both homotopy equivalent to a CW-complex with only a finite number of cells in each dimension. Deduce that their homology and cohomology groups are finitely generated in each degree.

Invariants of the homotopy type of  $\overline{\mathcal{M}}_{1,1}$ , such as its homotopy, homology and cohomology groups are defined to be those of its homotopy type.

**7.2. The fundamental group of  $\overline{\mathcal{M}}_{1,1}$ .** We can apply van Kampen's theorem to compute the fundamental group of  $\overline{\mathcal{M}}_{1,1}$ . It is the amalgamated free product

$$(18) \quad \mathrm{SL}_2(\mathbb{Z}) *_{C_2 \times \mathbb{Z}} C_2$$

obtained by pushing out the diagram

$$\mathrm{SL}_2(\mathbb{Z}) \xleftarrow{\text{inclusion}} C_2 \times \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \xrightarrow{\text{projection}} C_2$$

in the category of groups.

*Exercise 65.* Recall the definition of  $S, T, U \in \mathrm{SL}_2(\mathbb{Z})$  and the presentation (3) of  $\mathrm{SL}_2(\mathbb{Z})$ . Set

$$\widehat{T} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Verify that  $U = T^{-1}\widehat{T}$  and that  $S = T^{-1}\widehat{T}T^{-1}$ . Deduce that  $\mathrm{SL}_2(\mathbb{Z})$  is generated by  $T$  and  $\widehat{T}$ . Use this to show that the amalgamated free product (18) is the trivial group.

This proves:

**Proposition 7.2.** *The orbifold  $\overline{\mathcal{M}}_{1,1}$  is simply connected. Consequently,  $H_1(\overline{\mathcal{M}}_{1,1}; \mathbb{Z}) = 0$ .*

**7.3. Chern classes.** Orbifold vector bundles over  $\mathcal{M}_{1,1}$  give rise to genuine vector bundles over its homotopy type  $B\mathrm{SL}_2(\mathbb{Z})$ . Similarly, an orbifold vector bundle over  $\overline{\mathcal{M}}_{1,1}$  determines a genuine vector bundle over its homotopy type. One can therefore define Chern classes

$$c_j(E) \in H^{2j}(\mathcal{M}_{1,1}; \mathbb{Z}), \quad c_j(F) \in H^{2j}(\overline{\mathcal{M}}_{1,1}; \mathbb{Z})$$

of orbifold vector bundles  $E$  over  $\mathcal{M}_{1,1}$  and  $F$  over  $\overline{\mathcal{M}}_{1,1}$ . In particular, we have Chern class homomorphisms

$$c_1 : \mathrm{Pic} \mathcal{M}_{1,1} \rightarrow H^2(\mathcal{M}_{1,1}; \mathbb{Z}) \quad \text{and} \quad c_1 : \mathrm{Pic} \overline{\mathcal{M}}_{1,1} \rightarrow H^2(\overline{\mathcal{M}}_{1,1}; \mathbb{Z})$$

such that the diagram

$$\begin{array}{ccc} \text{Pic } \overline{\mathcal{M}}_{1,1} & \xrightarrow{c_1} & H^2(\overline{\mathcal{M}}_{1,1}; \mathbb{Z}) \\ \downarrow & & \downarrow \\ \text{Pic } \mathcal{M}_{1,1} & \xrightarrow{c_1} & H^2(\mathcal{M}_{1,1}; \mathbb{Z}) \end{array}$$

commutes.

*Exercise 66.* Show that  $c_1 : \text{Pic } \mathcal{M}_{1,1} \rightarrow H^2(\mathcal{M}_{1,1}; \mathbb{Z})$  is an isomorphism.

**7.4. Low dimensional cohomology of  $\overline{\mathcal{M}}_{1,1}$ .** The homology and cohomology groups of  $\overline{\mathcal{M}}_{1,1}$  can be computed using the Mayer-Vietoris sequence

$$\begin{aligned} \cdots \rightarrow H^k(\overline{\mathcal{M}}_{1,1}) &\rightarrow H^k(\mathcal{M}_{1,1}) \oplus H^k(BC_2) \\ &\rightarrow H^k(\mathbb{D}^* \times BC_2) \rightarrow H^{k+1}(\overline{\mathcal{M}}_{1,1}) \rightarrow \cdots \end{aligned}$$

associated to the covering (17) or use the ‘‘Gysin sequence’’

$$\cdots \rightarrow H^k(\overline{\mathcal{M}}_{1,1}) \rightarrow H^k(\mathcal{M}_{1,1}) \rightarrow \tilde{H}^{k-1}(BC_2) \rightarrow H^{k+1}(\overline{\mathcal{M}}_{1,1}) \rightarrow \cdots$$

associated to the cofibration sequence

$$\mathcal{M}_{1,1} \rightarrow \overline{\mathcal{M}}_{1,1} \rightarrow (\mathbb{D}, S^1) \times BC_2.$$

*Exercise 67.* Justify these sequences.

Since  $\overline{\mathcal{M}}_{1,1}$  is simply connected,  $H_1(\overline{\mathcal{M}}_{1,1})$  and  $H^1(\overline{\mathcal{M}}_{1,1})$  vanish with all coefficients.

**Proposition 7.3.** *The first Chern class*

$$c_1 : \text{Pic } \overline{\mathcal{M}}_{1,1} \rightarrow H^2(\overline{\mathcal{M}}_{1,1}; \mathbb{Z})$$

*is an isomorphism. Consequently,  $H^2(\overline{\mathcal{M}}_{1,1}; \mathbb{Z})$  is infinite cyclic.*

*Proof.* Consider the diagram

(19)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}[\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(\infty)] & \longrightarrow & \text{Pic } \overline{\mathcal{M}}_{1,1} & \longrightarrow & \text{Pic } \mathcal{M}_{1,1} \longrightarrow 0 \\ & & \downarrow e & & \downarrow c_1 & & \downarrow c_1 \\ 0 & \longrightarrow & H^0(BC_2; \mathbb{Z}) & \longrightarrow & H^2(\overline{\mathcal{M}}_{1,1}; \mathbb{Z}) & \longrightarrow & H^2(\mathcal{M}_{1,1}; \mathbb{Z}) \longrightarrow 0 \end{array}$$

The top row is exact by Theorem 6.9. The second row is a portion of the Gysin sequence. It is exact as  $H^1(\mathcal{M}_{1,1}; \mathbb{Z}) = 0$  and as  $H^1(BC_2; \mathbb{Z}) = \text{Hom}(C_2, \mathbb{Z}) = 0$ . Since  $c_1 : \text{Pic } \mathcal{M}_{1,1} \rightarrow H^2(\mathcal{M}_{1,1}; \mathbb{Z})$  is

an isomorphism, there is a map  $e : \mathbb{Z}[\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(\infty)] \rightarrow H^0(BC_2; \mathbb{Z})$  making the diagram commute. This implies that  $H^2(\overline{\mathcal{M}}_{1,1}; \mathbb{Z})$  is infinite cyclic. To see that the middle vertical map is an isomorphism requires more work. To complete the proof we will sketch a proof that  $e$  is an isomorphism.

Consider the portion

$$\begin{array}{ccc} \mathbb{Z}[\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(\infty)] & \longrightarrow & \text{Pic } \overline{\mathcal{M}}_{1,1} \\ e \downarrow & & c_1 \downarrow \\ H^0(\{\infty\}; \mathbb{Z}) & \longrightarrow & H^2(\overline{\mathcal{M}}_{1,1}; \mathbb{Z}) \end{array}$$

of the analogue of the diagram (19) for  $(\overline{\mathcal{M}}_{1,1}, M_{1,1})$ . Since  $(\overline{\mathcal{M}}_{1,1}, M_{1,1})$  is isomorphic to  $(\mathbb{P}^1, \mathbb{C})$ , all four maps in this diagram are isomorphisms. The map  $\pi : (\overline{\mathcal{M}}_{1,1}, \mathcal{M}_{1,1}) \rightarrow (\overline{\mathcal{M}}_{1,1}, M_{1,1})$  induces a morphism of Gysin sequences that is compatible with Chern classes. It maps the commutative square in this diagram to the left-hand square in (19). The map on the top left corner is an isomorphism. The map on the bottom left hand corner is the map

$$H^0(\{\infty\}; \mathbb{Z}) \cong H^1(\infty \times (\mathbb{D}, S^1)) \rightarrow H^1(BC_2 \times (\mathbb{D}, S^1)) \cong H^0(BC_2)$$

which is an isomorphism. It follows that  $e$  is an isomorphism as claimed.  $\square$

## 8. CONCLUDING REMARKS

Our goal in this final section is to tie together several loose ends to explain how the moduli space  $\overline{\mathcal{M}}_{1,1}$  can be viewed as a Deligne-Mumford stack in the category of schemes over  $\mathbb{Q}$ . Along the way, we identify the fundamental group of several moduli spaces of elliptic curves with the braid group on 3 strings, the group of the trefoil knot, and with a canonical central extension of  $SL_2(\mathbb{Z})$ .

**8.1. The moduli space  $\mathcal{M}_{1,\vec{1}}$ .** In this section we will consider the problem of determining the moduli space  $\mathcal{M}_{1,\vec{1}}$  of triples  $(X, P, \vec{v})$ , where  $(X, P)$  is an elliptic curve and  $\vec{v} \in T_P X$  is a non-zero holomorphic tangent vector to  $X$  at  $P$ . Since the holomorphic cotangent bundle of  $X$  is trivial, such a triple is determined by and determines a triple  $(X, P, \omega)$ , where  $\omega$  is a non-zero holomorphic differential on  $X$ . The correspondence is given by insisting that  $\langle \omega, \vec{v} \rangle = 1$ .

It follows from Lemma 5.13 that  $\mathcal{M}_{1,\vec{1}} = \mathcal{L}^*$  where  $\mathcal{L}^*$  is the  $\mathbb{C}^*$ -bundle obtained by removing the zero section from  $\mathcal{L}$ . This is a genuine complex surface, not just an orbifold.



*Exercise 68.* Show that the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathbb{C}^* \times \mathfrak{h}$

$$(20) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u, \tau) \mapsto ((c\tau + d)u, (a\tau + b)/(c\tau + d))$$

is fixed point free. Deduce that  $\mathcal{L}^*$  is a genuine complex surface whose universal covering is  $\mathbb{C} \times \mathfrak{h}$  and whose fundamental group is an extension

$$0 \rightarrow \mathbb{Z} \rightarrow \pi_1(\mathcal{L}^*) \rightarrow \mathrm{SL}_2(\mathbb{Z}) \rightarrow 1.$$

*Exercise 69.* Show that the complex surface  $\mathcal{M}_{1, \bar{1}}$  is a fine moduli space of triples  $(X, P, \omega)$ , where  $(X, P)$  is an elliptic curve and  $\omega$  is a non-zero holomorphic differential on  $X$ .

The moduli space  $\mathcal{M}_{1, \bar{1}}$  has a natural *partial* compactification. Namely

$$\overline{\mathcal{M}}_{1, \bar{1}} := \overline{\mathcal{L}}^*,$$

the  $\mathbb{C}^*$ -bundle associated to  $\overline{\mathcal{L}}^*$ .<sup>30</sup>

**8.2. The topology of  $\mathcal{M}_{1, \bar{1}}$ .** Since the maximal compact subgroup of  $\mathrm{SL}_2(\mathbb{R})$  is the circle  $\mathrm{SO}(2)$ , its fundamental group is isomorphic to  $\mathbb{Z}$ . This implies that the universal covering group of  $\mathrm{SL}_2(\mathbb{R})$  is an extension

$$0 \rightarrow \mathbb{Z} \rightarrow \widetilde{\mathrm{SL}}_2(\mathbb{R}) \rightarrow \mathrm{SL}_2(\mathbb{R}) \rightarrow 1.$$

Denote the inverse image of  $\mathrm{SL}_2(\mathbb{Z})$  in  $\widetilde{\mathrm{SL}}_2(\mathbb{R})$  by  $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ . It is an extension

$$0 \rightarrow \mathbb{Z} \rightarrow \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}) \rightarrow 1.$$

**Proposition 8.1.** *There is a natural isomorphism*

$$\pi_1(\mathcal{M}_{1, \bar{1}}, p) \xrightarrow{\cong} \widetilde{\mathrm{SL}}_2(\mathbb{Z}),$$

where  $p : \mathbb{C} \times \mathfrak{h} \rightarrow \mathbb{C}^* \times \mathfrak{h} \rightarrow \mathcal{M}_{1, \bar{1}}$  is the base point that takes  $(v, \tau)$  to the  $\mathrm{SL}_2(\mathbb{Z})$  orbit of  $(e^v, \tau) \in \mathbb{C}^* \times \mathfrak{h}$ .

*Proof.* The group  $\mathrm{SL}_2(\mathbb{R})$  acts on  $\mathbb{C} \times \mathfrak{h}$  by the formula (20). It preserves the metric  $\|(u, \tau)\| = |u| \mathrm{Im}(\tau)^{-1/2}$  and therefore restricts to an action on  $S^1 \times \mathfrak{h}$ . This action is easily checked to be transitive. The isotropy group of  $(1, i)$  is trivial. The mapping

$$\mathrm{SL}_2(\mathbb{R}) \rightarrow S^1 \times \mathfrak{h}; \quad g \mapsto g(1, i)$$

<sup>30</sup>This is *not* compact. However,  $\mathcal{M}_{1, \bar{1}}$  does admit a natural smooth compactification by adding the sections 0 and  $\infty$  to  $\overline{\mathcal{L}}^*$ . Explicitly, this is:

$$\overline{\mathcal{L}} \cup_{\overline{\mathcal{L}}^*} \overline{\mathcal{L}}_{-1} = \mathbb{P}(\overline{\mathcal{L}} \oplus \mathcal{O}_{\overline{\mathcal{M}}_{1,1}})$$

which is a  $\mathbb{P}^1$ -bundle over  $\overline{\mathcal{M}}_{1,1}$ . We will not use this compactification.

is therefore a diffeomorphism. It is also  $\mathrm{SL}_2(\mathbb{R})$ -equivariant with respect to the two natural left  $\mathrm{SL}_2(\mathbb{R})$ -actions. The mapping therefore lifts to a diffeomorphism

$$\widetilde{\mathrm{SL}}_2(\mathbb{R}) \rightarrow \mathbb{R} \times \mathfrak{h},$$

which implies that  $\widetilde{\mathrm{SL}}_2(\mathbb{R})$  is contractible.

The  $\mathrm{SL}_2(\mathbb{R})$ -action on  $S^1 \times \mathfrak{h}$  can be lifted to an  $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ -action on  $\mathbb{R} \times \mathfrak{h}$  by defining the previous mapping to be  $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ -equivariant. It follows that the unit circle bundle of  $\mathcal{L}$  is the quotient

$$\widetilde{\mathrm{SL}}_2(\mathbb{Z}) \backslash \widetilde{\mathrm{SL}}_2(\mathbb{R}) \cong \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \backslash (\mathbb{R} \times \mathfrak{h}) \cong \mathrm{SL}_2(\mathbb{Z}) \backslash (S^1 \times \mathfrak{h}).$$

Since the inclusion of the unit circle bundle into  $\mathcal{L}^*$  is a homotopy equivalence, it follows that the fundamental group of  $\mathcal{L}^*$  is isomorphic to  $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ .  $\square$

**8.3. Plane cubics and  $\mathcal{M}_{1,\bar{1}}$ .** Consider the universal family of cubics  $E \rightarrow \mathbb{C}^2$  where

$$E = \{([x, y, z], (a, b)) \in \mathbb{P}^2 \times \mathbb{C}^2 : zy^2 = 4x^3 - axz^2 - bz^3\}.$$

The total space  $E$  is smooth except over the origin  $a = b = 0$ . The point  $[0, 1, 0]$  lies in each fiber and therefore defines a section of  $E \rightarrow \mathbb{C}^2$ .

Recall that  $D(a, b) = a^3 - 27b^2$  is the discriminant of the cubic  $4x^3 - ax - b$ . Let  $\Delta$  be the divisor in  $\mathbb{C}^2$  defined by  $D = 0$ . It is called the *discriminant locus*. Over  $\mathbb{C}^2 - \Delta$  the fibers of  $E \rightarrow \mathbb{C}^2$  are smooth; over  $\Delta - \{0\}$  they are nodal cubics; and over the origin the fiber is the cuspidal cubic. As we have seen in Section 5.2 the rational differential  $dx/y$  on  $E$  restricts to a non-zero holomorphic differential on each smooth fiber of  $E$ . Since  $\mathcal{M}_{1,\bar{1}}$  is a fine moduli space for triples  $(X, P; \omega)$  (Cf. Exercise 69), there is a holomorphic mapping

$$F : \mathbb{C}^2 - \Delta \rightarrow \mathcal{M}_{1,\bar{1}}$$

that classifies the tautological family of cubics  $E$  over  $\mathbb{C}^2 - \Delta$  and the differential  $dx/y$ .

Define  $\mathbb{C}^*$ -actions on  $\mathbb{C}^2$  and  $\mathcal{M}_{1,\bar{1}}$  by

$$\lambda \cdot (a, b) := (\lambda^{-4}a, \lambda^{-6}b) \text{ and } \lambda \cdot [X, P; \omega] := [X, P; \lambda\omega],$$

respectively. The  $\mathbb{C}^*$ -action restricts to an action on  $\mathbb{C}^2 - \Delta$ .

**Proposition 8.2.** *The mapping  $F$  is a  $\mathbb{C}^*$ -equivariant biholomorphism.*

*Proof.* Proposition 5.2 and the results in Section 5.2 imply that  $F$  is a bijection. We use modular forms to construct the inverse of  $F$ .

Define  $\tilde{G} : \mathbb{C}^* \times \mathfrak{h} \rightarrow \mathbb{C}^2 - D^{-1}(0)$  by

$$f(u, \tau) = (u^{-4}g_2(\tau), u^{-6}g_3(\tau)).$$

It is  $\mathbb{C}^*$ -invariant and also  $\mathrm{SL}_2(\mathbb{Z})$ -equivariant with respect to the action

$$\gamma : (u, \tau) \mapsto ((c\tau + d)u, \gamma(\tau)).$$

It therefore induces a holomorphic function

$$G : \mathcal{L}^* \rightarrow \mathbb{C}^2 - \Delta.$$

This is an inverse of  $F$  as it is  $\mathbb{C}^*$ -equivariant and  $(1, \tau) \in \mathbb{C} \times \mathfrak{h}$  corresponds to  $(\mathbb{C}/\Lambda_\tau; dz)$  and  $G(1, \tau)$  corresponds to the curve  $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$  with the differential  $dx/y$  which corresponds to  $dz$  by Proposition 5.4.  $\square$

**Corollary 8.3.** *The fundamental group of  $\mathcal{M}_{1, \bar{1}}$  is isomorphic to the braid group  $B_3$  on 3 strings and also to the fundamental group of the complement of the trefoil knot in the 3-sphere. Both groups are isomorphic to  $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ .*

*Sketch of Proof.* Since  $\mathcal{M}_{1, \bar{1}}$  is the variety  $\mathbb{C}^2$  minus the cusp  $D(a, b) = 0$ , we need only compute the fundamental group of the space  $\mathbb{C}^2 - \Delta$ . The intersection  $L$  of the cusp  $D = 0$  with the unit sphere  $S^3$  in  $\mathbb{C}^2$  is the trefoil knot. This can be seen by writing  $S^3$  as the union of the two solid tori that intersect along the 2-torus  $|a| = c|b|$  for suitably chosen  $c$ . The discriminant locus  $a^3 = 27b^2$  intersects this in the torus knot  $\theta \mapsto (1 + c^2)^{-1/2}(ce^{2i\theta}, e^{3i\theta})$  of type  $(2, 3)$  — the trefoil knot.

The  $\mathbb{R}_+$ -action

$$t : (a, b) \mapsto (t^2a, t^3b)$$

restricts to an action on  $\mathbb{C}^2 - \Delta$ . Each orbit intersects  $S^3$  transversely in a unique point, which implies that the action induces a diffeomorphism

$$\mathbb{R}_+ \times (S^3 - L) \xrightarrow{\cong} \mathbb{C}^2 - \Delta.$$

The inclusion  $S^3 - L \hookrightarrow \mathbb{C}^2 - \Delta$  is therefore a homotopy equivalence so that  $\pi_1(\mathcal{M}_{1, \bar{1}})$  is isomorphic to  $\pi_1(S^3 - L)$ .

The braid group  $B_n$  is the fundamental group of the quotient of

$$Y_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : \lambda_1 + \dots + \lambda_n = 0, \lambda_j \neq \lambda_k \text{ when } j \neq k\}$$

by the natural action of the symmetric group  $S_n$ . The quotient is the space of polynomials

$$X_n := \{p(T) = T^n + a_{n-2}T^{n-2} + \dots + a_0 : \text{discriminant of } p(T) \neq 0\}.$$

The coordinate  $a_j$  of  $X_n$  is  $(-1)^j$  the  $j$ th elementary symmetric function of the “roots”  $\lambda_j$  of  $p(T)$ .

Specializing to the case  $n = 3$ , we see that  $\pi_1(\mathbb{C}^2 - D) \cong B_3$ .  $\square$

*Remark 8.4.* The decomposition of  $S^3$  into two solid tori described in the proof restricts to a decomposition of  $S^3 - L$ . Van Kampen's Theorem then gives a presentation

$$\widetilde{\mathrm{SL}}_2(\mathbb{Z}) \cong \pi_1(\mathbb{C}^2 - \Delta) \cong \pi_1(S^3 - L) \cong \langle S, U : S^2 = U^3 \rangle$$

where  $U$  and  $S$  represent the positive generators of the circles  $a = 0$  and  $b = 0$ , respectively, in  $S^3$ . These map to the generators  $S$  and  $U$  of  $\mathrm{SL}_2(\mathbb{Z})$  given in the presentation (3) of  $\mathrm{SL}_2(\mathbb{Z})$ . The kernel of the homomorphism to  $\pi_1(\mathcal{M}_{1,1}) \cong \mathrm{SL}_2(\mathbb{Z})$  is generated by the central element  $S^4 = U^6$ .

The braid group  $B_3$  has presentation

$$B_3 = \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

An isomorphism with  $\pi_1(S^3 - L)$  is given by  $S \mapsto \sigma_1\sigma_2\sigma_1$  and  $T \mapsto \sigma_1\sigma_2$ . The center of  $B_3$  is generated by the full twist  $S^3$ ; the kernel of the homomorphism to  $\mathrm{SL}_2(\mathbb{Z})$  is generated by the square of this — a double twist.

*Exercise 70.* Show that the orbifold  $\mathcal{L}_2^*$  is isomorphic to the orbifold quotient  $C_2 \backslash \mathcal{L}^*$  of the complex manifold  $\mathcal{L}^*$  by the trivial  $C_2$ -action. Deduce that  $\mathcal{L}_2^* = C_2 \backslash (\mathbb{C}^2 - \Delta)$  where  $C_2$  acts trivially on  $\mathbb{C}^2 - \Delta$ .

*Remark 8.5.* It is not difficult to show that, as *stacks*,  $\mathcal{M}_{1,1}$  is isomorphic to the quotient of  $\mathcal{L}^*$  by the natural  $\mathbb{C}^*$ -action. Combining this with Proposition 8.2, we have a stack isomorphism

$$\mathcal{M}_{1,1} \cong \mathbb{C}^* \backslash (\mathbb{C}^2 - \Delta).$$

This is significant for two reasons. First, it shows that  $\mathcal{M}_{1,1}$  is the quotient of an affine variety by an algebraic action of a reductive group. Second, this description works over any field of characteristic not equal to 2 or 3 to give an algebraic description of the moduli stack of elliptic curves. Below we shall explain briefly how to generalize this to write  $\overline{\mathcal{M}}_{1,1}$  as a stack over  $\mathbb{Q}$ .

Note that the  $\mathbb{C}^*$ -action on  $\mathbb{C}^2 - \Delta$  factors through the homomorphism  $\mathbb{C}^* \rightarrow \mathbb{C}^*$  that takes  $u$  to  $u^2$ . This is related to the fact that the automorphism group of every point of  $\mathcal{M}_{1,1}$  contains  $C_2$ .

*Exercise 71.* Show that if  $G$  is an algebraic group and if  $X$  is a variety over an algebraically closed field  $F$  of characteristic zero on which  $G$  acts transitively, then for each  $x \in X$ , the natural mapping

$$G_x \backslash \{x\} \rightarrow G \backslash X$$

is an isomorphism of stacks, where  $G_x$  denotes the isotropy group of  $x$ . (That is, it is an equivalence of categories when viewed as a functor of

groupoids.) In particular, if  $\mathbb{G}_m$  acts on itself by the character  $u \mapsto u^d$ , then the stacks  $\mathbb{G}_m \backslash \mathbb{G}_m$  and  $\boldsymbol{\mu}_d \backslash \text{Spec}(F)$  are isomorphic, where  $\boldsymbol{\mu}_d$  acts trivially on  $\text{Spec } F$ .

8.4.  $\overline{\mathcal{M}}_{1,1}$  as a stack over  $\mathbb{Q}$ . The results in this section lead to a construction of  $\overline{\mathcal{M}}_{1,1}$  as a Deligne-Mumford stack in the category of schemes over  $\mathbb{Q}$ . The starting point is the statement that  $\mathcal{M}_{1,1}$  is isomorphic to the stack  $\mathbb{C}^* \backslash (\mathbb{C}^2 - \Delta)$  where the  $\mathbb{C}^*$ -action on  $\mathbb{C}^2$  is defined by  $\lambda \cdot (a, b) = (\lambda^{-4}a, \lambda^{-6}b)$ . One can show that if  $F$  is a field of characteristic not equal to 2 or 3, then the moduli stack  $\mathcal{M}_{1,1/F}$  of smooth elliptic curves over  $F$  is

$$\mathcal{M}_{1,1/F} \cong \mathbb{G}_{m/F} \backslash (\mathbb{A}_F^2 - \Delta)$$

where  $\mathbb{G}_{m/F}$  denotes the multiplicative group over  $F$ .

The next observation is that, over  $\mathbb{C}$ , there is a stack isomorphism

$$\overline{\mathcal{M}}_{1,1} \cong \mathbb{C}^* \backslash (\mathbb{C}^2 - \{0\}).$$

This is a Deligne-Mumford stack. That is, for each isomorphism class  $[E] \in \overline{\mathcal{M}}_{1,1}$ , there is a morphism  $T \rightarrow \mathbb{C}^2 - \{0\}$  from a smooth algebraic curve  $T$  that is transverse to each  $\mathbb{C}^*$ -orbit. Such a morphism  $T = \mathbb{C} \rightarrow \mathbb{C}^2 - \{0\}$  corresponds to the family

$$E_t : \quad y^2 = 4x(x-1)(x-t), \quad t \in \mathbb{C}$$

of cubics, each with differential  $dx/y$ . This family is considered to be an “étale neighbourhood” of  $[E_t]$  in  $\overline{\mathcal{M}}_{1,1}$  for each  $t \in \mathbb{C}$ .

This construction works equally well over any field of characteristic not equal to 2 or 3 to give a construction of the moduli stack  $\overline{\mathcal{M}}_{1,1/F}$  of stable elliptic curves in the category of schemes over  $F$ :

$$\overline{\mathcal{M}}_{1,1/F} \cong \mathbb{G}_{m/F} \backslash (\mathbb{A}_F^2 - \{0\})$$

It is a Deligne-Mumford stack.

## APPENDIX A. BACKGROUND ON RIEMANN SURFACES

This is a very brief summary of some basic facts about Riemann surfaces. Detailed expositions can be found in [4, 5, 6].

**A.1. Topology.** Riemann surfaces, like all complex manifolds, have a natural orientation.

*Exercise 72.* Denote the complex parameter on the disk  $\mathbb{D}$  by  $z$ . Write  $z = x + iy$  where  $x$  and  $y$  are real. Show that if  $\omega$  is a non-vanishing holomorphic 1-form on  $\mathbb{D}$ , then  $i\omega \wedge \overline{\omega}$  is a positive multiple of  $dx \wedge dy$ . Deduce that every Riemann surface has a natural orientation which

locally agrees with the standard orientation of the complex plane. Deduce that if  $\omega$  is a holomorphic 1-form on a compact Riemann surface, then

$$i \int_X \omega \wedge \bar{\omega} \geq 0$$

with equality if and only if  $\omega = 0$ . This is equivalent to Riemann's second bilinear relation.

*Exercise 73.* Suppose that  $X$  is a Riemann surface and that  $\omega$  is a holomorphic 1-form on  $X$ . Show that  $\omega$  is closed and therefore determines an element of

$$H^1(X; \mathbb{C}) := \text{Hom}_{\mathbb{Z}}(H_1(X), \mathbb{C})$$

by the formula

$$\omega : \gamma \mapsto \int_{\gamma} \omega.$$

Show that if  $\omega$  is an exact 1-form, then  $\omega = df$ , where  $f : X \rightarrow \mathbb{C}$  is holomorphic. Deduce that if  $X$  is compact, then the mapping

$$H^0(X, \Omega_X^1) \rightarrow H^1(X; \mathbb{C})$$

is injective, and therefore that  $H^0(X, \Omega_X^1)$  is finite dimensional.

*Exercise 74.* Suppose that  $\phi : X \rightarrow \mathbb{C}$  is a smooth function. Show that if

$$d\phi = \omega_1 + \bar{\omega}_2$$

where  $\omega_1$  and  $\omega_2$  are holomorphic 1-forms, then  $\phi$  is harmonic. Deduce that if  $X$  is compact, then  $\phi$  is constant and  $\omega_1 = \omega_2 = 0$ . Use this to show that the  $\mathbb{C}$ -linear mapping

$$(21) \quad H^0(X, \Omega_X^1) \oplus \overline{H^0(X, \Omega_X^1)} \rightarrow H^1(X; \mathbb{C}),$$

where overlining denotes complex conjugation, that takes  $(\omega_1, \bar{\omega}_2)$  to

$$\gamma \mapsto \int_{\gamma} (\omega_1 + \bar{\omega}_2)$$

is injective.

The *genus* of a compact Riemann surface<sup>31</sup>  $X$  is, by definition, the dimension of its space of holomorphic 1-forms:

$$g(X) := \dim H^0(X, \Omega_X^1).$$

The topological surface that underlies  $X$  is orientable, and is therefore determined up to diffeomorphism by its first Betti number

$$b_1(X) := \text{rank } H_1(X; \mathbb{Z}).$$

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<sup>31</sup>Our Riemann surfaces are assumed to be connected.

A basic fact in the theory of Riemann surfaces, which can be proved using Hodge theory, is that  $b_1(X) = 2g(X)$ . This equality is equivalent to the statement that (21) is an isomorphism and is a special case of the Hodge theorem for compact Kähler manifolds.

**A.2. Local structure of holomorphic mappings.** Recall from complex analysis that if  $w = f(z)$  is meromorphic and defined in a neighbourhood of  $z = 0$ , then we can write

$$f(z) = z^k g(z)$$

where  $g$  is holomorphic near  $z = 0$  and  $g(0) \neq 0$ . The integer  $k$  is called the *order of  $f$  at  $z = 0$* . We shall write  $k = \text{ord}_0 f$ .

If  $f(z)$  is holomorphic and satisfies  $f(0) = 0$ , then the order  $k$  of  $f$  at  $z = 0$  is positive. In this case, by basic complex analysis, there is a holomorphic function  $\phi(z)$  defined in a neighbourhood of the origin such that  $g(z) = \phi(z)^k$ . Then  $u = z\phi(z)$  is a local holomorphic coordinate defined in a neighbourhood of the origin and, with respect to the new coordinate  $u$ ,  $f$  is given by  $w = u^k$ .

From this it follows that every non-constant holomorphic mapping between Riemann surfaces is locally of the form  $z \mapsto z^k$  for some positive integer  $k$ . More precisely, given  $x \in X$ , there is a local holomorphic coordinate  $z$  about  $x \in X$  and a local holomorphic coordinate  $w$  about  $y \in Y$  such that  $f$  is locally given by  $w = z^k$  in terms of these coordinates. We call  $k$  the *local degree* of  $f$  at  $x$  and denote it by  $\nu_x(f)$ .

*Exercise 75.* Prove that every non-constant holomorphic mapping  $f : X \rightarrow Y$  between Riemann surfaces is open. (That is, the image of every open set is open.)

*Exercise 76.* Show that if  $X$  is compact and  $Y$  is connected, then every holomorphic mapping  $f : X \rightarrow Y$  is surjective. Deduce that every holomorphic function  $f : X \rightarrow Y$  is constant when  $Y$  is non-compact.

*Exercise 77.* Show that if  $f : X \rightarrow Y$  is a non-constant mapping between compact Riemann surfaces, then the function

$$y \mapsto \sum_{x \in f^{-1}(y)} \nu_x(f)$$

is locally constant and therefore constant if  $Y$  is connected. The common value is called the *degree* of  $f$ . Show that if  $\nu_x(f) = 1$  for all  $x \in X$ , then  $f$  is a covering map.

*Exercise 78 (Riemann-Hurwitz formula).* Suppose that  $f : X \rightarrow Y$  is a non-constant mapping of degree  $d$  between compact Riemann surfaces.

For each  $x \in X$ , define

$$b = \sum_{x \in X} (\nu_x(f) - 1).$$

(This is well define as  $\nu_x(f) = 1$  for all but a finite number of  $x \in X$ .) Show that

$$\chi(Y) = d\chi(X) - b.$$

(Hint: Triangulate  $Y$  so that each critical value is a vertex. Lift this triangulation to  $X$ . Compute Euler characteristics.) Deduce that  $b$  is even. Show that if  $Y = \mathbb{P}^1$ , then the genus of  $X$  is

$$g(X) = b/2 + 1 - d.$$

In particular, if  $g = 1$  and  $d = 2$ , then  $b = 4$ .

*Exercise 79.* Show that a 1-1 holomorphic mapping  $f : X \rightarrow Y$  between compact Riemann surfaces is a biholomorphism. (That is,  $f$  has a holomorphic inverse  $g : Y \rightarrow X$ .)

**A.3. Divisors and line bundles.** A divisor  $D$  on a Riemann surface  $X$  is a locally finite formal linear combination

$$\sum_{x \in X} n_x [x]$$

of points of  $X$ . We say that  $D$  is *effective* and write  $D \geq 0$  when each  $n_x \geq 0$ . When  $X$  is compact, the sum is finite. Thus one can define the *degree*  $\deg D$  of a divisor  $D$  on a compact Riemann surface to be the sum of its coefficients:

$$\deg D = \sum_{x \in X} n_x.$$

To a holomorphic line bundle  $L \rightarrow X$  with a meromorphic section  $s$ , we may associate the divisor

$$\operatorname{div} s := \sum_{x \in X} \operatorname{ord}_x s.$$

The section  $s$  is holomorphic precisely when  $\operatorname{div} s \geq 0$ . Every other meromorphic section of  $L$  is of the form  $fs$ , where  $f$  is a meromorphic function. The space of holomorphic sections  $H^0(X, L)$  of  $L$  thus equals

$$L(D) := \{\text{meromorphic functions } f \text{ on } X: \operatorname{div} f + D \geq 0\}.$$

The corresponding sheaf is denoted by  $\mathcal{O}_X(D)$ . Its space of sections  $\mathcal{O}_X(D)(U)$  over the open subset  $U$  of  $X$  is defined by

$$\mathcal{O}_X(D)(U) = \{\text{meromorphic functions } f \text{ on } U: \operatorname{div} f + D|_U \geq 0\}$$



where the restriction  $D|_U$  of  $D$  to  $U$  is defined by

$$D|_U := \sum_{x \in U} n_x [x].$$

Evidently,  $L(D) = H^0(X, \mathcal{O}_X(D))$ .

Two divisors  $D$  and  $D'$  on a Riemann surface are *linearly equivalent* if there is a non-zero meromorphic function  $f$  on  $X$  such that

$$D' = D + \operatorname{div} f.$$

A divisor of the form  $\operatorname{div} f$  is said to be *principal*. Every principal divisor on a compact Riemann surface has degree 0. Thus linearly equivalent divisors have the same degree.

*Exercise 80.* Suppose that  $f : X \rightarrow \mathbb{P}^1$  is a non-constant meromorphic function. To each  $a \in \mathbb{P}^1$ , define

$$D_a = \sum_{x \in f^{-1}(a)} \nu_x(f)[x].$$

Show that if  $a \neq \infty$ , then

$$D_a - D_\infty = \operatorname{div}(f - a)$$

Deduce that any two fibers  $D_a$  of  $f$  are linearly equivalent. Note that the degree of each  $D_a$  equals the degree of  $f : X \rightarrow \mathbb{P}^1$ .

It is a fact that every holomorphic line bundle on a Riemann surface has a non-zero meromorphic section.<sup>32</sup>

*Exercise 81.* Show that for every divisor  $D$  on a Riemann surface  $X$  there is a holomorphic line bundle  $\mathcal{L} \rightarrow X$  with a meromorphic section  $s$  whose divisor is  $D$ . Show that the map

$$\mathcal{O}_X(D)(U) \rightarrow H^0(U, \mathcal{L})$$

that takes  $f$  to  $fs$  induces an isomorphism between  $\mathcal{O}_X(D)$  and the sheaf of holomorphic sections of  $\mathcal{L}$ . In particular,  $L(D)$  is isomorphic to  $H^0(X, \mathcal{L})$ .

Suppose that  $\mathcal{L}$  is a line bundle over  $X$  and that  $D$  is a divisor on  $X$ . Define  $\mathcal{L}(D) = \mathcal{L} \otimes \mathcal{O}_X(D)$ .

*Exercise 82.* Show that the group (under tensor product) of holomorphic line bundles on  $X$  is isomorphic to the group of divisors on  $X$  modulo principal divisors.

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<sup>32</sup>If  $X$  is not compact, then every holomorphic line bundle has a holomorphic section.

**A.4. Riemann-Roch formula.** The canonical divisor class  $K_X$  of a compact Riemann surface  $X$  is the divisor class associated to it holomorphic cotangent bundle. In concrete terms, the canonical class is the divisor class of a non-zero meromorphic 1-form on  $X$ .

For a divisor  $D$  on  $X$ , define

$$\ell(D) := \dim L(D) \in \mathbb{N} \cup \{\infty\}.$$

*Exercise 83.* Note that  $g(X) = \ell(K_X)$ . Suppose that  $X$  is a compact Riemann surface of genus  $g$  and that  $D$  is a divisor on  $X$ . Show that

- (i) if  $\deg D = 0$ , then  $\ell(D) = 0$  or  $1$  and that  $\ell(D) = 1$  if and only if  $D$  is principal;
- (ii) if  $\deg D < 0$ , then  $\ell(D) = 0$ ;
- (iii) if  $P \in X$ , then  $\ell(D) \leq \ell(D + P) \leq 1 + \ell(D)$ ;
- (iv) if  $D$  is effective, then  $\ell(D) \leq 1 + \deg D$ ;
- (v)  $\ell(D)$  is finite for all  $D$ .

**Theorem A.1** (Riemann-Roch formula). *If  $X$  is a compact Riemann surface of genus  $g$  and  $D$  is a divisor on  $X$ , then*

$$\ell(D) - \ell(K_X - D) = \deg D + 1 - g.$$

*Exercise 84.* Show that  $\deg K_X = 2g - 2$ .

#### A.5. Moduli of genus 0 Riemann surfaces.

*Exercise 85.* Show that if  $X$  is a compact Riemann surface of genus 0, then there is a degree 1 holomorphic mapping  $f : X \rightarrow \mathbb{P}^1$ . Deduce that  $f$  is a biholomorphism.

*Exercise 86.* Show that the group of biholomorphisms of  $\mathbb{P}^1$  is isomorphic to  $\mathrm{PSL}_2(\mathbb{C})$ , where  $\mathrm{SL}_2(\mathbb{C})$  acts on  $\mathbb{P}^1$  via fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

Show that  $\mathrm{PSL}_2(\mathbb{C})$  acts 3-transitively on  $\mathbb{P}^1$ . That is, given two sets of distinct points  $\{x_1, x_2, x_3\}$  and  $\{y_1, y_2, y_3\}$ , there exists  $\phi \in \mathrm{PSL}_2(\mathbb{C})$  such that  $\phi(x_j) = y_j$ ,  $j = 1, 2, 3$ . Show that  $\phi$  is unique.

*Exercise 87.* Suppose that  $x_1, \dots, x_n \in \mathbb{P}^1$  are distinct. Define

$$\mathrm{Aut}(\mathbb{P}^1, \{x_1, \dots, x_n\}) = \{\phi \in \mathrm{Aut} \mathbb{P}^1 : \phi(x_j) = x_j, j = 1, \dots, n\}.$$

Show that this group is finite (resp. trivial) if and only if  $n \geq 3$ .

**A.6. The action of  $\mathrm{SL}_2(\mathbb{Z})$  on the upper half plane.** The group  $\mathrm{SL}_2(\mathbb{Z})$  acts on the upper half plane  $\mathfrak{h}$  by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

The boundary of the upper half plane is  $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ . This is a circle on the Riemann sphere which forms the boundary of  $\mathfrak{h}$ . Let  $\bar{\mathfrak{h}}$  be the closure of  $\mathfrak{h}$  in the Riemann sphere  $\mathbb{P}^1$ ; it is the union of  $\mathfrak{h}$  and  $\mathbb{P}^1(\mathbb{R})$ . Recall that every non-trivial element of  $\mathrm{PSL}_2(\mathbb{C})$  has at most two fixed points in  $\mathbb{P}^1$ . Note that the fixed points of elements of  $\mathrm{PSL}_2(\mathbb{R})$  are real or occur in complex conjugate pairs. Consequently, each element of  $\mathrm{PSL}_2(\mathbb{R})$  has at most one fixed point in  $\mathfrak{h}$ .

*Exercise 88.* Suppose that  $A \in \mathrm{SL}_2(\mathbb{Z})$  is not a scalar matrix. Show that  $A$  has exactly

- (i) one fixed point in  $\mathfrak{h}$  if and only if  $|\mathrm{tr} A| < 2$ ;
- (ii) one fixed point in  $\mathbb{P}^1(\mathbb{R})$  if and only if  $|\mathrm{tr} A| = 2$ ;
- (iii) two fixed points in  $\mathbb{P}^1(\mathbb{R})$  if and only if  $|\mathrm{tr} A| > 2$ .

Show that  $T \in \mathrm{SL}_2(\mathbb{R})$  has finite order if and only if  $A$  has a fixed point in  $\mathfrak{h}$ .

Suppose that  $m \in \mathbb{N}$ . The *level  $m$  congruence subgroup* of  $\mathrm{SL}_2(\mathbb{Z})$  is defined by

$$\mathrm{SL}_2(\mathbb{Z})[m] := \{A \in \mathrm{SL}_2(\mathbb{Z}) : A \equiv \mathrm{id} \pmod{m}\}.$$

*Exercise 89.* Show that if  $m > 0$ , then  $\mathrm{SL}_2(\mathbb{Z})[m]$  has finite index in  $\mathrm{SL}_2(\mathbb{Z})$  and that  $\mathrm{SL}_2(\mathbb{Z})[m]$  is torsion free when  $m \geq 3$ . Show that the torsion subgroup of  $\mathrm{SL}_2(\mathbb{Z})[2]$  is its center  $C_2 = \{\pm I\}$ . Use the fact that  $\mathrm{SL}_2(\mathbb{Z})[m] \backslash \mathfrak{h}$  is a non-compact Riemann surface to prove that  $\mathrm{SL}_2(\mathbb{Z})[m]$  is a free group for all  $m \geq 3$ .

**A.7. Quotients by discrete group actions.** The action of a discrete group  $\Gamma$  on a topological space  $X$  is said to be *properly discontinuous* if each  $x \in X$  has a neighbourhood  $U$  such that if  $\gamma \in \Gamma$ , then  $\gamma U \cap U \neq \emptyset$  implies that  $\gamma x = x$ . An action of  $\Gamma$  on  $X$  is *free* or *fixed point free* if  $x \in X$  and  $\gamma \in \Gamma$ , then  $\gamma x = x$  implies that  $\gamma$  is the identity. If the action of  $\Gamma$  on  $X$  is both properly discontinuous and free, then  $X \rightarrow \Gamma \backslash X$  is a covering projection. (Exercise: prove this.)

*Exercise 90.* Prove that the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathfrak{h}$  is properly discontinuous and that  $\mathrm{SL}_2(\mathbb{Z})[m]$  acts fixed point freely (and properly discontinuously) on  $\mathfrak{h}$  when  $m \geq 3$ .

*Exercise 91.* Suppose that  $p : X \rightarrow Y$  is a Galois (i.e., normal or regular) covering map with Galois group (i.e., group of deck transformations)  $\Gamma$ . Suppose that  $X$  is a Riemann surface and that  $\Gamma$  acts on  $X$  as a group of biholomorphisms. Show that  $Y$  has a unique complex structure such that  $p$  is holomorphic. Deduce that if  $X$  is a Riemann surface and  $\Gamma$  is a subgroup of  $\text{Aut } X$  that acts properly discontinuously and fixed point freely on  $X$ , then  $\Gamma \backslash X$  has a unique Riemann surface structure such that the covering projection  $X \rightarrow \Gamma \backslash X$  is holomorphic.

Coverings of Riemann surfaces with punctures can be extended across the punctures. This is a local problem.

*Exercise 92.* Show that all finite coverings of the punctured disk  $\mathbb{D}^*$  are isomorphic to  $p_n : \mathbb{D}^* \rightarrow \mathbb{D}^*$  where  $p_n(z) = z^n$ . Deduce that all such coverings  $U \rightarrow \mathbb{D}^*$  can be completed to a proper holomorphic map  $X \rightarrow \mathbb{D}$  where  $X$  is a Riemann surface containing  $U$  as an open dense subset.

*Exercise 93.* Suppose that  $Y$  is a compact Riemann surface and that  $F$  is a finite subset of  $Y$ . Set  $Y' = Y - F$ . Show that if  $f : X' \rightarrow Y'$  is a finite, unramified covering, there exists a compact Riemann surface  $X$ , a finite subset  $F_X$  of  $X$  and a holomorphic mapping  $\tilde{f} : X \rightarrow Y$  such that  $X' = X - F_X$  and the restriction of  $\tilde{f}$  to  $X'$  is  $f$ .

*Exercise 94.* Suppose that  $X$  is a Riemann surface and that  $P \in X$ . Let

$$\text{Aut}(X, P) = \{\phi \in \text{Aut } X : \phi(P) = P\}.$$

Show that taking  $\phi$  to its derivative at  $P$  defines a homomorphism  $\rho : \text{Aut}(X, P) \rightarrow \mathbb{C}^*$ . Denote its kernel by  $\text{Aut}_0(X, P)$ .

- (i) Show that  $\text{Aut}_0(X, P)$  is torsion free. (Hint: use power series.)
- (ii) Deduce that if  $\Gamma$  is a finite subgroup of  $\text{Aut}(X, P)$  of order  $d$ , then the restriction of  $\rho$  to  $\Gamma$  is injective and that  $\rho(\Gamma)$  is the group  $\mu_d$  of  $d$ th roots of unity.
- (iii) With  $\Gamma$  as above, show that there is a holomorphic coordinate  $z$  in  $X$ , centered at  $P$ , such that (for  $Q$  in a neighbourhood of  $P$ ) the action of  $\Gamma$  is given by  $\gamma : z \mapsto \rho(\gamma)z$ . More precisely,

$$z(\gamma(Q)) = \rho(\gamma)z(Q).$$

Hint: Let  $w$  be any holomorphic coordinate centered at  $P$  and consider how  $\Gamma$  acts on a  $d$ th root of  $\prod_{\gamma \in \Gamma} \gamma^* w$ .

- (iv) Show that  $\Gamma \backslash X$  has a natural Riemann surface structure such that the projection  $X \rightarrow \Gamma \backslash X$  is holomorphic. (Hint: Localize about each fixed point of  $\Gamma$ .)

Recall that the action of a group  $\Gamma$  on a set  $X$  is *virtually free* if  $\Gamma$  has a finite index subgroup  $\Gamma'$  such that the restriction of the action to  $\Gamma'$  is free.

*Exercise 95.* Show that if the discrete group  $\Gamma$  acts properly discontinuously and virtually freely on a Riemann surface  $X$ , then  $\Gamma \backslash X$  has a unique Riemann surface structure such that the projection  $X \rightarrow \Gamma \backslash X$  is holomorphic.

*Exercise 96.* Show that the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathfrak{h}$  is virtually free. Deduce that  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$  has a unique Riemann surface structure such that the projection  $\mathfrak{h} \rightarrow \mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$  is holomorphic.

## APPENDIX B. A VERY BRIEF INTRODUCTION TO STACKS

This appendix is a very brief and informal introduction to stacks. The book [9] by Laumon and Moret-Bailly is a standard reference. There are also the notes [2] by Fulton et al. Recall that a *groupoid* is a category in which every morphism is an isomorphism. A starting observation, explained in Remark 3.2, is that an orbifold may be viewed as a groupoid in the category of (say) topological spaces.

Suppose that  $\mathcal{C}$  is a category in which fibered products always exist, such as the category of complex manifolds, the category of varieties over a field, or the category of schemes over a fixed base. A *stack*  $\mathcal{X}$  in  $\mathcal{C}$  is a groupoid in  $\mathcal{C}$ . A groupoid in  $\mathcal{C}$  consists of two objects  $U$  and  $R$  of  $\mathcal{C}$  together with five morphisms  $s, t, e, m, i$  called the source, target, identity, multiplication, and inverse:

$$R \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} U, \quad U \xrightarrow{e} R, \quad R_t \times_s R \xrightarrow{m} R, \quad R \xrightarrow{i} R.$$

These satisfy natural axioms which can be worked out from the example in Remark 3.2, where  $U = X$ ,  $R = \Gamma \times X$ , and

$$(22) \quad s(\gamma, x) = x, \quad t(\gamma, x) = \gamma x, \quad e(x) = (\mathrm{id}, x), \quad i(\gamma, x) = (\gamma^{-1}, \gamma x)$$

and

$$(23) \quad m((\mu, \gamma x), (\gamma, x)) = (\mu\gamma, x).$$

The structure  $(U, R, s, t, e, m, i)$  is called an *atlas* on  $\mathcal{X}$ . It is the analogue of an open covering of a topological space. One can define equivalence classes of atlases and consider a stack to be an equivalence class of atlases, just as one can consider a manifold to be an equivalence class of atlases. Roughly speaking, an equivalence of atlases is induced by an equivalence of categories that induces the identity on isomorphism classes and which has “good descent properties”. Morphisms of stacks in  $\mathcal{C}$  are induced by functors in  $\mathcal{C}$  from one atlas to another.

*Exercise 97.* Show that if  $m \geq 3$ , then the atlases of  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$  and  $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) \backslash \mathcal{M}_{1,1}[m]$  are equivalent, where  $\mathcal{M}_{1,1}[m]$  is the Riemann surface defined in Section 4.2.

Basic orbifolds are stacks in the category of topological spaces (or smooth manifolds, complex manifolds, etc.) In particular, we may regard  $\mathcal{M}_{1,1}$  as a stack in the category of Riemann surfaces.

The rules (22) and (23) above can be used to define the quotient  $\Gamma \backslash X$  of an object  $X$  of a category  $\mathcal{C}$  by a group object  $\Gamma$  of  $\mathcal{C}$ . A typical example is taking the quotient of a variety (or scheme)  $X$  by the action of an algebraic group  $\Gamma$ .

**B.1. The stack  $\overline{\mathcal{M}}_{1,1}$ .** Crudely speaking,  $\overline{\mathcal{M}}_{1,1}$  is the stack in the category of Riemann surfaces that is obtained by attaching the disk  $\mathbb{D}$  to  $\mathcal{M}_{1,1}$  along the morphism  $\mathbb{D}^* \rightarrow \mathcal{M}_{1,1}$  constructed in Exercise 27:

$$\overline{\mathcal{M}}_{1,1} = \mathcal{M}_{1,1} \cup_{\mathbb{D}^*} \mathbb{D}.$$

The coordinate in the disk will be denoted by  $q$ . It is related to the coordinate  $\tau$  of  $\mathfrak{h}$  by  $q = \exp(2\pi i\tau)$ .

Set  $R = e^{-2\pi}$  and  $\mathfrak{h}_a = \{\tau : \mathrm{Im} \tau > a\}$ . Denote the open  $q$ -disk of radius  $R$  by  $\mathbb{D}_R$ . The mapping  $q = \exp(2\pi i\tau)$  defines a covering  $\mathfrak{h}_1 \rightarrow \mathbb{D}_R^*$ .

The analytic stack  $\overline{\mathcal{M}}_{1,1}$  is defined by the atlas where

$$U = \mathfrak{h} \dot{\cup} \mathbb{D}_R$$

and

$$R = \mathrm{Iso}(\mathfrak{h}, \mathfrak{h}) \dot{\cup} \mathrm{Iso}(\mathbb{D}_R, \mathfrak{h}) \dot{\cup} \mathrm{Iso}(\mathfrak{h}, \mathbb{D}_R) \dot{\cup} \mathrm{Iso}(\mathbb{D}_R, \mathbb{D}_R)$$

where these and the source and target maps are defined by

$$\begin{aligned} \mathrm{Iso}(\mathfrak{h}, \mathfrak{h}) &= \mathrm{SL}_2(\mathbb{Z}) \times \mathfrak{h}, & (s, t) &: (\gamma, \tau) \mapsto (\tau, \gamma\tau) \\ \mathrm{Iso}(\mathbb{D}, \mathbb{D}) &= C_2 \times \mathbb{D}_R, & (s, t) &: q \mapsto q \\ \mathrm{Iso}(\mathfrak{h}, \mathbb{D}_R) &= C_2 \times \mathbb{Z} \times \mathfrak{h}_1, & (s, t) &: (n, \tau) \mapsto (\tau, q(\tau)) \\ \mathrm{Iso}(\mathbb{D}_R, \mathfrak{h}) &= C_2 \times \mathbb{Z} \times \mathfrak{h}_1, & (s, t) &: (n, \tau) \mapsto (q(\tau), \tau + n). \end{aligned}$$

The identity maps

$e : \mathbb{D}_R \rightarrow \mathrm{Iso}(\mathbb{D}_R, \mathbb{D}_R) = C_2 \times \mathbb{D}_R$  and  $e : \mathfrak{h} \rightarrow \mathrm{Iso}(\mathfrak{h}, \mathfrak{h}) = \mathrm{SL}_2(\mathbb{Z}) \times \mathfrak{h}$  are  $q \mapsto (\mathrm{id}, q)$  and  $\tau \mapsto (\mathrm{id}, \tau)$ , respectively.

*Exercise 98.* Define the composition mappings

$$m : \mathrm{Iso}(Y, Z) \times \mathrm{Iso}(X, Y) \rightarrow \mathrm{Iso}(X, Z)$$

where  $X, Y, Z \in \{\mathfrak{h}, \mathbb{D}\}$ .

Two other constructions of  $\overline{\mathcal{M}}_{1,1}$  are sketched in these notes. The construction given in Section 4.2 is as the stack quotient of the compact Riemann surface (algebraic curve)  $\overline{\mathcal{M}}_{1,1}[m]$ , where  $m \geq 3$ . The construction given in Section 8.4 is as a quotient  $\mathbb{C}^* \backslash (\mathbb{C}^2 - \{0\})$ . Each construction has advantages and disadvantages: the construction above makes clear the connection with modular forms, but is transcendental; the other two constructions are as quotients of an algebraic variety by an algebraic group and lie within algebraic geometry; the third works over any field of characteristic not equal to 2 or 3.

*Exercise 99.* Show that if  $m \geq 3$ , then  $\overline{\mathcal{M}}_{1,1}$  is isomorphic to the stack  $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) \backslash \overline{\mathcal{M}}_{1,1}[m]$ .

*Exercise 100.* Construct stack morphisms  $\mathcal{M}_{1,1} \rightarrow \overline{\mathcal{M}}_{1,1}$  and  $\mathbb{D} \rightarrow \overline{\mathcal{M}}_{1,1}$  such that the diagram

$$\begin{array}{ccc} \mathbb{D}^* & \longrightarrow & \mathbb{D} \\ \downarrow & & \downarrow \\ \mathcal{M}_{1,1} & \longrightarrow & \overline{\mathcal{M}}_{1,1} \end{array}$$

where the left hand vertical mapping is the one constructed in Exercise 27.

**B.2. Bundles over stacks.** A vector bundle  $\mathcal{V}$  over a stack  $\mathcal{X}$  in  $\mathcal{C}$  with atlas

$$R \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} U$$

consists of a vector bundle  $V$  over  $U$  in  $\mathcal{C}$  together with isomorphisms (the “transition functions”)

$$(24) \quad s^*V \rightarrow t^*V$$

of vector bundles over  $R$  whose pullback along the identity section  $e : U \rightarrow R$  is the identity. This can be thought of as a family of linear isomorphisms  $f_* : V_{s(f)} \rightarrow V_{t(f)}$  between the fibers of  $V$  over  $s(f)$  and  $t(f)$ , indexed by the  $f \in R$ . It is the identity when  $f$  is. The map (24) is also required to be compatible with multiplication in the sense that if  $x, y, z \in U$  and  $f, g \in R$  such that

$$x \xrightarrow{f} y \xrightarrow{g} z$$

are morphisms, then the diagram

$$\begin{array}{ccc} V_x & \xrightarrow{f_*} & V_y \\ & \searrow m(g,f)_* & \downarrow g_* \\ & & V_z \end{array}$$

commutes.

*Exercise 101.* Show that the orbifold line bundle  $\overline{\mathcal{L}}_k \rightarrow \overline{\mathcal{M}}_{1,1}$  is a line bundle when  $\overline{\mathcal{M}}_{1,1}$  is viewed as a stack.

Bundles with other structure groups can be defined similarly.

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