MAT 566: Differential Topology

Problem Set 7

Here is a final collection of exercises, which will not be collected: 19-A, 19-B, x, xi, xii below.

Problem (x). For this problem, assume Hurewicz Theorem (standard and torsion versions), homotopy l.e.s. for fibration, and that $\pi_i(S^{2n-1})$ is finite unless $i=2n-1$. Let

$$W^{2n-1} = \big\{(v,w)\in\mathbb{R}^{n+1}\colon |v|,|w|=1,\ v\perp w\big\}.$$

(a) If n is odd, to what simpler space is W^{2n-1} diffeomorphic to? What is its homology?
(b) Suppose n is even. Determine the cohomology and homology of W^{2n-1}, at least mod torsion. Determine $\pi_i(W^{2n-1})$ for $i\le 2n-1$, at least mod torsion. Determine $\pi_i(S^n)$ for $i\le 2n-1$, at least mod torsion.

Problem (xi). Suppose $n\ge 2$, $\pi_i(X)=0$ and $\pi_i(Y)=0$ for all $i<n$, and the Hurewicz homomorphisms

$$h_i\colon \pi_i(X)\longrightarrow H_i(X;\mathbb{Z}) \qquad \text{and} \qquad \pi_i(Y)\longrightarrow H_i(Y;\mathbb{Z})$$

are isomorphisms mod torsion for all $i<2n-1$. Show that

$$h_i\colon \pi_i(X\vee Y)\longrightarrow H_i(X\vee Y;\mathbb{Z})$$

is also an isomorphism mod torsion for all $i<2n-1$.
Note: This is part of the proof of Theorem 18.3, but the book's proof is rather sketchy.

Problem (xii). Show that $\pi_n(S^1\vee S^n)$ is not finitely generated for all $n\ge 2$.