MAT 614: Enumerative Geometry
Fall 2007

Course Information

Course Instructor

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Course Schedule

The class is currently scheduled to be held MW 2:20-3:40pm, but I'd be happy to discuss a different schedule that may be more convenient for everyone. If the schedule stays as it is, I am tentatively planning to teach replacement classes on Fridays 2:20-3:40pm during the weeks that I am out of town either on Monday or Wednesday. The first class will be held on Wednesday, September 5, 2:20-3:40pm, in Earth&Space 177, as scheduled.

Course Website

All updates, including schedule and references, will be posted on the course website,

`http://math.sunysb.edu/~azinger/mat614`.

Please visit this website regularly.

Prerequisites

MAT 530/531 are required. If you are solid on Chapter 0 of Griffiths&Harris and familiar with chern classes, you should be fine. Otherwise, please contact me before registering for the course.

Grading

Your grade will be based on class participation, in all possible forms. While there will be no formal homework assignments, you should write up solutions to some of the exercises that will appear in the lecture. The exercises you choose should vary in flavor and difficulty and should be handed in within two weeks of the first time they are stated. However, please turn in solutions to *no more* than one exercise in any given week (even if the exercises are from different weeks); so please do not delay until the end of the term to hand in a pile of solutions. The exercises will be an essential part of this course; so please try to work out as many of them as possible, in addition to the ones you hand in. I'd suggest that you discuss a number of problems together and then choose distinct individual problems to write up.

Course Description

Contrary to what the official title might suggest, this is not a course in algebraic geometry and no background in algebraic geometry is needed. The course will instead be a combination of algebraic topology and complex geometry, with a touch of combinatorics.

As the name might suggest, enumerative geometry is concerned with counting various geometric objects. In this course we will deal only with counting complex curves (i.e. Riemann surfaces, but possibly with various singularities) in complex manifolds, primarily the complex projective space $\mathbb{P}^n$ and hypersurfaces in $\mathbb{P}^n$ of low degree. Since there are often infinitely many such curves, one imposes enough conditions to expect a finite number, such as passing through some constraints, order of tangencies to constraints, and/or singularities of the curve. For example, there is exactly 1 line through any 2 distinct points in $\mathbb{C}^n$ (or $\mathbb{P}^n$) and there is exactly 1 line through 2 lines and 1 point in general position in $\mathbb{C}^3$ (or $\mathbb{P}^3$). While enumerative geometry is a classical subject, dating back at least to the 19th century, its recent interactions with symplectic topology and string theory have led to a variety of new developments and much progress in the subject.

Algebraic topology will play a central role in this course as counts of curves can often be expressed as the number of zeros of a vector bundle section. The following classical, but nontrivial, topological fact will be either used directly in or be the motivation (but not always apparent) behind much done in the class.

Theorem: *If M is a compact oriented manifold, $V \longrightarrow M$ is an oriented vector bundle, and s is a generic section of V, then $s^{-1}(0)$ is a smooth oriented submanifold of M and its Poincare dual in M is $e(V)$.*

Following an introduction and overview on the first day, the course will tentatively have four, mostly independent, parts. Most of the time will be spent on Parts II and IV below.

Part I: *Grassmannians, Schubert Calculus, Pseudocycles, and Counting Low-Degree Curves in Projective Hypersurfaces*
Counts of lines in $\mathbb{P}^n$, and more generally in projective hypersurfaces, can be expressed as intersections of homology classes in the Grassmannian of 2-planes in $\mathbb{C}^{n+1}$, $G(2, n+1)$. These homology classes are called *Schubert cycles*, and Schubert Calculus describes their intersections, or equivalently the cup products of their Poincare Duals. Schubert cycles are generally not smooth manifolds, but are complex varieties that may have singularities. These are special cases of pseudocycles, all of which represent homology classes. Schubert calculus, along with some observations from algebraic geometry, can also be used to count other low-degree curves in projective hypersurfaces.

This part of the course is mostly foundational, though Schubert Calculus and related topics are still subject of much research, which often intertwines combinatorics and geometry. Some questions we will encounter are: *How many lines pass through* 4 *general lines in* 3*-space? How many lines lie on a cubic surface in $\mathbb{P}^3$? More generally, how many lines lie in a hypersurface X (or even complete intersection) in $\mathbb{P}^n$ and pass through a points, b lines, etc.? How many degree-two curves pass through* 4 *points in $\mathbb{P}^3$ that do not lie in a* 2*-plane? How many degree-two curves pass through* 3 *points and* 2 *lines in general position in $\mathbb{P}^3$?*

References:

E. Arbarello, M. Cornalba, P Griffiths, and J. Harris, Geometry of Algebraic Curves, Chapter III.
P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Chapter 1, Section 5.
S. Katz, *Enumerative Geometry and String Theory*, Chapters 1,2,4-7,9 (intended for advanced undergraduates; you'll likely find this quite easy to follow).
S. Katz, *On the Finiteness of Rational Curves on Quintic Threefolds.*
R. Vakil, slides for 2005 MAA Invited Address R. Vakil, *A Geometric Littlewood-Richardson Rule.*
A. Zinger, *Pseudocycles and Integral Homology.*

Part II: *Local Excess Intersection and Counting Curves in $\mathbb{P}^2$*
Counts of plane curves can often be written as the (signed) cardinality of the zero set of a transverse section s of a (complex) vector bundle V over a complex manifold M. If M is compact, the desired number is then the euler class of V evaluated on the fundamental class of M, $\langle e(V), M\rangle$. However, in most cases M is not compact, but does admit a nice compactification $\bar{M}$ along with extensions of V, still denoted by V, and $\bar{s}$ of s. The latter may have additional zeros, which may not be transverse, and $\bar{s}^{-1}(0)$ may be of positive dimension. In such a case, the cardinality of $s^{-1}(0)\subset\bar{s}^{-1}(0)$ is $\langle e(V), \bar{M}\rangle$ *minus* the contribution from $\bar{s}^{-1}(0)-s^{-1}(0)$, $\mathtt{C}_{\bar{s}^{-1}(0)-s^{-1}(0)}(\bar{s})$. In many cases, the latter admits a stratification by smooth manifolds $\bigsqcup_i \mathcal{Z}_i$ such that

$$\mathtt{C}_{\bar{s}^{-1}(0)-s^{-1}(0)}(\bar{s}) = \sum_i \mathtt{C}_{\mathcal{Z}_i}(\bar{s}), \qquad \text{with} \qquad \mathtt{C}_{\mathcal{Z}_i}(\bar{s}) = m_i N(\alpha_i),$$

where $m_i\in\mathbb{Z}$ and $N(\alpha_i)$ is the number of zeros of an affine bundle map between two vector bundles over $\mathcal{Z}_i$, with the linear term α_i. The number $N(\alpha_i)$ can itself be written as the number of zeros of a vector bundle section s_i over a (possibly non-compact) manifold, but with the rank of the bundle reduced by at least 1. Thus, this process eventually terminates, giving a tree of chern classes that sum up to the cardinality of $s^{-1}(0)$.

While this approach may be rather laborious in specific applications, it is quite straightforward and has a wide range of applications, including beyond counting curves in $\mathbb{P}^2$. Most importantly, it works in many cases, including some with no known alternative.

Some questions we will encounter in this part are: *How many genus-zero cubic curves pass through 8 points in general position in $\mathbb{P}^2$? How many cuspidal cubics pass through 7 general points in $\mathbb{P}^2$? How many genus-zero quartics pass through general 11 points in $\mathbb{P}^2$?*

References:

S. Katz, *Enumerative Geometry and String Theory*, Chapter 8.
A. Zinger, *Counting Plane Rational Curves: Old and New Approaches*, Sections 1,2,3,A.
A. Zinger, *Enumeration of Genus-Two Curves with a Fixed Complex Structure in $\mathbb{P}^2$ and $\mathbb{P}^3$*, Section 3.

Part III: *Rudiments of Gromov-Witten Theory and Counting Genus-Zero Curves in* $\mathbb{P}^n$
The modern theory of GW-invariants, originating in symplectic topology and string theory, turned out to be a powerful tool in enumerative geometry as well. The discussion in this section will focus on genus-zero GW-invariants of projective spaces and corresponding moduli spaces of stable holomorphic maps. The former leads to a simple solution of the first question below (originating in the 19th century, but not solved until the 1990s). The latter, combined with the method of Part II, can be used to solve many other enumerative problems, especially those involving genus-zero and genus-one curves in projective spaces.

Some questions we will encounter in this part are: *How many genus-zero degree-d curves pass through* $3d-1$ *general points in* $\mathbb{P}^2$*? How many genus-zero cuspidal degree-d curves pass through* $3d-2$ *general points in* $\mathbb{P}^2$*?*

References:
A. Zinger, *Counting Plane Rational Curves: Old and New Approaches*, Section 4.
K. Hori, et. al., *Mirror Symmetry*, Chapters 21-26.
R. Vakil, *The Enumerative Geometry of Rational and Elliptic Curves in Projective Space.*
S. Katz, *Enumerative Geometry and String Theory*, Chapter 3.
A. Zinger, *Pseudocycles and Integral Homology.*
A. Zinger, *Counting Rational Curves of Arbitrary Shape in Projective Spaces.*

Part IV: *Genus-Zero Gromov-Witten Invariants of Projective Hypersurfaces and Mirror Symmetry*
There is a duality principle in string theory called mirror symmetry. It states that the *A Model* potential (something involving GW-invariants of a Kahler manifold M) is related to the *B Model* potential (something involving the moduli space of complex structures on the "dual" of M). This principle led to stunning predictions for genus-zero GW-invariants of a quintic threefold and later for higher-genus invariants of other manifolds. The former were verified about 10 years ago using the classical Atiyah-Bott localization theorem; many similar genus-zero statements have been proved as well. On the other hand, very few positive-genus predictions have been mathematically confirmed. This part of the course will generally follow the reference *Mirror Symmetry* below in confirming the original mirror symmetry prediction. The localization theorem will be proved as well.

Some questions we will encounter in this part are: *Find the euler characteristics of Grassmannians and other characteristic numbers of manifolds admitting groups actions. Find all genus-zero GW-invariants of a quintic threefold and more generally of any complete intersection?*

References:
P. Candelas, et.al., A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory.
S. Katz, Enumerative Geometry and String Theory, Chapters 10-14.
M. Atiyah and R. Bott, The Moment Map and Equivariant Cohomology, Sections 1-3.
K. Hori, et. al., Mirror Symmetry, Chapters 27, 29,30.
I will try to prepare additional notes on this.

If time permits, positive-genus GW-invariants, their relations with enumerative geometry, and/or integrality predictions for GW-invariants (these are usually rational numbers) will be discussed.