MAT 566: Differential Topology
Fall 2006

Notes on the Splitting Principle

The purpose of these notes is to formally state the Splitting Principle and comment on its justification. There are two possible proofs, but one of them is based on a fact that we have not proved. Therefore, one should consider the other approach as the proof.

Throughout these notes, all vector bundles should be assumed to be complex, all cohomology rings are with arbitrary coefficients, $\mathbb{P}^\infty$ denotes the infinite complex projective space $\mathbb{C}P^\infty$, and $\mathbb{G}_n$ is the infinite complex Grassmannian $\mathrm{Gr}_n\mathbb{C}^\infty$. Alternatively, all vector bundles should be assumed to be real, all cohomology rings are with $\mathbb{Z}_2$-coefficients, $\mathbb{P}^\infty$ denotes the infinite real projective space $\mathbb{R}P^\infty$, and $\mathbb{G}_n$ is the infinite real Grassmannian $\mathrm{Gr}_n\mathbb{R}^\infty$. A base space B will be assumed to be paracompact. Let $H^{\Pi}(B)$ be the product (rather than just sum) of all cohomology groups of B. So, an element of $H^{\Pi}(B)$ is a possibly infinite series

$$a_0 + a_1 + \ldots, \qquad \text{where} \qquad a_i \in H^i(B).$$

Basic Splitting Principle: Suppose for every vector bundle $E \longrightarrow B$ of rank k we have assigned classes $p(E), q(E) \in H^{\Pi}(B)$ that are natural with respect to continuous maps. In other words,

$$p(f^*E) = f^*p(E) \in H^{\Pi}(B') \qquad \text{and} \qquad q(f^*E) = f^*q(E) \in H^{\Pi}(B')$$

for every continuous map $f : B' \longrightarrow B$ and vector bundle $E \longrightarrow B$. If $p(E) = q(E)$ for every *split* vector bundle E (over every base B), then $p(E) = q(E)$ for every vector bundle E.

General Splitting Principle: Suppose for every r-tuple of vector bundles $(E_1, \ldots, E_r)$ of ranks $(k_1, \ldots, k_r)$ over every base B we have assigned classes

$$p(E_1, \ldots, E_r),\ q(E_1, \ldots, E_r) \in H^{\Pi}(B)$$

that are natural with respect to continuous maps. In other words,

$$\begin{aligned} p\big(f^*E_1, \ldots, f^*E_r\big) &= f^*p(E_1, \ldots, E_r) \in H^{\Pi}(B') \qquad \text{and} \\ q\big(f^*E_1, \ldots, f^*E_r\big) &= f^*q(E_1, \ldots, E_r) \in H^{\Pi}(B') \end{aligned}$$

for every continuous map $f : B' \longrightarrow B$ and r-tuple of vector bundles $E_1, \ldots, E_r \longrightarrow B$ (of ranks $k_1, \ldots, k_r$). If

$$p(E_1, \ldots, E_r) = q(E_1, \ldots, E_r)$$

for every r-tuple of *split* vector bundles $E_1, \ldots, E_r$ (over every base B), then

$$p(E_1, \ldots, E_r) = q(E_1, \ldots, E_r)$$

for every r-tuple of vector bundles $E_1, \ldots, E_r$.

Approach I: The first proof of the splitting principle is based on the following claim, which we have not proved.

Claim (basic version): For every vector bundle $E \longrightarrow B$, there exists a topological space $\tilde{B}$ and a continuous map $\pi\colon \tilde{B} \longrightarrow B$ such that the homomorphism

$$\pi^*\colon H^*(B) \longrightarrow H^*(\tilde{B})$$

is injective and the vector bundle $\pi^*E \longrightarrow \tilde{B}$ splits.

Claim (general version): For every r-tuple of vector bundles $E_1, \ldots, E_r \longrightarrow B$, there exists a topological space $\tilde{B}$ and a continuous map $\pi\colon \tilde{B} \longrightarrow B$ such that the homomorphism

$$\pi^*\colon H^*(B) \longrightarrow H^*(\tilde{B})$$

is injective and the vector bundle $\pi^*E_i \longrightarrow \tilde{B}$ splits for every $i=1, \ldots, r$.

Assuming the basic version of the claim, the basic Splitting Principle is proved as follows. Given a vector bundle $E \longrightarrow B$, let $\pi\colon \tilde{B} \longrightarrow B$ be as in the claim. Since $\pi^*E \longrightarrow \tilde{B}$ splits, $p(\pi^*E)=q(\pi^*E)$. Thus, by the naturality of p and q,

$$\pi^*p(E) = p(\pi^*E)=q(\pi^*E) = \pi^*q(E) \in H^{\Pi}(\tilde{B}).$$

Since the homomorphism π^* is injective, it follows that

$$p(E) = q(E) \in H^{\Pi}(B).$$

The general splitting principle is proved in exactly the same way using the general version of the claim.

Why is the claim true? For every vector bundle $E \longrightarrow B$, there exists a fibration $\mathbb{P}E \longrightarrow B$ called the projectivization of E. It is obtained by replacing each fiber E_b of E by its projectivization (taken over $\mathbb{C}$ or $\mathbb{R}$ as appropriate), i.e. $\mathbb{P}E_x$. Since a (linear) isomorphism between vector spaces V and W induces a diffeomorphism between $\mathbb{P}V$ and $\mathbb{P}W$, the linear trivialization (transition) maps for E induce trivialization (transition) maps for $\mathbb{P}E$. If k is the rank E, the fibers of the fibration $p\colon \mathbb{P}E \longrightarrow B$ are the $(k-1)$-dimensional projective spaces. Under our assumptions on the coefficient ring, the fibration $p\colon \mathbb{P}E \longrightarrow B$ admits a cohomology extension of the fiber

$$\theta\colon H^*(\mathbb{P}^{k-1}) \longrightarrow H^*(\mathbb{P}E).$$

This means that θ is a homomorphism such that

$$\iota_b^* \circ \theta\colon H^*(\mathbb{P}^{k-1}) \longrightarrow H^*(\mathbb{P}E_b)$$

is an isomorphism for every $b \in B$, where $\iota_b\colon E_b \longrightarrow E$ is the inclusion map. Thus, by the Thom Isomorphism Theorem, the homomorphism

$$\Phi\colon H^*(B) \otimes H^*(\mathbb{P}^{k-1}) \longrightarrow H^*(\mathbb{P}E), \qquad \alpha \otimes \beta \longrightarrow p^*(\alpha) \cup \theta(\beta),$$

is an isomorphism. In particular, the homomorphism

$$p^*\colon H^*(B) \longrightarrow H^*(\mathbb{P}E)$$

is injective. So, the key missing argument is a construction of θ. This is not very simple; in fact, θ need not be unique.

Let's assume the conclusion of the previous paragraph. Suppose $E \longrightarrow B$ is a vector bundle of rank k. Let

$$\pi_1\colon \mathbb{P}E \longrightarrow B$$

be its projectivization. The vector bundle $\pi_1^*E \longrightarrow \mathbb{P}E$ contains the tautological line bundle:

$$\gamma_1 \equiv \gamma = \big\{(\ell, v) \in \pi_1^*E \subset \mathbb{P}E \times E\colon v \in \ell \subset E_{\pi_1(\ell)}\big\}.$$

Since B is paracompact, we obtain a splitting

$$\pi_1^*E = E_1 \oplus \gamma_1,$$

where E_1 is a vector bundle of rank $k-1$. If $k=2$, we are done, as π_1^*E is a split vector bundle and the homomorphism

$$\pi_1^*\colon H^*(B) \longrightarrow H^*(\mathbb{P}E)$$

is injective. If $k>2$, let $\pi_2\colon \mathbb{P}E_1 \longrightarrow \mathbb{P}E$ be the projectivization of E_1. Then,

$$\pi_2^*E_1 = E_2 \oplus \gamma_2 \qquad \Longrightarrow \qquad \pi_2^*\pi_1^*E = E_2 \oplus \gamma_2 \oplus \pi_2^*\gamma_1$$

for some vector bundle $E_2 \longrightarrow \mathbb{P}E_2$ of rank $k-2$. After taking $k-1$ projectivizations, we obtain a fibration

$$\pi \equiv \pi_1 \circ \ldots \circ \pi_{k-1}\colon \tilde{B} \equiv \mathbb{P}E_{k-2} \longrightarrow B$$

such that

$$\pi^*E = E_{k-1} \oplus \gamma_{k-1} \oplus \pi_{k-1}^*\gamma_{k-2} \oplus \pi_{k-1}^*\pi_{k-2}^*\gamma_{k-3} \oplus \ldots \oplus \pi_{k-1}^* \ldots \pi_2^*\gamma_1$$

is a sum of line bundles and the homomorphism

$$\pi^* = \pi_{k-1}^* \circ \ldots \circ \pi_1^* : H^*(B) \longrightarrow H^*(\tilde{B})$$

is injective.

The last paragraph implies the basic case of the claim (assuming θ exists). The general case is obtained by repeating the same construction for vector bundles $E_2, \ldots, E_r$ pull-backed to $\tilde{B}$. So, we have to do the construction of the previous paragraph r times.

Approach II: Let $\gamma_k \longrightarrow \mathbb{G}_k$ be the tautological k-plane bundle. The second proof of the splitting principle is based on the following claim.

Claim (basic version): There exists a continuous map $f : (\mathbb{P}^\infty)^k \longrightarrow \mathbb{G}_k$ such that the homomorphism

$$f^*\colon H^*(\mathbb{G}_k) \longrightarrow H^*\big((\mathbb{P}^\infty)^k\big)$$

is injective and

$$f^*\gamma_k = (\gamma_1)^k = \bigoplus_{j=1}^{j=k} \pi_j^*\gamma_1 \longrightarrow (\mathbb{P}^\infty)^k.$$

Claim (general version): Let $k_1, \ldots, k_r$ be positive integers. There exists a continuous map

$$f\colon (\mathbb{P}^\infty)^{k_1+\ldots+k_r} \longrightarrow \mathbb{G}_{k_1} \times \ldots \times \mathbb{G}_{k_r}$$

such that the homomorphism

$$f^*\colon H^*\big(\mathbb{G}_{k_1} \times \ldots \times \mathbb{G}_{k_r}\big) \longrightarrow H^*\big((\mathbb{P}^\infty)^{k_1+\ldots+k_r}\big)$$

is injective and the vector bundle $f^*(\pi_i^*\gamma_{k_i}) \longrightarrow (\mathbb{P}^\infty)^{k_1+\ldots+k_r}$ splits for every $i = 1, \ldots, r$.

Assuming the basic version of this claim, the basic Splitting Principle is proved as follows. By assumption, $p(\gamma_1^k) = q(\gamma_1^k)$. Since $\gamma_1^k = f^*\gamma_k$, by naturality of p and q, we have

$$f^*p(\gamma_k) = p(\gamma_1^k) = q(\gamma_1^k) = f^*q(\gamma_k) \in H^\Pi\big((\mathbb{P}^\infty)^k\big).$$

Since the homomorphism f^* is injective, it follows that

$$p(\gamma_k) = q(\gamma_k) \in H^\Pi(\mathbb{G}_k).$$

If $E \longrightarrow B$ is any vector bundle of rank k, $E = g^*\gamma_k$ for some continuous map $g\colon B \longrightarrow \mathbb{G}_k$. Since $p(\gamma_k) = q(\gamma_k)$ and p and q are natural with respect to continuous maps,

$$p(E) = g^*p(\gamma_k) = g^*q(\gamma_k) = q(E) \in H^\Pi(B).$$

The general Splitting Principle follows in a similar way from the general version of the claim. In particular, we first obtain that

$$p\big(\pi_1^*\gamma_{k_1}, \ldots, \pi_r^*\gamma_{k_r}\big) = q\big(\pi_1^*\gamma_{k_1}, \ldots, \pi_r^*\gamma_{k_r}\big) \in H^\Pi\big(\mathbb{G}_{k_1} \times \ldots \times \mathbb{G}_{k_r}\big).$$

If $E_1, \ldots, E_r \longrightarrow B$ are vector bundles of ranks $k_1, \ldots, k_r$, respectively, for each i there exists a continuous map $g_i\colon B \longrightarrow \mathbb{G}_{k_i}$ such that $E_i = g_i^*\gamma_{k_i}$. Let

$$g = g_1 \times \ldots \times g_r\colon B \longrightarrow \mathbb{G}_{k_1} \times \ldots \times \mathbb{G}_{k_r}.$$

Since $g_i = \pi_i \circ g$, $E_i = g^*\pi_i^*\gamma_{k_i}$. Thus, by the naturality of p and q, we have

$$\begin{aligned} p\big(E_1, \ldots, E_r\big) &= p\big(g^*\pi_1^*\gamma_{k_1}, \ldots, g^*\pi_r^*\gamma_{k_r}\big) = g^*p\big(\pi_1^*\gamma_{k_1}, \ldots, \pi_r^*\gamma_{k_r}\big) \\ &= g^*q\big(\pi_1^*\gamma_{k_1}, \ldots, \pi_r^*\gamma_{k_r}\big) = q\big(g^*\pi_1^*\gamma_{k_1}, \ldots, g^*\pi_r^*\gamma_{k_r}\big) \\ &= q\big(E_1, \ldots, E_r\big) \in H^\Pi(B), \end{aligned}$$

as needed.

In contrast to the key claim in Approach I, the key claim in Approach II has been proved. Recall that the last condition on f in the basic case of the latter claim determines $f\colon (\mathbb{P}^\infty)^k \longrightarrow \mathbb{G}_k$ up to homotopy and thus the homomorphism

$$f^*\colon H^*(\mathbb{G}_k) \longrightarrow H^*\big((\mathbb{P}^\infty)^k\big)$$

uniquely (see Milnor's Theorems 5.6, 5.7 for the real case; Theorem 14.6 for the complex case). Furthermore,

$$H^*(\mathbb{G}_k) \approx R[c_1, \ldots, c_k],$$

where $c_i = c_i(\gamma_k)$ in the complex case and $c_i = w_i(\gamma_k)$ in the real case (see 14.5 and 7.1, respectively). By the product formula for chern (Stiefel-Whitney) classes,

$$f^* c_i \in H^*\big((\mathbb{P}^\infty)^k\big) \approx R[a_1, \ldots, a_k]$$

is the i-th elementary symmetric polynomials in $a_1, \ldots, a_k$, where $a_i = \pi_i^* c_1(\gamma_1)$ in the complex case and $a_i = \pi_i^* w_1(\gamma_1)$ in the real case. Since the k elementary symmetric polynomials $\sigma_1, \ldots, \sigma_k$ are algebraically independent in $R[a_1, \ldots, a_k]$, it follows that f^* is injective.

The general case of the claim follows from the basic one and the Kunneth formula (see Theorem A.6). We can simply take

$$f = f_{k_1} \times \ldots \times f_{k_r}\colon (\mathbb{P}^\infty)^{k_1} \times \ldots \times (\mathbb{P}^\infty)^{k_r} \longrightarrow \mathbb{G}_{k_1} \times \ldots \times \mathbb{G}_{k_r},$$

where $f_{k_i}\colon (\mathbb{P}^\infty)^{k_i} \longrightarrow \mathbb{G}_{k_i}$ are the maps provided by the basic case of the claim. We then find that

$$f^* \pi_i^* \gamma_{k_i} = p_i^* \gamma_1^{k_i} \longrightarrow (\mathbb{P}^\infty)^{k_1} \times \ldots \times (\mathbb{P}^\infty)^{k_r},$$

where

$$p_i\colon (\mathbb{P}^\infty)^{k_1} \times \ldots \times (\mathbb{P}^\infty)^{k_r} \longrightarrow (\mathbb{P}^\infty)^{k_i}$$

is the projection onto the i-th factor.

This second proof also shows that for the basic Splitting Principle it is sufficient to require that

$$p(\gamma_1^k) = q(\gamma_1^k) \in H^{\Pi}\big((\mathbb{P}^\infty)^k\big),$$

i.e. just for the split bundle $E = \gamma_1^k$. In the general case, we need to check only that

$$p\big(p_1^* \gamma_1^{k_1}, \ldots, p_r^* \gamma_1^{k_r}\big) = q\big(p_1^* \gamma_1^{k_1}, \ldots, p_r^* \gamma_1^{k_r}\big) \in H^{\Pi}\big((\mathbb{P}^\infty)^{k_1} \times \ldots \times (\mathbb{P}^\infty)^{k_r}\big).$$

Application: If $E \longrightarrow B$ is a complex vector bundle of rank k,

$$c_1\big(\Lambda_{\mathbb{C}}^{\mathrm{top}} E\big) \equiv c_1\big(\Lambda_{\mathbb{C}}^k E\big) = c_1(E).$$

If E is real, this equality holds with c_1 and $\Lambda_{\mathbb{C}}$ replaced by w_1 and $\Lambda_{\mathbb{R}}$.

For every vector bundle $E \longrightarrow B$ (over every paracompact base B), let

$$p(E) = c_1\big(\Lambda_{\mathbb{C}}^{\mathrm{top}}E\big) \in H^2(B;\mathbb{Z}) \qquad \text{and} \qquad q(E) = c_1(E) \in H^2(B;\mathbb{Z}).$$

If $f\colon B' \longrightarrow B$ is any continuous map and $E \longrightarrow B$ is a vector bundle of rank k, then

$$\begin{aligned} p(f^*E) &\equiv c_1\big(\Lambda_{\mathbb{C}}^{\mathrm{top}}(f^*E)\big) = c_1\big(f^*(\Lambda_{\mathbb{C}}^{\mathrm{top}}E)\big) = f^*c_1\big(\Lambda_{\mathbb{C}}^{\mathrm{top}}E\big) \equiv f^*p(E) \in H^2(B';\mathbb{Z}); \\ q(f^*E) &\equiv c_1\big(f^*E\big) = f^*c_1(E) \equiv f^*q(E) \in H^2(B';\mathbb{Z}). \end{aligned}$$

Thus, p and q are natural with respect to smooth maps.

On the other hand, if $E = L_1 \oplus \ldots \oplus L_k$ is a sum of line bundles, then

$$\begin{gathered} \Lambda_{\mathbb{C}}^{\mathrm{top}}E = L_1 \otimes \ldots \otimes L_k \qquad p(E) = c_1\big(L_1 \otimes \ldots \otimes L_k\big) = c_1(L_1) + \ldots + c_1(L_k); \\ c(E) = \big(1 + c_1(L_1)\big) \ldots \big(1 + c_1(L_k)\big) \qquad \Longrightarrow \qquad q(E) \equiv c_1(E) = c_1(L_1) + \ldots + c_1(L_k). \end{gathered}$$

Thus, $p(E) = q(E)$ for every split vector bundle E of rank k. Since p and q are natural with respect to continuous maps, it follows that $p(E) = q(E)$ for every vector bundle E of rank k.

Previous Applications: Your first application of the Splitting Principle was to compute $w(E \otimes F)$ (Problem 7-C). In this case, $r = 2$. You used the same case to check that $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\mathrm{ch}(F)$ (Problem 16-B). The basic Splitting Principle was used to express $e(\mathrm{Sym}^2 E)$ and $e(\mathrm{Sym}^3 E)$ for a complex rank-two vector bundle E in terms of the chern classes of E (Problem v).